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SMALL UNIVERSAL COVERS FOR SETS OF UNIT DIAMETER

ABSTRACT. The current smallest convex universal cover for sets of unit diameter is described. This reduction of Sprague's cover is by $4 \cdot 10^{-11}$ and results in an asymmetrical cover. Another small universal cover of sets of unit diameter with an axis of symmetry reduces Sprague's cover by 0.0019. An indication is given of how to use computers in the solution of this kind of problem.

1. INTRODUCTION

In 1914 Lebesgue posed the problem of finding a universal cover of least area in the plane, which is a set having a subset congruent to any given plane set of unit diameter.

In the majority of work on this problem the cover is required to be convex, in which case a solution must exist. The most successful tradition is based on a regular hexagon, which has been reduced by cutting off the corners: J. Pál in 1920 [1]; R. Sprague in 1936 [2]; and H. C. Hansen in 1975 [3].

A further reduction was obtained by G. F. D. Duff in 1980 [4], who, for the first time, introduced a non-convex cover.

In this paper I shall remain in the convex tradition and make an improvement on my 1975 result using almost the same line of reasoning. Just as Sprague was mistaken in believing his cover to be minimal, I was wrong in conjecturing the minimality of my 1975 cover. I proposed a general conjecture stating that Reuleaux n-gons can be covered inside the regular hexagon in essentially only one way, at least for n less than 10. Computer simulation indicates that the conjecture is correct for $n=3$, but for $n=5$ and upwards an n-gon can often be covered in at least three essentially different positions.

I shall give a few hints showing how to make computer simulations on Lebesgue's problem. Furthermore I shall prove that an essential reduction of the area is possible if we restrict ourselves to sets with an axis of symmetry, which is a slight improvement on my result of 1981 [5].

2. A SMALL CONVEX UNIVERSAL COVER

Let $ABCDEF$ be a regular hexagon circumscribed about a circle of unit diameter. Inside the hexagon, $A_1A_2B_1B_2C_1C_2D_1D_2E_1E_2F_1F_2$ is the regular 12-gon circumscribed about the same circle, as shown in Figure 1.

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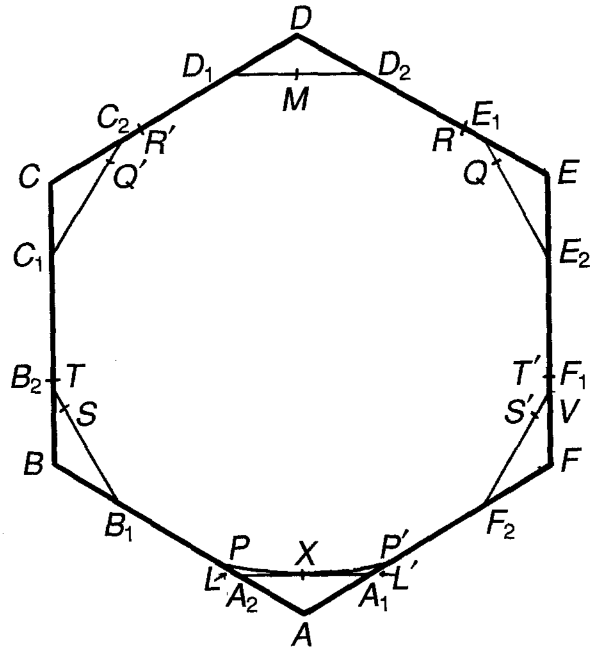

Fig. 1.

Pál proved that the hexagon with the corners B_1B_2B and F_1F_2F removed was a universal cover. Sprague observed that near D the part that is outside the circle of radius 1 and centre B_1 could be removed as well as what is outside the circle of radius 1 and centre F_2. I proved that two tiny corners at B_2 and F_1 could be removed. However, my result can be improved:

Let M be the midpoint of D_1D_2. Let the circle of radius 1 and centre in M intersect AB in P and AF in P'. The circle of radius 1 and centre P is tangent to DE in R and intersects E_1E_2 in Q. The circle of radius 1 and centre Q intersects BC in T and is tangent to B_1B_2 in S. We obtain R', Q', T', S' by a symmetric construction about the axis DA. We shall prove that either the area B_2TS or $F_1T'S'$ can be removed from the cover of Sprague. In doing so we shall restrict ourselves to sets of constant width 1, which is permissible as every set of unit diameter is contained in a set of constant width.

Let $\mathscr{F}$ be a set of constant width 1. $\mathscr{F}$ can be covered by Sprague's cover. Suppose that $\mathscr{F}$ has a point H in the interior of B_2TS. Let the circle of radius 1 and centre H enter (anti-clockwise) the triangle EE_1E_2 at a point I on E_1E_2 and leave at a point J, which may be on E_1E (as shown in Figure 2) or on E_1E_2.

Let $\mathscr{G}$ be the intersection of the closed disk of radius 1 and centre H with

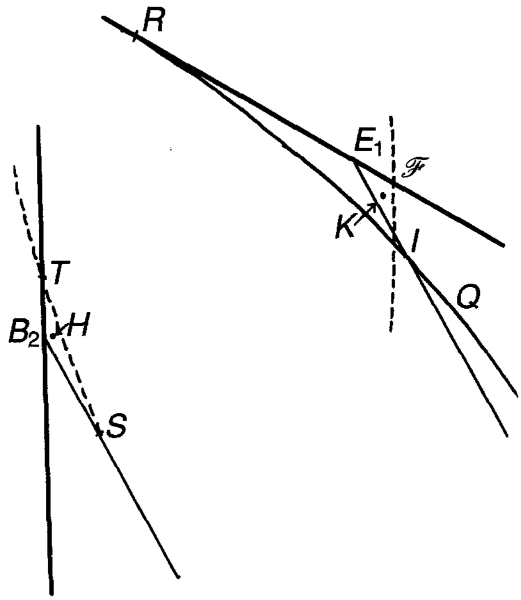

Fig. 2.

the closed triangle EE_1E_2. As $HQ > 1$, I is on E_1Q. If J is on E_1E (as in Figure 2) $\mathscr{G}$ is obviously outside the circle arc RQ. If J is on E_1E_2 then J is on E_1I and again $\mathscr{G}$ is obviously outside the circle arc RQ.

However, $\mathscr{F}$ has some point K in the closed triangle EE_1E_2 as $\mathscr{F}$ of constant width has no points in the triangle BB_1B_2 and the distance between the parallel lines B_1B_2 and E_1E_2 is 1. Because H is a point of $\mathscr{F}$, K must be inside the set $\mathscr{G}$, i.e. K is outside the circle arc RQ or, expressed otherwise, P is outside the circle of radius 1 and centre K. However, $\mathscr{F}$ of constant width must have a point L on AB which thus must be in the open section PA in order not to have a distance to K exceeding 1.

By the exact same argument we can prove that if $\mathscr{F}$ has points in the interior of $F_1T'S'$ then $\mathscr{F}$ has a point L' on the interior of AP'. So if $\mathscr{F}$ has points in both $F_1T'S'$ and B_2TS then the circle arc of radius 1 connecting L and L' has a part inside the triangle AA_1A_2, because the arc connecting P and P' is tangent to A_1A_2 and the new arc is in a lower position. As a figure of constant width 1 always has a circle arc of radius 1 connecting two of its points as a subset, $\mathscr{F}$ must have points in AA_1A_2. This means that $\mathscr{F}$ has no points inside the opposite triangle DD_1D_2.

We can now rotate $\mathscr{F}$ by 120 degrees anti-clockwise about the centre of the hexagon to $\mathscr{F}'$, in which position it is still covered by the hexagon. As $\mathscr{F}$ had no points in DD_1D_2 or in BB_1B_2, $\mathscr{F}'$ will have no points in BB_1B_2 or in FF_1F_2. Nor will $\mathscr{F}'$ have points in B_2TS or $F_1T'S'$, as $\mathscr{F}$ is far from B_1 and D_2: $\mathscr{F}$ curves away from AB at least as fast as the circle arc of radius 1 tangent to AB in L, giving $\mathscr{F}$ a safe distance to B and the above argument shows that the midpoint X of A_1A_2 is in $\mathscr{F}$ which keeps $\mathscr{F}$ inside the circle of radius 1 and centre X, giving a safe distance to D_2. As Sprague's reduction only depends on the fact that a set of constant width has points on AB and on AF, we can conclude that B_2TS can be removed from Sprague's cover.

The argument does not permit us to remove $F_1T'S'$ as well. However we can, near F_1, remove what is outside the circle of radius 1 and centre T, as any figure of constant width covered by our cover has a point on the side TC. If we let the intersection of this circle with F_1F_2 be V and we make a numerical calculation of the whole construction, the final result is:

REDUCED CONVEX UNIVERSAL COVER. Sprague's cover can be reduced by two areas in the corners at B_2 and F_1 in Figure 1: B_2TS and $F_1T'V$.

B_2TS: B_2S and B_2T are line segments of lengths $6 \cdot 10^{-4}$ and $4 \cdot 10^{-7}$ resp. TS is a circle arc of radius 1 tangent to B_2S at S. The area is $4 \cdot 10^{-11}$.

$F_1T'V$: F_1T' and F_1V are line segments of lengths $4 \cdot 10^{-7}$ and $2 \cdot 10^{-10}$ resp. $T'V$ is a circle arc of radius 1 tangent to $T'F$ at T'. The area is $6 \cdot 10^{-18}$.

Comments on the result. The area of $6 \cdot 10^{-18}$ is the one I found in 1973 as the size of either of the two corners that could be cut off. The improvement to $4 \cdot 10^{-11}$ for one corner is obtained by giving up the idea that the cover should be symmetrical. Duff also introduced an asymmetrical and even smaller non-convex cover in 1980. This indicates that further progress should look for asymmetrical covers.

As to the set-theoretical minimality of the present cover, I doubt it. The figure that suggests itself to span the cover near the new cut-offs can be covered in three essentially different positions inside the hexagon – that is, in a total of 18 positions, some of which allow bigger cut-offs even after slight alterations to the figure.

3. Computer simulation of the cover problem

It is my guess that for further progress in Lebesgue's cover problem better insight into the behaviour of figures of constant width is required. Computer simulation could be of importance in finding new results and in testing hypotheses. In fact it is not difficult to set up a simulation system like this, and I shall indicate how it is done if we focus on the following problem:

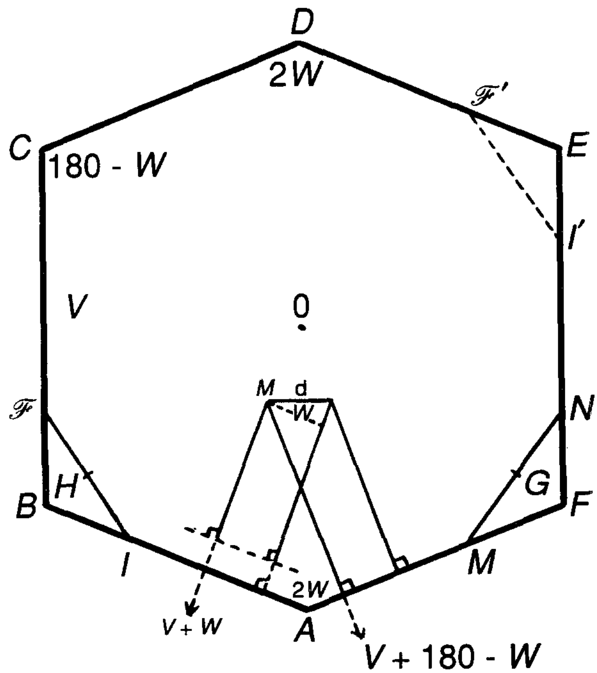

Fig. 3.

Let, as in Figure 3, $ABCDEF$ be a hexagon circumscribed about the circle of diameter 1 and centre O. The angles at A and D are $2w$, while the remaining angles are $180-w$. Let IJ be a tangent to the circle at the point H on OB and let MN be a tangent at the point G on OF. We would like to investigate whether $\mathscr{H} = AIJCDENM$ is a universal cover of figures of unit diameter. If $w=60$ degrees we obtain Pal's cover. If, however, we find covers for other values of w, such as $w=64$, we can reduce Pal's cover, as I shall prove in the last section.

3.1. *Input of Figures*

Any figure of constant width unity can be approximated by a Reuleaux n-gon, whence we can restrict ourselves to consider such figures.

Start with one diameter $P(2)P(1)$ in a coordinate system. $P(2)=(0,0)$ and $P(1)=(1,0)$ as in Figure 4. For a given or randomly selected set of angles $v(i)$ we determine the coordinates $(x(i), y(i))$ of the vertices $P(i)$ as follows, starting with $u=0$ and $i=1$:

Repeat until distance $(P(i), P(1)) > 1$ or distance $(P(i), P(2)) > 1$

$$i = i+1,\ u = u + v(i).$$

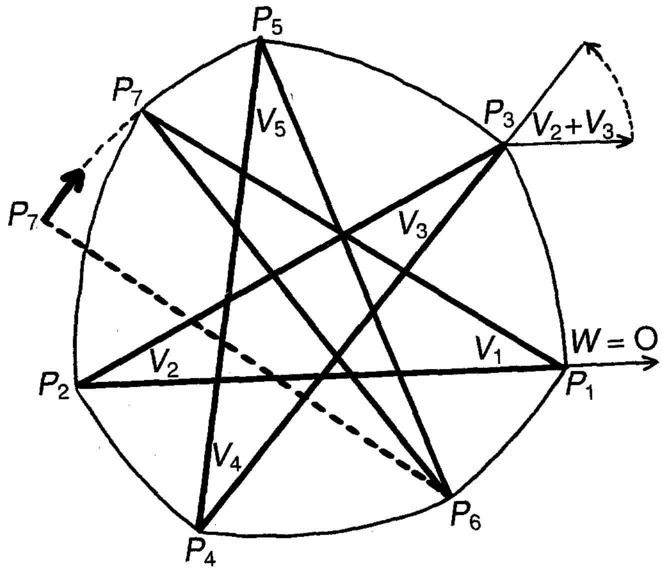

Fig. 4.

If i is odd, let

$$x(i) = x(i-1) + \cos(u), \qquad y(i) = y(i-1) + \sin(u+180).$$

If i is even, let

$$x(i) = x(i-1) + \cos(u+180), \qquad y(i) = y(i-1) + \sin(u+180).$$

End repeat/loop.

If one of the distance conditions in the repeat-line is met we reduce $v(i)$ to make that distance equal to unity, calculate $P(i+1)$, $v(i+1)$ and $v(0)$ and the input polygon $\mathscr{P}$ is now constructed. $P(1)$ may disappear as a vertex, but can be formally kept as such with $v(1) = 0$.

3.2. *The Support Function*

We can chose $M = (0.5, 0)$, the midpoint of $P(1)P(2)$, as the base point for our support function, $d(v)$, which measures the distance from M to the tangent of the polygon in direction perpendicular to the v-direction. In the direction of $P(2)$, v is put equal to 0. Comparing a given v to the stepwise accumulated sums $v(2) + v(3) + \cdots$ tells us which side or vertex of the polygon is touched by the tangent; then it is easy to calculate the point of contact, and as the tangent has slope $= -\cot(v)$ we can easily calculate the distance from M to the tangent, giving us $d(v)$ as a subroutine in the program.

3.3. *The Covering Condition*

We now want to find out if the polygon $\mathscr{P}$ we put into the program can be covered by $\mathscr{H}$. The computational information is contained in the following theorem, which is also of purely theoretical interest: $h=(1/\cos(w/2)-1)/2$.

THEOREM 1. *Let*

$$f(v)=d(v+w)-d(v+180-w)-\cos(w)(d(v)-d(v+180)),$$

$$g(v)=d\left(v+\frac{w}{2}\right)-\frac{d(v)+d(v+w)}{2\cos(w/2)}+h,$$

$$h(v)=d\left(v+180-\frac{w}{2}\right)-\frac{d(v+180-w)+d(v+180)}{2\cos(w/2)}+h.$$

If $f(v)=0$, $g(v)\leqslant 0$ and $h(v)\leqslant 0$ for some v then and only then can the Reuleaux polygon $\mathscr{P}$ be covered by $\mathscr{H}$.

Proof. We shall in fact prove that the polygon can be covered in a position where BC in Figure 3 is perpendicular to the direction v and tangent to the polygon. $\mathscr{P}$ can certainly be accommodated with its v-diameter between BC and EF, and we can suppose that FA is tangent to $\mathscr{P}$.

We then must prove that the condition $f(v)=0$ is equivalent to the statement that BA is a tangent to $\mathscr{P}$ as well, which will prove that $\mathscr{P}$ is accommodated inside the hexagon.

First we observe that $f(v)$ is independent of the choice of M. Referring to Figure 3, a vertical displacement d of M only affects $d(v+w)$ and $d(v+180-w)$, to both of which will be added $d\cdot\sin(w)$, which will not change $f(v)$. A horizontal displacement d of M will increase the difference $d(v)-d(v+180)$ by $2d$ and increase the difference $d(v+w)-d(v+180-w)$ by $2d\cdot\cos(w)$, as is easily seen. Again $f(v)$ remains unchanged. Hence we can chose M to be the centre O of the circle inscribed in the hexagon. In that case $d(v)=d(v+180)=0.5$ and $d(v+180-w)=0.5$ as FA was tangent to $\mathscr{P}$. The condition $f(v)=0$ then reduces to $d(v+w)=0.5$, which means that AB is tangent to $\mathscr{P}$ as we set out to prove.

Secondly, we prove that the condition $g(v)\leqslant 0$ ensures that we can remove the corner triangle BIJ. Again we start by observing that $g(v)$ is independent of the choice of M considering displacements parallel to IJ and perpendicular to IJ separately. Thus, we can choose $M=0$ in which case $d(v)=0.5$ and $d(v+w)=0.5$ as BC and AB are tangents to the circle. $g(v)\leqslant 0$ now reduces to $d(v+w/2)-(0.5+0.5)/(2\cos(w/2))+0.5(1/\cos(w/2)-1)\leqslant 0$ or $d(v+w/2)\leqslant 0.5$, which is equivalent to the statement that $\mathscr{P}$ has no points in BIJ.

That $h(v)\leqslant 0$ permits us to remove the triangle NFH is proved in exactly the same way.

3.4. *Results of a Simulation*

If $\mathscr{H}$ is a universal cover of figures of unit diameter for some w in the interval $(60, 66)$ then Pal's cover can be reduced, as a calculation of the area with Sprague's improvements will reveal. However, a computer simulation using Theorem 1 falsifies the hypothesis of any universal cover $\mathscr{H}$ in this interval. Using random input it can take hundreds of different input figures before such a hypothesis is falsified for a given w. The figure found as counter-example for a given w is, however, also a counter-example in a certain interval around w, because the conditions in Theorem 1 are continuous as functions of w. In this way it is possible to falsify the hypothesis for the whole interval.

The simulation also tells us that, for $n > 3$, an n-gon is sometimes covered in at least three essentially different positions in the regular hexagon ($w = 60$). In fact this is a purely theoretical consequence of Theorem 1: The regular pentagon with labelled vertices is covered in five different positions but they are all equal/symmetrical from a geometrical point of view. Each position corresponds to a v for which $f(v) = 0$, $g(v) < 0$ and $h(v) < 0$; moreover, $f'(v) \neq 0$. As $d(v)$ is continuous as a function of $\mathscr{P}$, so are $f(v)$, $h(v)$ and $g(v)$. It then follows from Theorem 1 that any figure of constant width close to the regular pentagon can also be covered in five different positions, and these will be essentially different, if the figure has no symmetry.

4. A small universal cover for symmetric figures

If we restrict ourselves to cover figures with an axis of symmetry we can find universal covers of substantially smaller areas. In the above notation we can state the main result.

THEOREM 2. *$\mathscr{H}$ is a universal cover of symmetric figures of unit diameter for any positive w less than or equal to 90 degrees.*

Proof. As a symmetric figure of unit diameter is a subset of a symmetric figure of constant width unity, we only have to prove the theorem for these figures.

Let $\mathscr{F}$ be a symmetric figure of constant width unity. It can obviously be placed inside the hexagon in Figure 3, if we put the axis of symmetry along AD. If $\mathscr{F}$ has points in the triangle IJB then it cannot have points in the triangle $I'J'E$ symmetrically situated about O, as IJ and $I'J'$ are parallels at unit distance. Because of the horizontal symmetry of the hexagon we can thus assume that $\mathscr{F}$ has no points in IJB. As $\mathscr{F}$ is symmetric about AD it does not have points in MNF either, i.e. is covered by $\mathscr{H}$ for any w.

THEOREM 3. *The minimal area of* $\mathscr{H}$ *with Sprague's cut-offs at D is* 0.84236 *and is obtained for* $w = 63.5$ *degrees.*

Proof. As $\mathscr{F}$ has no points on BI or FM in Figure 3, it cannot, near D, have points outside the circle of radius 1 and centre I or outside the circle with centre M. These are the Sprague cut-offs. The numerical calculation is elementary.

This result is an essential reduction of Sprague's minimum of 0.84414. The result can be slightly improved. In $\mathscr{H}$ the two isosceles triangles BJI and FNM have base angles equal to $w/2$. If we still let IJ and MN be tangents of the circle but reduce the lower angles JIB and NMF by v, we can obtain Sprague cut-offs larger than the reduction in the area of the triangles. The referee pointed out to me that cut-offs are also possible at the vertices C and E in Figure 3, as the angle here is larger than 120 degrees. In fact we can use Sprague's reasoning: Since any figure of constant width covered by $\mathscr{H}$ has points on BA and on AF, it cannot have points outside the circle of radius 1 and centre A in the corners at C and E. It is not difficult to make a small computer program calculating the resulting area as function of w and v. The most favourable situation, obtained for $w = 63.776$ and $v = 1.86$, gives an area of 0.842203809, which could be the minimal area for covers of symmetric figures of unit diameter.

As stated in Section 3, these covers are not universal covers for figures of unit diameter in general.

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