MOCK ϑ-FUNCTIONS AND REAL ANALYTIC MODULAR FORMS

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Abstract. In this paper we examine three examples of Ramanujan's third order mock ϑ-functions and relate them to Rogers' false ϑ-series and to a real-analytic modular form of weight 1/2.

1. Introduction

Mock ϑ-functions were introduced by S. Ramanujan in the last letter he wrote to G.H. Hardy, dated January, 1920. For a photocopy of the mathematical part of this letter see [Ra, pp. 127–131] (also reproduced in [A2]). In this letter he provided a list of 17 mock ϑ-functions (4 of "order three", 10 of "order five" and 3 of "order seven"), together with identities they satisfy.

In [AH] we find a definition of the concept of a mock ϑ-function. Slightly rephrased it reads: a mock ϑ-function is a function f of the complex variable q, defined by a q-series of a particular type (Ramanujan calls this the Eulerian form), which converges for $|q| < 1$ and satisfies the following conditions:

(1) infinitely many roots of unity are exponential singularities,
(2) for every root of unity ξ there is a ϑ-function $\vartheta_\xi(q)$ such that the difference $f(q) - \vartheta_\xi(q)$ is bounded as $q \to \xi$ radially,
(3) there is no ϑ-function that works for all ξ, i.e. f is not the sum of two functions, one of which is a ϑ-function and the other a function which is bounded in all roots of unity.

(When Ramanujan refers to ϑ-functions, he means sums, products, and quotients of series of the form $\sum_{n\in\mathbf{Z}} \epsilon^n q^{an^2+bn}$ with $a, b \in \mathbf{Q}$ and $\epsilon = -1, 1$).

The 17 functions given by Ramanujan indeed satisfy condition (1) and (2) (see [W1], [W2] and [S]). However no proof has ever been given that they also satisfy condition (3). Watson (see [W1]) proved a very weak form of condition (3) for the "third order" mock ϑ-functions, namely, that they are not equal to ϑ-functions.

In section 3 we will see that condition (3) is not satisfied if we weaken it slightly. Indeed, we shall discuss a vector-valued third order mock ϑ-function F for which there is a real analytic modular form H such that $F - H$ is bounded in all roots of unity.

Before that, we discuss in the next section a connection between mock ϑ-functions and Rogers' false ϑ-series. Again we look at the behaviour of a mock theta function when q approaches a root of unity radially. But now we extend the function across the unit circle.

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2. False ϑ-series

We will consider the mock ϑ-function ν, which is not mentioned in Ramanujan's letter, but which was found by Watson in [W1], and can also be found in Ramanujan's "lost" notebook [Ra]:

$$\nu(q) = \sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(-q;q^2)_{n+1}} = \frac{1}{1+q} + \frac{q^2}{(1+q)(1+q^3)} + \frac{q^6}{(1+q)(1+q^3)(1+q^5)} + \ldots \tag{2.1}$$

We can easily see that the defining sum for ν converges not only for $|q| < 1$, but also for $|q| > 1$. We will now study the function that is defined by the sum outside the unit disk. In order to do so, we replace q by q^{-1} in the sum, take $|q| < 1$ and call this new function ν_-. We get

$$\nu_-(q) = \sum_{n=0}^{\infty} \frac{q^{n+1}}{(-q;q^2)_{n+1}} = \frac{q}{1+q} + \frac{q^2}{(1+q)(1+q^3)} + \frac{q^3}{(1+q)(1+q^3)(1+q^5)} + \ldots \tag{2.2}$$

In Ramanujan's "lost" notebook [Ra] we find the following identity for $|q| < 1$ (which was proved by Andrews in [A1]):

$$\nu_-(q) = \sum_{n=0}^{\infty} (-1)^n q^{6n^2+4n+1}(1+q^{4n+2}) \tag{2.3}$$

$$= \left(\sum_{n=0}^{\infty} - \sum_{n=-\infty}^{-1} \right) (-1)^n q^{6n^2+4n+1} = q^{\frac{1}{3}} \sum_{n=0}^{\infty} (-1)^{n+1} \left(\frac{-3}{n} \right) q^{\frac{2}{3}n^2} \tag{2.4}$$

From these identities we see that ν_- has a very simple power series expansion. This expansion looks very much like a ϑ-function, only the signs are somewhat different. Rogers uses the term *false ϑ-series* for this type of functions (see [Ro, pp. 328]).

The following proposition (see [LZ]) shows that for every root of unity ξ the function ν_- is bounded as $q \to \xi$ radially. We can even compute the complete asymptotic expansion.

Proposition 2.1. *Let $C : \mathbf{Z} \to \mathbf{C}$ be a periodic function with mean value 0. Then the associated L-series $L(s,C) = \sum_{n=1}^{\infty} C(n) n^{-s}$ $(\mathrm{Re}(s) > 1)$ extends holomorphically to $\mathbf{C}$. The two functions $\sum_{n=1}^{\infty} C(n) e^{-nt}$ and $\sum_{n=1}^{\infty} C(n) e^{-n^2 t}$ $(t > 0)$ have the asymptotic expansions*

$$\sum_{n=1}^{\infty} C(n) e^{-nt} \sim \sum_{r=0}^{\infty} L(-r,C) \frac{(-t)^r}{r!}, \qquad \sum_{n=1}^{\infty} C(n) e^{-n^2 t} \sim \sum_{r=0}^{\infty} L(-2r,C) \frac{(-t)^r}{r!} \tag{2.5}$$

as $t \searrow 0$. The numbers $L(-r,C)$ are given explicitly by

$$L(-r,C) = -\frac{M^r}{r+1} \sum_{n=1}^{M} C(n) B_{r+1}\left(\frac{n}{M}\right) \qquad (r = 0, 1, \ldots) \tag{2.6}$$

where $B_k(x)$ denotes the k^{th} Bernoulli polynomial and M is any period of the function C.

In order to get the asymptotic expansion of ν_- as $q \to \xi$ radially, with ξ a root of unity, we write $q = \xi e^{-t}$. Thus we have to find the asymptotic expansion of $\sum_{n=0}^{\infty}(-1)^{n+1}\left(\frac{-3}{n}\right)\xi^{\frac{2}{3}n^2}e^{-\frac{2}{3}tn^2}$ as $t \searrow 0$. We can now use the proposition provided we check that $C(n) := (-1)^{n+1}\left(\frac{-3}{n}\right)\xi^{\frac{2}{3}n^2}$ is a periodic function with mean value 0. Indeed, if K is the order of ξ then $6K$ is a period for C, while $C(6K-n) = -C(n)$. Hence the mean value of C is zero.

The behaviour of ν outside the unit circle is thus completely known. A question that now arises is whether the behaviour of ν outside the unit circle is related to the behaviour of ν inside the unit circle. Numerical computations in this and related examples led me to the following:

Conjecture 2.2. *If ξ is a root of unity where ν is bounded (as $q \to \xi$ radially inside the unit circle), for example $\xi = 1$, then ν is C^∞ over the line radially through ξ.*

If ξ is a root of unity where ν is not bounded, for example $\xi = -1$, then the asymptotic expansion of the bounded term in condition (2) in the introduction is the same as the asymptotic expansion of ν as $q \to \xi$ radially outside the unit circle.

Let us proceed a bit, assuming this conjecture. Let $\tilde{\nu}$ be a function which is defined in- and outside the unit circle and also at all roots of unity, such that (a) $\tilde{\nu}$ is holomorphic in- and outside the unit circle, (b) $\tilde{\nu}$ is C^∞ over all radial lines through roots of unity and (c) $\tilde{\nu} = \nu$ outside the unit circle. If we can find such a function $\tilde{\nu}$, then $\nu - \tilde{\nu}$ is zero outside the unit circle, it has asymptotic expansion zero for $q \to \xi$ if ξ is a root of unity where ν is bounded, and the bounded term in condition (2) for mock ϑ- functions also has asymptotic expansion zero for $q \to \xi$. Because of this one might expect $\nu - \tilde{\nu}$ to be modular. If indeed this is the case we have written ν as the sum of two functions $\nu - \tilde{\nu}$ and $\tilde{\nu}$, one of which is a ϑ-function and the other a function which is bounded in all roots of unity. This contradicts condition (3) in the definition of a mock ϑ-function.

Ramanujan probably had this idea in mind when he wrote in his letter to Hardy: "... I have constructed a number of examples in which it is inconceivable to construct a ϑ-function to cut out the singulartities of the original function. Also I have shown that if *it is necessarily so* then it leads to the following assertion—viz. it is possible to construct two power series in x, namely $\sum a_n x^n$ and $\sum b_n x^n$, both of which have *essential singularities* on the unit circle, are convergent when $|x| < 1$, and tend to *finite limits at every point* $x = e^{2i\pi r/s}$, and that at the same time the limit of $\sum a_n x^n$ at the point $x = e^{2i\pi r/s}$ is equal to the limit of $\sum b_n x^n$ at the point $x = e^{-2i\pi r/s}$."

Although it's possible to construct two such power series (see [A2, pp. 284]), it might not be possible to construct a function $\tilde{\nu}$ that satisfies the conditions (a), (b) and (c).

3. Mock ϑ-Functions and Real Analytic Modular Forms

In this section we will consider the following third order mock ϑ-functions:

$$\begin{aligned} f(q) &= \sum_{n=0}^{\infty} \frac{q^{n^2}}{(-q;q)_n^2} \\ &= 1 + \frac{q}{(1+q)^2} + \frac{q^4}{(1+q)^2(1+q^2)^2} + \dots \\ \omega(q) &= \sum_{n=0}^{\infty} \frac{q^{2n^2+2n}}{(q;q^2)_{n+1}^2} \\ &= \frac{1}{(1-q)^2} + \frac{q^4}{(1-q)^2(1-q^3)^2} + \frac{q^{12}}{(1-q)^2(1-q^3)^2(1-q^5)^2} + \dots \end{aligned} \tag{3.1}$$

Ramanujan mentioned f in his letter, and ω can be found in [W1] and [Ra].

Definition 3.1. Define $F = (f_0, f_1, f_2)^T$ by:

$$\begin{aligned} f_0(\tau) &= q^{-\frac{1}{24}} f(q) \\ f_1(\tau) &= 2q^{\frac{1}{3}} \omega(q^{\frac{1}{2}}) \\ f_2(\tau) &= 2q^{\frac{1}{3}} \omega(-q^{\frac{1}{2}}), \end{aligned} \tag{3.2}$$

with $q = e^{2\pi i\tau}$, $\tau \in \mathcal{H}$.

In [W1] Watson gave the modular transformation properties of f and ω. If we rewrite them in terms of F we get

Lemma 3.2. *For $\tau \in \mathcal{H}$ we have*

$$F(\tau+1) = \begin{pmatrix} \zeta_{24}^{-1} & 0 & 0 \\ 0 & 0 & \zeta_3 \\ 0 & \zeta_3 & 0 \end{pmatrix} F(\tau) \tag{3.3}$$

and

$$\frac{1}{\sqrt{-i\tau}} F(-1/\tau) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} F(\tau) + R(\tau), \tag{3.4}$$

with $\zeta_n = e^{2\pi i/n}$, $R(\tau) = 4\sqrt{3}\sqrt{-i\tau}(j_2(\tau), -j_1(\tau), j_3(\tau))^T$, where

$$\begin{aligned} j_1(\tau) &= \int_0^\infty e^{3\pi i\tau x^2} \frac{\sin 2\pi\tau x}{\sin 3\pi\tau x} dx \\ j_2(\tau) &= \int_0^\infty e^{3\pi i\tau x^2} \frac{\cos \pi\tau x}{\cos 3\pi\tau x} dx \\ j_3(\tau) &= \int_0^\infty e^{3\pi i\tau x^2} \frac{\sin \pi\tau x}{\sin 3\pi\tau x} dx. \end{aligned} \tag{3.5}$$

Proof. The transformation formula for $\tau \to \tau + 1$ is trivial.

If we take the first formula from the set of transformation formulae on p. 78 in [W1], with $\alpha = -2\pi i\tau$, and multiply both sides by -1, we get

$$\frac{1}{\sqrt{-i\tau}} f_1(-1/\tau) - f_0(\tau) = -\frac{4\sqrt{3}}{\sqrt{-i\tau}} J_1(-2\pi i\tau) = -\frac{4\sqrt{3}}{\sqrt{-i\tau}} j_1(\tau), \tag{3.6}$$

which is the second component of equation (3.4).

If we take the last formula from the set of transformation formulae on p. 78 in [W1], with $\alpha = -\pi i\tau$, and multiply both sides by -2, we get

$$\frac{1}{\sqrt{-i\tau}} f_0(-1/\tau) - f_1(\tau) = \frac{2\sqrt{3}}{\sqrt{-i\tau}} J_2(-\frac{\pi i\tau}{2}) = \frac{4\sqrt{3}}{\sqrt{-i\tau}} j_2(\tau), \tag{3.7}$$

where we have replaced x by $2x$ in the integral. This equation is the first component of equation (3.4).

If we take the formula on the middle of p. 79 in [W1], with $\alpha = -\pi i\tau$, and multiply both sides by 2, we get

$$\frac{1}{\sqrt{-i\tau}} f_2(-1/\tau) + f_2(\tau) = \frac{4\sqrt{3}}{\sqrt{-i\tau}} J_3(-\pi i\tau) = \frac{4\sqrt{3}}{\sqrt{-i\tau}} j_3(\tau), \tag{3.8}$$

which is the third component of equation (3.4). □

In a moment we will define a (nonholomorphic) function G that satisfies the same modular transformation properties as F. Before that, we rewrite R in terms of period integrals of the following theta functions of weight $3/2$:

$$\begin{aligned} g_0(z) &= \sum_{n\in\mathbf{Z}} (-1)^n (n+1/3) e^{3\pi i(n+\frac{1}{3})^2 z} \\ g_1(z) &= -\sum_{n\in\mathbf{Z}} (n+1/6) e^{3\pi i(n+\frac{1}{6})^2 z} \\ g_2(z) &= \sum_{n\in\mathbf{Z}} (n+1/3) e^{3\pi i(n+\frac{1}{3})^2 z}. \end{aligned} \tag{3.9}$$

These theta functions have the following modular transformation properties, which can be verified using standard methods:

$$\begin{pmatrix} g_0(z+1) \\ g_1(z+1) \\ g_2(z+1) \end{pmatrix} = \begin{pmatrix} 0 & 0 & \zeta_6 \\ 0 & \zeta_{24} & 0 \\ \zeta_6 & 0 & 0 \end{pmatrix} \begin{pmatrix} g_0(z) \\ g_1(z) \\ g_2(z) \end{pmatrix} \tag{3.10}$$

and

$$\begin{pmatrix} g_0(-1/z) \\ g_1(-1/z) \\ g_2(-1/z) \end{pmatrix} = -(-iz)^{3/2} \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} g_0(z) \\ g_1(z) \\ g_2(z) \end{pmatrix}. \tag{3.11}$$

From these transformation properties and the Fourier expansions, we see that the g_j's are cusp forms.

Lemma 3.3. *For $\tau \in \mathcal{H}$ we have*

$$R(\tau) = -2i\sqrt{3} \int_0^{i\infty} \frac{g(z)}{\sqrt{-i(z+\tau)}}\, dz, \tag{3.12}$$

where g is the vector $(g_0, g_1, g_2)^T$, and we have to integrate each component of the vector.

Proof. (sketch)
If we replace τ by $-1/\tau$ in equation (3.4), multiply both sides by $\frac{1}{\sqrt{-i\tau}} \left(\begin{smallmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -1 \end{smallmatrix}\right)$

and subtract equation (3.4), then we see that

$$R(\tau)=\frac{-1}{\sqrt{-i\tau}}\begin{pmatrix}0&1&0\\1&0&0\\0&0&-1\end{pmatrix}R(-1/\tau). \tag{3.13}$$

If we now take $\tau=it$ with $t\in\mathbf{R}$, $t>0$, we have

$$R(it)=\frac{-1}{\sqrt{t}}\begin{pmatrix}0&1&0\\1&0&0\\0&0&-1\end{pmatrix}R(i/t)=\frac{4\sqrt{3}}{t}\begin{pmatrix}j_1(i/t)\\-j_2(i/t)\\j_3(i/t)\end{pmatrix}. \tag{3.14}$$

We now consider the first component:

$$\frac{4\sqrt{3}}{t}j_1(i/t)=\frac{4\sqrt{3}}{t}\int_0^\infty e^{-3\pi x^2/t}\frac{\sinh 2\pi x/t}{\sinh 3\pi x/t}\,dx=4\sqrt{3}\int_0^\infty e^{-3\pi t y^2}\frac{\sinh 2\pi y}{\sinh 3\pi y}\,dy, \tag{3.15}$$

where we have substituted $x=ty$ in the integral.

From the theory of partial fraction decompositions (see [WW, pp. 134–136]) we get

$$\frac{\sinh 2\pi y}{\sinh 3\pi y}=-\frac{i\sqrt{3}}{6\pi}\sum_{n\in\mathbf{Z}}\frac{(-1)^n}{y-i(n+\frac{1}{3})}-\frac{i\sqrt{3}}{6\pi}\sum_{n\in\mathbf{Z}}\frac{(-1)^n}{-y-i(n+\frac{1}{3})}. \tag{3.16}$$

Using this we see

$$\begin{aligned}\frac{4\sqrt{3}}{t}&j_1(i/t)\\&=-\frac{2i}{\pi}\int_0^\infty e^{-3\pi t y^2}\left(\sum_{n\in\mathbf{Z}}\frac{(-1)^n}{y-i(n+\frac{1}{3})}+\sum_{n\in\mathbf{Z}}\frac{(-1)^n}{-y-i(n+\frac{1}{3})}\right)dy\\&=-\frac{2i}{\pi}\int_{-\infty}^\infty e^{-3\pi t y^2}\left(\sum_{n\in\mathbf{Z}}\frac{(-1)^n}{y-i(n+\frac{1}{3})}\right)dy\\&=-\frac{2i}{\pi}\sum_{n\in\mathbf{Z}}(-1)^n\int_{-\infty}^\infty\frac{e^{-3\pi t y^2}}{y-i(n+\frac{1}{3})}\,dy.\end{aligned} \tag{3.17}$$

It's not immediately clear that interchanging the order of integration and summation in the last equation is justified. However, it can be proven rigorously if we consider $\int_{-\infty}^\infty e^{-3\pi t y^2}\left(\sum_{n\in\mathbf{Z}}(-1)^n\left(\frac{1}{y-i(n+\frac{1}{3})}+\frac{1}{i(n+\frac{1}{3})}\right)\right)dy$ (here we can interchange the order of integration and summation because of absolute convergence).

We have for $r\in\mathbf{R}$, $r\neq 0$

$$\int_{-\infty}^\infty\frac{e^{-\pi t y^2}}{y-ir}\,dy=\pi i r\int_0^\infty\frac{e^{-\pi r^2 u}}{\sqrt{u+t}}\,du, \tag{3.18}$$

(both sides are solutions of $(-\frac{\partial}{\partial t}+\pi r^2)f(t)=\frac{\pi i r}{\sqrt{t}}$ and have the same limit 0 if $t\to\infty$, and hence are equal). If we use this with $r=(n+1/3)$ and t replaced by $3t$, we obtain

$$\begin{aligned}\frac{4\sqrt{3}}{t}j_1(i/t) &= 2\sum_{n\in\mathbf{Z}}(-1)^n(n+1/3)\int_0^\infty \frac{e^{-\pi(n+1/3)^2u}}{\sqrt{u+3t}}\,du \\ &= 2\int_0^\infty \frac{\sum_{n\in\mathbf{Z}}(-1)^n(n+1/3)e^{-\pi(n+\frac{1}{3})^2u}}{\sqrt{u+3t}}\,du.\end{aligned} \tag{3.19}$$

Again it's not immediately clear that interchanging the order of integration and summation in the last step is justified. It can be proven rigorously by first using partial integration on the integral

(3.20)

$$\int_0^\infty \frac{e^{-\pi(n+1/3)^2u}}{\sqrt{u+3t}}\,du = \frac{1}{\pi(n+1/3)^2}\frac{1}{\sqrt{3t}} - \frac{1}{2\pi(n+1/3)^2}\int_0^\infty \frac{e^{-\pi(n+1/3)^2u}}{(u+3t)^{3/2}}\,du,$$

then interchanging the order of integration and summation, which is justified by absolute convergence, and finally using partial integration again. By partial integration we introduce some "boundary terms". To get rid of them we have to use Abel's theorem on continuity up to the circle of convergence, see [WW, pp. 57–58].

If we now substitute $u=-3iz$ in the integral we get

$$\frac{4\sqrt{3}}{t}j_1(i/t) = -2i\sqrt{3}\int_0^{i\infty}\frac{g_0(z)}{\sqrt{-i(z+it)}}\,dz, \tag{3.21}$$

so we have proven the first component of equation (3.12) for $\tau = it$. Since both sides are analytic on $\mathcal{H}$, the identity holds for all $\tau\in\mathcal{H}$.

The second and third component of equation (3.12) can be proven along the same lines. Here we have to use

$$\begin{aligned}\frac{\cosh \pi y}{\cosh 3\pi y} &= -\frac{i\sqrt{3}}{6\pi}\sum_{n\in\mathbf{Z}}^{*}\frac{1}{y-i(n+\frac{1}{6})} - \frac{i\sqrt{3}}{6\pi}\sum_{n\in\mathbf{Z}}^{*}\frac{1}{-y-i(n+\frac{1}{6})} \\ \frac{\sinh \pi y}{\sinh 3\pi y} &= -\frac{i\sqrt{3}}{6\pi}\sum_{n\in\mathbf{Z}}^{*}\frac{1}{y-i(n+\frac{1}{3})} - \frac{i\sqrt{3}}{6\pi}\sum_{n\in\mathbf{Z}}^{*}\frac{1}{-y-i(n+\frac{1}{3})},\end{aligned} \tag{3.22}$$

where $\sum_{n\in\mathbf{Z}}^{*}$ means $\lim_{m\to\infty}\sum_{n=-m}^{m}$. □

Definition 3.4. For $\tau\in\mathcal{H}\cup\mathbf{Q}$ we define

$$G(\tau) := 2i\sqrt{3}\int_{-\overline{\tau}}^{i\infty}\frac{(g_1(z),g_0(z),-g_2(z))^T}{\sqrt{-i(z+\tau)}}\,dz. \tag{3.23}$$

The integrals converge, even if $\tau\in\mathbf{Q}$, because the g_j's are cusp forms.

The function G satisfies the same modular transformation properties as F:

Lemma 3.5. *For $\tau\in\mathcal{H}$ we have*

$$G(\tau+1) = \begin{pmatrix}\zeta_{24}^{-1} & 0 & 0\\ 0 & 0 & \zeta_3\\ 0 & \zeta_3 & 0\end{pmatrix}G(\tau) \tag{3.24}$$

and

$$\frac{1}{\sqrt{-i\tau}}G(-1/\tau) = \begin{pmatrix}0 & 1 & 0\\ 1 & 0 & 0\\ 0 & 0 & -1\end{pmatrix}G(\tau) + R(\tau). \tag{3.25}$$

Proof. The first equation follows from equation (3.10) if we replace z by $z-1$ in the integral.

We have

$$\begin{aligned}\frac{1}{\sqrt{-i\tau}}G(-1/\tau) &= \frac{2i\sqrt{3}}{\sqrt{-i\tau}}\int_{1/\overline{\tau}}^{i\infty}\frac{(g_1(z),g_0(z),-g_2(z))^T}{\sqrt{-i(z-1/\tau)}}\,dz\\ &= 2i\sqrt{3}\int_0^{-\overline{\tau}}\frac{(g_1(-1/z),g_0(-1/z),-g_2(-1/z))^T}{\sqrt{1+\tau/z}}\frac{dz}{(-iz)^2},\end{aligned}\tag{3.26}$$

where we have replaced z by $-1/z$ in the integral. If we now use equation (3.11) we get

$$\frac{1}{\sqrt{-i\tau}}G(-1/\tau) = -2i\sqrt{3}\int_0^{-\overline{\tau}}\frac{g(z)}{\sqrt{-i(z+\tau)}}\,dz;\tag{3.27}$$

hence we get

$$\begin{aligned}&\frac{1}{\sqrt{-i\tau}}G(-1/\tau) - \begin{pmatrix}0&1&0\\1&0&0\\0&0&-1\end{pmatrix}G(\tau)\\ &= -2i\sqrt{3}\int_0^{-\overline{\tau}}\frac{g(z)}{\sqrt{-i(z+\tau)}}\,dz - 2i\sqrt{3}\int_{-\overline{\tau}}^{i\infty}\frac{g(z)}{\sqrt{-i(z+\tau)}}\,dz\\ &= -2i\sqrt{3}\int_0^{i\infty}\frac{g(z)}{\sqrt{-i(z+\tau)}}\,dz = R(\tau),\end{aligned}\tag{3.28}$$

by Lemma 3.3. □

Theorem 3.6. *The function H definied by*

$$H(\tau) = F(\tau) - G(\tau),\tag{3.29}$$

is a (vector-valued) real-analytic modular form of weight $1/2$, satisfying

$$\begin{aligned}H(\tau+1) &= \begin{pmatrix}\zeta_{24}^{-1}&0&0\\0&0&\zeta_3\\0&\zeta_3&0\end{pmatrix}H(\tau),\\ \frac{1}{\sqrt{-i\tau}}H(-1/\tau) &= \begin{pmatrix}0&1&0\\1&0&0\\0&0&-1\end{pmatrix}H(\tau),\end{aligned}\tag{3.30}$$

and $H$is an eigenfunction of the Casimir operator $\Omega_{1/2} = -4y^2\frac{\partial^2}{\partial\tau\partial\overline{\tau}} + iy\frac{\partial}{\partial\overline{\tau}} + \frac{3}{16}$ with eigenvalue $\frac{3}{16}$, where $\tau = x+iy$, $\frac{\partial}{\partial\tau} = \frac{1}{2}\left(\frac{\partial}{\partial x} - i\frac{\partial}{\partial y}\right)$ and $\frac{\partial}{\partial\overline{\tau}} = \frac{1}{2}\left(\frac{\partial}{\partial x} + i\frac{\partial}{\partial y}\right)$.

Proof. The modular transformation properties of H are a direct consequence of the transformation properties of F and G given in Lemma 3.2 and Lemma 3.5.

Since F is a holomorphic function of τ, we have $\frac{\partial}{\partial\overline{\tau}}F(\tau)=0$; hence

$$\begin{aligned}\frac{\partial}{\partial\overline{\tau}}H(\tau) = -\frac{\partial}{\partial\overline{\tau}}G(\tau) &= -2i\sqrt{3}\frac{(g_1(-\overline{\tau}),g_0(-\overline{\tau}),-g_2(-\overline{\tau}))^T}{\sqrt{-i(\tau-\overline{\tau})}}\\ &= -\frac{i\sqrt{6}}{\sqrt{y}}(g_1(-\overline{\tau}),g_0(-\overline{\tau}),-g_2(-\overline{\tau}))^T.\end{aligned}\tag{3.31}$$

We see that $\sqrt{y}\frac{\partial}{\partial\bar{\tau}}H(\tau)$ is anti-holomorphic, so

$$\frac{\partial}{\partial\tau}\sqrt{y}\frac{\partial}{\partial\bar{\tau}}H(\tau)=0. \tag{3.32}$$

We can write the operator $\Omega_{1/2}=-4y^2\frac{\partial^2}{\partial\tau\partial\bar{\tau}}+iy\frac{\partial}{\partial\bar{\tau}}+\frac{3}{16}$ as

$$\Omega_{1/2}=\frac{3}{16}-4y^{3/2}\frac{\partial}{\partial\tau}\sqrt{y}\frac{\partial}{\partial\bar{\tau}}. \tag{3.33}$$

Hence

$$\Omega_{1/2}H=\frac{3}{16}H. \tag{3.34}$$

□

If we now write F as $H+G$, we get the following:

Corollary 3.7. *The vector-valued third order mock ϑ-function F can be written as the sum of a real analytic modular form H and a function G that is bounded in all rational points.*

4. Other mock ϑ-functions

In the previous section we have only dealt with the third order mock ϑ-functions f and ω. However, I have found similar results for most other mock ϑ-functions, and I expect that it can be done for all known ones. I hope to present these results, and the details omitted in the previous section, in my Ph.D.-thesis, which should appear somewhere near the end of 2002.

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