ON THE DE BRUIJN-NEWMAN CONSTANT

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Abstract. If $\lambda^{(0)}$ denotes the infimum of the set of real numbers λ such that the entire function Ξ_λ represented by

$$\Xi_\lambda(t) = \int_0^\infty e^{\frac{\lambda}{4}(\log x)^2 + \frac{it}{2}\log x} \left(x^{5/4} \sum_{n=1}^\infty \left(2n^4\pi^2 x - 3n^2\pi\right) e^{-n^2\pi x} \right) \frac{dx}{x}$$

has only real zeros, then the de Bruijn-Newman constant Λ is defined as $\Lambda = 4\lambda^{(0)}$. The Riemann hypothesis is equivalent to the inequality $\Lambda \le 0$. The fact that the non-trivial zeros of the Riemann zeta-function ζ lie in the strip $\{s : 0 < \operatorname{Re} s < 1\}$ and a theorem of de Bruijn imply that $\Lambda \le 1/2$. In this paper, we prove that all but a finite number of zeros of Ξ_λ are real and simple for each $\lambda > 0$, and consequently that $\Lambda < 1/2$.

1. Introduction

This paper is concerned with the zeros of certain entire functions that are related with the Riemann zeta-function.

Let the functions ψ, φ and Φ be defined respectively as

$$\psi(x) = \sum_{n=1}^\infty e^{-n^2\pi x}, \quad \varphi(x) = x^{5/4}\left(2x\psi''(x) + 3\psi'(x)\right), \quad \text{and} \quad \Phi(u) = 2\varphi\left(e^{2u}\right).$$

We immediately see that $\varphi(x) \sim 2\pi^2 x^{9/4} e^{-\pi x}$ for $x \to \infty$ and $\Phi(u) \sim 4\pi^2 \exp\left(\frac{9u}{2} - \pi e^{2u}\right)$ for $u \to \infty$. The functional equations

$$2\psi(x) + 1 = x^{-1/2}\left[2\psi\left(\frac{1}{x}\right) + 1\right], \quad \varphi(x) = \varphi(1/x), \quad \text{and} \quad \Phi(u) = \Phi(-u)$$

are well known. In fact, the first one is a consequence of the Poisson summation formula; the second follows from the first by differentiating it twice and rearranging the terms; and the third is equivalent to the second one.

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We define the function Ξ_λ by

$$\Xi_\lambda(t) = \int_0^\infty e^{\frac{\lambda}{4}(\log x)^2 + \frac{it}{2}\log x} \varphi(x) \frac{dx}{x} = \int_{-\infty}^\infty e^{\lambda u^2} \Phi(u) e^{itu} du, \tag{1.1}$$

where λ is a constant. The last expression shows that Ξ_λ is the Fourier transform of an even function of u which tends to 0 very rapidly as $u \to \infty$. As a consequence, Ξ_λ is an even entire function of order 1 and maximal type by the Paley-Wiener theorem and [17, pp. 9–10]. In particular, Ξ_λ has infinitely many zeros, by Hadamard's factorization theorem. If $\lambda \in \mathbb{R}$, then Ξ_λ assumes only real values on the real axis, and by Pólya's criterion [16] it has infinitely many real zeros.

If we put $s = \frac{1}{2} + it$, then Ξ_0 and the Riemann zeta-function ζ are related through

$$\Xi_0(t) = \frac{s(s-1)}{2} \Gamma\left(\frac{s}{2}\right) \pi^{-s/2} \zeta(s).$$

It is known that the zeros of Ξ_0 lie in the strip $\{t : |\operatorname{Im} t| < 1/2\}$, and the Riemann hypothesis is equivalent to the statement that Ξ_0 has only real zeros [18].

If $\lambda_1 < \lambda_2$, then the zeros of Ξ_{λ_2} lie closer to the real axis than those of Ξ_{λ_1}.

Proposition A. *If $\lambda_1 \le \lambda_2$, $\Delta \ge 0$, and the zeros of Ξ_{λ_1} lie in $\{t : |\operatorname{Im} t| \le \Delta\}$, then those of Ξ_{λ_2} lie in $\{t : |\operatorname{Im} t| \le \widetilde{\Delta}\}$, where $\widetilde{\Delta} = \sqrt{\max\{\Delta^2 - 2(\lambda_2 - \lambda_1), 0\}}$.*

This proposition is easily proved by applying a theorem of N. G. de Bruijn [1, Theorem 13] to the integral representation (1.1) of Ξ_λ. See also Theorem 2.3 below, which generalizes de Bruijn's theorem.

Since the zeros of Ξ_0 lie in $\{t : |\operatorname{Im} t| < 1/2\}$, Proposition A implies that Ξ_λ has only real zeros when $\lambda \ge 1/8$. It also implies that if $\lambda_1 < \lambda_2$ and Ξ_{λ_1} has only real zeros, then so does Ξ_{λ_2}. In [14], C. M. Newman proved that Ξ_λ has non-real zeros for some negative constant λ. Consequently, there is a real constant $\lambda^{(0)} \le 1/8$ such that Ξ_λ has only real zeros when $\lambda^{(0)} \le \lambda$ but has non-real zeros when $\lambda < \lambda^{(0)}$ [14, Theorem 3]. The Riemann hypothesis is equivalent to the inequality $\lambda^{(0)} \le 0$. On the other hand, Newman conjectured the opposite inequality $0 \le \lambda^{(0)}$, and called his conjecture a quantitative version of the dictum that the Riemann hypothesis, if true, is only barely so [14, p. 247].

The de Bruijn-Newman constant Λ is defined as $\Lambda = 4\lambda^{(0)}$ [7]. Thus we have $\Lambda \le 1/2$, and the Riemann hypothesis and Newman's conjecture are equivalent to the inequalities $\Lambda \le 0$ and $0 \le \Lambda$, respectively. The first explicit lower bound for Λ was given by G. Csordas, T. S. Norfolk and R. S. Varga in 1988 [7]: They proved that

$$-50 < \Lambda.$$

Since then, this lower bound has been greatly improved in favor of Newman's conjecture by many authors [15, Section 1]. However, it seems that no upper bounds for Λ better than $\Lambda \le 1/2$ have been published.

In this paper, we improve the inequality $\Lambda \le 1/2$ very slightly.

Theorem 1.1. *The de Bruijn-Newman constant* Λ *is less than* $1/2$.

It should be remarked that our method of proving this theorem gives no explicit upper bound for Λ less than $1/2$.

It follows from Hadamard's factorization theorem that the zeros of $\Xi_0^{(m)}$ lie in the strip $\{t : |\mathrm{Im}\, t| < 1/2\}$ for every $m \geq 0$, and that if the Riemann hypothesis were true, then $\Xi_0^{(m)}$ would have only real zeros for every $m \geq 0$. (See Theorems 2.5 and 2.6 below.) However, it is known that the "proportion" of real zeros of $\Xi_0^{(m)}$ tends to 1 as $m \to \infty$ [4]. As we shall see in Section 2, the function $\Xi_\lambda^{(m)}$ is obtained by applying to $\Xi_0^{(m)}$ a certain differential operator of infinite order, and Proposition A still holds if we replace Ξ_λ by $\Xi_\lambda^{(m)}$. Motivated by this observation, we consider the sequence $\{\lambda^{(m)}\}$ defined by

$$\lambda^{(m)} = \inf\left\{\lambda : \Xi_\lambda^{(m)} \text{ has only real zeros}\right\} \qquad (m = 0, 1, 2, \dots).$$

Theorem 1.2. *The sequence* $\{\lambda^{(m)}\}$ *is non-increasing, and its limit is* ≤ 0.

We remark that, by Newman's theorem [14, Theorem 2], we have $-\infty < \lambda^{(m)}$ for all m. In the course of proving Theorems 1.1 and 1.2, the following theorem plays a crucial role.

Theorem 1.3. *For every* $\lambda > 0$ *all but a finite number of zeros of* Ξ_λ *are real and simple.*

Suppose $\lambda > 0$. Then Theorem 1.3 states that there is a positive constant T_λ such that Ξ_λ has finitely many zeros in the vertical strip $\{t : |\mathrm{Re}\, t| < T_\lambda\}$ and all the zeros of Ξ_λ that lie outside the strip are real and simple. Since Ξ_λ has infinitely many zeros, we may denote the zeros of Ξ_λ in the closed half plane $\{t : \mathrm{Re}\, t \geq T_\lambda\}$ by

$$\gamma_{(\lambda,1)} < \gamma_{(\lambda,2)} < \gamma_{(\lambda,3)} < \cdots.$$

For $T > 0$ let $N_\lambda(T)$ denote the number of zeros of Ξ_λ in $\{t : 0 \leq \mathrm{Re}\, t \leq T\}$.

Theorem 1.4. *If* $\lambda > 0$, *then*

$$\lim_{n\to\infty} \left(\gamma_{(\lambda,n+1)} - \gamma_{(\lambda,n)}\right) \frac{\log \gamma_{(\lambda,n)}}{2\pi} = 1$$

and

$$N_\lambda(T) = \frac{T}{2\pi} \log \frac{T}{2\pi} - \frac{T}{2\pi} + \frac{\lambda}{4} \log \frac{T}{2\pi} + O(1) \qquad (T \to \infty).$$

This theorem, in particular, shows that for $\lambda > 0$ the distribution of the gaps between consecutive real zeros of Ξ_λ is very different from the generally conjectured one for Ξ_0. On the Riemann hypothesis, Montgomery's pair correlation conjecture [13] implies that

$$\liminf_{n\to\infty} (\gamma_{n+1} - \gamma_n) \frac{\log \gamma_n}{2\pi} = 0 \quad \text{and} \quad \limsup_{n\to\infty} (\gamma_{n+1} - \gamma_n) \frac{\log \gamma_n}{2\pi} = \infty,$$

where $\gamma_1, \gamma_2, \gamma_3, \dots$ denote the positive zeros of Ξ_0 with $\gamma_1 \leq \gamma_2 \leq \gamma_3 \leq \cdots$. In this sense, the functions Ξ_λ, $\lambda > 0$, seem to be quite different from Ξ_0, although $\Xi_\lambda \to \Xi_0$ as $\lambda \to 0$

uniformly on compact sets in the complex plane. In fact, as we shall see in the sequel, there is a one-parameter family $\{H_\lambda\}$ of entire functions which is analogous to $\{\Xi_\lambda\}$ and the zero-distributions of the functions H_λ, $\lambda > 0$, are very different from those of H_λ, $\lambda \le 0$, in respect that the analogues of Theorems 1.1 through 1.4 are all true for the functions H_λ, $\lambda > 0$, but H_λ has no real zeros at all when $\lambda \le 0$.

This paper is composed as follows. In Section 2, we introduce some notations and general theorems that are needed in later discussions; and prove that Theorem 1.3 implies Theorems 1.1 and 1.2. We prove Theorem 1.3 in Section 3. In proving Theorem 1.3, we will require some technical results, namely Propositions 3.3 and 3.4. They are proved in Section 4. Section 5 is devoted to proving Theorem 1.4. Finally, in Section 6, we give some examples to which our method of obtaining Theorems 1.1 through 1.4 applies. There one finds such one-parameter families of entire functions mentioned.

2. Zeros of Real Entire Functions and Proof of Theorems 1.1 and 1.2

Suppose μ is a constant and f is an entire function of order less than 2. Then the series

$$\sum_{n=0}^{\infty} \frac{\mu^n}{n!} f^{(2n)}$$

converges absolutely and uniformly on compact sets in the complex plane [12, Lemma 2.1]. Hence it represents a new entire function. We denote it by $e^{\mu D^2} f$. In this case, the entire functions f and $e^{\mu D^2} f$ are of the same order and type, and we have

$$e^{\lambda D^2}\left(e^{\mu D^2} f\right) = e^{(\lambda+\mu)D^2} f$$

for every constant λ [12, Lemmas 3.4 and 3.5; 3, Theorem 1.1].

By an elementary argument, we see that

$$\Xi_\lambda^{(m)} = e^{-\lambda D^2} \Xi_0^{(m)}$$

holds for every constant λ and for every non-negative integer m. A straightforward calculation leads to the following lemma.

Lemma 2.1. *Let g be an entire function of order less than 2, a a constant, and $f(z) = (z-a)g(z)$. If λ is a constant and $h = e^{-\lambda D^2} g$, then*

$$e^{-\lambda D^2} f(z) = (z-a)h(z) - 2\lambda h'(z).$$

If $\lambda \ne 0$, then

$$(z-a)h(z) - 2\lambda h'(z) = -2\lambda \exp\left(\frac{(z-a)^2}{4\lambda}\right) \frac{d}{dz} \exp\left(-\frac{(z-a)^2}{4\lambda}\right) h(z).$$

An entire function is said to be a *real entire function* if it assumes only real values on the real axis. If f is a real entire function and a is a zero of f, then so is $\bar{a}$. In view of the following theorems we know that if f is a real entire function of order less than 2 and $\lambda > 0$, then the zeros of $e^{-\lambda D^2} f$ lie closer to the real axis than those of f.

Theorem 2.2. *If f is a real entire function of order less than 2 and $\lambda > 0$, then the number of non-real zeros of $e^{-\lambda D^2} f$ does not exceed that of f.*

Proof. The theorem is an immediate consequence of Theorem 9a of [1] and Proposition 3.1 of [12]. □

Theorem 2.3. *Let $\lambda, \Delta > 0$ and f be a real entire function of order less than 2. If the zeros of f lie in $\{z : |\operatorname{Im} z| \leq \Delta\}$, then those of $e^{-\lambda D^2} f$ lie in $\{z : |\operatorname{Im} z| \leq \widetilde{\Delta}\}$, where $\widetilde{\Delta} = \sqrt{\max\{\Delta^2 - 2\lambda, 0\}}$. If $\Delta^2 < 2\lambda$, then all the zeros of $e^{-\lambda D^2} f$ are (real and) simple.*

Proof. See Section 3 of [12]. See also [5, Theorem 3.10]. □

Theorem 2.4. *Suppose that f is a real entire function of order less than 2, and that for every $\epsilon > 0$ all but a finite number of zeros of f lie in $\{z : |\operatorname{Im} z| \leq \epsilon\}$. If $\lambda > 0$, then all but a finite number of zeros of $e^{-\lambda D^2} f$ are real and simple.*

Proof. The theorem is a special case of Theorem 2.2 in [12]. □

A real entire function f is said to be of *genus* 1^* if there are a real constant $\alpha \geq 0$ and a real entire function g of genus at most 1 such that $f(z) = e^{-\alpha z^2} g(z)$.

Theorem 2.5 (Jensen's Theorem). *If f is a real entire function of genus 1^* and z_1 is a non-real zero of f', then f has a non-real zero z_0 such that $|z_1 - \operatorname{Re} z_0| \leq \operatorname{Im} z_0$.*

Proof. If $c \in \mathbb{C}$ is such that $|c - \operatorname{Re} a| > |\operatorname{Im} a|$ for every non-real zero a of f, then $\operatorname{Im}\,(f'(c)/f(c)) \operatorname{Im} c < 0$. □

Corollary. *Let h be a real entire function of order less than 2, $\lambda > 0$, $a \in \mathbb{R}$, and $h_1(z) = (z - a)h(z) - 2\lambda h'(z)$. If z_1 is a non-real zero of h_1, then h has a non-real zero z_0 such that $|z_1 - \operatorname{Re} z_0| \leq \operatorname{Im} z_0$.*

Proof. Since h is of order less than 2, it is of genus at most 1, by Hadamard's factorization theorem. Therefore the result is an immediate consequence of Lemma 2.1 and Theorem 2.5. □

Theorem 2.6 (The Pólya-Wiman Theorem). *Suppose f is a real entire function of genus 1^* and f has finitely many non-real zeros. Then f' is again a real entire function of genus 1^* and the number of non-real zeros of f' does not exceed that of f. Moreover, there is a positive integer N such that $f^{(N)}$ has only real zeros.*

Proof. See Section 2 of [11]. See also [6] and [10]. □

If $\lambda, \Delta > 0$ and $f(z) = z^2 + \Delta^2$, then $e^{-\lambda D^2} f(z) = z^2 + \Delta^2 - 2\lambda$. Thus Theorem 2.3 cannot be improved in the general case; however, in a certain case, it is possible.

Theorem 2.7. *Suppose that f is a real entire function of order less than 2, f has finitely many non-real zeros, and the number of non-real zeros of f in the upper half plane $\{z : \operatorname{Im} z > 0\}$ does not exceed the number of real zeros of f. Suppose also that $\Delta_0 > 0$ and the zeros of f lie in $\{z : |\operatorname{Im} z| \leq \Delta_0\}$. If $0 < \lambda < \Delta_0^2/2$, then the zeros of $e^{-\lambda D^2} f$ lie in $\{z : |\operatorname{Im} z| \leq \Delta\}$ for some $\Delta < \sqrt{\Delta_0^2 - 2\lambda}$.*

Proof. Suppose $0 < \lambda < \Delta_0^2/2$, and put $\Delta_1 = \sqrt{\Delta_0^2 - 2\lambda}$. By Theorem 2.2, $e^{-\lambda D^2} f$ has at most a finite number of non-real zeros; and by Theorem 2.3, the non-real zeros lie in

$\{z : |\operatorname{Im} z| \le \Delta_1\}$. Hence it is enough to show that $e^{-\lambda D^2} f$ has no zeros on the line $\{z : \operatorname{Im} z = \Delta_1\}$.

Let N denote the number of non-real zeros of f in the upper half plane. From the assumption, we may write

$$f(z) = (z - a_1) \cdots (z - a_N) g(z), \tag{2.1}$$

where $a_1, \dots, a_N$ are real zeros of f, and g is a real entire function of order less than 2.

The functions f and g have the same non-real zeros. Hence $e^{-\lambda D^2} g$ has at most N non-real zeros in the upper half plane, and the non-real zeros lie in $\{z : |\operatorname{Im} z| \le \Delta_1\}$. Let $h_0 = e^{-\lambda D^2} g$, and define $h_1, \dots, h_N$ by

$$h_n(z) = (z - a_n) h_{n-1}(z) - 2\lambda h'_{n-1}(z) \qquad (n = 1, \dots, N).$$

An inductive argument shows that $h_0, h_1, \dots, h_N$ are real entire functions of order less than 2. By (2.1) and Lemma 2.1, we have $h_N = e^{-\lambda D^2} f$.

Suppose, to obtain a contradiction, that h_N has a zero z_N on the line $\{z : \operatorname{Im} z = \Delta_1\}$. Then, by the corollary to Theorem 2.5, there are complex numbers $z_0, \dots, z_{N-1}$ in the upper half plane such that $h_n(z_n) = 0$ and

$$|z_{n+1} - \operatorname{Re} z_n| \le \operatorname{Im} z_n \tag{2.2}$$

for $n = 0, 1, \dots, N - 1$.

It follows from (2.2) that $\operatorname{Im} z_{n+1} \le \operatorname{Im} z_n$, and that $\operatorname{Im} z_{n+1} = \operatorname{Im} z_n$ if and only if $z_{n+1} = z_n$. Since $h_0(z_0) = 0$ and $h_0 = e^{-\lambda D^2} g$, we have $\operatorname{Im} z_0 \le \Delta_1$, and we are assuming that $\operatorname{Im} z_N = \Delta_1$. Hence $z_0 = z_1 = \cdots = z_N$, and we conclude, by Lemma 2.1, that z_0 is a zero of h_0 whose multiplicity is greater than N. This is the desired contradiction, because h_0 $(=e^{-\lambda D^2} g)$ has at most N non-real zeros in the upper half plane. $\square$

Proof of Theorem 1.1. To prove the theorem, it is enough to show that there is a $\lambda_1 < 1/8$ such that the zeros of Ξ_{λ_1} lie in $\{t : |\operatorname{Im} t| \le \Delta\}$ for some $\Delta < \sqrt{\frac{1}{4} - 2\lambda_1}$: If such a λ_1 exists, then Ξ_λ, with $\lambda = \frac{1}{2}\Delta^2 + \lambda_1$ $(< 1/8)$, has only real zeros by Theorem 2.3. In fact, we will show that if $0 < \lambda < 1/8$, then the zeros of Ξ_λ lie in $\{t : |\operatorname{Im} t| \le \Delta\}$ for some $\Delta < \sqrt{\frac{1}{4} - 2\lambda}$.

Suppose $0 < \lambda < 1/8$. Choose λ_0 so that $0 < \lambda_0 < \lambda$, and put $\Delta_0 = \sqrt{\frac{1}{4} - 2\lambda_0}$. Since the zeros of Ξ_0 lie in $\{t : |\operatorname{Im} t| \le 1/2\}$ and $\Xi_{\lambda_0} = e^{-\lambda_0 D^2} \Xi_0$, Theorem 2.3 implies that the zeros of Ξ_{λ_0} lie in $\{t : |\operatorname{Im} t| \le \Delta_0\}$. By Theorem 1.3, all but a finite number of zeros of Ξ_{λ_0} are real; and Ξ_λ has infinitely many zeros for arbitrary constant λ. Hence Theorem 2.7 implies that the zeros of Ξ_λ $(= e^{-(\lambda - \lambda_0) D^2} \Xi_{\lambda_0})$ lie in $\{t : |\operatorname{Im} t| \le \Delta\}$ for some $\Delta < \sqrt{\Delta_0^2 - 2(\lambda - \lambda_0)}$ $(= \sqrt{\frac{1}{4} - 2\lambda})$. $\square$

It may be remarked that, by Theorems 1.1 and 2.3, all the zeros of $\Xi_{1/8}$ are real and simple.

Proof of Theorem 1.2. By Hadamard's factorization theorem, the functions $\Xi_\lambda, \Xi'_\lambda, \Xi''_\lambda, \ldots$ are real entire functions of genus 1^* whenever $\lambda \in \mathbb{R}$. If $\lambda \in \mathbb{R}$ and $\Xi_\lambda^{(m)}$ has only real zeros, then so does $\Xi_\lambda^{(m+1)}$, by Theorem 2.6. Thus $\{\lambda^{(m)}\}$ is non-increasing. Let $\lambda > 0$ be arbitrary. By Theorem 1.3, Ξ_λ has a finite number of non-real zeros; hence, by Theorem 2.6, there is a positive integer m such that $\Xi_\lambda^{(m)}$ has only real zeros, so that $\lambda^{(m)} \leq \lambda$. Therefore

$$\lim_{m\to\infty} \lambda^{(m)} \leq 0. \quad \square$$

3. Proof of Theorem 1.3

In this section, we assume that λ is a fixed positive constant and prove the following.

Proposition 3.1. *For every $\epsilon > 0$ all but a finite number of zeros of Ξ_λ lie in $\{t : |\mathrm{Im}\, t| \leq \epsilon\}$.*

Since λ is arbitrary, this proposition together with Theorem 2.4 implies Theorem 1.3.

We need some preparations. First of all, to simplify the expressions, we put

$$H(t) = \int_0^\infty \varphi(x) e^{\lambda(\log x)^2 + i\pi t \log x} \frac{dx}{x},$$

that is, $H(t) = \Xi_{4\lambda}(2\pi t)$, and will show that for every $\epsilon > 0$ all but a finite number of zeros of H lie in $\{t : |\mathrm{Im}\, t| \leq \epsilon\}$. When a and b are complex numbers, we denote by $\int_a^b f(x)dx$ the integral of f over the path parameterized as $x = a + (b-a)u$, $0 \leq u \leq 1$, and by $\int_a^{a+\infty} f(x)dx$ the one over the infinite path parameterized as $x = a + u$, $0 \leq u < \infty$.

We define the function $\psi^{(-1)}$ by

$$\psi^{(-1)}(x) = \sum_{n=1}^\infty \frac{-1}{n^2\pi} e^{-n^2\pi x}.$$

Thus

$$\frac{d}{dx}\psi^{(-1)}(x) = \psi(x) \qquad (\mathrm{Re}\, x > 0),$$

$\psi^{(-1)}(x) \sim -\pi^{-1}e^{-\pi x}$ for $\mathrm{Re}\, x \to \infty$, $\psi^{(-1)}$ is continuous in the closed right half plane $\{x : \mathrm{Re}\, x \geq 0\}$, and $|\psi^{(-1)}(x)| \leq \pi/6$ for $\mathrm{Re}\, x \geq 0$. If m is a non-negative integer, we put

$$I_m(t) = \int_i^{i+\infty} x^{-7/4}(\log x)^m e^{\lambda(\log x)^2 + i\pi t \log x} \psi^{(-1)}(x) dx.$$

Here, log denotes of course the principal branch of the logarithm. By an elementary argument, we see that I_m is an entire function.

Define the polynomials $P_0, P_1, P_2, \ldots,$ by $P_0(u) \equiv 1$ and

$$P_{n+1}(u) = (u-n)P_n(u) + 2\lambda P'_n(u) \qquad (n = 0, 1, 2, \ldots).$$

Let $\mathcal{I} = \{(l,m) : l,m = 0,1,2,3 \text{ and } l+m \leq 3\}$, define the coefficients $a_{(l,m)}$, $(l,m) \in \mathcal{I}$, by

$$2P_3\left(2\lambda \log x + i\pi t + \frac{5}{4}\right) - 3P_2\left(2\lambda \log x + i\pi t + \frac{1}{4}\right) = \sum_{(l,m)\in\mathcal{I}} a_{(l,m)} t^l (\log x)^m,$$

and put

$$L(t) = \sum_{(l,m)\in\mathcal{I}} a_{(l,m)} t^l I_m(t).$$

Proposition 3.2. *There is a real polynomial R of degree ≤ 2 such that*

$$H(t) = R(t)e^{-\frac{\lambda\pi^2}{4} - \frac{\pi^2}{2}t} - L(t) - \overline{L(\bar{t})}.$$

Proof. Suppose $0 < \theta < \pi/2$. Put

$$K_\theta(t) = \int_{e^{i\theta}}^{e^{i\theta}+\infty} \varphi(x) e^{\lambda(\log x)^2 + i\pi t \log x} \frac{dx}{x},$$

and denote by γ the path parameterized by $x = \left(e^{-i\theta} + u\right)^{-1}$, $0 \leq u < \infty$. By applying Cauchy's theorem we obtain

$$H(t) = K_\theta(t) - \int_\gamma \varphi(x) e^{\lambda(\log x)^2 + i\pi t \log x} \frac{dx}{x},$$

because φ is analytic in the right half plane $\{x : \operatorname{Re} x > 0\}$,

$$\varphi(x) = 2\pi^2 x^{9/4} e^{-\pi x}\left(1 + O\left(x^{-1}\right)\right) \qquad (\operatorname{Re} x > \epsilon)$$

for every $\epsilon > 0$, and

$$\varphi(x) = 2\pi^2 x^{-9/4} e^{-\pi/x}\left(1 + O\left(x\right)\right) \qquad (|x - r| < r)$$

for every $r > 0$. Since $\varphi(x) = \varphi(1/x) = \overline{\varphi(\bar{x})}$, we see that

$$-\int_\gamma \varphi(x) e^{\lambda(\log x)^2 + i\pi t \log x} \frac{dx}{x} = \overline{K_\theta(\bar{t})}.$$

Hence

$$H(t) = K_\theta(t) + \overline{K_\theta(\bar{t})}.$$

We have defined the function $\psi^{(-1)}$ so that

$$\frac{d}{dx}\psi^{(n)}(x) = \psi^{(n+1)}(x) \qquad (\operatorname{Re} x > 0)$$

and

$$\psi^{(n)}(x) \sim (-\pi)^n e^{-\pi x} \qquad (\mathrm{Re}\, x \to \infty)$$

hold for every $n \geq -1$. The polynomials $P_0, P_1, P_2, \dots$ have the property that

$$\left(\frac{d}{dx}\right)^n \left(x^a e^{\lambda(\log x)^2 + i\pi t \log x}\right) = x^{a-n} P_n \left(2\lambda \log x + i\pi t + a\right) e^{\lambda(\log x)^2 + i\pi t \log x}$$
$$(n = 0, 1, 2 \dots,\ a \in \mathbb{C}).$$

We express $K_\theta(t)$ as

$$K_\theta(t) = 2 \int_{e^{i\theta}}^{e^{i\theta}+\infty} x^{5/4} e^{\lambda(\log x)^2 + i\pi t \log x} \psi''(x) dx$$
$$+ 3 \int_{e^{i\theta}}^{e^{i\theta}+\infty} x^{1/4} e^{\lambda(\log x)^2 + i\pi t \log x} \psi'(x) dx,$$

and apply integration by parts three times to the first integral and twice to the second one to obtain

$$K_\theta(t) = Q_\theta(t) e^{-\lambda\theta^2 - \pi t\theta} - L_\theta(t),$$

where

$$Q_\theta(t) = -2 \sum_{n=0}^{2} (-1)^n e^{\left(\frac{5}{4}-n\right)i\theta} P_n \left(2\lambda i\theta + i\pi t + \frac{5}{4}\right) \psi^{(1-n)}\left(e^{i\theta}\right)$$
$$- 3 \sum_{n=0}^{1} (-1)^n e^{\left(\frac{1}{4}-n\right)i\theta} P_n \left(2\lambda i\theta + i\pi t + \frac{1}{4}\right) \psi^{(-n)}\left(e^{i\theta}\right),$$

and

$$L_\theta(t) = \sum_{(l,m)\in\mathcal{I}} a_{(l,m)} t^l \int_{e^{i\theta}}^{e^{i\theta}+\infty} x^{-7/4} (\log x)^m e^{\lambda(\log x)^2 + i\pi t \log x} \psi^{(-1)}(x) dx.$$

Since $\deg P_n = n$ for all n, Q_θ is a polynomial of degree ≤ 2. If we define R_θ by $R_\theta(t) = Q_\theta(t) + \overline{Q_\theta(\overline{t})}$, then R_θ is a real polynomial of degree ≤ 2 and

$$H(t) = R_\theta(t) e^{-\lambda\theta^2 - \pi t\theta} - L_\theta(t) - \overline{L_\theta(\overline{t})}.$$

On the other hand, we have

$$\lim_{\theta \to \pi/2} L_\theta(t) = L(t)$$

for every t. Hence there is a real polynomial R of degree ≤ 2 such that

$$\lim_{\theta \to \pi/2} R_\theta(t) = R(t)$$

for every t, and this proves the proposition. $\square$

If $m \geq 0$ and $n \geq 1$ are integers, we put

$$I_{(m,n)}(t) = \int_i^{i+\infty} x^{-7/4}(\log x)^m e^{\lambda(\log x)^2 - n^2\pi x + i\pi t \log x} dx$$

and

$$\tilde{I}_{(m,n)}(t) = \int_i^{i+\infty} x^{-7/4}(\log x)^m e^{\lambda(\log x)^2 + i\pi t \log x} \psi_n^{(-1)}(x) dx,$$

where

$$\psi_n^{(-1)}(x) = \sum_{k=n}^{\infty} \frac{-1}{k^2\pi} e^{-k^2\pi x}.$$

For $S \in \mathbb{R}$ and $T > 0$ we put

$$F(S,T) = T^{-\frac{5}{4} - \pi S} e^{\lambda(\log T)^2 - \frac{\pi^2}{2}T}.$$

The following two propositions will be proved in the next section.

Proposition 3.3. *Suppose $m \geq 0$ and $n \geq 1$ are integers, and Δ is a positive constant. Then there is a positive constant T_1 such that*

$$\left|I_{(m,n)}(T + iS)\right| = c_n n^{2\pi S} T^{-4\lambda \log n} \left|\log \frac{iT}{n^2}\right|^m F(S,T)\left(1 + O\left(T^{-1}(\log T)^2\right)\right)$$
$$(-\Delta \leq S \leq \Delta,\ T \geq T_1),$$

where

$$c_n = \sqrt{2} n^{3/2} e^{\lambda\left(4(\log n)^2 - \frac{\pi^2}{4}\right)}.$$

Proposition 3.4. *Suppose $m \geq 0$ and $n \geq 1$ are integers, and Δ is a positive constant. Then there are positive constants C and T_1 such that*

$$\left|\tilde{I}_{(m,n)}(T + iS)\right| \leq C T^{\frac{1}{2} - 4\lambda \log n} \left|\log \frac{iT}{n^2}\right|^m F(S,T) \qquad (-\Delta \leq S \leq \Delta,\ T \geq T_1).$$

Now we prove Proposition 3.1.

Proof of Proposition 3.1. Let $\epsilon > 0$ be arbitrary. We must show that H has at most a finite number of zeros outside $\{t : |\mathrm{Im}\, t| \leq \epsilon\}$. By Proposition A, there is a positive constant Δ $(< \frac{1}{4\pi})$ such that the zeros of H lie in $\{t : |\mathrm{Im}\, t| \leq \Delta\}$, because $H(t) = \Xi_{4\lambda}(2\pi t)$, $\lambda > 0$ and the zeros of Ξ_0 lie in $\{t : |\mathrm{Im}\, t| < 1/2\}$. We may assume that $0 < \epsilon < \Delta$. Since Ξ_0 is an even real entire function and $\lambda \in \mathbb{R}$, so is H. If a is a zero of H, then so are $-a$, $\bar{a}$ and $-\bar{a}$. By Proposition 3.2,

$$H(t) = E(t) - L(t) - \overline{L(\bar{t})},$$

where

$$E(t) = R(t) e^{-\frac{\lambda\pi^2}{4} - \frac{\pi^2}{2}t}$$

and R is a polynomial. To prove the proposition, it is enough to show that there is a positive constant T_0 such that

$$|E(T+iS)| + |L(T+iS)| < |L(T-iS)| \qquad (\epsilon \le S \le \Delta,\ T \ge T_0). \tag{3.1}$$

Suppose $S \in \mathbb{R}$ and $T > 0$. Put $t = T + iS$. Let N be a positive integer such that

$$\frac{1}{2} - 4\lambda \log N \le -4\lambda \log 2.$$

By Propositions 3.3 and 3.4, we can find positive constants C and T_0 depending only on N and Δ such that the inequalities

$$\left| \frac{I_{(m,n)}(t)}{I_{(m,1)}(t)} \right| \le CT^{-4\lambda \log n} \qquad (m = 0,1,2,3,\ n = 2,3,\dots,N-1), \tag{3.2}$$

$$\left| \frac{\tilde{I}_{(m,N)}(t)}{I_{(m,1)}(t)} \right| \le CT^{-4\lambda \log 2} \qquad (m = 0,1,2,3) \tag{3.3}$$

and

$$\left| \frac{t^l I_{(m,1)}(t)}{t^3 I_{(0,1)}(t)} \right| \le CT^{-1} \log T \qquad ((l,m) \in \mathcal{I} \setminus \{(3,0)\}) \tag{3.4}$$

hold whenever $-\Delta \le S \le \Delta$ and $T \ge T_0$.

Since

$$\psi^{(-1)}(x) = -\sum_{n=1}^{N-1} \frac{1}{n^2\pi} e^{-n^2\pi x} + \psi_N^{(-1)}(x),$$

we have, by (3.2) and (3.3),

$$I_m(t) = -\sum_{n=1}^{N-1} \frac{1}{n^2\pi} I_{(m,n)}(t) + \tilde{I}_{(m,N)}(t) = \frac{-1}{\pi} I_{(m,1)}(t) \left(1 + O\left(T^{-4\lambda \log 2}\right)\right) \qquad (-\Delta \le S \le \Delta,\ T \ge T_0). \tag{3.5}$$

Since

$$L(t) = \sum_{(l,m)\in\mathcal{I}} a_{(l,m)} t^l I_m(t)$$

and $a_{(3,0)} = -2i\pi^3$, (3.4) and (3.5) imply

$$L(t) = 2i\pi^2 t^3 I_{(0,1)}(t) \left(1 + O\left(T^{-4\lambda \log 2}\right)\right) \qquad (-\Delta \le S \le \Delta,\ T \ge T_0)$$

in the case when $4\lambda \log 2 < 1$, and

$$L(t) = 2i\pi^2 t^3 I_{(0,1)}(t) \left(1 + O\left(T^{-1} \log T\right)\right) \qquad (-\Delta \le S \le \Delta,\ T \ge T_0)$$

in the case when $4\lambda\log 2 \geq 1$. If we put $A = 2^{3/2}\pi^2 e^{-\lambda\pi^2/4}$, then, by Proposition 3.3, we obtain

$$|L(T+iS)| = AT^{\frac{7}{4}-\pi S} e^{\lambda(\log T)^2 - \frac{\pi^2}{2}T}\left(1 + O\left(T^{-4\lambda\log 2}\right)\right) \qquad (-\Delta \leq S \leq \Delta,\ T \geq T_0)$$

in the case when $4\lambda\log 2 < 1$, and

$$|L(T+iS)| = AT^{\frac{7}{4}-\pi S} e^{\lambda(\log T)^2 - \frac{\pi^2}{2}T}\left(1 + O\left(T^{-1}(\log T)^2\right)\right) \qquad (-\Delta \leq S \leq \Delta,\ T \geq T_0)$$

in the case when $4\lambda\log 2 \geq 1$. Now, it is clear that if we take T_0 sufficiently large, then (3.1) holds. $\square$

4. Proof of Propositions 3.3 and 3.4

We begin this section by introducing some functions which will be used in our proof of the propositions.

The relation $e^z - 1 - z = \frac{1}{2}u^2$ defines a unique one-to-one conformal mapping $z = g(u)$ from the region

$$\Omega = \{u : \mathrm{Re}\, u > -\sqrt{2\pi} \quad \text{or} \quad |\mathrm{Im}\, u|\mathrm{Re}\, u \neq -2\pi\}$$

onto the strip $\{z : |\mathrm{Im}\, z| < 2\pi\}$ such that $g(0) = 0$ and $g'(0) = 1$. If we put

$$\Omega_1 = \{u : |\mathrm{Im}\, u|\mathrm{Re}\, u > \pi\},$$

then $\Omega_1 \subset \Omega$ and $g(\Omega_1) = \{z : |\mathrm{Im}\, z| < \pi\}$. We define the function G by

$$G(u) = e^{g(u)} - 1. \tag{4.1}$$

The function G is analytic in Ω and one-to-one in Ω_1. We have $G(0) = 0$, $G'(0) = 1$, $G(\Omega_1) = \{w : \mathrm{Re}\, w > -1 \quad \text{or} \quad \mathrm{Im}\, w \neq 0\}$, and

$$G(u) - \log(1 + G(u)) = \frac{1}{2}u^2 \qquad (u \in \Omega_1). \tag{4.2}$$

Lemma 4.1. *If we put $\alpha = e^{-\pi i/4}$, then there is a constant ρ_0 with $0 < \rho_0 < 2\sqrt{\pi}$ such that the inequalities*

$$\frac{\rho}{2} \leq (-1)^j \mathrm{Re}\, g\left((-1)^j \rho\alpha\right),\ (-1)^{j+1}\mathrm{Im}\, g\left((-1)^j \rho\alpha\right) \leq \rho \qquad (j = 1, 2)$$

hold for $0 \leq \rho \leq \rho_0$.

Proof. The result is a consequence of the facts that g is analytic in the disk $\{u : |u| < 2\sqrt{\pi}\}$, $g(0) = 0$, $g'(0) = 1$, $\mathrm{Re}\,\alpha = -\mathrm{Im}\,\alpha = 1/\sqrt{2}$, and $1/2 < 1/\sqrt{2} < 1$. $\square$

Now we prove Propositions 3.3 and 3.4. Let $m \geq 0$ and $n \geq 1$ be fixed integers, and λ be a fixed positive constant. Suppose $S \in \mathbb{R}$ and $T > 0$. Put $t = T + iS$, $\xi = iT/n^2$, $h(x) = e^{n^2\pi x}\psi_n^{(-1)}(x)$ and

$$\begin{aligned} U(x) = U(x;t) &= x^{-7/4}(\log x)^m e^{\lambda(\log x)^2 - n^2\pi x + i\pi t \log x} \\ &= x^{-\frac{7}{4} - \pi S}(\log x)^m e^{\lambda(\log x)^2 - n^2\pi x + i\pi T \log x}. \end{aligned}$$

Thus

$$I_{(m,n)}(t) = \int_i^{i+\infty} U(x)dx, \quad \tilde{I}_{(m,n)}(t) = \int_i^{i+\infty} U(x)h(x)dx,$$

and

$$|\xi U(\xi)| = 2^{-1/2} c_n n^{2\pi S} T^{\frac{1}{2} - 4\lambda \log n} \left| \log \frac{iT}{n^2} \right|^m F(S,T).$$

To prove Propositions 3.3 and 3.4, it is enough to show that for every $\Delta > 0$ there is a positive constant T_1 such that

$$I_{(m,n)}(t) = e^{-\pi i/4}(T/2)^{-1/2}\xi U(\xi)\left(1 + O\left(T^{-1}(\log T)^2\right)\right) \tag{4.3}$$

and

$$|\tilde{I}_{(m,n)}(t)| = O\left(|\xi U(\xi)|\right) \tag{4.4}$$

hold for $-\Delta \leq S \leq \Delta$ and $T \geq T_1$.

The function h is analytic in the right half plane $\{x : \operatorname{Re} x > 0\}$, continuous in the closed right half plane $\{x : \operatorname{Re} x \geq 0\}$, and $|h(x)| \leq \pi/6$ for all x in the closed right half plane. On the other hand, U is analytic in the region $\{x : \operatorname{Re} x > 0 \text{ or } \operatorname{Im} x \neq 0\}$ which properly contains the closed right half plane.

Let ρ_0 be as in Lemma 4.1,

$$M = \max_{|u|=1} |g(u)| \quad \text{and} \quad \rho = \min\{1/2, \rho_0, (8\lambda M)^{-1}\}.$$

We put $\alpha = e^{-\pi i/4}$, $x_1 = \xi(1 + G(-\rho\alpha))$, $x_2 = \xi(1 + G(\rho\alpha))$, and $x_0 = x_1/|x_1|$. By (4.1), we have $x_1 = \xi e^{g(-\rho\alpha)}$ and $x_2 = \xi e^{g(\rho\alpha)}$. If we write $x_1 = i|x_1|e^{i\theta_1}$ and $x_2 = i|x_2|e^{i\theta_2}$, then Lemma 4.1 implies that $|\xi|e^{-\rho} \leq |x_1| \leq |\xi|e^{-\rho/2}$, $|\xi|e^{\rho/2} \leq |x_2| \leq |\xi|e^{\rho}$, $\rho/2 \leq \theta_1 \leq \rho$ and $-\rho \leq \theta_2 \leq -\rho/2$. In particular, x_0 and x_1 lie in the second quadrant $\{x : x \neq 0 \text{ and } \pi/2 < \arg x < \pi\}$, and x_2 in the first quadrant $\{x : x \neq 0 \text{ and } 0 < \arg x < \pi/2\}$. By applying Cauchy's theorem, we obtain

$$\int_i^{i+\infty} U(x)dx = \int_i^{x_0} U(x)dx + \int_{x_0}^{x_1} U(x)dx + \int_{x_1}^{x_2} U(x)dx + \int_{x_2}^{x_2+\infty} U(x)dx$$

and

$$\int_i^{i+\infty} U(x)h(x)dx = \int_i^{\xi} U(x)h(x)dx + \int_{\xi}^{x_2} U(x)h(x)dx + \int_{x_2}^{x_2+\infty} U(x)h(x)dx.$$

Now, the propositions are proved by a routine application of the saddle point method [2], however, for the reader's convenience, we present a detailed proof. We prove the propositions by showing that for every $\Delta > 0$ there is a $T_1 > 0$ such that the following hold for $-\Delta \le S \le \Delta$ and $T \ge T_1$:

$$\int_{x_1}^{x_2} U(x)dx = \alpha(T/2)^{-1/2}\xi U(\xi)\left(1 + O\left(T^{-1}(\log T)^2\right)\right), \tag{4.5}$$

$$\int_{i}^{x_0} U(x)dx = O\left(|U(\xi)|\, T^{\frac{7}{4}+\pi\Delta+4\lambda\log n} e^{-\lambda(\log T)^2}\right), \tag{4.6}$$

$$\int_{x_0}^{x_1} U(x)dx = O\left(|U(\xi)|\, e^{-\pi T\rho^2/2}\right), \tag{4.7}$$

$$\int_{x_2}^{x_2+\infty} U(x)dx = O\left(|U(\xi)|\, T^{2\lambda\rho} e^{-\pi T\rho^2/2}\right), \tag{4.8}$$

$$\int_{i}^{\xi} U(x)h(x)dx = O\left(|\xi U(\xi)|\right), \tag{4.9}$$

$$\int_{\xi}^{x_2} U(x)h(x)dx = O\left(|U(\xi)|\, T^{3/4}\right), \tag{4.10}$$

and

$$\int_{x_2}^{x_2+\infty} U(x)h(x)dx = O\left(|U(\xi)|\, T^{2\lambda\rho} e^{-\pi T\rho^2/2}\right). \tag{4.11}$$

Thus (4.3) is a consequence of (4.5) through (4.8), and (4.4) is a consequence of (4.9) through (4.11).

In the remainder of this section, we assume that Δ is a fixed positive constant and that $-\Delta \le S \le \Delta$.

Proof of (4.5) and (4.10). If we put $x(u) = \xi e^{g(\alpha u)}$, then Lemma 4.1 implies that $x(u)$ lies in the second quadrant when $-\rho \le u < 0$ and in the first quadrant when $0 < u \le \rho$. By (4.1), we have $x(u) = \xi\left(1 + G(\alpha u)\right)$. Hence

$$\int_{x_1}^{x_2} U(x)dx = \alpha\xi \int_{-\rho}^{\rho} G'(\alpha u) U\left(\xi e^{g(\alpha u)}\right) du$$

and

$$\int_{\xi}^{x_2} U(x)h(x)dx = \alpha\xi \int_{0}^{\rho} G'(\alpha u) U\left(\xi e^{g(\alpha u)}\right) h\left(\xi e^{g(\alpha u)}\right) du.$$

Since $\xi = iT/n^2$, we have $|\log \xi| \geq \pi/2$. In particular, $\log \xi \neq 0$. Hence

$$U\left(\xi e^{g(\alpha u)}\right) = U(\xi)\left(1 + (\log \xi)^{-1} g(\alpha u)\right)^m e^{2\lambda g(\alpha u)\log \xi + \lambda g(\alpha u)^2 - \left(\frac{7}{4}+\pi S\right) g(\alpha u)} e^{-\pi T u^2/2}. \tag{4.12}$$

If we put

$$V(u) = V(u;t) = G'(u)\left(1 + (\log \xi)^{-1} g(u)\right)^m e^{2\lambda g(u)\log \xi + \lambda g(u)^2 - \left(\frac{7}{4}+\pi S\right) g(u)}, \tag{4.13}$$

then V is analytic in $\{u : |u| < 2\sqrt{\pi}\}$,

$$\int_{x_1}^{x_2} U(x)dx = \alpha \xi U(\xi) \int_{-\rho}^{\rho} V(\alpha u) e^{-\pi T u^2/2} du \tag{4.14}$$

and

$$\int_{\xi}^{x_2} U(x)h(x)dx = \alpha \xi U(\xi) \int_0^{\rho} V(\alpha u) h\left(\xi e^{g(\alpha u)}\right) e^{-\pi T u^2/2} du. \tag{4.15}$$

We have

$$\max_{|u|\leq r} |g(u)| \leq rM \qquad (0 \leq r \leq 1),$$

because $g(0) = 0$ and $|g(u)| \leq M$ for $|u| = 1$. We also have $|\log \xi| \geq \pi/2$ and $-\Delta \leq S \leq \Delta$. Hence there is a positive constant C_1 such that

$$\max_{|u|\leq r} |V(u)| \leq C_1 T^{2\lambda r M} \qquad (0 \leq r \leq 1). \tag{4.16}$$

Since $\rho = \min\{1/2, \rho_0, (8\lambda M)^{-1}\}$, it follows that $|V(u)| \leq C_1 T^{1/4}$ for $|u| \leq \rho$, and we have seen that $|h(x)| \leq \pi/6$ for $\operatorname{Re} x \geq 0$. From

$$\int_0^{\infty} u^k e^{-\pi T u^2/2} du = \frac{1}{2}\left(\frac{\pi T}{2}\right)^{-\frac{k+1}{2}} \Gamma\left(\frac{k+1}{2}\right) \qquad (k > -1) \tag{4.17}$$

and (4.15), we obtain

$$\left|\int_{\xi}^{x_2} U(x)h(x)dx\right| \leq \frac{T}{n^2}|U(\xi)| \int_0^{\rho} \left|V(\alpha u) h\left(\xi e^{g(\alpha u)}\right)\right| e^{-\pi T u^2/2} du = O\left(|U(\xi)|\, T^{3/4}\right).$$

This proves (4.10).

We have $g(0) = 0$ and $G'(0) = 1$. Hence $V(0) = 1$; and V is analytic in $\{u : |u| < 2\sqrt{\pi}\}$. We may express V as

$$V(u) = 1 + b_1 u + b_2 u^2 + B(u) u^3 \qquad (|u| < 2\sqrt{\pi}),$$

where $b_1 = V'(0)$, $b_2 = \frac{1}{2}V''(0)$, and

$$B(u) = \frac{1}{2\pi i}\int_{|z|=r} \frac{V(z)}{z^3(z-u)}dz \qquad (|u| < r < 2\sqrt{\pi}). \tag{4.18}$$

By (4.13), we see that $|b_2| \leq C_2(\log T)^2$ for some positive constant C_2; and by (4.18), we have

$$\max_{|u|\leq\rho} |B(u)| \leq \frac{1}{4\rho^3} \max_{|u|=2\rho} |V(u)|.$$

Hence, by (4.16), there is a positive constant C_3 such that $|B(u)| \leq C_3 T^{1/2}$ for $|u| \leq \rho$. Thus we obtain (4.5) from (4.14), (4.17),

$$\int_\rho^\rho V(\alpha u)e^{-\pi T u^2/2}du = 2\int_0^\infty e^{-\pi T u^2/2}du$$
$$- 2\int_\rho^\infty e^{-\pi T u^2/2}du + 2\alpha^2 b_2 \int_0^\rho u^2 e^{-\pi T u^2/2}du + \alpha^3 \int_{-\rho}^\rho B(u)u^3 e^{-\pi T u^2/2}du$$

and

$$\int_\rho^\infty e^{-\pi T u^2/2}du < (\rho\pi T)^{-1}e^{-\pi T\rho^2/2}. \quad \square$$

Proof of (4.6). By straightforward calculation, we obtain

$$|U(i)| = |U(\xi)|\left(T/n^2\right)^{\frac{7}{4}+\pi S}\left(1 + \frac{4}{\pi^2}\left(\log\frac{T}{n^2}\right)^2\right)^{-m/2} e^{-\lambda\left(\log\frac{T}{n^2}\right)^2},$$

so that

$$|U(i)| \leq C|U(\xi)|T^{\frac{7}{4}+\pi\Delta+4\lambda\log n}e^{-\lambda(\log T)^2}$$

for some positive constant C. If we write $x_0 = ie^{i\theta_1}$, then $\rho/2 \leq \theta_1 \leq \rho$ $(< \pi/2)$. Thus

$$\left|\int_i^{x_0} U(x)dx\right| \leq \int_{\frac{\pi}{2}}^{\frac{\pi}{2}+\theta_1} \left|U\left(e^{i\theta}\right)\right| d\theta.$$

We have

$$\frac{d}{d\theta}\log\left|U\left(e^{i\theta}\right)\right| = \frac{m}{\theta} - 2\lambda\theta + n^2\pi\sin\theta - \pi T.$$

If $T \geq \left(\frac{2m}{\pi^2} + n^2 + \frac{1}{\pi}\right)$, then the right hand side is less than -1 for every $\theta \geq \pi/2$, so that

$$\int_{\frac{\pi}{2}}^{\frac{\pi}{2}+\theta_1} \left|U\left(e^{i\theta}\right)\right| d\theta \leq |U(i)|,$$

and hence (4.6) holds. $\square$

Proof of (4.7). Suppose $T \geq n^2 e$, so that $|x_1| > 1$. We have $|\log \xi| \geq \pi/2$, $x_1 = \xi e^{g(-\rho\alpha)}$, $-\rho \leq \operatorname{Re} g(-\rho\alpha) \leq -\rho/2$ and $\rho/2 \leq \operatorname{Im} g(-\rho\alpha) \leq \rho$. By (4.12), there is a positive constant C such that

$$|U(x_1)| \leq C|U(\xi)|e^{-\pi T\rho^2/2}.$$

Since $x_0 = x_1/|x_1|$,

$$\left|\int_{x_0}^{x_1} U(x)dx\right| \leq \int_1^{|x_1|} |U(rx_0)|dr.$$

Let c be a constant such that $0 < c < n^2\pi\sin(\rho/2)$. If we put $x_0 = ie^{i\theta_1}$, then

$$\begin{aligned}\frac{d}{dr}\log|U(rx_0)| &= -\left(\frac{7}{4}+\pi S\right)r^{-1} + \frac{m\log r}{r\left((\log r)^2 + \left(\theta_1+\frac{\pi}{2}\right)^2\right)} + \frac{2\lambda\log r}{r} + n^2\pi\sin\theta_1 \\ &> n^2\pi\sin\frac{\rho}{2} - \left(\frac{7}{4}+\pi\Delta\right)r^{-1} \qquad (r \geq 1).\end{aligned}$$

Hence there is a positive constant b_1 such that

$$\int_1^b |U(rx_0)|dr \leq c^{-1}|U(bx_0)| \qquad (b \geq b_1).$$

Since $|x_1| > |\xi|e^{-\rho} = n^{-2}e^{-\rho}T$, (4.7) holds whenever $T \geq n^2 e^\rho b_1$. $\square$

Proof of (4.8) and (4.11). From (4.12),

$$|U(x_2)| \leq C_1|U(\xi)|T^{2\lambda\rho}e^{-\pi T\rho^2/2}$$

for some positive constant C_1. It is clear that

$$\left|\int_{x_2}^{x_2+\infty} U(x)dx\right|,\ \frac{6}{\pi}\left|\int_{x_2}^{x_2+\infty} U(x)h(x)dx\right| \leq \int_0^\infty |U(x_2+x)|dx.$$

Let

$$U_1(x) = e^{-n^2\pi x + i\pi T\log x} \quad \text{and} \quad U_2(x) = x^{-\frac{7}{4}-\pi S}(\log x)^m e^{\lambda(\log x)^2},$$

so that $U(x) = U_1(x)U_2(x)$.

Since $|x_2| \geq e^{\rho/2}n^{-2}T$ and $0 \leq \frac{\pi}{2} - \rho \leq \arg x_2 \leq \frac{\pi}{2} - \frac{\rho}{2}$, we have

$$\frac{d}{dx}\log|U_1(x_2+x)| = -n^2\pi + \frac{\pi T \operatorname{Im} x_2}{|x_2+x|^2} \leq -n^2\pi\left(1 - e^{-\rho/2}\cos\frac{\rho}{2}\right) \qquad (x \geq 0).$$

If $|x_2| \geq e$, then

$$\begin{aligned}\frac{d}{dx}\log|U_2(x_2+x)| &= \operatorname{Re}\frac{U_2'(x_2+x)}{U_2(x_2+x)} \leq \left|\frac{U_2'(x_2+x)}{U_2(x_2+x)}\right| \\ &\leq |x_2|^{-1}\left(\frac{7}{4}+\pi\Delta+m+2\lambda\log|x_2|+\lambda\pi\right) \qquad (x \geq 0).\end{aligned}$$

Let c be a constant such that $0 < c < n^2\pi\left(1 - e^{-\rho/2}\cos(\rho/2)\right)$. Since $|x_2| \geq e^{\rho/2}n^{-2}T$, there is a positive constant T_1 such that

$$|x_2|^{-1}\left(\frac{7}{4} + \pi\Delta + m + 2\lambda\log|x_2| + \lambda\pi\right) \leq n^2\pi\left(1 - e^{-\rho/2}\cos\frac{\rho}{2}\right) - c$$

holds whenever $T \geq T_1$. Now, suppose that $T \geq T_1$. Then

$$\frac{d}{dx}\log|U(x_2+x)| = \frac{d}{dx}\log|U_1(x_2+x)| + \frac{d}{dx}\log|U_2(x_2+x)| \leq -c \qquad (x \geq 0),$$

so that

$$\int_0^\infty |U_2(x_2+x)|dx \leq c^{-1}|U(x_2)|;$$

hence (4.8) and (4.11) hold. $\square$

Proof of (4.9). We have

$$\begin{aligned}|U(ir)h(ir)| &\leq \frac{\pi}{6}\,|U(ir)| \\ &= \frac{\pi}{6}\left((\log r)^2 + \frac{\pi^2}{4}\right)^{m/2}\exp\left(\lambda(\log r)^2 - \left(\frac{7}{4} + \pi S\right)\log r - \left(\frac{\lambda\pi^2}{4} + \frac{\pi^2}{2}T\right)\right)\end{aligned}$$

for $r > 0$. If $T \geq n^2 e^{\lambda^{-1}\left(\frac{7}{4}+\pi\Delta\right)}$, that is, $\log|\xi| \geq \lambda^{-1}\left(\frac{7}{4} + \pi\Delta\right)$, then we have

$$|U(ir)h(ir)| \leq \frac{\pi}{6}\,|U(\xi)| \qquad (1 \leq r \leq |\xi|),$$

so that

$$\left|\int_i^\xi U(x)h(x)dx\right| \leq \int_1^{|\xi|} |U(ir)h(ir)|\,dr \leq \frac{\pi}{6}(|\xi| - 1)\,|U(\xi)|\,,$$

and hence (4.9) holds. $\square$

5. Proof of Theorem 1.4

Suppose $\lambda > 0$. As in Section 3, we put $H(t) = \Xi_{4\lambda}(2\pi t)$. Let L and R be as in Proposition 3.2, and put

$$H_0(t) = R(t)e^{-\frac{\lambda\pi^2}{4} - \frac{\pi^2}{2}t} - 2L(t),$$

so that $H(t) = \operatorname{Re} H_0(t)$ for all $t \in \mathbb{R}$. Throughout this section, T denotes a positive variable, and the asymptotic expressions are understood as $T \to \infty$.

Since

$$L(t) = \sum_{(l,m)\in\mathcal{I}} a_{(l,m)}t^l I_m(t)$$

and

$$I_m(t) = \int_i^{i+\infty} x^{-7/4}(\log x)^m e^{\lambda(\log x)^2 + i\pi t \log x}\psi^{(-1)}(x)dx,$$

we have

$$L'(t) = \sum_{(l,m)\in\mathcal{I}} a_{(l,m)} \left(i\pi t^l I_{m+1}(t) + lt^{l-1}I_m(t)\right).$$

By the same argument as in the proof of Proposition 3.1, we obtain

$$H_0(T) = -4i\pi^2 T^3 I_{(0,1)}(T)\,(1+o(1))$$

and

$$H_0'(T) = 4\pi^3 T^3 I_{(1,1)}(T)\,(1+o(1)).$$

If we put

$$U(x) = U(x;T) = x^{-7/4}e^{\lambda(\log x)^2 - \pi x + i\pi T \log x},$$

then, by (4.3), we have

$$I_{(0,1)}(T) = e^{-\pi i/4}(T/2)^{-1/2} iTU(iT)\,(1+o(1))$$

and

$$I_{(1,1)}(T) = (\log iT) I_{(0,1)}(T)\,(1+o(1)).$$

In particular, there is a positive constant T_0 such that $H_0(T) \neq 0$ for $T > T_0$.

If we write

$$H_0(T) = |H_0(T)|e^{i\pi\theta(T)},$$

then we have

$$H(T) = |H_0(T)| \cos \pi\theta(T).$$

We also have

$$\frac{d}{dT}\theta(T) = \frac{1}{\pi}\mathrm{Im}\,\frac{H_0'(T)}{H_0(T)}, \quad \frac{H_0'(T)}{H_0(T)} = i\pi(\log iT)\,(1+o(1)) \qquad (T > T_0),$$

and

$$\theta(T) = T\log T - T + \lambda \log T + O(1).$$

Hence we obtain Theorem 1.4, because $H(t) = \Xi_{4\lambda}(2\pi t)$.

6. Examples

We begin with a general argument. Suppose $\{a_n\}$ is a sequence of real numbers with $a_1 \neq 0$, and $\{b_n\}$ is a non-decreasing sequence of positive numbers with $b_1 < b_2$ such that

$$\sum_{n=1}^{\infty} |a_n| b_n^{-A} < \infty$$

for some positive constant A. Put

$$L(s) = \sum_{n=1}^{\infty} a_n b_n^{-s} \quad \text{and} \quad \psi(x) = \sum_{n=1}^{\infty} a_n e^{-b_n x}.$$

Then L is analytic in the half plane $\{s : \operatorname{Re} s > A\}$, ψ is analytic in the right half plane $\{x : \operatorname{Re} x > 0\}$, $\psi(x) = \overline{\psi(\overline{x})}$ for all x there, and $\psi(x) \sim a_1 e^{-b_1 x}$ for $\operatorname{Re} x \to \infty$. One can find that these two functions are related through

$$\Gamma(s)L(s) = \int_0^{\infty} x^s \psi(x) \frac{dx}{x} \qquad (\operatorname{Re} s > A). \tag{6.1}$$

Since $a_1 \neq 0$, $\{b_n\}$ is non-decreasing and $b_1 < b_2$, there is a constant $B(\geq A)$ such that L has no zeros in the half plane $\{s : \operatorname{Re} s > B\}$. If we define the functions $\psi^{(-1)}, \psi^{(-2)}, \dots$ by

$$\psi^{(-l)}(x) = \sum_{n=1}^{\infty} a_n (-b_n)^{-l} e^{-b_n x} \qquad (l = 1, 2, \dots),$$

then these functions are analytic in the right half plane, and $\psi^{(-l)}$ is continuous and bounded in the closed half plane $\{x : \operatorname{Re} x \geq 0\}$ whenever $l \geq A$.

Suppose there are constants $\delta \in \{0, 1\}$ and $k, c_0, \dots, c_N \in \mathbb{R}$, with $c_N \neq 0$, such that

$$(-1)^{\delta} x^k \sum_{n=0}^{N} c_n x^n \psi^{(n)}(x) = \sum_{n=0}^{N} \frac{c_n}{x^n} \psi^{(n)}\left(\frac{1}{x}\right) \qquad (\operatorname{Re} x > 0). \tag{6.2}$$

If we put

$$h(s) = \int_0^{\infty} x^s \left(\sum_{n=0}^{N} c_n x^n \psi^{(n)}(x) \right) \frac{dx}{x},$$

then h becomes an entire function satisfying the functional equation

$$h(s) = (-1)^{\delta} h(k - s),$$

and, by (6.1), there is a real polynomial P of degree N such that

$$h(s) = P(s)\Gamma(s)L(s) \qquad (\operatorname{Re} s > A). \tag{6.3}$$

Define the function φ by

$$\varphi(x) = x^{k/2} \sum_{n=0}^{N} c_n x^n \psi^{(n)}(x),$$

and for arbitrary constant λ define the function H_λ by

$$H_\lambda(t) = i^{\delta} \int_0^{\infty} e^{\lambda (\log x)^2 + it \log x} \varphi(x) \frac{dx}{x}.$$

We see that φ is analytic in the right half plane, $\varphi(x) = \overline{\varphi(\overline{x})} = (-1)^\delta \varphi(1/x)$ for all x in the right half plane,

$$\varphi(x) \sim C x^{\frac{k}{2}+N} e^{-b_1 x} \qquad (\mathrm{Re}\, x \to \infty)$$

holds for some constant $C \neq 0$, $H_0(t) = i^\delta h\left(\frac{k}{2} + it\right)$, H_λ is an entire function of order 1 and maximal type satisfying $H_\lambda(-t) = (-1)^\delta H_\lambda(t)$, and H_λ is a real entire function whenever $\lambda \in \mathbb{R}$. Since L has no zeros in the half plane $\{s : \mathrm{Re}\, s > B\}$, (6.3) implies that the zeros of H_0 lie in $\{t : |\mathrm{Im}\, t| \leq \Delta\}$ for some constant $\Delta \geq 0$. There is now no difficulty in extending the results of this paper to the functions H_λ, $\lambda > 0$.

Example 1. Let χ be a primitive Dirichlet character modulo $q(> 2)$, and denote the Gauss sum by $\tau(\chi)$:

$$\tau(\chi) = \sum_{n=1}^{q} \chi(n) e^{2\pi i n/q}.$$

We have $|\tau(\chi)| = \sqrt{q}$ and

$$x^{\frac{1}{2}+a} \sum_{n=1}^{\infty} n^a \chi(n) e^{-n^2 \pi x/q} = \frac{(-i)^a \tau(\chi)}{\sqrt{q}} \sum_{n=1}^{\infty} n^a \overline{\chi(n)} e^{-n^2 \pi x^{-1}/q}, \tag{6.4}$$

where $a = 0$ if $\chi(-1) = 1$ and $a = 1$ otherwise [9, p.85]. Choose a constant ω so that $(-i)^a \tau(\chi) \omega^2 = \sqrt{q}$ and put

$$\psi(x) = \sum_{n=1}^{\infty} n^a \mathrm{Re}\,(\omega \chi(n)) e^{-n^2 \pi x/q}.$$

Then (6.4) implies that

$$x^{\frac{1}{2}+a} \psi(x) = \psi(1/x).$$

Thus (6.2) is satisfied with $\delta = 0$, $k = \frac{1}{2} + a$ and $N = 0$, and we obtain the analogues of Theorems 1.1 through 1.4 for the functions H_λ, $\lambda > 0$, defined by

$$H_\lambda(t) = \int_0^\infty e^{\lambda (\log x)^2 + it \log x} \varphi(x) \frac{dx}{x},$$

where $\varphi(x) = x^{\frac{1}{4}+\frac{a}{2}} \psi(x)$.

If χ is real, that is, $\chi(n) \in \mathbb{R}$ for all n, then it is known that $(-i)^a \tau(\chi) = \sqrt{q}$ [9, p.49], and hence we may take $\omega = 1$. In this case, H_0 and the Dirichlet L-function are related through

$$H_0(t) = \Gamma(s)(q/\pi)^s L(2s - a, \chi) \qquad (s = \tfrac{1}{4} + \tfrac{a}{2} + it),$$

so that the zeros of H_0 lie in $\{t : |\mathrm{Im}\, t| < 1/4\}$.

If χ is not real, then H_0 may have infinitely many non-real zeros. A concrete example is found in Section 10.25 of [18].

Example 2. Let the functions Λ and ψ be defined by

$$\Lambda(s) = (2\pi)^{-s}\Gamma(s)\zeta(s)\zeta(s-k+1) \quad \text{and} \quad \psi(x) = \sum_{n=1}^{\infty} \sigma_{k-1}(n) e^{-2n\pi x},$$

where k is an even integer, ζ is the Riemann zeta-function and

$$\sigma_{k-1}(n) = \sum_{d|n} d^{k-1}.$$

These functions are related through

$$\Lambda(s) = \int_0^\infty x^s \psi(x) \frac{dx}{x} \qquad (\operatorname{Re} s > \max\{1, k\}).$$

The function Λ is meromorphic in the whole s-plane and has a finite number of poles. The poles lie in the set $\{0, -1, -2, \dots\} \cup \{1, k\}$, and they are symmetrically located with respect to the point $s = k/2$. The non-real zeros of Λ lie in $\{s : 0 < \operatorname{Re} s < 1 \text{ or } k-1 < \operatorname{Re} s < k\}$. If $k \le 4$, Λ has no real zeros; otherwise it has exactly $\frac{k}{2} - 2$ real zeros and they are at the odd integers in the interval $[3, k-3]$. From well known properties of ζ and Γ, we see that Λ satisfies the functional equation

$$\Lambda(s) = (-1)^{k/2}\Lambda(k-s),$$

and Λ is bounded in the vertical strip $\{s : a < \operatorname{Re} s < b\}$ whenever $-\infty < a < b < \infty$. Let P denote the monic real polynomial of the smallest degree such that $P\Lambda$ becomes an entire function. Then there are constants $\delta \in \{0, 1\}$ and $c_0, \dots, c_N \in \mathbb{R}$, with $c_N = (-1)^N$, such that

$$P(s)\Lambda(s) = \int_0^\infty x^s \left(\sum_{n=0}^{N} c_n x^n \psi^{(n)}(x) \right) \frac{dx}{x} \qquad (s \in \mathbb{C})$$

and (6.2) hold; hence we obtain the analogues of the results of this paper for the functions $e^{-\lambda D^2} H$, $\lambda > 0$, where

$$H(t) = i^\delta P\left(\frac{k}{2} + it\right) \Lambda\left(\frac{k}{2} + it\right). \tag{6.5}$$

For instance, if $k \ge 4$, then we have

$$(-1)^{k/2} x^k \left(\frac{\zeta(k)\Gamma(k)}{(2\pi i)^k} + \psi(x) \right) = \frac{\zeta(k)\Gamma(k)}{(2\pi i)^k} + \psi\left(\frac{1}{x}\right) \qquad (\operatorname{Re} x > 0)$$

[8, pp. 10–14], $P(s) = s(s-k)$,

$$s(s-k)\Lambda(s) = \int_0^\infty x^s \left((k+1)x\psi'(x) + x^2\psi''(x) \right) \frac{dx}{x} \qquad (s \in \mathbb{C}),$$

and

$$(-1)^{k/2} x^k \left((k+1)x\psi'(x) + x^2\psi''(x) \right) = \frac{k+1}{x}\psi'\left(\frac{1}{x}\right) + \frac{1}{x^2}\psi''\left(\frac{1}{x}\right) \qquad (\operatorname{Re} x > 0).$$

Finally, we remark that if $k \le 4$ or k is a multiple of 4, then $\delta = 0$ and the function H defined by (6.5) is an even real entire function having no real zeros at all, and hence so are the functions $e^{-\lambda D^2} H$, $\lambda < 0$, by Lemma 3.2 of [12].

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