Chordal cubic spline interpolation is fourth order accurate

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Abstract: It is well known that complete cubic spline interpolation of functions with four continuous derivatives is fourth order accurate. In this paper we show that this kind of interpolation, when used to construct parametric spline curves through sequences of points in any space dimension, is again fourth order accurate if the parameter intervals are chosen by chord length. We also show how such chordal spline interpolants can be used to approximate the arc length derivatives of a curve and its length.

Keywords: curve parameterization, arc length, spline interpolation, approximation order.

1. Introduction

This paper continues a study, started in [**4**], of the effect of parameterization on the rate of convergence of parametric curve interpolation. While results for polynomial interpolation of arbitrary degree were obtained in [**4**], we begin here with spline interpolation, and study cubic spline interpolation based on the chord length parameterization. While this method for curve fitting has been in widespread use for a long time, and was suggested as early as 1967 by Ahlberg, Nilson, and Walsh [**1**], little seems to be known about its approximation order. Roughly speaking, we obtain results analogous to those found in [**4**] for cubic polynomials. The main point is that chordal cubic spline interpolation has fourth order accuracy, which provides a sound justification for choosing the chord length parameterization rather than some other parameterization such as uniform or centripetal [**6**]: the latter two were found to lead at best to second order accuracy in numerical tests on polynomial interpolation in [**4**].

Let $\mathbf{f} : [a, b] \to \mathbb{R}^d$, $d \geq 2$, be a regular parametric curve, parameterized with respect to arc length, i.e., $\mathbf{f}$ is a continuously differentiable function such that $|\mathbf{f}'(s)| = 1$, for all $s \in [a, b]$, where $|\cdot|$ is the Euclidean norm in $\mathbb{R}^d$. We will study cubic spline interpolation with derivative end conditions. Thus suppose that corresponding to some parameter values

$$a = s_0 < s_1 < \cdots < s_n = b,$$

we are given the points $\mathbf{f}(s_0), \ldots, \mathbf{f}(s_n)$ and the (unit) derivatives $\mathbf{f}'(s_0)$ and $\mathbf{f}'(s_n)$. In many applications the values s_i are unknown, and so we are forced to choose some appropriate parameter values

$$t_0 < t_1 < \cdots < t_n,$$

in order to fit a cubic spline interpolant $\boldsymbol{\sigma} : [t_0, t_n] \to \mathbb{R}^d$, by which we mean the unique C^2 curve whose restriction to each interval $[t_i, t_{i+1}]$, $i = 0, 1, \ldots, n-1$, is a cubic polynomial curve, and such that

$$\begin{aligned} \boldsymbol{\sigma}(t_i) &= \mathbf{f}(s_i), \qquad i = 0, 1, \ldots, n, \\ \boldsymbol{\sigma}'(t_i) &= \mathbf{f}'(s_i), \qquad i = 0, n. \end{aligned} \tag{1.1}$$

Our main result is to show that under the *chord length parameterization*

$$t_{i+1} - t_i = |\mathbf{f}(s_{i+1}) - \mathbf{f}(s_i)|, \qquad i = 0, 1, \ldots, n-1, \tag{1.2}$$

if $\mathbf{f} \in C^4[a,b]$ then the spline $\boldsymbol{\sigma}$ converges to the curve $\mathbf{f}$ at the rate of h_s^4 as $h_s \to 0$, where

$$h_s := \max_{0 \le i \le n-1} (s_{i+1} - s_i).$$

We later show that the derivatives of $\mathbf{f}$ are approximated by the *arc-length* derivatives of $\boldsymbol{\sigma}$ to the same accuracy as for functions. We also show that the length of $\mathbf{f}$ is approximated by the length of $\boldsymbol{\sigma}$ with fourth order accuracy. At the end of the paper we discuss how the derivatives of $\mathbf{f}$ are even approximated *directly* by the derivatives of $\boldsymbol{\sigma}$ and that the parameterization of $\boldsymbol{\sigma}$ converges to arc length. However, the convergence orders of the latter approximations are not optimal and can be raised by one by improving the parameterization.

2. Error between curves

Our starting point is an estimate for the difference between the length $\Delta s_i = s_{i+1} - s_i$ of the curve piece $\mathbf{f}|_{[s_i, s_{i+1}]}$ and the chord length $|\mathbf{f}(s_{i+1}) - \mathbf{f}(s_i)|$. It was established in [**4**] that if $\mathbf{f} \in C^2[a,b]$ then for $i = 0, 1, \ldots, n-1$,

$$0 \le \Delta s_i - |\mathbf{f}(s_{i+1}) - \mathbf{f}(s_i)| \le \frac{1}{12} (\Delta s_i)^3 \|\mathbf{f}''\|^2_{[s_i, s_{i+1}]},$$

where $\|\cdot\|_A = \sup_{x \in A} |\cdot(x)|$, and so with the values t_i of (1.2),

$$|\Delta t_i - \Delta s_i| = O\big((\Delta s_i)^3\big) \quad \text{as } h_s \to 0. \tag{2.1}$$

This implies, among other things, that for h_s small enough, $\Delta s_i/2 \le \Delta t_i \le \Delta s_i$, and therefore, $h_s/2 \le h_t \le h_s$. Hence, as $h_s \to 0$, we can freely interchange expressions of the form $O((\Delta s_i)^k)$ with $O((\Delta t_i)^k)$ and $O(h_s^k)$ with $O(h_t^k)$.

To measure the distance between the curves $\boldsymbol{\sigma}$ and $\mathbf{f}$ we use the metric [**7**]

$$d_P(\mathbf{f}, \boldsymbol{\sigma}) := \inf_{\phi} \|\mathbf{f} \circ \phi - \boldsymbol{\sigma}\|_{[t_0, t_n]}, \tag{2.2}$$

the infimum taken over continuously differentiable functions $\phi : [t_0, t_n] \to [s_0, s_n]$ with $\phi'(t) > 0$, $t_0 \le t \le t_n$, and $\phi(t_0) = s_0$ and $\phi(t_n) = s_n$. We will prove

Theorem 2.1. *Suppose $\boldsymbol{\sigma}$ is the cubic spline interpolant (1.1) based on the chord length parameterization (1.2). If $\mathbf{f} \in C^4[a,b]$ then*

$$d_P(\mathbf{f}, \boldsymbol{\sigma}) = O\big(h_s^4\big) \quad \text{as } h_s \to 0.$$

The main idea of the proof is to study the reparameterization

$$\mathbf{g}(t) := \mathbf{f}(\phi(t)), \qquad t \in [t_0, t_n], \tag{2.3}$$

where $\phi : [t_0, t_n] \to \mathbb{R}$ is the C^2 cubic spline function satisfying

$$\begin{aligned} \phi(t_i) &= s_i, \qquad i = 0, 1, \ldots, n, \\ \phi'(t_i) &= 1, \qquad i = 0, n. \end{aligned} \tag{2.4}$$

It turns out that when h_s is small enough, ϕ is monotonically increasing, in which case $\mathbf{g}$ is well-defined, and also that

$$\|\mathbf{g} - \boldsymbol{\sigma}\|_{[t_0,t_n]} = O\big(h_s^4\big) \quad \text{as } h_s \to 0. \tag{2.5}$$

To establish both properties, we begin by deriving some bounds on the derivatives of ϕ. Using the fact that the parameter values t_i are chord length, (2.1) implies

$$|\Delta t_i - \Delta s_i| = O\big((\Delta t_i)^3\big) \quad \text{as } h_s \to 0,$$

and, dividing by Δt_i, we deduce that for small enough h_s there is some constant $C > 0$ such that

$$|[t_i, t_{i+1}]\phi - 1| \le C(\Delta t_i)^2, \qquad i = 0, 1, \ldots, n-1. \tag{2.6}$$

Lemma 2.2. *If (2.6) holds then, for $i = 0, 1, \ldots, n-1$,*

$$\|\phi' - 1\|_{(t_i,t_{i+1})} \le 4C\Delta t_i h_t, \quad \|\phi''\|_{(t_i,t_{i+1})} \le 6Ch_t, \quad \|\phi'''\|_{(t_i,t_{i+1})} \le 12C\frac{h_t}{\Delta t_i}. \tag{2.7}$$

Proof: Setting $\phi_i = \phi(t_i)$, $\phi_i' = \phi'(t_i)$, and $\phi_i'' = \phi''(t_i)$, it is convenient to represent ϕ in the interval $[t_i, t_{i+1}]$, $i = 0, 1, \ldots, n-1$, in terms of its values and second derivatives at the end points, giving

$$\phi(t) = (1-u)\phi_i + u\phi_{i+1} - \frac{(\Delta t_i)^2}{6}\Big((2u - 3u^2 + u^3)\phi_i'' + (u - u^3)\phi_{i+1}''\Big), \tag{2.8}$$

where $u = (t - t_i)/\Delta t_i$. Since

$$\phi'(t) = [t_i, t_{i+1}]\phi - \frac{\Delta t_i}{6}\Big((2 - 6u + 3u^2)\phi_i'' + (1 - 3u^2)\phi_{i+1}''\Big), \tag{2.9}$$

the continuity of ϕ' at the nodes $t_1, \ldots, t_{n-1}$ leads to the familiar linear system in the second derivatives [**8**], ([**1**], p. 10),

$$\frac{\Delta t_{i-1}}{\Delta t_{i-1} + \Delta t_i}\phi_{i-1}'' + 2\phi_i'' + \frac{\Delta t_i}{\Delta t_{i-1} + \Delta t_i}\phi_{i+1}'' = 6[t_{i-1}, t_i, t_{i+1}]\phi, \quad i = 1, \ldots, n-1, \tag{2.10}$$

while the end conditions are captured in the two additional equations

$$2\phi_0'' + \phi_1'' = 6[t_{-1}, t_0, t_1]\phi, \qquad \phi_{n-1}'' + 2\phi_n'' = 6[t_{n-1}, t_n, t_{n+1}]\phi, \tag{2.11}$$

in which $t_{-1} = t_0$, hence $[t_{-1}, t_0]\phi = \phi'(t_0)$, and similarly $t_{n+1} = t_n$. The diagonal dominance of the matrix of this linear system implies that

$$|\phi_i''| \le 6 \max_{0 \le j \le n} |[t_{j-1}, t_j, t_{j+1}]\phi|, \qquad 0 \le i \le n.$$

Then by (2.6) and the end conditions $\phi_0' = \phi_n' = 1$,

$$|[t_{j-1}, t_j, t_{j+1}]\phi| \le \frac{|[t_j, t_{j+1}]\phi - 1| + |[t_{j-1}, t_j]\phi - 1|}{\Delta t_{j-1} + \Delta t_j} \le C\frac{(\Delta t_{j-1})^2 + (\Delta t_j)^2}{\Delta t_{j-1} + \Delta t_j} \le Ch_t,$$

for $0 \le j \le n$, hence

$$\max_{0\le i\le n} |\phi_i''| \le 6Ch_{\mathbf{t}}, \tag{2.12}$$

and since ϕ'' is linear in each interval $[t_i, t_{i+1}]$, this proves the second inequality in (2.7). Next consider the first derivative of ϕ in $[t_i, t_{i+1}]$. Writing equation (2.9) as

$$\begin{aligned}\phi'(t) - 1 = ([t_i, t_{i+1}]\phi - 1) - \frac{\Delta t_i}{6}\Big((1-u)^2(2\phi_i'' + \phi_{i+1}'') \\ + 2u(1-u)(\phi_{i+1}'' - \phi_i'') - u^2(\phi_i'' + 2\phi_{i+1}'')\Big),\end{aligned}$$

inequalities (2.6) and (2.12) imply

$$|\phi'(t) - 1| \le C(\Delta t_i)^2 + \frac{\Delta t_i}{2}\Big((1-u)^2 + 2u(1-u) + u^2\Big) \max_{0\le j\le n} |\phi_j''| \le 4C\Delta t_i h_t.$$

Using (2.12) and the linearity of ϕ'' in $[t_i, t_{i+1}]$ we also deduce

$$\|\phi'''\|_{(t_i,t_{i+1})} = \frac{1}{\Delta t_i}|\phi_{i+1}'' - \phi_i''| \le \frac{1}{\Delta t_i}(|\phi_i''| + |\phi_{i+1}''|) \le 12C\frac{h_t}{\Delta t_i}.$$

■

Proof of Theorem 2.1: The equation for ϕ' in (2.7) shows that ϕ in (2.4) is increasing when h_s is small enough and it remains to establish (2.5). Due to (2.4),

$$\begin{aligned}\boldsymbol{\sigma}(t_i) &= \mathbf{g}(t_i), \qquad i = 0, 1, \ldots, n, \\ \boldsymbol{\sigma}'(t_i) &= \mathbf{g}'(t_i), \qquad i = 0, n,\end{aligned} \tag{2.13}$$

and since $\mathbf{g} \in C^2[t_0, t_n]$, we can bound the error between $\boldsymbol{\sigma}$ and $\mathbf{g}$ using the well-known error analysis for cubic spline interpolation developed by Birkhoff and de Boor [**2**], Sharma and Meir [**8**], Hall and Meyer [**5**], and others, much of which is summarized by de Boor in Chapter V of [**3**]. The following error estimate, which derives from the best approximation property of the complete spline interpolation (2.13) is sufficient for our needs:

$$\|\mathbf{g}'' - \boldsymbol{\sigma}''\|_{[t_0,t_n]} \le 4\mathrm{dist}_\infty(\mathbf{g}'', S_1^0),$$

where S_1^0 is the space of piecewise linear spline curves $[t_0, t_n] \to \mathbb{R}^d$ over the partition $t_0 < t_1 < \cdots < t_n$. This is a straightforward vector-valued generalization of the estimate of ([**3**], p. 68). If $\boldsymbol{\tau} \in S_1^0$ is the piecewise linear interpolant to $\mathbf{g}''$ at the t_i we obtain (via the Hermite-Genocchi formula)

$$\|\mathbf{g}'' - \boldsymbol{\sigma}''\|_{[t_0,t_n]} \le 4\|\mathbf{g}'' - \boldsymbol{\tau}\|_{[t_0,t_n]} \le \frac{1}{2}\max_{0\le i\le n-1}(\Delta t_i)^2\|\mathbf{g}^{(4)}\|_{(t_i,t_{i+1})}. \tag{2.14}$$

Consider then the (local) fourth derivative of $\mathbf{g}$ in terms of the derivatives of $\mathbf{f}$ and ϕ. The chain rule (with $\mathbf{g}$ and ϕ evaulated at t and $\mathbf{f}$ evaluated at $\phi(t)$) gives

$$\begin{aligned}
\mathbf{g}' &= \phi'\mathbf{f}', \\
\mathbf{g}'' &= (\phi')^2\mathbf{f}'' + \phi''\mathbf{f}', \\
\mathbf{g}''' &= (\phi')^3\mathbf{f}''' + 3\phi'\phi''\mathbf{f}'' + \phi'''\mathbf{f}', \\
\mathbf{g}^{(4)} &= (\phi')^4\mathbf{f}^{(4)} + 6(\phi')^2\phi''\mathbf{f}''' + (3(\phi'')^2 + 4\phi'\phi''')\mathbf{f}'' + \phi^{(4)}\mathbf{f}'.
\end{aligned} \tag{2.15}$$

Since $\phi^{(4)} = 0$ in each interval (t_i, t_{i+1}) and since, by Lemma 2.2,

$$\|\phi'\|_{(t_i,t_{i+1})} = O(1), \qquad \|\phi''\|_{(t_i,t_{i+1})} = O(1), \qquad \|\phi'''\|_{(t_i,t_{i+1})} = O\Big(\frac{h_t}{\Delta t_i}\Big),$$

as $h_s \to 0$, we deduce from the expression for $\mathbf{g}^{(4)}$ in (2.15) that

$$\|\mathbf{g}^{(4)}\|_{(t_i,t_{i+1})} = O\Big(\frac{h_t}{\Delta t_i}\Big), \qquad i = 0, 1, \ldots, n-1. \tag{2.16}$$

Hence (2.14) implies that

$$\|\mathbf{g}'' - \boldsymbol{\sigma}''\|_{[t_0,t_n]} = O(h_t^2). \tag{2.17}$$

Since ([**3**], Chap. V)

$$\|\mathbf{g} - \boldsymbol{\sigma}\|_{[t_0,t_n]} \le \frac{1}{8}h_t^2\|\mathbf{g}'' - \boldsymbol{\sigma}''\|_{[t_0,t_n]},$$

this gives

$$d_P(\mathbf{f}, \boldsymbol{\sigma}) \le \|\mathbf{g} - \boldsymbol{\sigma}\|_{[t_0,t_n]} = O(h_t^4) = O(h_s^4).$$

■

3. Error between arc-length derivatives

We have seen that the spline curve $\boldsymbol{\sigma}$ approximates the reparameterization $\mathbf{g}$ of the curve $\mathbf{f}$ to fourth order for $\mathbf{f} \in C^4[a, b]$. Similarly, the derivatives of $\boldsymbol{\sigma}$ approximate the derivatives of $\mathbf{g}$. Indeed, from inequality (2.17),

$$\|\mathbf{g}^{(k)} - \boldsymbol{\sigma}^{(k)}\|_{[t_0,t_n]} = O(h_s^{4-k}) \quad \text{as } h_s \to 0, \qquad k = 0, 1, 2. \tag{3.1}$$

The important question though is how to approximate the derivatives of $\mathbf{f}$ since these are fixed (arc length) derivatives. One approach is to use the *arc length* derivatives of $\boldsymbol{\sigma}$, and we get the same accuracy as for functional interpolation. To see this, first observe that the derivatives of $\mathbf{f}$, being arc length, are the same as the arc length derivatives of $\mathbf{g}$. Thus it is enough to look at the error between the arc length derivatives of $\boldsymbol{\sigma}$ and the arc length derivatives of $\mathbf{g}$. Denoting arc length derivatives by dots, the first derivatives are

$$\dot{\mathbf{g}}(t) = \frac{\mathbf{g}'(t)}{|\mathbf{g}'(t)|}, \qquad \dot{\boldsymbol{\sigma}}(t) = \frac{\boldsymbol{\sigma}'(t)}{|\boldsymbol{\sigma}'(t)|}.$$

It follows that

$$\dot{\mathbf{g}} - \dot{\boldsymbol{\sigma}} = \frac{\mathbf{g}' - \boldsymbol{\sigma}'}{|\mathbf{g}'|} + (|\boldsymbol{\sigma}'| - |\mathbf{g}'|)\frac{\boldsymbol{\sigma}'}{|\mathbf{g}'||\boldsymbol{\sigma}'|}.$$

Now since $|\mathbf{g}'| = \phi'$, equations (2.7) and, with $k = 1$, (3.1) imply that both $|\mathbf{g}'|$ and $|\boldsymbol{\sigma}'|$ are bounded away from zero as $h_s \to 0$. Thus from (3.1),

$$\|\mathbf{f}' \circ \phi - \dot{\boldsymbol{\sigma}}\|_{[t_0,t_n]} = \|\dot{\mathbf{g}} - \dot{\boldsymbol{\sigma}}\|_{[t_0,t_n]} = O(h_s^3).$$

Similarly for second derivatives, using the fact that

$$\ddot{\mathbf{g}} = \frac{1}{|\mathbf{g}'|^2}\mathbf{g}'' - \frac{\mathbf{g}' \cdot \mathbf{g}''}{|\mathbf{g}'|^4}\mathbf{g}',$$

and similarly for $\ddot{\boldsymbol{\sigma}}$, we find from (3.1) that

$$\|\mathbf{f}'' \circ \phi - \ddot{\boldsymbol{\sigma}}\|_{[t_0,t_n]} = \|\ddot{\mathbf{g}} - \ddot{\boldsymbol{\sigma}}\|_{[t_0,t_n]} = O(h_s^2). \tag{3.2}$$

Specializing to the interpolation points gives estimates that are independent of ϕ:

Theorem 3.1. *If* $\mathbf{f} \in C^4[a,b]$, *then*

$$\mathbf{f}'(s_i) - \dot{\boldsymbol{\sigma}}(t_i) = O(h_s^3), \quad \mathbf{f}''(s_i) - \ddot{\boldsymbol{\sigma}}(t_i) = O(h_s^2), \qquad \text{as } h_s \to 0, \qquad i = 0, 1, \ldots, n.$$

Note that the second equation implies that the curvature of $\boldsymbol{\sigma}$ at t_i approximates the curvature of $\mathbf{f}$ at s_i to order $O(h_s^2)$.

We even get approximation in the arc length third derivatives if the global mesh ratio $\beta_s := h_s / \min_{0 \le i \le n-1} \Delta s_i$ is bounded. Let $\mathbf{p}_1$ be the linear polynomial interpolating $\mathbf{e}''$ at $t = t_i, t_{i+1}$, where $\mathbf{e} := \mathbf{g} - \boldsymbol{\sigma}$. Then

$$|\mathbf{e}'''(t) - \mathbf{p}_1'(t)| \le 2\Delta t_i \|\mathbf{e}^{(4)}\|_{(t_i,t_{i+1})} = 2\Delta t_i \|\mathbf{g}^{(4)}\|_{(t_i,t_{i+1})}, \qquad t_i < t < t_{i+1},$$

and since $\mathbf{p}_1'(t) = (\mathbf{e}''(t_{i+1}) - \mathbf{e}''(t_i))/\Delta t_i$, it follows from (2.16) and (2.17) that

$$\|\mathbf{e}'''\|_{(t_i,t_{i+1})} \le \|\mathbf{p}_1'\|_{(t_i,t_{i+1})} + 2\Delta t_i \|\mathbf{g}^{(4)}\|_{(t_i,t_{i+1})} = O\Big(\frac{h_t^2}{\Delta t_i}\Big).$$

A similar approach to that used to derive (3.2) then gives

$$\|\mathbf{f}''' \circ \phi - \dddot{\boldsymbol{\sigma}}\|_{(t_i,t_{i+1})} = \|\dddot{\mathbf{g}} - \dddot{\boldsymbol{\sigma}}\|_{(t_i,t_{i+1})} = O\Big(\frac{h_s^2}{\Delta s_i}\Big),$$

and so

$$\mathbf{f}'''(s_i) - \dddot{\boldsymbol{\sigma}}(t_i\pm) = O(\beta_s h_s) \quad \text{as } h_s \to 0, \qquad i = 0, 1, \ldots, n.$$

4. Error in curve length

A well known way of estimating the length $L(\mathbf{f})$ of the curve $\mathbf{f}$ given only the points $\mathbf{f}(s_0), \ldots, \mathbf{f}(s_n)$ is to compute the length

$$L(P) = \sum_{i=0}^{n-1} |\mathbf{f}(s_{i+1}) - \mathbf{f}(s_i)|$$

of the polygon P passing through them. If $\mathbf{f} \in C^2[a,b]$, the error between $L(\mathbf{f})$ and $L(P)$ is of order $O(h_s^2)$. In fact Proposition 3.1 of [**4**] gives the explicit bound

$$|L(P) - L(\mathbf{f})| \le \frac{1}{12}\|\mathbf{f}''\|_{[a,b]}^2 \sum_{i=0}^{n-1} (\Delta s_i)^3 \le \frac{1}{12}\|\mathbf{f}''\|_{[a,b]}^2 L(\mathbf{f}) h_s^2.$$

We next show that the length of the spline interpolant $\boldsymbol{\sigma}$ gives a higher order estimate. The main point is that the length of $\mathbf{f}$ is the same as the length of $\mathbf{g}$, and so it is sufficient to compare the lengths of $\boldsymbol{\sigma}$ and $\mathbf{g}$.

Theorem 4.1. *If* $\mathbf{f} \in C^4[a,b]$, *then*

$$L(\boldsymbol{\sigma}) - L(\mathbf{f}) = O(h_s^4) \quad \text{as } h_s \to 0.$$

Proof: From (3.1) we have

$$\|\mathbf{e}\| = O(h_s^4), \qquad \|\mathbf{e}'\| = O(h_s^3), \tag{4.1}$$

as $h_s \to 0$, where $\mathbf{e}(t) := \mathbf{g}(t) - \boldsymbol{\sigma}(t)$. Since both $|\boldsymbol{\sigma}'|$ and $|\mathbf{g}'|$ are bounded away from zero for small enough h_s, the bound on $\mathbf{e}'$ in (4.1) implies that

$$L(\boldsymbol{\sigma}) - L(\mathbf{g}) = \int_{t_0}^{t_n} |\boldsymbol{\sigma}'(t)| - |\mathbf{g}'(t)|\, dt = -2\int_{t_0}^{t_n} \frac{\mathbf{e}'(t)\cdot\mathbf{g}'(t)}{|\boldsymbol{\sigma}'(t) + |\mathbf{g}'(t)|}\, dt + O\big(h_s^6\big),$$

where we have used the identity

$$|\boldsymbol{\sigma}'| - |\mathbf{g}'| = \frac{-2\mathbf{e}'\cdot\mathbf{g}' + \mathbf{e}'\cdot\mathbf{e}'}{|\boldsymbol{\sigma}'| + |\mathbf{g}'|}.$$

But since $\mathbf{e}(t_0) = \mathbf{e}(t_n) = \mathbf{0}$, integration by parts implies

$$-\int_{t_0}^{t_n} \frac{\mathbf{e}'(t)\cdot\mathbf{g}(t)}{|\boldsymbol{\sigma}'(t)| + |\mathbf{g}'(t)|}\, dt = \int_{t_0}^{t_n} \mathbf{e}(t)\cdot\frac{d}{dt}\Big(\frac{\mathbf{g}'(t)}{|\boldsymbol{\sigma}'(t)| + |\mathbf{g}'(t)|}\Big)\, dt.$$

Since by (2.15) and (3.1), $\mathbf{g}'$, $\mathbf{g}''$, $\boldsymbol{\sigma}'$, and $\boldsymbol{\sigma}''$ are bounded as $h_s \to 0$, so too is

$$\frac{d}{dt}\Big(\frac{\mathbf{g}'(t)}{|\boldsymbol{\sigma}'(t) + |\mathbf{g}'(t)|}\Big)$$

and the result follows from the bound on $\mathbf{e}$ in (4.1). ∎

5. Error between derivatives

We showed in Section 3 that the *arc length* derivatives of $\boldsymbol{\sigma}$ approximate the (arc length) derivatives of $\mathbf{f}$ for $\mathbf{f} \in C^4[a,b]$. Interestingly, though, it turns out that the *derivatives of* $\boldsymbol{\sigma}$ *themselves* also approximate the derivatives of $\mathbf{f}$, albeit with the loss of one order of accuracy. We can use the triangle inequality to get the estimate

$$\|\mathbf{f}^{(k)} \circ \phi - \boldsymbol{\sigma}^{(k)}\|_{[t_0,t_n]} \le \|\mathbf{f}^{(k)} \circ \phi - \mathbf{g}^{(k)}\|_{[t_0,t_n]} + \|\mathbf{g}^{(k)} - \boldsymbol{\sigma}^{(k)}\|_{[t_0,t_n]},$$

and the first term on the right-hand side can be estimated from (2.15) and Lemma 2.2. Since the first two equations of (2.15) imply

$$\begin{aligned} \mathbf{f}'(\phi(t)) - \mathbf{g}'(t) &= (1-\phi'(t))\mathbf{f}'(\phi(t)), \\ \mathbf{f}''(\phi(t)) - \mathbf{g}''(t) &= (1-(\phi'(t))^2)\mathbf{f}'(\phi(t) - \phi''(t)\mathbf{f}(\phi(t)), \end{aligned} \tag{5.1}$$

the bounds on ϕ in Lemma 2.2 imply

$$\|\mathbf{f}^{(k)} \circ \phi - \mathbf{g}^{(k)}\|_{[t_0,t_n]} = O(h_s^{3-k}), \qquad k = 1,2, \tag{5.2}$$

and we deduce from (3.1) that

$$\|\mathbf{f}^{(k)} \circ \phi - \boldsymbol{\sigma}^{(k)}\|_{[t_0,t_n]} = O(h_s^{3-k}), \qquad k = 1,2. \tag{5.3}$$

Theorem 5.1. *If* $\mathbf{f} \in C^4[a,b]$ *then*

$$\mathbf{f}'(s_i) - \boldsymbol{\sigma}'(t_i) = O(h_s^2), \quad \mathbf{f}''(s_i) - \boldsymbol{\sigma}''(t_i) = O(h_s), \quad \text{as } h_s \to 0, \qquad i = 0,1,\ldots,n.$$

Notice that the case $k = 1$ in (5.3) implies that

$$\||\boldsymbol{\sigma}'| - 1\|_{[t_0,t_n]} = \||\boldsymbol{\sigma}'| - |\mathbf{f}' \circ \phi|\|_{[t_0,t_n]} = O(h_s^2), \tag{5.4}$$

which means that the parameterization of $\boldsymbol{\sigma}$ *converges to arc length*. This makes chordal cubic spline interpolation an attractive method for *reparameterizing* a given parametric curve, giving an approximate curve whose parameterization is close to arc length. This is certainly not a new idea, but equation (5.4) seems to be the first mathematical justification for this method.

6. Improving the parameterization

A natural question raised by the results of the last section is whether there are parameterizations which are even better than the chordal one in the sense that the approximation orders of Theorem 5.1 and (5.4) are one higher. The answer is 'yes' if we use the approach of iterative improvement, studied in [**4**]. Suppose we can construct a parameterization $\hat{t}_0 < \hat{t}_1 < \cdots < \hat{t}_n$, where each $\hat{t}_i$ is viewed as a function of $\mathbf{f}(s_0),\ldots,\mathbf{f}(s_n)$ and $\mathbf{f}'(s_0), \mathbf{f}'(s_n)$, such that

$$|\Delta\hat{t}_i - \Delta s_i| = O\big((\Delta s_i)^3 h_s\big), \qquad i = 0,1,\ldots,n-1, \tag{6.1}$$

and suppose $\hat{\boldsymbol{\sigma}} : [\hat{t}_0, \hat{t}_n] \to \mathbb{R}^d$ is the cubic spline interpolant resulting from the replacement of the t_i in (1.1) by the $\hat{t}_i$. Then all the bounds on ϕ in Lemma 2.2 are raised by one power of h_s and from (5.1) we can replace (5.2) by

$$\|\mathbf{f}^{(k)} \circ \hat{\phi} - \hat{\mathbf{g}}^{(k)}\|_{[\hat{t}_0, \hat{t}_n]} = O(h_s^{4-k}), \qquad k = 1, 2,$$

and therefore (5.3) by

$$\|\mathbf{f}^{(k)} \circ \hat{\phi} - \hat{\boldsymbol{\sigma}}^{(k)}\|_{[\hat{t}_0, \hat{t}_n]} = O(h_s^{4-k}), \qquad k = 1, 2.$$

Thus

$$|\mathbf{f}^{(k)}(s_i) - \hat{\boldsymbol{\sigma}}_3^{(k)}(\hat{t}_i)| = O(h_s^{4-k}), \qquad k = 1, 2, \quad i = 0, 1, \ldots, n,$$

and

$$\||\hat{\boldsymbol{\sigma}}'| - 1\|_{[\hat{t}_0, \hat{t}_n]} = O(h_s^3).$$

There are at least two ways of finding a parameterization satisfying (6.1). One is to let

$$\hat{t}_{i+1} = \hat{t}_i + L(\mathbf{p}_i|_{[t_i, t_{i+1}]}), \qquad i = 0, 1, \ldots, n-1,$$

where $\mathbf{p}_i$ is the (chordal) cubic polynomial such that $\mathbf{p}_i(t_j) = \mathbf{f}(s_j)$, $j = i-1, i, i+1, i+2$; a Lagrange interpolant for $1 \le i \le n-2$ and a Hermite one for $i = 0, n-1$. Then by Theorem 4.1 of [**4**],

$$|\Delta \hat{t}_i - \Delta s_i| = O\big((\Delta s_i)^3 (s_{i+1} - s_{i-1})(s_{i+2} - s_i)\big),$$

from which (6.1) follows. In practice it would be sufficient [**4**] to approximate $L(\mathbf{p}_i|_{[t_i, t_{i+1}]})$ using for example Simpson's rule or the 2-point Gauss rule to approximate the integral of $|\mathbf{p}_i'|$ over the interval $[t_i, t_{i+1}]$. Another way is to use $\boldsymbol{\sigma}$. If we set

$$\hat{t}_{i+1} = \hat{t}_i + L(\boldsymbol{\sigma}|_{[t_i, t_{i+1}]}), \qquad i = 0, 1, \ldots, n-1,$$

then

$$|\Delta \hat{t}_i - \Delta s_i| = O\big((\Delta s_i)^3 h_s^2\big), \tag{6.2}$$

and so (6.1) is satisfied. To see this, we can use the fact that

$$\Delta \hat{t}_i - \Delta s_i = L(\boldsymbol{\sigma}|_{[t_i, t_{i+1}]}) - L(\mathbf{g}|_{[t_i, t_{i+1}]}).$$

By treating each interval $[t_i, t_{i+1}]$ in (2.17) separately, we obtain the estimates

$$\|\mathbf{e}\|_{[t_i, t_{i+1}]} = O((\Delta t_i)^2 h_t^2), \qquad \|\mathbf{e}'\|_{[t_i, t_{i+1}]} = O(\Delta t_i h_t^2),$$

and (6.2) follows from the same method of proof as for Theorem 4.1.

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