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Disciplined, efficient, generalised folds for nested datatypes

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Abstract. Nested (or non-uniform, or non-regular) datatypes have recursive definitions in which the type parameter changes. Their folds are restricted in power due to type constraints. Bird and Paterson introduced *generalised folds* for extra power, but at the cost of a loss of efficiency: folds may take more than linear time to evaluate. Hinze introduced *efficient generalised folds* to counter this inefficiency, but did so in a pragmatic way, at the cost of a loss of reasoning power: without categorical or equivalent underpinnings, there are no universal properties for manipulating folds. We combine the efficiency of Hinze's construction with the powerful reasoning tools of Bird and Paterson's.

Keywords: Nested datatype, non-uniform datatype, non-regular datatype, polymorphic recursion, fold, universal property, functor category.

1. Introduction

The fold operator of functional programming can be defined on any regular datatype [BdM97], but there is a problem when extending its definition to nested datatypes [BM98] because some of its parameters are required to be polymorphic. A solution to this problem, proposed by Bird and Paterson [BP99b], was to introduce a new kind of operator called a *generalised fold*. Like an ordinary fold, the generalised fold of a given initial algebra is characterised uniquely by its defining equation, and this characterisation gives rise to useful fusion laws. Unfortunately generalised folds still lack some of the other well-known properties of folds: they cannot be used to define map functions, and the map fusion law is subject to a side condition. A more serious shortcoming, observed by Hinze [Hin00], is that generalised folds are inefficient for some datatypes. So Hinze has introduced a different kind of fold, called an *efficient generalised fold*, which suffers none of the problems described above. It is specified as a generalised fold composed with a map, but the terminology is slightly misleading since Hinze's generalised folds are different from Bird and Paterson's.

The contribution of this paper is to draw together the folds of [Hin00] and [BP99b]. We will show how an efficient fold similar to that in [Hin00] can be given a unique characterisation in terms of initial algebras. This universal property provides a proof principle that can be used to show that an efficient fold can be expressed as a Bird/Paterson generalised fold after a map, and consequently we obtain the fusion laws for free. Perhaps

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more importantly, the initial algebra definition can be used to determine the type of the efficient fold and its parameters on any nested datatype: this was not possible with the definition in [Hin00]. Unfortunately there are still some problems with the implementation of these types in the programming language Haskell [PJH99].

We begin, in Section 2, by recalling some of the well-known properties of folds on regular datatypes, exemplified by the familiar datatype of lists. In Section 3, we discuss the relative merits of ordinary folds, generalised folds and efficient folds on the nested datatype of perfectly balanced binary trees. The time complexity of the latter two folds is then compared on a simple random access list [Oka98] in Section 4. Sections 5, 6 and 7 contain a short description of the semantics of nested datatypes, generalised folds and efficient folds respectively. Some applications of efficient folds are given in Section 8.

2. Fold on Regular Datatypes

We will use the datatype of lists to exhibit some of the intrinsic properties of folds on regular datatypes. The most fundamental of these is the universal property of fold, from which the fold fusion law for program transformation can be derived. In addition, the map function of any regular datatype can be expressed as a fold. As such, it can be fused with any other fold, yielding a second law for program derivation, usually known as the map fusion law.

The datatype of lists can be defined in Haskell by

$$\textbf{data}\ \mathit{List}\ a = \mathit{Nil} \mid \mathit{Cons}\ a\ (\mathit{List}\ a)$$

and its fold operation can be defined recursively by

$$\begin{array}{lcl} \mathit{fold} :: b \to (a \to b \to b) \to \mathit{List}\ a \to b & & \\ \mathit{fold}\ e\ f\ \mathit{Nil} & = & e \\ \mathit{fold}\ e\ f\ (\mathit{Cons}\ x\ xs) & = & f\ x\ (\mathit{fold}\ e\ f\ xs) \end{array}$$

The universal property of fold states that, for finite lists, the function $\mathit{fold}\ e\ f$ is the unique solution of this pair of equations: if $u : \mathit{List}\ a \to b$, $e : b$ and $f : a \to b \to b$,

$$\Leftrightarrow \quad \begin{array}{l} u = \mathit{fold}\ e\ f \\ \\ u\ \mathit{Nil} = e \text{ and } u\ (\mathit{Cons}\ x\ xs) = f\ x\ (u\ xs) \text{ for all } x, xs \end{array} \tag{1}$$

This property is a useful proof principle for program transformation since it captures the familiar pattern of inductive proof for finite lists. In particular, it can be used to derive the following fusion law for lists, which gives conditions under which the composition of an arbitrary function and a fold can be fused into a single fold: if $k : b \to b'$, $e : b$, $f : a \to b \to b$, $e' : b'$ and $f' : a \to b' \to b'$, then

$$\Rightarrow \quad \begin{array}{l} k\ e = e' \text{ and } k\ (f\ x\ y) = f'\ x\ (k\ y) \quad \text{for all } x, y \\ \\ k \cdot \mathit{fold}\ e\ f = \mathit{fold}\ e'\ f' \end{array} \tag{2}$$

The map fusion law for lists can be derived from this law, together with the definition of the map function as a fold. We will refer to the map function on lists as *list* for consistency with later examples. It can be expressed as a fold in the following way:

$$\begin{array}{lcl} \mathit{list} :: (a \to b) \to \mathit{List}\ a \to \mathit{List}\ b & & \\ \mathit{list}\ f & = & \mathit{fold}\ \mathit{Nil}\ (\mathit{Cons} \cdot f) \end{array}$$

This definition might not be as readable as one written using explicit recursion, but it does have the advantage that properties of *list* can be proved from more fundamental ones of *fold*. The map fusion law below is an example of this. It would normally have been proved by induction, but is now almost immediate from the fold fusion law (2): if $e : b$, and $f : a \to b \to b$, then for all $k : a' \to a$,

$$(\mathit{fold}\ e\ f) \cdot (\mathit{list}\ k) \quad = \quad \mathit{fold}\ e\ (f \cdot k) \tag{3}$$

This law does not hold for generalised folds, but it does hold for efficient folds, as we shall see in Sections 3 and 7.

3. Folds on Nested Datatypes

Nested datatypes are recursively defined parameterised datatypes in which the parameter of the datatype changes in the recursive call. For example, the *Nest* datatype, which represents perfectly balanced binary trees, can be defined in Haskell by

$$\begin{array}{lcl} \textbf{data}\ \mathit{Nest}\ a & = & \mathit{Nil} \mid \mathit{Cons}\ a\ (\mathit{Nest}\ (\mathit{Pair}\ a)) \\ \textbf{type}\ \mathit{Pair}\ a & = & (a,a) \end{array} \tag{4}$$

This is a nested datatype because the *Nest* constructor is given the parameter a on the left hand side of this equation and a different argument, *Pair a*, on the right. The example below shows how this datatype represents perfectly balanced trees:

$$\begin{array}{lcl} \mathit{example} & :: & \mathit{Nest}\ \mathit{Int} \\ \mathit{example} & = & \mathit{Cons}\ 1\ (\mathit{Cons}\ (2,3)\ (\mathit{Cons}\ ((4,5),(6,7))\ \mathit{Nil})) \end{array} \tag{5}$$

3.1. Ordinary Folds

One standard function that we might wish to apply to a nest of integers is the *sum* function, to sum the values stored in it. However, it is not immediately obvious to see how such a function should be defined. This is one reason why higher order operations that capture common patterns of computation, like *fold*, are particularly important for nested datatypes. The initial algebra semantics of nested datatypes proposed in [BM98, MJ01] leads to the following definition of *fold* on this datatype:

$$\begin{array}{lcl} \mathit{fold} :: (\forall a.\ n\ a) \to (\forall a.\ a \to n\ (\mathit{Pair}\ a) \to n\ a) \to \mathit{Nest}\ b \to n\ b & & \\ \mathit{fold}\ e\ f\ \mathit{Nil} & = & e \\ \mathit{fold}\ e\ f\ (\mathit{Cons}\ x\ xs) & = & f\ x\ (\mathit{fold}\ e\ f\ xs) \end{array}$$

The universal property associated with this definition is identical to that for lists (1) apart from the type declarations. The problem with this fold is that its type signature means that the second parameter must always be polymorphic. So it can be used to implement some polymorphic functions, like that of flattening a nest to a list, but not usually for monomorphic functions such as *sum*. Even polymorphic functions cannot necessarily be written as folds, just as they cannot on regular datatypes [GHA01]. We can illustrate the difficulty in expressing *sum* as a fold on the above example (5). Writing the parameter f as an infix operator $\oplus$ and using the Haskell syntax of writing the prefix form of an infix operator $\oplus$ in parentheses $(\oplus)$, we have

$$\mathit{fold}\ e\ (\oplus)\ \mathit{example} \quad = \quad 1 \oplus ((2,3) \oplus (((4,5),(6,7)) \oplus e))$$

For type reasons, there is no value that could be assigned to $\oplus$ in Haskell to sum the values in the nest.

3.2. Generalised Folds

Generalised folds provide a solution to the type problem encountered with ordinary folds, but their definitions depend on those of the corresponding map functions. The map function for *Nest* is:

$$\begin{array}{lcl} \mathit{nest} :: (l\ a \to m\ b) \to \mathit{Nest}\ (l\ a) \to \mathit{Nest}\ (m\ b) & & \\ \mathit{nest}\ f\ \mathit{Nil} & = & \mathit{Nil} \\ \mathit{nest}\ f\ (\mathit{Cons}\ x\ xs) & = & \mathit{Cons}\ (f\ x)\ (\mathit{nest}\ (\mathit{pair}\ f)\ xs) \end{array}$$

where the map for *Pair* is simply

$$\begin{array}{lcl} \mathit{pair} :: (a \to b) \to (\mathit{Pair}\ a \to \mathit{Pair}\ b) & & \\ \mathit{pair}\ f\ (x,y) & = & (f\ x, f\ y) \end{array}$$

The generalised fold on nests is then defined below. It has one more parameter than the ordinary fold, and this extra parameter is repeatedly mapped over the datatype at each recursive call; this is the cause of the inefficiency observed in [Hin00] which is reiterated here in Section 4.

$$\begin{array}{lcl} \mathit{gfold} :: (\forall a.\ n\ a) \to (\forall a.\ m\ a \to n\ (\mathit{Pair}\ a) \to n\ a) \to (\forall a.\ \mathit{Pair}\ (m\ a) \to m\ (\mathit{Pair}\ a)) \to (\mathit{Nest}\ (m\ a) \to n\ a) & & \\ \mathit{gfold}\ e\ f\ g\ \mathit{Nil} & = & e \\ \mathit{gfold}\ e\ f\ g\ (\mathit{Cons}\ x\ xs) & = & f\ x\ (\mathit{gfold}\ e\ f\ g\ (\mathit{nest}\ g\ xs)) \end{array}$$

An ordinary fold is a special case of this, obtained by setting function parameter g to the identity function and type parameter m to the identity functor. The difference between the two folds is most clearly shown by applying a generalised fold to the example nest of integers again. Writing the parameter g as an infix operator $\otimes$, and using the Haskell function $uncurry : (a \to (b \to c)) \to ((a,b) \to c)$ we have

$$gfold\ e\ (\oplus)\ (uncurry\ (\otimes))\ example \quad = \quad 1 \oplus ((2 \otimes 3) \oplus (((4 \otimes 5) \otimes (6 \otimes 7)) \oplus e)) \tag{6}$$

It is now evident that the *sum* function can be written as:

$$sum = gfold\ 0\ (+)\ (uncurry\ (+))$$

The *sum* function cannot be implemented in Haskell with the most general type signature for the generalised fold: the type must be instantiated to different values for different applications. This is due to weaknesses in Haskell's type checker, which uses a kinded first-order unification rather than full higher-order unification [Jon95], and is one complication that is not removed by using efficient folds.

3.3. Fusion

We have now shown how two of the three problems with generalised folds that were listed in the introduction manifest themselves on the datatype of nests: the generalised fold is defined in terms of the map function, rather than vice versa, and the generalised fold can be inefficient because of the use of the map function in the recursive call. The third problem concerned the map fusion law, which we will consider in this section: the law is subject to a side condition. Before doing so, we will observe that the fold fusion law for generalised folds can be derived from the universal property, just as it can for ordinary folds on regular datatypes.

Fold Fusion

The universal property of generalised folds on nests states that for all $u : \forall a.\ Nest\ (m\ a) \to (n\ a)$, $e : \forall a.\ n\ a$, $f : \forall a.\ m\ a \to n\ (Pair\ a) \to n\ a$ and $g : \forall a.\ Pair\ (m\ a) \to m\ (Pair\ a)$,

$$\Leftrightarrow \quad \begin{array}{l} u = gfold\ e\ f\ g \\ \\ u\ Nil = e \text{ and } u\ (Cons\ x\ xs) = f\ x\ (u\ (nest\ g\ xs)) \end{array} \tag{7}$$

A fusion law identical to that for lists (2) can be derived from this universal property alone, but a stronger law is also derivable by observing that the naturality of u implies that for functors m and n and for all $k' : a \to b$,

$$u \cdot nest\ (m\,k') \quad = \quad (n\,k') \cdot u \tag{8}$$

Combining (8) with the universal property (7) gives the following law: if $e' : \forall a.\ n'\ a$, $f' : \forall a.\ m\ (m'\ a) \to n'\ (Pair\ a) \to n'\ a$, $k : \forall a.\ n\ a \to n'\ a$, and $k' : \forall a.\ Pair\ (m'\ a) \to m'\ (Pair\ a)$ then

$$\Rightarrow \quad \begin{array}{l} k\ e = e' \text{ and } k\ (f\ x\ y) = f'\ x\ (k\ (n\ k'\ y)) \\ \\ k \cdot gfold\ e\ f\ g = gfold\ e'\ f'\ (m\ k' \cdot g) \end{array} \tag{9}$$

The simpler law corresponding to (2) is recaptured by setting k' to the identity function and m' to the identity functor.

Map Fusion

Recall that the map fusion law for lists (3) was derived from the fold fusion law together with the expression of the map function as a fold. Clearly this is not possible for generalised folds, because their definition depends on that of the map function; indeed it is not always the case that a generalised fold composed with a map can be rewritten as another generalised fold, as the following example illustrates:

$$sumsquares \quad = \quad gfold\ 0\ (+)\ (uncurry\ (+)) \cdot nest\ (\backslash x \to x * x)$$

There are cases where fusion is still possible though, such as the *revels* function below which reverses the order of the labels in each level of a tree of pairs:

$$revels \quad = \quad gfold\ Nil\ Cons\ swap \cdot nest\ swap$$

where $swap\ (x,y) = (y,x)$. This can be fused using the map fusion law for nests of [BP99b] which is easily derivable from (7) and gives the following sufficient condition for fusion: if $k :: \forall a.\ m'\ a \to m\ a$ and $g' :: \forall a.\ Pair\ (m'\ a) \to m'\ (Pair\ a)$, then

$$\Rightarrow \begin{array}{l} g \cdot pair\ k = k \cdot g' \\ \\ gfold\ e\ f\ g \cdot nest\ k = gfold\ e\ (f \cdot k)\ g' \end{array} \tag{10}$$

This condition is certainly not necessary for fusing a map with a generalised fold. For example, it applies to neither of the following functions, yet both can be expressed as a single generalised fold:

$$\begin{array}{lcl} flatten & = & gfold\ [\,]\ (+\!\!+)\ (uncurry\ (+\!\!+)) \cdot nest\ (\backslash x \to [x]) \\ size & = & gfold\ 0\ (+)\ (uncurry\ (+)) \cdot nest\ (\backslash x \to 1) \end{array}$$

(Here, $+\!\!+$ denotes list concatenation.) We will see in Section 8 that fusion laws (9) and (10) can be combined to give a single *fold equivalence* law that does apply to examples like *flatten* and *size*, but we have yet to derive a generic version of the law.

3.4. Efficient Folds

Now we will introduce the efficient fold on nests, and see that it is not necessary to put any conditions on its map fusion law. The equational definition given below is the same as that in [Hin00]. The type signature is not the most general, since Haskell does not yet support the most general type declaration for efficient folds, but it is no less general than that of the corresponding generalised fold; the types of the first three parameters correspond.

$$\begin{array}{lcl} efold :: (\forall a.\ n\ a) \to (\forall a.\ m\ a \to n\ (Pair\ a) \to n\ a) \to (\forall a.\ Pair\ (m\ a) \to m\ (Pair\ a)) \to (l\ a \to m\ b) \\ \qquad\qquad \to (Nest\ (l\ a) \to n\ b) \\ efold\ e\ f\ g\ h\ Nil & = & e \\ efold\ e\ f\ g\ h\ (Cons\ x\ xs) & = & f\ (h\ x)\ (efold\ e\ f\ g\ (g \cdot pair\ h)\ xs) \end{array}$$

This definition is shorter than the corresponding definition of a generalised fold, since it does not require the definition of the the *nest* function. Indeed, *nest* can be implemented as an efficient fold: for all $h : l\ a \to m\ b$,

$$nest\ h \quad = \quad efold\ Nil\ Cons\ id\ h \tag{11}$$

Clearly the identity function is a special case of this. This definition of *nest* is not surprising when we consider the action of the efficient fold on the previous example of a nest of integers:

$$efold\ e\ (\oplus)\ (uncurry\ (\otimes))\ h\ example = h\ 1 \oplus ((h\ 2 \otimes h\ 3) \oplus (((h\ 4 \otimes h\ 5) \otimes (h\ 6 \otimes h\ 7)) \oplus e))$$

Comparing this with (6) suggests the following more general property of the efficient fold on nests: for all e, f, g and h of appropriate type,

$$efold\ e\ f\ g\ h \quad = \quad gfold\ e\ f\ g \cdot nest\ h \tag{12}$$

A property similar to this was given as the specification of efficient folds in [Hin00]: the only difference was that the generalised fold was not that same as that of [BP99b]. Instead, we observe that both (11) and (12) can be derived from the following universal property of efficient folds: for all $u : \forall a.\ (l\ a \to m\ (y\ a)) \to Nest\ (l\ a) \to (n\ (y\ a))$, $e : \forall a.\ n\ a$, $f : \forall a.\ m\ a \to n\ (Pair\ a) \to n\ a$ and $g : \forall a.\ Pair\ (m\ a) \to m\ (Pair\ a)$,

$$\Leftrightarrow \begin{array}{l} u = efold\ e\ f\ g \\ \\ u\ h\ Nil = e \text{ and } u\ h\ (Cons\ x\ xs) = f\ (h\ x)\ (u\ (g \cdot pair\ h)\ xs) \end{array}$$

for all $h : l\ a \to m\ (y\ a)$. This property is extended to an initial algebra definition of efficient folds on arbitrary nested datatypes in Section 7, where it is proved that they can always be written as a generalised fold after a map. It follows that any function that could be written as a generalised fold could also be written as an efficient fold. Moreover, it is now clear from (12), together with the functorial property of *nest*, that in addition to fusion laws corresponding to (9) and (10), efficient folds satisfy the following simple map fusion law: for all $k :: \forall a.\ l'\ a \to l\ a$,

$$efold\ e\ f\ g\ h \cdot nest\ k \quad = \quad efold\ e\ f\ g\ (h \cdot k) \tag{13}$$

This law is similar to the map fusion law for ordinary folds, which suggests that the pattern of computation captured by ordinary folds may be more closely mimicked by efficient folds than generalised folds. Notice however that the derivation of law (13) is quite different from its regular counterpart; in fact, neither (10) nor (13) can be deduced solely from the universal property of efficient folds.

4. Efficiency

Efficient folds were described on the datatype of nests in Section 3 because it is one of the most simple non-regular datatypes that can be defined. Unfortunately it is not a suitable datatype with which to demonstrate how performance can be increased by using efficient folds, because they are not significantly faster than generalised folds on nests. It was observed in [Hin00] that one datatype for which efficient folds do give a worthwhile improvement is that of de Bruijn terms [BP99a]. In this section we will describe how a similar benefit is gained on a datatype that is more closely related to nests: random access lists. Random access lists were invented by Okasaki [Oka98] as a means of improving the efficiency of certain operations on traditional lists. The definition given here was introduced by Hinze [Hin98], as a slight variation of Okasaki's original definition.

The Haskell definition of a random access list given below contains the type constructors *Nil* and *One* corresponding to *Nil* and *Cons* in the definition of *Nest* (4), together with an additional type constructor *Zero*.

$$\textbf{data}\ \mathit{Ral}\ a \quad = \quad \mathit{Nil} \mid \mathit{Zero}\ (\mathit{Ral}\ (\mathit{Pair}\ a)) \mid \mathit{One}\ a\ (\mathit{Ral}\ (\mathit{Pair}\ a))$$

As such, this datatype can store a collection of values of any size, unlike the *Nest* datatype which is restricted to storing $2^k - 1$ values for some k. A collection of data of size n is stored as a sequence of tuples of sizes 2^i for each i corresponding to a 1 in the binary representation of n. For example, the sequence $[0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10]$ has length 11, so it could be stored as:

$$\mathit{One}\ 0\ (\mathit{One}\ (1,2)\ (\mathit{Zero}\ (\mathit{One}\ (((3,4),(5,6)),((7,8),(9,10)))\ \mathit{Nil})))$$

Notice how the constructors $\mathit{One}, \mathit{One}, \mathit{Zero}, \mathit{One}$ spell out the binary expansion of 11, least significant bit first. Like binary numbers, random access lists can contain any number of leading zeros. So, for example, any such list x is equivalent to $\mathit{Zero}^n\ x$ for any n. In particular, the empty collection of values could be represented by *Nil* or by $\mathit{Zero}^n\ \mathit{Nil}$ for any n. We will use this empty collection to compare the relative efficiency of the generalised and efficient folds on this datatype. In order to do so, we must first give their definitions, together with that of the map function. The types have been omitted from the definitions below for conciseness, since it is the pattern of computation, rather than the types, that is relevant to efficiency.

$$\begin{array}{lcl}
\mathit{ral}\ f\ \mathit{Nil} & = & \mathit{Nil} \\
\mathit{ral}\ f\ (\mathit{Zero}\ xs) & = & \mathit{Zero}\ (\mathit{ral}\ (\mathit{pair}\ f)\ xs) \\
\mathit{ral}\ f\ (\mathit{One}\ x\ xs) & = & \mathit{One}\ (f\ x)\ (\mathit{ral}\ (\mathit{pair}\ f)\ xs)
\end{array}$$

$$\begin{array}{lcl}
\mathit{gfold}\ e\ f\ g\ h\ \mathit{Nil} & = & e \\
\mathit{gfold}\ e\ f\ g\ h\ (\mathit{Zero}\ xs) & = & f\ (\mathit{gfold}\ e\ f\ g\ h\ (\mathit{ral}\ h\ xs)) \\
\mathit{gfold}\ e\ f\ g\ h\ (\mathit{One}\ x\ xs) & = & g\ x\ (\mathit{gfold}\ e\ f\ g\ h\ (\mathit{ral}\ h\ xs))
\end{array}$$

$$\begin{array}{lcl}
\mathit{efold}\ e\ f\ g\ h\ k\ \mathit{Nil} & = & e \\
\mathit{efold}\ e\ f\ g\ h\ k\ (\mathit{Zero}\ xs) & = & f\ (\mathit{efold}\ e\ f\ g\ h\ (h \cdot \mathit{pair}\ k)\ xs) \\
\mathit{efold}\ e\ f\ g\ h\ k\ (\mathit{One}\ x\ xs) & = & g\ (k\ x)\ (\mathit{efold}\ e\ f\ g\ h\ (h \cdot \mathit{pair}\ k)\ xs)
\end{array}$$

At first sight, these definitions may seem rather daunting, because they have such a large number of parameters. In this case, however, the following less general versions are often useful:

$$\begin{array}{lcl}
\mathit{greduce}\ e\ g & = & \mathit{gfold}\ e\ \mathit{id}\ g\ (\mathit{uncurry}\ g) \\
\mathit{ereduce}\ e\ g\ h & = & \mathit{efold}\ e\ \mathit{id}\ g\ (\mathit{uncurry}\ g)\ h
\end{array}$$

The *sum* function can be implemented by either of these, taking e, g and h to be 0, $(+)$ and *id* respectively, but the first program takes quadratic time whereas the second is linear. This can be seen by first considering the evaluation of $\mathit{greduce}\ 0\ (+)\ (\mathit{Zero}^n\ \mathit{Nil})$. Let $\mathit{plus} = \mathit{uncurry}\ (+)$; then at the kth step, the expression

$$(\mathit{greduce}\ 0\ (+)) \cdot \mathit{ral}\ \mathit{plus} \cdot \mathit{ral}\ (\mathit{pair}\ \mathit{plus}) \cdot \ldots \cdot \mathit{ral}\ (\mathit{pair}^{k-1}\ \mathit{plus})\ (\mathit{Zero}^{n-k}\ \mathit{Nil})$$

must be simplified to

$$(\mathit{greduce}\ 0\ (+)) \cdot \mathit{Zero} \cdot \mathit{ral}\ (\mathit{pair}\ \mathit{plus}) \cdot \mathit{ral}\ (\mathit{pair}^2\ \mathit{plus}) \cdot \ldots \cdot \mathit{ral}\ (\mathit{pair}^k\ \mathit{plus})\ (\mathit{Zero}^{n-k-1}\ \mathit{Nil})$$

in order for computation to proceed. Since this takes $O(k)$ time, the total running time is $O(n^2)$. Now consider the evaluation of *ereduce* $0\ (+)\ id\ (Zero^n\ Nil)$. At the kth step, the expression

$$ereduce\ 0\ (+)\ (plus \cdot pair\ (plus \cdot (pair\ (\ldots))))\ (Zero^{n-k}\ Nil)$$

must be simplified to

$$ereduce\ 0\ (+)\ (plus \cdot pair\ (plus \cdot pair\ (plus \cdot (pair\ (\ldots)))))\ (Zero^{n-k-1}\ Nil)$$

in order for computation to proceed. Since this takes constant time, the total running time is $O(n)$.

5. Semantics of Nested Datatypes

Having motivated the introduction of efficient generalised folds, we turn now to their formalisation. We base our presentation on the semantics for nested datatypes presented in [MJ01].

Suppose that **C** is a ω-cocomplete category with initial object and all finite products and coproducts, where $\omega = \{0 \to 1 \to 2 \to 3 \to \ldots\}$. Then the class of nested datatypes is the closure under initial algebras of the class of higher-order functors or *hofunctors* defined below. Let **Coc(C)** denote the category with objects all ω-cocontinuous endofunctors on **C** and arrows all natural transformations between them. Then we will call a *nested* hofunctor **Coc(C)**→ **Coc(C)** any hofunctor which can be constructed from the identity hofunctor and constant hofunctors, through the use of the horizontal composition, product or coproduct operators, as listed below in decreasing order of precedence.

1. $\mathscr{I}d$, the identity functor;
2. $\mathscr{K}_a$, the constant functor for each object a;
3. $\mathscr{F} \dot{\star} \mathscr{G}$, the horizontal composition of $\mathscr{F}$ and $\mathscr{G}$;
4. $\mathscr{F} \ddot{\times} \mathscr{G}$, the product of $\mathscr{F}$ and $\mathscr{G}$;
5. $\mathscr{F} \ddot{+} \mathscr{G}$, the coproduct of $\mathscr{F}$ and $\mathscr{G}$.

The last three operators are defined pointwise in terms of the corresponding bifunctors in $\mathbf{Coc}(\mathbf{C})$. Given any bifunctor $\oplus : \mathbf{C} \times \mathbf{C} \to \mathbf{C}$, the bifunctor $\dot{\oplus} : \mathbf{Coc}(\mathbf{C}) \times \mathbf{Coc}(\mathbf{C}) \to \mathbf{Coc}(\mathbf{C})$ is defined for all objects F, G and arrows θ, ψ of $\mathbf{Coc}(\mathbf{C})$ by

$$\begin{array}{lcl} (F \dot{\oplus} G)\, c & = & F\, c \oplus G\, c \\ (F \dot{\oplus} G)\, f & = & F\, f \oplus G\, f \\ (\theta \dot{\oplus} \psi)\, c & = & \theta\, c \oplus \psi\, c \end{array}$$

for objects c and arrows f of **C**. We use the notation $\ddot{\oplus}$ to denote operators that are lifted twice. So, for example, the *Nest* datatype can be defined as the initial algebra of the hofunctor $\mathscr{N} : \mathbf{Coc}(\mathbf{C}) \to \mathbf{Coc}(\mathbf{C})$ given by

$$\mathscr{N} = \mathscr{K}_{K_1} \ddot{+} (\mathscr{K}_{Id} \ddot{\times} (\mathscr{I}d \dot{\star} \mathscr{K}_{Pair})) \tag{14}$$

where the parentheses are redundant, thanks to the precedences chosen above. Since *Nest* is an initial algebra of $\mathscr{N}$, it satisfies $\mathscr{N}\ (Nest) \cong Nest$, because the initial algebra of any functor is a fixed point of that functor. This equation corresponds to the recursive definition of nests in Haskell (4).

6. Generalised Folds

The standard definition of the fold function (see eg [BdM97]) states that if **C** is a category an $F : \mathbf{C} \to \mathbf{C}$ is a functor with least fixed point T, then there exists and initial F-algebra $\alpha : FT \to T$ such that for all $v : FN \to N$ there exists a unique arrow $fold_F\, v$ where

$$fold_F\, v : T \to N$$

is characterised by the universal property that for all $u : T \to N$,

$$u = fold_F\, v \quad \Leftrightarrow \quad u \cdot \alpha = v \cdot F\, u$$

This definition was generalised in [BP99b] to the category **Coc(C)**. We will use different notation from [BP99b] but the definition is essentially the same. Let $\mathscr{F} : \mathbf{Coc}(\mathbf{C}) \to \mathbf{Coc}(\mathbf{C})$ be a hofunctor with least fixed point T

and initial algebra $\alpha : \mathscr{F}T \to T$. Let $\overline{v}$ denote a collection of parameters $v_0\ v_1 \ldots v_n$, then, given functors M and N, the generalised fold $gfold\,\overline{v}$ has type

$$gfold_{\mathscr{F}}\,\overline{v} : T \cdot M \to N$$

and is characterised by the property that for all $u : T \cdot M \to N$

$$u = gfold_{\mathscr{F}}\,\overline{v} \quad \Leftrightarrow \quad u \cdot \alpha_M = \Psi_{\mathscr{F}}\,\overline{v}\,u \tag{15}$$

where the types of the parameters $v_0, v_1, \ldots, v_m$ and definition of $\Psi_{\mathscr{F}}\,\overline{v} : (T \cdot M \to N) \to \mathscr{F}T \cdot M \to N$ are given below.

The definition of $\Psi_{\mathscr{F}}$ given in [BP99b] is top-down, but the bottom-up definition given below is equivalent, and more convenient for the purposes here. We will give an inductive definition of a function $\Phi_{\mathscr{F}}\,\overline{v} : (T \cdot M \to N) \to \mathscr{F}T \cdot M \to \mathscr{R}_{\mathscr{F}}(M,N)$, where $\mathscr{R}_{\mathscr{F}}$ is a hofunctor that is independent of M and N, and is determined by the definition of $\Phi_{\mathscr{F}}$ below. An empty collection of parameters will be denoted by $\langle\rangle$, the single parameter r by $\langle r\rangle$, and if $\overline{w}$ represents the collection $w_0, w_1, \ldots, w_n$, then $\overline{v}^\frown\overline{w}$ is the collection $v_0, v_1, \ldots, v_m, w_0, w_1, \ldots, w_n$. We now define

$$\Psi_{\mathscr{F}}\,(\langle v_0\rangle^\frown\overline{w})\,u = v_0 \cdot \Phi_{\mathscr{F}}\,\overline{w}\,u \tag{16}$$

for all $u : T \cdot M \to N$ and $v_0 : \mathscr{R}_{\mathscr{F}}(M,N) \to N$. The definition of Φ is as follows:

$$\begin{array}{llcll}
\text{(a)} & \Phi_{\mathscr{K}_Q}\,\langle\rangle\,u & = & id_{Q\cdot M} & \\
\text{(b)} & \Phi_{\mathscr{I}d}\,\langle r\rangle\,u & = & u \cdot T\,r & r : M \to M \\
\text{(c)} & \Phi_{\mathscr{F}\ddot{+}\mathscr{G}}\,(\overline{v}^\frown\overline{w})\,u & = & \Phi_{\mathscr{F}}\,\overline{v}\,u + \Phi_{\mathscr{G}}\,\overline{w}\,u & \\
\text{(d)} & \Phi_{\mathscr{F}\ddot{\times}\mathscr{G}}\,(\overline{v}^\frown\overline{w})\,u & = & \Phi_{\mathscr{F}}\,\overline{v}\,u \times \Phi_{\mathscr{G}}\,\overline{w}\,u & \\
\text{(e)} & \Phi_{\mathscr{K}_Q\dot{\star}\mathscr{G}}\,(\langle s\rangle^\frown\overline{w})\,u & = & Q\,(s \cdot \Phi_{\mathscr{G}}\,\overline{w}\,u) & s \cdot \Phi_{\mathscr{G}}\,\overline{w}\,u : \mathscr{G}T \cdot M \to M \cdot \mathscr{G}N \\
\text{(f)} & \Phi_{\mathscr{I}d\dot{\star}\mathscr{G}}\,(\langle s\rangle^\frown\overline{w})\,u & = & u_{\mathscr{G}N} \cdot T(\,s \cdot \Phi_{\mathscr{G}}\,\overline{w}\,u) & s \cdot \Phi_{\mathscr{G}}\,\overline{w}\,u : \mathscr{G}T \cdot M \to M \cdot \mathscr{G}N
\end{array}$$

where $s = id_M$ if $\mathscr{G} = \mathscr{K}_{Id}$. It is not necessary to consider any other cases, since for all $\mathscr{F}$, $\mathscr{G}$ and $\mathscr{H}$

$$\begin{array}{lcl}
(\mathscr{F}\ddot{+}\mathscr{G})\dot{\star}\mathscr{H} & = & (\mathscr{F}\dot{\star}\mathscr{H})\ddot{+}(\mathscr{G}\dot{\star}\mathscr{H}) \\
(\mathscr{F}\ddot{\times}\mathscr{G})\dot{\star}\mathscr{H} & = & (\mathscr{F}\dot{\star}\mathscr{H})\ddot{\times}(\mathscr{G}\dot{\star}\mathscr{H})
\end{array}$$

6.1. Examples of generalised folds

First we will show how the generalised fold on the datatype *Nest* given in Section 3 can be derived from the inductive definition above. Then we will give one more example: the generalised fold on the datatype *Bush*. Two further examples are given in [BP99b].

6.1.1. Nest

The *Nest* datatype is an initial algebra of the hofunctor $\mathscr{N}$ defined in equation (14) by

$$\mathscr{N} \quad = \quad \mathscr{K}_{K_1} \ddot{+} \mathscr{K}_{Id} \ddot{\times} \mathscr{I}d \dot{\star} \mathscr{K}_{Pair}$$

The calculation below determines both the value of $\Phi_{\mathscr{N}}\,\overline{w}\,u$ and the types of the parameters in the collection $\overline{w}$.

$$\begin{array}{ll}
 & \Phi_{\mathscr{N}}\ \overline{w}\,u \\
= & \quad \text{definition of } \mathscr{N} \\
 & \Phi_{\mathscr{K}_{K_1}\ddot{+}\mathscr{K}_{Id}\ddot{\times}\mathscr{I}d\dot{\star}\mathscr{K}_{Pair}}\ \overline{w}\,u \\
= & \quad \text{(a), (c) and (d)} \\
 & \Phi_{\mathscr{K}_{K_1}}\ \langle\rangle\,u + \Phi_{\mathscr{K}_{Id}}\ \langle\rangle\,u \times \Phi_{\mathscr{I}d\dot{\star}\mathscr{K}_{Pair}}\ \overline{w}\,u \\
= & \quad \text{(a) and (f)} \\
 & id_{K_1} + id_M \times \Phi_{\mathscr{I}d\dot{\star}\mathscr{K}_{Pair}}\,\overline{w}\,u \\
= & \quad \text{(f), with } \overline{w} = \langle s\rangle \\
 & id_{K_1} + id_M \times u_{Pair} \cdot nest\,(s \cdot \Phi_{\mathscr{K}_{Pair}}\langle\rangle\,u) \qquad s : Pair \cdot M \to M \cdot Pair \\
= & \quad \text{(a)} \\
 & id_{K_1} + id_M \times u_{Pair} \cdot nest\,s
\end{array}$$

We can deduce the type of parameter v_0 of equation (16) from the type of $\Phi_{\mathscr{N}}\ \overline{w}\ u$:

$$v_0 : 1 + M \times N \cdot Pair \to N$$

If we write v_0 as a coproduct $[e, f]$, take $s = g$ and suppose that $\overline{v}$ represents the parameters e, f and g, then

$$gfold_{\mathscr{F}}\ \overline{v} \cdot \alpha \quad = \quad [e, f] \cdot (id_{K_1} + id_M \times (gfold_{\mathscr{F}}\ \overline{v}) \cdot nest\ g)$$

which corresponds to the Haskell definition of the generalised fold in equation (3.2) of Section 3, except that parameter f was curried in the Haskell definition.

6.1.2. *Bush*

The datatype *Bush* was introduced in [BM98]. Its definition is the following:

$$\textbf{data}\ Bush\ a \quad = \quad Nil \mid Cons\ a\ (Bush\ (Bush\ a))$$

This datatype is the initial algebra of the following hofunctor:

$$\mathscr{B} \quad = \quad \mathscr{K}_{K_1} \ddot{+} \mathscr{K}_{Id} \ddot{\times} (\mathscr{I}d \dot{\star} \mathscr{I}d) \tag{17}$$

Therefore the map function for *Bush* is defined by:

$$\begin{array}{lcl} bush :: (l\ a \to m\ b) \to Bush\ (l\ a) \to Bush\ (m\ b) & & \\ bush\ f\ Nil & = & Nil \\ bush\ f\ (Cons\ x\ xs) & = & Cons\ (f\ x)\ (bush\ (bush\ f)\ xs) \end{array}$$

The calculation of $\Phi_{\mathscr{B}}\ \overline{w}\ u$ is similar to that of $\Phi_{\mathscr{N}}\ u$ given above, so some of the steps are omitted:

$$\begin{array}{ll} & \Phi_{\mathscr{B}}\ \overline{w}\ u \\ = & \quad \text{(a), (c), (d) and (f), with } \overline{w} = \langle s \rangle \\ & id_{K_1} + id_M \times u_N \cdot bush(s \cdot \Phi_{\mathscr{I}d}\ \langle\rangle\ u) \qquad s : N \to M \cdot N \\ = & \quad \text{(b)} \\ & id_{K_1} + id_M \times u_N \cdot bush(s \cdot u \cdot bush\ r) \qquad r : M \to M \end{array}$$

So this time the v_0 parameter has type

$$v_0 : 1 + M \times N \cdot N \to N$$

If we write v_0 as a coproduct $[e, f]$, take $s = g$ and $r = h$ and suppose that $\overline{v}$ represents the parameters e, f, g and h, then we have

$$gfold_{\mathscr{F}}\ \overline{v} \cdot \alpha \quad = \quad [e, f] \cdot (id_{K_1} + id_M \times (gfold_{\mathscr{F}}\ \overline{v}) \cdot bush(g \cdot (gfold_{\mathscr{F}}\ \overline{v}) \cdot bush\ h))$$

which corresponds to the following Haskell definition:

$$\begin{array}{lcl} gfold :: (\forall a.\ n\ a) \to (\forall a.\ m\ a \to n(n\ a) \to n\ a) \to (\forall a.\ n\ a \to m(n\ a)) \to (\forall a.\ m\ a \to m\ a) & & \\ \qquad \to Bush\ (m\ b) \to n\ b & & \\ gfold\ e\ f\ g\ h\ Nil & = & e \\ gfold\ e\ f\ g\ h\ (Cons\ x\ xs) & = & f\ x\ ((gfold\ e\ f\ g\ h \cdot bush\ (g \cdot gfold\ e\ f\ g\ h \cdot bush\ h))\ xs) \end{array}$$

The ordinary fold on bushes is then a special case of this definition, when specialised to the appropriate type:

$$fold\ e\ f \quad = \quad gfold\ e\ f\ id\ id$$

Unlike the ordinary fold on nests, this fold can be used to sum a bush, but it has the restrictive type $Bush\ a \to a$ which makes it still rather inflexible. The generalised fold in [Hin00] has the same definition as this ordinary fold, but it has a different type signature.

7. An Efficient Fold

We will now give the definition of the efficient fold. The definition is given inductively, like the definition of generalised fold.

Let $\mathscr{F} : \mathbf{Coc}(\mathbf{C}) \to \mathbf{Coc}(\mathbf{C})$ be a hofunctor with least fixed point T and initial algebra $\alpha : \mathscr{F}T \to T$, and let

$\overline{v}$ denote a collection of parameters $v_0\ v_1\ \ldots\ v_m$. Then, given functors M and N, the efficient fold $efold_{\mathscr{F}}\ \overline{v}$ has type

$$efold_{\mathscr{F}}\ \overline{v} : (\forall L, X.\ (L \to M \cdot X) \to T \cdot L \to N \cdot X)$$

and is characterised by the universal property that for all $u : (\forall L, X.\ (L \to M \cdot X) \to T \cdot L \to N \cdot X)$

$$u = efold_{\mathscr{F}}\ \overline{v} \quad \Leftrightarrow \quad (\forall p : L \to M \cdot X : u\ p \cdot \alpha_L = \Pi_{\mathscr{F}}\ \overline{v}\ u\ p) \tag{18}$$

where

$$\Pi_{\mathscr{F}}\ \overline{v} : (\forall L, X.\ (L \to M \cdot X) \to T \cdot L \to N \cdot X) \to (L \to M \cdot X) \to \mathscr{F}T \cdot L \to N \cdot X$$

is defined by

$$\Pi_{\mathscr{F}}\ (\langle v_0 \rangle \frown \overline{w})\ u\ p \quad = \quad (v_0)_X \cdot \Theta_{\mathscr{F}}\ \overline{w}\ u\ p \tag{19}$$

for all $u : (\forall L, X.\ (L \to M \cdot X) \to T \cdot L \to N \cdot X)$, $v_0 : \mathscr{R}_{\mathscr{F}}(M, N) \to N$ and $p : L \to M \cdot X$. The types of the parameters $v_0,\ v_1,\ \ldots\ v_m$ are determined by the inductive definition of

$$\Theta_{\mathscr{F}}\ \overline{v} : (\forall L, X.\ (L \to M \cdot X) \to T \cdot L \to N \cdot X) \to (L \to M \cdot X) \to \mathscr{F}T \cdot L \to \mathscr{R}_{\mathscr{F}}(M, N) \cdot X$$

which is as follows:

$$\begin{array}{llcll}
\text{(a)} & \Theta_{\mathscr{K}_Q}\ \langle\rangle\ u\ p & = & Q\ p & \\
\text{(b)} & \Theta_{Id}\ \langle r \rangle\ u\ p & = & u\ (r \cdot p) & r :: M \cdot X \to M \cdot X \\
\text{(c)} & \Theta_{\mathscr{F} \ddot{+} \mathscr{G}}\ (\overline{v} \frown \overline{w})\ u\ p & = & \Theta_{\mathscr{F}}\ \overline{v}\ u\ p + \Theta_{\mathscr{G}}\ \overline{w}\ u\ p & \\
\text{(d)} & \Theta_{\mathscr{F} \ddot{\times} \mathscr{G}}\ (\overline{v} \frown \overline{w})\ u\ p & = & \Theta_{\mathscr{F}}\ \overline{v}\ u\ p \times \Theta_{\mathscr{G}}\ \overline{w}\ u\ p & \\
\text{(e)} & \Theta_{\mathscr{K}_Q \dot{\star} \mathscr{G}}\ (\langle s \rangle \frown \overline{w})\ u\ p & = & Q\ (s \cdot \Theta_{\mathscr{G}}\ \overline{w}\ u\ p) & s \cdot \Theta_{\mathscr{G}}\ \overline{w}\ u\ p :: \mathscr{G}T \cdot L \to M \cdot \mathscr{G}N \cdot X \\
\text{(f)} & \Theta_{\mathscr{I}d \dot{\star} \mathscr{G}}\ (\langle s \rangle \frown \overline{w})\ u\ p & = & u\ (s \cdot \Theta_{\mathscr{G}}\ \overline{w}\ u\ p) & s \cdot \Theta_{\mathscr{G}}\ \overline{w}\ u\ p :: \mathscr{G}T \cdot L \to M \cdot \mathscr{G}N \cdot X
\end{array}$$

where $s = id_M$ if $\mathscr{G} = \mathscr{K}_{Id}$.

7.1. Examples of efficient folds

7.1.1. Nest

We can use the definition of $\mathscr{N}$ in equation (14) to calculate the efficient fold on nests:

$$\begin{array}{ll}
 & \Theta_{\mathscr{N}}\ \overline{w}\ u\ p \\
= & \quad \text{definition of } \mathscr{N} \\
 & \Theta_{\mathscr{K}_{K_1} \ddot{+} \mathscr{K}_{Id} \ddot{\times} \mathscr{I}d \dot{\star} \mathscr{K}_{Pair}}\ \overline{w}\ u\ p \\
= & \quad \text{(a), (c) and (d)} \\
 & \Theta_{\mathscr{K}_{K_1}}\ \langle\rangle\ u\ p + \Theta_{\mathscr{K}_{Id}}\ \langle\rangle\ u\ p \times \Theta_{\mathscr{I}d \dot{\star} \mathscr{K}_{Pair}}\ \overline{w}\ u\ p \\
= & \quad \text{(a)} \\
 & id_{K_1} + p \times \Theta_{\mathscr{I}d \dot{\star} \mathscr{K}_{Pair}}\ \overline{w}\ u\ p \\
= & \quad \text{(a) and (f), with } \overline{w} = \langle s \rangle \\
 & id_{K_1} + p \times u\ (s \cdot \Theta_{\mathscr{K}_{Pair}} \langle\rangle\ u\ p) \qquad s \cdot \Theta_{\mathscr{K}_{Pair}} \langle\rangle\ u\ p : Pair \cdot L \to M \cdot Pair \cdot X \\
= & \quad \text{(a)} \\
 & id_{K_1} + p \times u\ (s \cdot pair\ p) \qquad s : Pair \cdot M \cdot X \to M \cdot Pair \cdot X
\end{array}$$

If the functor $X = Id$, $v_0 = [e, f]$, $s = g$ and $p = h$ and if we suppose that $\overline{v}$ represents the parameters e, f and g, then we have

$$efold_{\mathscr{F}}\ \overline{v}\ h \cdot \alpha \quad = \quad [e, f] \cdot (id_{K_1} + h \times (efold_{\mathscr{F}}\ \overline{v}\ (g \cdot pair\ h)))$$

which gives the definition in Section 3:

$$\begin{array}{lcl}
efold :: (\forall a.\ n\ a) \to (\forall a.\ m\ a \to n\ (Pair\ a) \to n\ a) \to (\forall a.\ Pair\ (m\ a) \to m\ (Pair\ a)) \to (l\ a \to m\ b) & & \\
\qquad \to (Nest\ (l\ a) \to n\ b) & & \\
efold\ e\ f\ g\ h\ Nil & = & e \\
efold\ e\ f\ g\ h\ (Cons\ x\ xs) & = & f\ (h\ a)\ (efold\ e\ f\ g\ (g \cdot pair\ h)\ xs)
\end{array}$$

Notice that the type signature of this Haskell definition is not as general as it should be. The fourth parameter h should have type $\forall a.\ l\ a \to m\ a$, but this is not acceptable to Haskell because it uses a kinded first-order unification rather than full higher-order unification. So we have avoided this problem by type specialising, and using functors of the form $L \cdot K_A$ and $M \cdot K_B$ instead of arbitrary L and M.

7.1.2. Bush

We can use the definition of $\mathscr{B}$ in equation (17) to calculate

$$\begin{array}{ll} & \Theta_{\mathscr{B}}\ \overline{w}\ u\ p \\ = & \quad \text{(a), (c), (d) and (f), with } \overline{w} = \langle s \rangle \\ & id_{K_1} + p \times u\ (s \cdot \Theta_{\mathscr{I}d}\ \langle\rangle\ u\ p) \qquad\qquad s \cdot \Theta_{\mathscr{I}d}\ \langle\rangle\ u\ p : T \cdot L \to M \cdot N \\ = & \quad \text{(b)} \\ & 1 + p \times u\ (s \cdot (u\ (r \cdot p))) \qquad\qquad r : M \to M,\ s : N \to M \cdot N \end{array}$$

Taking $v_0 = [e, f]$, $s = g$, $r = h$ and $p = k$ gives the definition:

$$\begin{array}{lcl} \multicolumn{3}{l}{efold :: (\forall a.\ n\ a) \to (\forall a.\ m\ a \to n\ (n\ a) \to n\ a) \to (\forall a.\ n\ a \to m\ (n\ a)) \to (\forall a.\ m\ a \to m\ a) \to (l\ a \to m\ b)} \\ \multicolumn{3}{l}{\qquad \to (Bush\ (l\ a) \to n\ b)} \\ efold\ e\ f\ g\ h\ k\ Nil & = & e \\ efold\ e\ f\ g\ h\ k\ (Cons\ x\ xs) & = & f\ (k\ x)\ (efold\ e\ f\ g\ h\ (g \cdot (efold\ e\ f\ g\ h\ (h \cdot k)))\ xs) \end{array}$$

This is different from the definition in [Hin00].

7.2. Uniqueness of efficient folds

We will now show that efficient folds are characterised uniquely by their defining equations (18). The corresponding property of generalised folds is deduced from a more general result in [BP99b], which is stated in Theorem 7.1 below. We will use the notation

$$A \to_{\mathbf{C}} B$$

to denote the (hom)set of all arrows from A to B in $\mathbf{C}$. The theorem below uses the contravariant hom-functor, which we will write as $__ \to_{\mathbf{C}} B$ for each object B in $\mathbf{C}$. It sends each object A to the set $A \to_{\mathbf{C}} B$ and each arrow $g : A \to A'$ to the function $\lambda f.f \cdot g$.

Theorem 7.1. Suppose that $F :: \mathbf{C} \to \mathbf{C}$ has initial algebra $\alpha : F\ T \to T$, and

$$\Psi :: \forall A.\ (P\ A \to_{\mathbf{D}} B) \to (P(F\ A) \to_{\mathbf{D}} B)$$

is a natural transformation for some functor $P : \mathbf{C} \to \mathbf{D}$. Then if F and P preserve colimits of chains and P preserves initiality, there is a unique $x :: P\ T \to B$ such that

$$x \cdot P\alpha \quad = \quad \Psi\ x. \tag{20}$$

Fortunately the following corollary shows that the uniqueness theorem for efficient folds is also an instance of Theorem 7.1.

Corollary 7.1. Suppose that $\mathscr{F} :: \mathbf{Coc}(\mathbf{C}) \to \mathbf{Coc}(\mathbf{C})$ has initial algebra $\alpha : \mathscr{F}\ T \to T$, and suppose that $u : \forall L, X.\ (L \to_{\mathbf{Coc}(\mathbf{C})} M \cdot X) \to T \cdot L \to_{\mathbf{Coc}(\mathbf{C})} N \cdot X$, then

$$u = efold\ \overline{v} \Leftrightarrow (\forall p : L \to M \cdot X : u\ p \cdot \alpha_L = \Pi_{\mathscr{F}}\ \overline{v}\ u\ p). \tag{21}$$

Proof. Let $\mathbf{D}$ denote the category whose objects are functions which map pairs of endofunctors on $\mathbf{Coc}(\mathbf{C})$ to a single endofunctor. So, if R is an object of $\mathbf{D}$, and $L, X : \mathbf{C} \to \mathbf{C}$ are functors, then $R(L, X) : \mathbf{C} \to \mathbf{C}$ is a functor. Each arrow $k : R \to S$ in $\mathbf{D}$ is a total function of type

$$k : (\forall L, X.\ (L \to_{\mathbf{Coc}(\mathbf{C})} M \cdot X) \to R(L, X) \to_{\mathbf{Coc}(\mathbf{C})} S(L, X)).$$

Let the composition $\circ$ of $k : R \to S$ and $l : S \to T$ in $\mathbf{D}$ be defined by:

$$l \circ k \quad = \quad \lambda p.(l\ p \cdot k\ p)$$

where $\cdot$ denotes composition in $\mathbf{Coc}(\mathbf{C})$. Define $\mathscr{P} : \mathbf{Coc}(\mathbf{C}) \to \mathbf{D}$ on each object A and arrow $\gamma : A \to B$ by

$$\begin{array}{lcl} \mathscr{P}\, A\, (L,X) & = & A \cdot L \\ \mathscr{P}\, \gamma\, (L,X) & = & \lambda p.\, \gamma_L. \end{array}$$

Then if $B(L,X) = N \cdot X$,

$$\Pi_{\mathscr{F}} :: \forall A.\ (\mathscr{P}\, A \to_{\mathbf{D}} B) \to (\mathscr{P}(\mathscr{F}\, A) \to_{\mathbf{D}} B)$$

is a natural transformation, so statement (21) follows by Theorem 7.1 provided that $\mathscr{P}$ preserves initiality and colimits of chains. It is left to the reader to check these properties of $\mathscr{P}$. $\square$

The uniqueness theorem can be used to prove the following:

Theorem 7.2. Suppose that $\mathscr{F} : \mathbf{Coc}(\mathbf{C}) \to \mathbf{Coc}(\mathbf{C})$ has initial algebra $\alpha : \mathscr{F}\, T \to T$. Then for all collections of parameters $\overline{v}$ of appropriate type, and all $p : L \to M \cdot X$,

$$efold_{\mathscr{F}}\ \overline{v}\, p \quad = \quad gfold_{\mathscr{F}}\ \overline{v} \cdot T\, p.$$

Proof Outline Suppose $\overline{v} = \langle v_0 \rangle ^\frown \overline{w}$, let $y = gfold_{\mathscr{F}}\ \overline{v}$ and define $u : (\forall L, X.\ (L \to M \cdot X) \to T \cdot L \to N \cdot X)$ by

$$u\, p = y_X \cdot T\, p$$

By the universal property of efficient folds (18) and definition of Π (19),

$$u = efold_{\mathscr{F}}\, \overline{v} \quad \Leftrightarrow \quad (\forall p : L \to M \cdot X : u\, p \cdot \alpha_L = (v_0)_X \cdot \Theta_{\mathscr{F}}\, \overline{w}\, u\, p)$$

We can calculate that

$$\begin{array}{ll} & u\, p \cdot \alpha_L \\ = & \quad \text{definition of } u \\ & y_X \cdot T\, p \cdot \alpha_L \\ = & \quad \text{naturality of } \alpha \\ & y_X \cdot \alpha_{M \cdot X} \cdot \mathscr{F}\, T\, p \\ = & \quad \text{universal property of generalised folds } (15) \\ & (\Psi_{\mathscr{F}}\, \overline{v}\, y)_X \cdot \mathscr{F}\, T\, p \\ = & \quad \text{definition of } \Psi\ (16) \\ & (v_0 \cdot \Phi_{\mathscr{F}}\, \overline{w}\, y)_X \cdot \mathscr{F}\, T\, p \end{array}$$

Hence it is sufficient to prove that for all $\mathscr{F}$ and $p : L \to M \cdot X$,

$$\Theta_{\mathscr{F}}\, \overline{w}\, u\, p \quad = \quad (\Phi_{\mathscr{F}}\, \overline{w}\, y)_X \cdot \mathscr{F}\, T\, p$$

which is easily verified by routine case analysis on each of the clauses (a) to (f) in the definitions of Φ and Θ. $\square$

7.3. Fusion laws for efficient folds

The following map fusion law is immediate from Theorem 7.2.

Law 7.1 (Map fusion I). Let $k : L' \to L$, then for all $p : L \to M$

$$efold_{\mathscr{F}}\ \overline{v}\, p \cdot T\, k = efold_{\mathscr{F}}\ \overline{v}\, (p \cdot k)$$

The fusion laws of [BP99b], together with Theorem 7.2 give two further fusion laws for efficient folds:

Law 7.2 (Map fusion II). Suppose that for each $i \in 0..m$,

$$v_i :: \mathscr{P}_i M \to \mathscr{Q}_i M \text{ and } v'_i :: \mathscr{P}_i L \to \mathscr{Q}_i L$$

where $\mathscr{P}_i$ and $\mathscr{Q}_i$ are hofunctors that are independent of M. Then if $p : L \to M$

$$(\forall i \in 1..m \bullet v_i \cdot \mathscr{P}_i p = \mathscr{Q}_i p \cdot v'_i) \Rightarrow efold_{\mathscr{F}}\ \overline{v}\, p = efold_{\mathscr{F}}\ \overline{w}\ id$$

where $w_0 = v_0 \cdot \mathscr{R}_{\mathscr{F}}(p, id_N)$ and for all $i \in 1..m$, $w_i = v'_i$.

Law 7.3 (Fold fusion). Let $k : N \to N'$, and suppose that for each $i \in 0..m$,

$$v_i :: \mathscr{P}_i N \to \mathscr{Q}_i N \text{ and } v'_i :: \mathscr{P}_i N' \to \mathscr{Q}_i N'$$

where $\mathscr{P}_i$ and $\mathscr{Q}_i$ are hofunctors that are independent of N. Then

$$(\forall i \in 0..m \bullet \mathscr{Q}_i k \cdot v_i = v'_i \cdot \mathscr{P}_i k) \Rightarrow k \cdot efold_{\mathscr{F}}\ v_0 \cdots v_m = efold_{\mathscr{F}}\ v'_0 \cdots v'_m.$$

This fold fusion law is an instance of a stronger law given for three specific types in [BP99b], and given here for nests in (9). The weaker law for nests, corresponding to the above, is obtained by taking k' to be the identity function in (9). The stronger law cannot be simply stated for arbitrary datatypes, so the reader is referred to [BP99b] for details.

8. Applications

The fusion laws of the previous section are not very meaningful until applied to a concrete example. We will now use them to give conditions under which an efficient fold can be rewritten as an ordinary fold for a small class of nested datatypes. We will then apply this result to functions on two different datatypes: nests and lists. We will show that there are two equivalent ways to use efficient folds to flatten a nest to a list. The two methods correspond to the inverse methods of building a tree: recursive, or top-down, and iterative, or bottom-up. The application to lists concerns Horner's rule for evaluating polynomials [BdM97]. We will show that this is a trivial consequence of the fusion laws.

We will only consider datatypes of the form:

$$\textbf{data}\ Collection\ a \quad = \quad Nil \mid Cons\ a\ (Collection\ (Fun\ a))$$

where *Fun* is a functor. So for example, if $Fun = Pair$, then *Collection* defines nests, and if $Fun = Id$ it defines lists.

The efficient fold on type *Collection* is identical to that on nests, except that *Pair* is replaced by the more general *Fun*:

$$\begin{array}{lcl} efold :: (\forall a.\ n\ a) \to (\forall a.\ m\ a \to n\ (Fun\ a) \to n\ a) \to (\forall a.\ Fun\ (m\ a) \to m\ (Fun\ a)) \to (l\ a \to m\ b) \\ \qquad\qquad \to (Unit\ (l\ a) \to n\ b) \\ efold\ e\ f\ g\ h\ Nil & = & e \\ efold\ e\ f\ g\ h\ (Cons\ x\ xs) & = & f\ (h\ x)\ (efold\ e\ f\ g\ (g \cdot fun\ h)\ xs) \end{array}$$

The fusion laws are also very similar to those for nests, but are restated here for easy reference in the proof of Lemma 8.1 below. The fold fusion law has one less parameter than its counterpart for nests (9) because it is the weaker version of the law, as given in Section 7.3. Assuming the types given to parameters e, f, g and h in the definition of the efficient fold above, the fold and map fusion laws of Section 7.3 can be instantiated to the following.

Law 8.1 (Map fusion I). Let $k' : \forall a.\ l'\ a \to l\ a$, then

$$efold\ e\ f\ g\ h \cdot fun\ k = efold\ e\ f\ g\ (h \cdot k)$$

Law 8.2 (Map fusion II). Let $g' : (\forall a.\ Fun\ (l\ a) \to l\ (Fun\ a))$, then

$$\begin{array}{l} \quad g \cdot fun\ h = h \cdot g' \\ \Rightarrow \\ \quad efold\ e\ f\ g\ h = efold\ e\ (f \cdot h)\ g'\ id \end{array}$$

Law 8.3 (Fold fusion). Let $e' : \forall a.\ n'\ a$, $f' : \forall a.\ m\ a \to n'\ (Fun\ a) \to n'\ a$ and $k : \forall a.\ n\ a \to n'\ a$, then

$$\begin{array}{l} \quad k\ e = e' \text{ and } k \cdot f = apply\ f'\ id\ k \\ \Rightarrow \\ \quad k \cdot efold\ e\ f\ g\ h = efold\ e'\ f'\ g\ h \end{array}$$

where $apply\ r\ s\ t\ (x,y) = r\ (s\,x)\ (t\,y)$.

These laws can be used together to deduce the following lemma:

Lemma 8.1 (Fold Equivalence Law). Let $e : \forall a.\, n\, a,\ f : m\, a \to n\, a \to n\, a,\ g : Fun\, (m\, a) \to n\, a,\ h : \forall a.\, a \to n\, a$ and suppose there exist $k : n\, (Fun\, a) \to n\, a$ and $k' : m\, (Fun\, a) \to m\, a$, then

$$\begin{array}{ll} & k\, e = e \quad k \cdot f = apply\, (f \cdot k')\, id\, k \quad g \cdot fun\, k' = k' \cdot g \ \text{ and } \ g \cdot fun\, h = k' \cdot h \\ \Rightarrow & \\ & efold\, e\, f\, g\, h = fold\, e\, (apply\, f\, h\, k) \end{array}$$

Proof. We have

$$\begin{array}{ll} & efold\, e\, f\, g\, h = fold\, e\, (apply f\, h\, k) \\ \Leftrightarrow & \quad \text{universal property of } fold \\ & efold\, e\, f\, g\, h\, Nil = e \text{ and } efold\, e\, f\, g\, h\, (Cons\, x\, xs) = f\ (h\, x)\ ((k \cdot efold\, e\, f\, g\, h)\, xs) \\ \Leftarrow & \quad \text{definition of } efold \\ & efold\, e\, f\, g\, (g \cdot fun\, h) = k \cdot efold\, e\, f\, g\, h \end{array}$$

This equality is established by appealing to the fusion laws:

$$\begin{array}{ll} & k \cdot efold\, e\, f\, g\, h \\ = & \quad \text{Law 8.3 (fold fusion)} \\ & efold\, e\, (f \cdot k')\, g\, h \\ = & \quad \text{Law 8.1 (map fusion I)} \\ & efold\, e\, (f \cdot k')\, g\, id \cdot fun\, h \\ = & \quad \text{Law 8.2 (map fusion II)} \\ & efold\, e\, f\, g\, k' \cdot fun\, h \\ = & \quad \text{Law 8.1 (map fusion I)} \\ & efold\, e\, f\, g\, (k' \cdot h) \\ = & \quad g \cdot fun\, h = k' \cdot h \\ & efold\, e\, f\, g\, (g \cdot fun\, h) \end{array}$$

□

We will give three applications of this lemma: two for nests and one for lists. The two functions on nests are *flatten* and *size*, which were cited in Section 3 as examples for which the map fusion law (10) did not directly apply. Writing these functions as efficient folds, we have

$$\begin{array}{lcl} flatten & = & efold\, [\,]\, (+\!\!+)\, (uncurry(+\!\!+))\, (\backslash x \to [x]) \\ size & = & efold\, 0\, (+)\, (uncurry(+))\, (\backslash x \to 1) \end{array}$$

The *flatten* function flattens a nest by a method that corresponds to the inverse method of recursively building a tree, but the fold equivalence law transforms it into a function corresponding to the inverse method of iteratively building a tree: if $m\, a = n\, a = [a]$, and $k = k' = concat \cdot map\, (\backslash (y,z) \to [y,z])$, then

$$flatten \quad = \quad fold\, [\,]\, (apply\, (+\!\!+)\, (\backslash x \to [x])\, k)$$

For the *size* function, we take $m\, a = n\, a = Int$, and $k = k' = (2*)$, which gives

$$size \quad = \quad fold\, 0\, (apply\, (+)\, (\backslash x \to 1)\, (2*))$$

Finally, we notice that Horner's rule for lists [BdM97] can be derived very simply by taking $m = n$, $k = k' = g$ and $h = id$, which leaves only two conditions:

$$\begin{array}{ll} & g\, e = e \text{ and } g \cdot f = apply\, f\, g\, g \\ \Rightarrow & \\ & efold\, e\, f\, g\, id = fold\, e\, (apply\, f\, id\, g) \end{array}$$

By applying this law to the classic example of polynomial evaluation, we can see that it is a restatement of Horner's rule: let $us = (Cons\ u\ (Cons\ v\ (Cons\ w\ Nil)))$, then

$$\begin{aligned} efold\ 0\ (+)\ (*x)\ id\ us &= u + v * x + w * x^2 \\ fold\ 0\ (apply\ (+)\ id\ (*x))\ us &= u + (v + (w + 0) * x) * x \end{aligned}$$

Horner's rule was generalised to arbitrary regular datatypes in [BdM97]; whether there is a similar generalisation for nested datatypes remains to be seen.

9. Conclusion

In this paper we have given a universal construction of a fold operator that was inspired by that of [Hin00]. The new construction resolves some of the shortcomings of the original definition: it can be used to calculate the types of the parameters of the fold on any nested datatype, and it comes equipped with a proof principle that we have used to relate efficient folds to generalised folds on arbitrary nested datatypes in Theorem 7.2. The performance improvement outlined in [Hin00] has been reiterated here through the example of random access lists, and the fusion laws of [BP99b] have been restated in the context of efficient folds. The laws have then been used to derive a fold equivalence law that gives conditions that are sufficient to rewrite an efficient fold as an ordinary fold. The fold equivalence law has yet to be generalised to arbirary datatypes.

We have observed that folds in general provide a useful definition principle for functions on nested datatypes. Without them it is difficult even to define a function as simple as *sum*. We have noted that efficient folds are not always more efficient than generalised ones, but they do capture a different pattern of computation. Furthermore, they are arguably more closely related than generalised folds to ordinary folds, because they have a similar map fusion law. Although the applications of nested datatypes have so far been quite limited, the example of Horner's rule in Section 8 suggests that the proof rules associated with efficient folds could still be useful for program derivation on regular datatypes. It might be interesting to investigate whether there is a class of problems on regular datatypes that can be specified as efficient folds, and then manipulated through the proof rules.

There are still a number of questions that we have yet to answer, such as when is a function on either regular or nested datatypes a fold [GHA01], or an efficient fold? Moreover, if a function can be expressed as an efficient fold, under what conditions does this give a significant improvement in performance? On the theoretical side, there are similarities between the initial algebra definitions of generalised and efficient folds and those of Mendler-style inductive types [UV99]. Closer inspection has shown that our definitions do not fit into the existing framework of [UV99], but it might be possible to extend the theory to include them. There seems little point in pursuing this theory yet though, until we have found some more interesting applications of efficient folds and nested datatypes.

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