

Optimal Power Control in Interference Limited Fading Wireless Channels with Outage Probability Specifications

Sunil Kandukuri and Stephen Boyd

Electrical Engineering Department, Stanford University

May 24, 2000

Abstract

We propose a new method of power control for interference limited wireless networks with Rayleigh fading of both the desired and interference signals. Our method explicitly takes into account the statistical variation of both the received signal and interference power, and optimally allocates power subject to constraints on the probability of fading induced outage for each transmitter/receiver pair. We establish several results for this type of problem.

For the case in which the only constraints are those on the outage probabilities, we give a fast iterative method for finding the optimal power allocation. We establish tight bounds that relate the outage probability caused by channel fading to the signal-to-interference margin calculated when the statistical variation of the signal and interference powers are ignored. This allows us to show that well-known methods for allocating power, based on Perron-Frobenius eigenvalue theory, can be used to determine power allocations that are provably close to achieving optimal (*i.e.*, minimal) outage probability.

In the most general case, which includes bounds on powers and other constraints, we show that the power control problem can be posed as a *geometric program*, which is a special type of optimization problem that can be transformed to a nonlinear convex optimization by a change of variables, and therefore solved globally and efficiently by recently developed interior-point methods.

1 Introduction

Wise allocation of power is critical in wireless networks for both longer battery life of the mobile devices, and for increased utilization of the limited wireless spectrum. Power control provides an intelligent way of determining transmitting power to achieve the Quality of Service (QoS) goals in wireless channels. Because of these benefits, it has been very well studied [GGF94, Mit93, ARZ96, ARZ97, YH95, Yat95, Bam98, BCP95]. Traditional power control schemes, whether centralized [Ari92, AN82, NA83, Aei73] or distributed [FM93, FM95, BCP95] always assume quasi-stationarity of the fading wireless channels and base their power control schemes on the observed signal-to-interference ration (SIR) at the receiver or the knowledge of the gains of all the links. Thus, the implicit assumption made is that the power control updates are made every time the fading state of the channel changes, *i.e.*, whenever the gain of any link changes. In wireless communication channels, which exhibit fast fading where the fades can change within milliseconds (at 900MHz, and mobile traveling at 60mph), this might not always be practical. Very frequent power updates can also consume a lot of signal processing energy.

In this paper, we propose a power control scheme in which the power does not need to be updated whenever the channel meanders from one fading state to another. Instead, we explicitly take into account the statistical variation of the signal-to-interference ratio of each transmitter/receiver pair, and optimally allocate power to minimize probability of fading-induced outage (which occurs when the SIR falls below a threshold SIR^{th}).

We find a global solution to this problem, by showing that it can be posed as a nonlinear convex optimization problem. Solutions methods for these problems not only produce the global optimum (efficiently), but also unambiguously determine feasibility. This enables us to make QoS guarantees, or to determine beforehand whether the services requested by the mobile user can be provided or not. Most importantly, our power control analysis allows power updates to be carried out at a time scale far larger than the Rayleigh fading time scale, which is often the lognormal shadowing time scale.

Clearly the probability of outage can be reduced by allocating power in such a way that each mobile has an extra margin of SIR, *i.e.*, its SIR is somewhat above the minimum value required for reception. Increasing the extra margin of SIR reduces the probability of outage, but costs extra power. Our method can be interpreted as an intelligent way to carry out this ad hoc method of giving extra SIR margins to the mobiles. Our method gives each mobile an extra margin of SIR that is directly based on the required probability of fading-induced outage.

The paper is organized as follows. In §2 we describe the system and fading model. In §3 we derive an expression for the probability that a mobile experiences fading-induced outage, and also some tight bounds that relate the probability of outage to a margin of SIR, ignoring statistical variation of the interference and signal powers. In §4, we formulate the problem of minimizing the outage probability, with no other constraints on the powers, and give an algorithm to solve it. In §5, we describe how the general the power control optimization problem is formulated and solved using geometric programming, and in §6, we give a simple illustrative example.

2 Rayleigh/Rayleigh fading environment

We consider the following setup. We have n transmitters, labeled $1, \dots, n$, which transmit at power level P_1, \dots, P_n , which are the variables in our optimization problem. We also have n receivers, labeled $1, \dots, n$; receiver i is meant to receive the signal from transmitter i . (By transmitter and receiver, we don't necessarily mean different physical transmitters and receivers; different receivers, for example, might refer to the same physical receiver, with different frequency channels, codes, or antenna beams in an antenna array.) The power received from transmitter j , at receiver i , is given by

$$G_{ij}F_jP_j. \tag{1}$$

The number G_{ij} , which is positive, represents the path gain (not including fading) from the j th transmitter to the i th receiver. This gain can be interpreted in many ways: it can represent the distance dependent power attenuation, log-normal shadowing, cross correlations between codes in a code division multiple access (CDMA) system, as well as the gains dependent on the antenna direction and size. In the analysis below, we assume that G_{ij} are *constant, i.e.*, don't change (much) with time. Therefore the analysis holds for a time scale over which the factors that determine G_{ij} are approximately constant: the distance between transmitters and receivers does not change much, the log-normal shadowing does not change much, direction dependent antenna gains do not change much.

The numbers F_j model *Rayleigh fading*. They are assumed to be independent, exponentially distributed random variables, with unit mean. (In a Rayleigh fading environment, the received signal envelope has a Rayleigh distribution; the received signal power has an exponential distribution [Stu97].) In other words, the power received at receiver i from transmitter j is an exponentially distributed random variable, with mean value

$$\mathbf{E}G_{ij}F_jP_j = G_{ij}P_j.$$

We refer to this situation, in which both desired signals and interference signals are subject to Rayleigh fading, as a *Rayleigh/Rayleigh fading environment*. The assumption behind the Rayleigh/Rayleigh fading environment is that the receiver gets no direct line-of-sight signal component, either from its own transmitter or from the interfering transmitters.

We will also assume that the interference from other transmitters is much larger than the white noise in the receivers, and therefore ignore receiver noise in our analysis. Both the Rayleigh/Rayleigh fading environment, and this assumption of interference-limited communication, are very realistic in urban wireless networking environments.

3 Outage probability and certainty-equivalent margin

3.1 SIR and outage probability

The signal power at the i th receiver is given by $G_{ii}F_iP_i$, and the total interference power is given by

$$\sum_{k \neq i} G_{ik}F_kP_k.$$

The *signal-to-interference ratio* (SIR) of the i th receiver (or transmitter) is given by

$$\text{SIR}_i = \frac{G_{ii}F_iP_i}{\sum_{k \neq i} G_{ik}F_kP_k}.$$

Note that in a Rayleigh/Rayleigh fading environment, SIR_i is a random variable with what would appear to be a very complex distribution, since it is the ratio of an exponential random variable to a sum of exponential random variables (with different means). (We will see later, however, that there is an analytical expression for its density.)

We assume that reception can occur provided the SIR exceeds a given threshold SIR^{th} . The *outage probability* of the i^{th} receiver/transmitter pair is given by

$$O_i = \mathbf{Prob}(\text{SIR}_i \leq \text{SIR}^{\text{th}}) = \mathbf{Prob}\left(G_{ii}F_iP_i \leq \text{SIR}^{\text{th}} \sum_{k \neq i} G_{ik}F_kP_k\right). \quad (2)$$

The outage probability O_i can be interpreted as the fraction of time the i^{th} transmitter/receiver pair experiences an outage due to fading. Note that in our expression for O_i , we take into account statistical variation of both received signal power and received interference power.

Surprisingly, the outage probability can be expressed in analytical form; it was derived in [YS90] (see also [YS92, Stu97]), although we will use an equivalent form that has not appeared in the literature, as far as we know. The analytic expression for O_i is derived from the following result: Suppose z_1, \dots, z_n are independent exponentially distributed random variables with means $\mathbf{E}z_i = 1/\lambda_i$. Then we have

$$\mathbf{Prob}\left(z_1 \leq \sum_{i=2}^n z_i\right) = 1 - \prod_{i=2}^n \left(\frac{1}{1 + \lambda_1/\lambda_i}\right).$$

We give a self-contained derivation of this result in §A.

Applying this result to (2), we find that the outage probability for the i^{th} transmitter/receiver pair can be expressed as

$$O_i = 1 - \prod_{k \neq i} \frac{1}{1 + \text{SIR}^{\text{th}} G_{ik}P_k / G_{ii}P_i}. \quad (3)$$

We define the worst outage probability, over all transmitter/receiver pairs, as

$$O = \max_i O_i,$$

and simply refer to O it as the *outage probability of the system*. (More accurately, it is the maximum of the outage probabilities of the transmitter/receiver pairs.) The outage probability O serves as a simple figure of merit for the system and power allocation.

3.2 Certainty equivalent margin

We now consider the *certainty-equivalent system*, in which we ignore all statistical variation of both signal and noise power, by replacing these random variables with their expected

values. The certainty-equivalent signal power at the i^{th} receiver is then $G_{ii}P_i$, and the certainty-equivalent interference power at the i^{th} receiver is given by $\sum_{k \neq i} G_{ik}P_k$. We define the certainty-equivalent signal-to-interference ratio at the i^{th} receiver as

$$\text{SIR}_i^{\text{ce}} = \frac{G_{ii}P_i}{\sum_{k \neq i} G_{ik}P_k}. \quad (4)$$

We can interpret SIR_i^{ce} as follows: it is what the signal-to-interference of the i^{th} transmitter/receiver pair would be, if the fading state of the system were $F_1 = \dots = F_n = 1$.

We also define

$$\text{SIR}^{\text{ce}} = \min_i \text{SIR}_i^{\text{ce}} = \min_i \frac{G_{ii}P_i}{\sum_{k \neq i} G_{ik}P_k},$$

which is the minimum certainty-equivalent SIR of the system, over all transmitter/receiver pairs. We refer to SIR^{ce} as, simply, the certainty-equivalent signal-to-interference ratio. Like the outage probability O , SIR^{ce} gives a figure of merit for the system and power allocation.

We define CEM, the *certainty-equivalent margin*, of the system and power allocation, as the ratio of the certainty-equivalent signal-to-interference ratio to the signal-to-interference reception threshold:

$$\text{CEM} = \frac{\text{SIR}^{\text{ce}}}{\text{SIR}^{\text{th}}} = \min_i \frac{G_{ii}P_i}{\text{SIR}^{\text{th}} \sum_{k \neq i} G_{ik}P_k}.$$

Clearly there is a relation between CEM and O : when CEM is large (which means that the SIR, ignoring statistical variation, is well above the minimum required for reception) we should have small O . The relation between CEM and O is the topic of the next section.

3.3 Relation between CEM and outage probability

In this section we derive some bounds between the certainty-equivalent margin and the outage probability. We use the following result (derived in §B): If $z_1, \dots, z_n \geq 0$, then

$$1 + \sum_{k=1}^n z_k \leq \prod_{k=1}^n (1 + z_k) \leq \exp \sum_{k=1}^n z_k. \quad (5)$$

By definition, we have

$$\begin{aligned} O &= \max_i \left(1 - \prod_{k \neq i} \frac{1}{1 + \text{SIR}^{\text{th}} G_{ik}P_k / G_{ii}P_i} \right) \\ &= 1 - \frac{1}{\max_i \prod_{k \neq i} (1 + \text{SIR}^{\text{th}} G_{ik}P_k / G_{ii}P_i)}. \end{aligned}$$

Using the righthand inequality in (5), we get

$$\begin{aligned} O &\leq 1 - \frac{1}{e^{\max_i \sum_{k \neq i} \text{SIR}^{\text{th}} G_{ik}P_k / G_{ii}P_i}} \\ &= 1 - e^{-1/\text{CEM}}. \end{aligned}$$

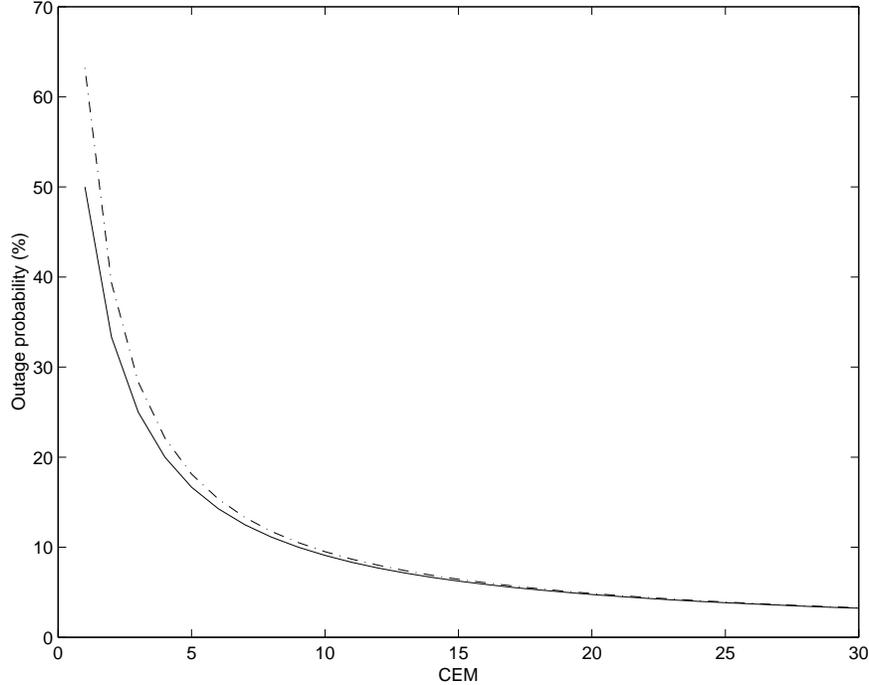


Figure 1: Upper and lower bound on outage probability O , as a function of certainty-equivalent margin (CEM).

In a similar way, using the lefthand inequality in (5), we have

$$\begin{aligned} O &\geq \frac{1}{1 + \max_i \sum_{k \neq i} \text{SIR}^{\text{th}} G_{ik} P_k / G_{ii} P_i} \\ &= \frac{1}{1 + \text{CEM}}. \end{aligned}$$

Putting these two inequalities together, we have the bounds

$$\frac{1}{1 + \text{CEM}} \leq O \leq 1 - e^{-1/\text{CEM}}. \quad (6)$$

A plot of these bounds is given in figure 1. From the plot it is clear that for outage probabilities of interest, *i.e.*, those smaller than 20% or so, the lower and upper bounds are very close, within about 5%. For larger CEM (and smaller outage probability), the bounds are much closer, confirming our intuition that CEM and outage probability are closely related. Figure 2 shows the ratio of the upper to the lower bound as a function of CEM. This plot shows that the bounds are very close for outage probabilities smaller than 10% or so, and not far from each other even for small CEM (and large O). For example, with CEM equal to one, the probability of outage is at least 50%, but no more than 63.3%.

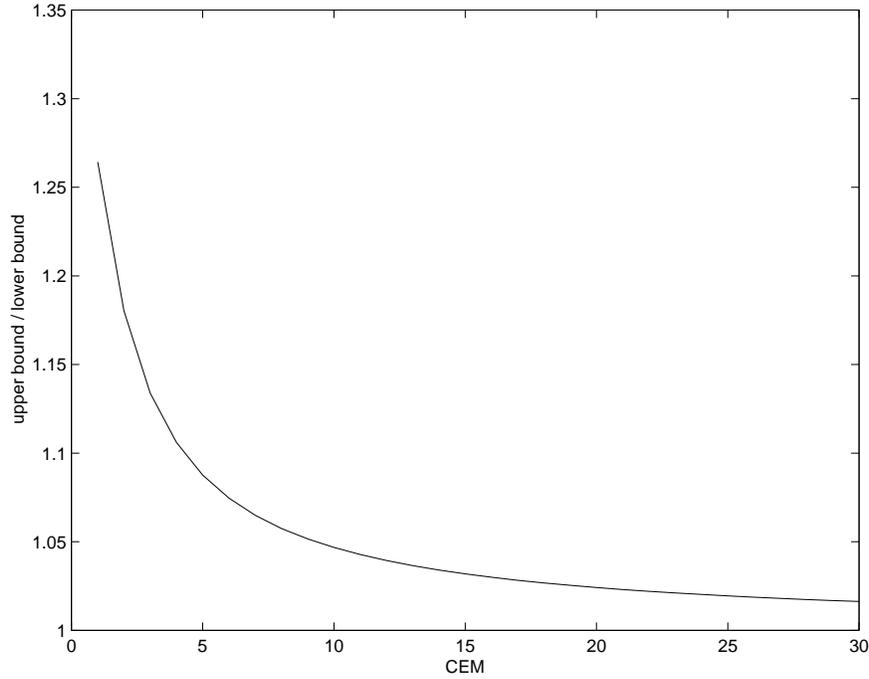


Figure 2: Ratio of upper to lower bound on outage probability O , as function of CEM.

4 Optimal power allocation

In this section we consider the problems of finding the power allocations that minimize the outage probability, and maximize CEM, respectively. The problem of minimizing outage probability can be expressed as

$$\begin{aligned} & \text{minimize} && \max_i \left(1 - \prod_{k \neq i} \frac{1}{1 + \text{SIR}^{\text{th}} G_{ik} P_k / G_{ii} P_i} \right) \\ & \text{subject to} && P_i > 0, \quad i = 1, \dots, n, \end{aligned} \quad (7)$$

and the problem of maximizing CEM can be expressed as the optimization problem

$$\begin{aligned} & \text{maximize} && \min_i \frac{G_{ii} P_i}{\text{SIR}^{\text{th}} \sum_{k \neq i} G_{ik} P_k} \\ & \text{subject to} && P_i > 0, \quad i = 1, \dots, n. \end{aligned} \quad (8)$$

In these problems, the variables are the powers P_1, \dots, P_n . The constants SIR^{th} and G_{ik} , $i, k = 1, \dots, n$, are problem parameters. We will assume that $G_{ik} > 0$.

We observe that the objective functions are homogeneous, *i.e.*, if we scale all powers by any (positive) scale factor, O and CEM remain the same. In other words, outage probability and certainty-equivalent margin depend only on the ratios of the powers. Since the constraints $P_i > 0$ are also homogeneous, it follows that if P is an optimal power allocation vector (for either problem), then so is αP , for any $\alpha > 0$.

We will let P^{out} denote a power allocation vector that is optimal for the problem (7), *i.e.*, that minimizes the outage probability. Similarly, we will let P^{cem} denote a power allocation vector that is optimal for the problem (8), *i.e.*, that maximizes the certainty-equivalent margin.

Our next observation is that in each problem, the optimum is achieved with the values of the maximum (for minimizing O) or minimum (for maximizing CEM) all equal. Let us first consider the problem (7) of minimizing the outage probability. We claim that at an optimal power allocation P^{out} , the outage probabilities of each transmitter/receiver pair must be equal. In other words, we have

$$O_i(P^{\text{out}}) = 1 - \prod_{k \neq i} \frac{1}{1 + \text{SIR}^{\text{th}} G_{ik} P_k^{\text{out}} / G_{ii} P_i^{\text{out}}} = O(P^{\text{out}}) = O^*, \quad i = 1, \dots, n,$$

where O^* denotes the minimal value of outage probability.

To establish the result, we first observe that O_i is monotone increasing in P_k for $k \neq i$, and monotone decreasing in P_i . Now suppose that not all $O_i(P^{\text{out}})$ are equal. Choose an index k for which $O_k < O^* = \max_i O_i$. Now if we decrease P_k^{out} , O_k increases, and all other O_i decrease. It follows that if we decrease P_k^{out} by a small amount, $O = \max_i O_i$ will decrease. But this contradicts the assumption that P^{out} minimizes O .

The analogous result holds for the problem (8) of maximizing CEM. In this problem, we observe that each CEM_i is monotonically increasing in P_i , and monotonically decreasing in P_k for $k \neq i$. Arguing exactly as above, we conclude that we must have

$$\text{CEM}_i(P^{\text{cem}}) = \frac{G_{ii} P_i}{\text{SIR}^{\text{th}} \sum_{k \neq i} G_{ik} P_k} = \text{CEM}(P^{\text{cem}}) = \text{CEM}^*, \quad i = 1, \dots, n,$$

where CEM^* is the maximal value of CEM.

4.1 Maximizing CEM

In the field of wireless networks, power control by maximizing CEM has been well studied and understood [Mit93, Bam98, YM94, YM95]. It is based on the Perron-Frobenius theorem for the maximum eigen value of a matrix which has non-negative elements [Mit93].

Using our observation that at the optimum, CEM_i are all equal, we can reformulate the problem (8) as

$$\begin{aligned} & \text{maximize} && t \\ & \text{subject to} && \frac{G_{ii} P_i}{\text{SIR}^{\text{th}} \sum_{k \neq i} G_{ik} P_k} = t, \quad i = 1, \dots, n, \\ & && P_i > 0, \quad i = 1, \dots, n, \end{aligned}$$

where t is another variable, whose optimal value is the optimal value of CEM. Substituting the variable $\tau = 1/t$, we can express this problem as

$$\begin{aligned} & \text{minimize} && \tau \\ & \text{subject to} && AP = \tau P \\ & && P_i > 0, \quad i = 1, \dots, n, \end{aligned}$$

where the matrix A is defined as

$$A_{ij} = \frac{G_{ij}}{\text{SIR}^{\text{th}} G_{ii}}, \quad i \neq j, \quad A_{ii} = 0.$$

We recognize the problem above as an eigenvalue problem, in which the matrix has all entries nonnegative. According to Perron-Frobenius theory, the eigenvalue λ of A that is largest in magnitude is real and positive, and has an associated eigenvector v all of whose components are positive. (Here we use the fact that A is not cyclic or reducible, which follows from $G_{ij} > 0$.) The eigenvector v (and associated eigenvalue λ) are called the Perron-Frobenius eigenvector (eigenvalue) of A . The Perron-Frobenius eigenvector v gives an optimal power allocation, *i.e.*, $P_i = v_i$ maximizes CEM. The optimal CEM is exactly $\text{CEM}^* = 1/\lambda$.

Eventhough the above optimal solution assumes a centralized controller, there exists distributed methods to achieve the same solution [FM93, FM95, Bam98].

4.2 Relation between CEM optimal and O optimal allocations

Using the bounds of §3, we can show that a power allocation P^{cem} that maximizes CEM (which can be found by computing the Perron-Frobenius eigenvector of an $n \times n$ matrix) is not too far from minimizing outage probability.

Let P denote an arbitrary power allocation (with $P_i > 0$). Then we have

$$\text{CEM}(P) \leq \text{CEM}(P^{\text{cem}})$$

since by definition, P^{cem} maximizes CEM. It follows that

$$\frac{1}{1 + \text{CEM}(P)} \geq \frac{1}{1 + \text{CEM}(P^{\text{cem}})}$$

(since the function mapping x into $1/(1+x)$ is monotone decreasing for $x \geq 0$). Combining this inequality with the lefthand bound in (6), we have

$$O(P) \geq \frac{1}{1 + \text{CEM}(P)} \geq \frac{1}{1 + \text{CEM}(P^{\text{cem}})}.$$

This inequality holds for all P , so we have

$$O^* \geq \frac{1}{1 + \text{CEM}(P^{\text{cem}})},$$

where O^* denotes the minimum possible outage probability, *i.e.*, the optimal value of the problem (7).

From this inequality we can make several conclusions. First of all, if we compute P^{cem} (by solving a Perron-Frobenius eigenvalue problem), then we can bracket O^* : it is certainly between the lower bound $1/(1 + \text{CEM}(P^{\text{cem}}))$ and the upper bound $O(P^{\text{cem}})$. These bounds are often extremely close, and in any case never far apart. Indeed, since

$$O(P^{\text{cem}}) \leq 1 - e^{-1/\text{CEM}},$$

we always have

$$\frac{1}{1 + \text{CEM}(P^{\text{cem}})} \leq O^* \leq 1 - e^{-1/\text{CEM}(P^{\text{cem}})}.$$

Since the ratio of these bounds is often near one, and never far from one, it follows that maximizing CEM is often very nearly the same as minimizing outage probability, and provably never very suboptimal.

4.3 Minimizing outage probability

In this section we consider the problem of minimizing outage probability. According to our observation above that at the optimum, all outage probabilities are equal, the problem can be expressed as

$$\begin{aligned} & \text{minimize} && t \\ & \text{subject to} && 1 - \prod_{k \neq i} \frac{1}{1 + \text{SIR}^{\text{th}} G_{ik} P_k / G_{ii} P_i} = t, \quad i = 1, \dots, n \\ & && P_i > 0, \quad i = 1, \dots, n, \end{aligned} \quad (9)$$

where t is another variable. (In fact, the condition that all of the outage probabilities be equal is not only necessary for optimality; it is also sufficient. This follows by examining the convex form of the problem, as explained in §5).

Several methods can be used to solve the problem (9). It can always be globally solved using geometric programming (see §5), for example. In the remainder of this section we describe a simple iterative algorithm that in our experience computes P^{out} within a few iterations, where each iteration consists of solving a Perron-Frobenius eigenvector problem. We do not have a proof that the method always converges, but we have never observed a case where it fails to converge in at most 4 or 5 iterations. (In any case, as mentioned above, P^{out} can always be computed using geometric programming.)

To motivate our iterative method, we start with the equality constraints

$$1 - \prod_{k \neq i} \frac{1}{1 + \text{SIR}^{\text{th}} G_{ik} P_k / G_{ii} P_i} = t, \quad i = 1, \dots, n,$$

where t is the variable to be minimized. This can be rewritten as

$$\prod_{k \neq i} \left(1 + \text{SIR}^{\text{th}} G_{ik} P_k / G_{ii} P_i \right) = \beta, \quad i = 1, \dots, n,$$

where $\beta = 1/(1 - t)$. Here, the objective is to minimize β .

We rewrite these equations in the form

$$\sum_{k \neq i} \log \left(1 + \text{SIR}^{\text{th}} G_{ik} P_k / G_{ii} P_i \right) = \gamma, \quad i = 1, \dots, n,$$

where $\gamma = \log \beta$ is to be minimized. This is equivalent to

$$\sum_{k \neq i} \left[\frac{P_i}{P_k} \log \left(1 + \text{SIR}^{\text{th}} G_{ik} P_k / G_{ii} P_i \right) \right] P_k = \gamma P_i, \quad i = 1, \dots, n,$$

which we express as $B(P)P = \gamma P$, where B is the matrix given by

$$B_{ik} = \frac{P_i}{P_k} \log \left(1 + \text{SIR}^{\text{th}} G_{ik} P_k / G_{ii} P_i \right), \quad i \neq k,$$

and $B_{ii} = 0$.

Now our problem can be stated as finding P (with positive entries) and γ , that minimize γ and satisfy the condition $B(P)P = \gamma P$. If we ignore the fact that B depends on P , this problem can be solved as a Perron-Frobenius eigenvector problem.

We can now describe our iterative method. We start with $P = P^{\text{cem}}$, then fix $B = B(P)$ and update P by solving the Perron-Frobenius eigenvector problem $BP = \gamma P$. This is repeated until P doesn't change, so we have $B(P)P = \gamma P$, which solves the problem of minimizing outage probability. In our experience the algorithm always converges in fewer than 5 or so steps, to an accuracy far exceeding any significance for the engineering problem (*i.e.*, to 10 significant figures).

5 Optimal power allocation via geometric and linear programming

In this section we show that the problem of power allocation with constraints on the outage probability, as well as other constraints such as limits on the individual powers, can be expressed as a special type of optimization problem called geometric programming.

5.1 Geometric programming

Let x_1, \dots, x_n be n real, positive variables, and x denote the vector of these n variables. A function $f : \mathbf{R}_+^n \rightarrow \mathbf{R}$ is called a *posynomial* function if it has the form

$$f(x_1, \dots, x_n) = \sum_{k=1}^t c_k x_1^{\alpha_{1k}} x_2^{\alpha_{2k}} \dots x_n^{\alpha_{nk}},$$

where $c_k \geq 0$ and $\alpha_{ij} \in \mathcal{R}$. Note that the coefficients c_k must be nonnegative, but the exponents α_{ij} can be any negative (or fractional) number. The function f is called a *monomial* function if $t = 1$ and $c_1 > 0$, *i.e.*, it consists of one nonzero term. Posynomials are closed under addition and multiplication.

A *geometric program* (GP) is an optimization problem of the form

$$\begin{aligned} & \text{minimize} && f_0(x) \\ & \text{subject to} && f_i(x) \leq 1, \quad i = 1, \dots, m, \\ & && g_i(x) = 1, \quad i = 1, \dots, p, \\ & && x_i > 0, \quad i = 1, \dots, n, \end{aligned} \tag{10}$$

where f_1, \dots, f_m are posynomial functions and g_1, \dots, g_p are monomial functions. Geometric programs were introduced by Duffin [DPZ67]; recent applications include wire and transistor

sizing for digital circuits [FD85] and op-amp design [HBL98]; see [BV99]. Using interior-point methods for nonlinear convex programming, originally developed by Nesterov and Nemirovsky [NN94], GPs can be solved with great efficiency. Indeed, very large GPs can be solved using primal-dual interior-point methods; see [Wri97, Van96, KXY96].

A geometric program can be reformulated as a *convex optimization problem*, *i.e.*, the problem of minimizing a convex function subject to convex inequality constraints and linear equality constraints, by a change of variables. Suppose that f is a posynomial, and define $y_i = \log x_i$, so that $x_i = e^{y_i}$ (which automatically enforces the positivity constraint on x_i). We define the function

$$h(y) = \log f(e^{y_1}, \dots, e^{y_n}) = \log \left(\sum_{k=1}^t e^{a_k^T y + b_k} \right),$$

where $a_k = (\alpha_{1k}, \dots, \alpha_{nk})$ and $b_k = \log c_k$. It can be shown that h is a *convex* function of the new variable y ; if the original function f were a monomial, then the function h is affine (*i.e.*, linear plus a constant). Applying this change of variable to the geometric program (10), we obtain

$$\begin{aligned} & \text{minimize} && \log f_0(e^{y_1}, \dots, e^{y_n}) \\ & \text{subject to} && \log f_i(e^{y_1}, \dots, e^{y_n}) \leq 0, \quad i = 1, \dots, m, \\ & && \log g_i(e^{y_1}, \dots, e^{y_n}) = 0, \quad i = 1, \dots, p. \end{aligned} \tag{11}$$

This is called the *convex form* of the geometric program. It is a (nonlinear) convex optimization problem, since the objective and inequality constraint functions are all convex, and the equality constraint functions are affine.

One important consequence is that we can solve GPs, globally, with great efficiency, using recently developed interior-point methods (see, *e.g.*, [NN94, BV99]).

5.2 Minimum total power with outage probability constraints

We start with an example power allocation problem: we minimize the total transmit power, subject to the constraint that each transmitter/receiver attain a maximum allowed outage probability (*i.e.*, a minimum allowed QoS), and subject to limits on the individual transmitter powers:

$$\begin{aligned} & \text{minimize} && P_1 + \dots + P_n \\ & \text{subject to} && P_i^{\min} \leq P_i \leq P^{\max}, \quad i = 1, \dots, n, \\ & && O_i \leq O_i^{\max}, \quad i = 1, \dots, n \end{aligned} \tag{12}$$

Here, P_i^{\min} and P_i^{\max} are the minimum and maximum transmitter power for transmitter i ; the maximum might be dependent on the transmitter hardware, and the minimum value guarantees that the white noise at receiver is overcome. The number O_i^{\max} is the maximum allowed outage probability for the i th transmitter/receiver. Note that these can be the same for each pair, or different, allowing different QoS to be assigned to different users.

Evidently the outage probability constraints $O_i \leq O_i^{\max}$ are the challenging ones, since O_i is a highly nonlinear function of the powers. Using (3), we can express the outage probability constraint $O_i \leq O_i^{\max}$ as

$$1 - O_i^{\max} \leq \prod_{k \neq i} \left(\frac{1}{1 + \text{SIR}^{\text{th}} G_{ik} P_k / G_{ii} P_i} \right),$$

which in turn, we can express as

$$(1 - O_i^{\max}) \prod_{k \neq i} (1 + \text{SIR}^{\text{th}} G_{ik} P_k / G_{ii} P_i) \leq 1. \quad (13)$$

Since each of the terms $1 + \text{SIR}^{\text{th}} G_{ik} P_k / G_{ii} P_i$ is a posynomial function of the powers, we conclude that the lefthand side of the inequality (13) is, in fact, a posynomial function of the powers P_1, \dots, P_n .

Using this result, we can express the problem (12) as

$$\begin{aligned} & \text{minimize} && P_1 + \dots + P_n \\ & \text{subject to} && P_i^{\min} / P_i \leq 1, \quad i = 1, \dots, n, \\ & && P_i / P_i^{\max} \leq 1, \quad i = 1, \dots, n, \\ & && (1 - O_i^{\max}) \prod_{k \neq i} (1 + \text{SIR}^{\text{th}} G_{ik} P_k / G_{ii} P_i) \leq 1, \quad i = 1, \dots, n. \end{aligned} \quad (14)$$

This is a geometric program in the variables P_1, \dots, P_n . Therefore we can solve the power allocation problem (12) globally and efficiently, using interior-point methods for geometric programming. Note that any other constraints that can be handled by geometric programming can be added to the power allocation problem.

5.3 Minimum outage probability with power constraints

As another example, we can also minimize the outage probability O by solving the GP

$$\begin{aligned} & \text{minimize} && \alpha \\ & \text{subject to} && P_i^{\min} / P_i \leq 1, \quad i = 1, \dots, n, \\ & && P_i / P_i^{\max} \leq 1, \quad i = 1, \dots, n, \\ & && (1/\alpha) \prod_{k \neq i} (1 + \text{SIR}^{\text{th}} G_{ik} P_k / G_{ii} P_i) \leq 1, \quad i = 1, \dots, n, \end{aligned} \quad (15)$$

with optimization variables P_1, \dots, P_n , and α . Here, α is an upper bound on $1/(1 - O_i^{\max})$, so when we solve the GP (15), the optimal value of α is $1/(1 - O^*)$, where O^* is the minimal value of the maximum outage probability.

6 Example

In this section we give a simple numerical example demonstrating the results of this paper. We consider a system with 50 transmitters and receivers, with Rayleigh/Rayleigh fading, and ambient white noise power that is insignificant compared to interference power.

We generated the gain matrix G in the following way. We take all the gains G_{ii} (from i th transmitter to i th receiver) to be one, and we generate the cross gains G_{ij} , $i \neq j$, as independent random variables, uniformly distributed between between 0 and 0.001. We varied SIR^{th} from 3 to 10 and for each value, computed P^{cem} and P^{out} . For each value of SIR^{th} , we also computed $O(P^{\text{cem}})$, the outage probability achieved by P^{cem} , as well as O^* (which is $O(P^{\text{out}})$), and the lower bound $1/(1 + O(P^{\text{cem}}))$. The results are shown in figure 3.

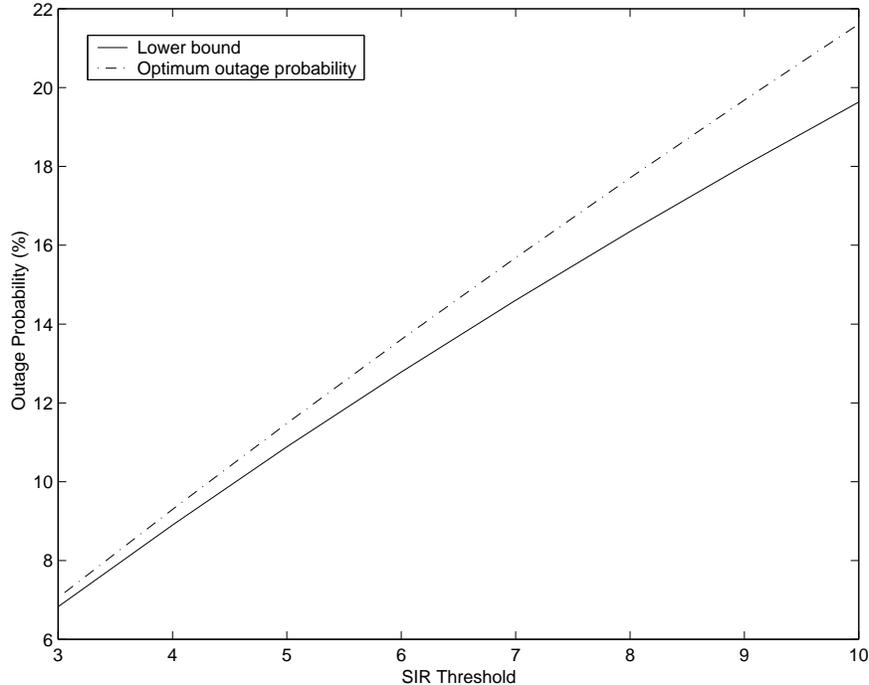


Figure 3: Outage probability versus SIR^{th} , for a system with 50 wireless links. The dotted curve shows the outage probability achieved both by the exact optimal power allocation and the outage probability achieved by the power allocation that maximizes CEM (the difference is negligible). The solid curve shows the lower bound on optimal outage probability based on CEM.

Our first observation is that P^{cem} , the power allocation that maximizes certainty-equivalent margin, also minimizes outage probability, for any practical purpose. The differences in outage probabilities obtained by the power allocation that maximizes CEM and those obtained by the power allocation that exactly minimize outage probability were insignificantly small.

7 Conclusions

We have considered the problem of allocating power in a wireless system, taking into account the statistical fluctuation in SIR induced by Rayleigh fading. In the general case, we establish that this problem can be cast as a geometric programming problem, hence efficiently solved. We establish that when the only constraints are those on probability of outage, the problem of minimizing probability of outage is for all practical purposes solved by maximizing the certainty-equivalent margin, which can be done using Perron-Frobenius eigenvalue methods, or other iterative methods developed for this problem. While maximizing this margin is certainly a natural heuristic for minimizing outage probability, we prove a rigorous bound on how suboptimal this heuristic can be.

The benefit of our method of allocating power is that it allows power allocation to be done

on the far longer time scale of log-normal shadowing, instead of the time scale of Rayleigh fading. The disadvantage is a positive probability of fading-induced outage. (Of course, this disadvantage is also present in a power allocation method that attempts to track fading state: for some fading states, allocating power to guarantee reception for all transmitter/receiver pairs is impossible.)

Acknowledgments

The authors are grateful to Professors Bambos, Prabhakar, and Goldsmith for helpful discussions.

A Derivation of probability expression

In this section we give a self-contained derivation of the following result: Suppose z_1, \dots, z_n are independent exponentially distributed random variables with means $\mathbf{E}z_i = 1/\lambda_i$. Then we have

$$\mathbf{Prob} \left(z_1 \leq \sum_{i=2}^n z_i \right) = 1 - \prod_{i=2}^n \left(\frac{1}{1 + \lambda_1/\lambda_i} \right). \quad (16)$$

To prove this, we note that

$$\begin{aligned} \mathbf{Prob} \left(z_1 > \sum_{i=2}^n z_i \right) &= \int_{t_2=0}^{\infty} \cdots \int_{t_n=0}^{\infty} \mathbf{Prob} \left(z_1 > \sum_{i=2}^n t_i \right) \prod_{i=2}^n \lambda_i e^{-\lambda_i t_i} dt_2 \cdots dt_n \\ &= \int_{t_2=0}^{\infty} \cdots \int_{t_n=0}^{\infty} e^{-\lambda_1(t_2+\cdots+t_n)} \prod_{i=2}^n \lambda_i e^{-\lambda_i t_i} dt_2 \cdots dt_n \\ &= \prod_{i=2}^n \int_{t_i=0}^{\infty} \lambda_i e^{-(\lambda_1+\lambda_i)t_i} dt_i \\ &= \prod_{i=2}^n \frac{\lambda_i}{\lambda_1 + \lambda_i}. \end{aligned}$$

Subtracting this expression from one yields (16).

B Derivation of bounds on $\prod_{k=1}^n (1 + z_k)$

In this section we derive the following inequalities: If $z_1, \dots, z_n \geq 0$, then

$$1 + \sum_{k=1}^n z_k \leq \prod_{k=1}^n (1 + z_k) \leq \exp \sum_{k=1}^n z_k.$$

To establish the lefthand inequality, we expand the middle expression as

$$\prod_{k=1}^n (1 + z_k) = 1 + \sum_{k=1}^n z_k + \sum_{k=1}^n \sum_{j>k}^n z_k z_j + \cdots$$

The first and second terms are the lefthand side of the inequality we wish to establish; the third and other remaining terms are nonnegative, since they consist of sums of products of z_i , which are nonnegative.

To establish the righthand inequality, we will derive the equivalent inequality

$$\sum_{k=1}^n \log(1 + z_k) \leq \sum_{k=1}^n z_k.$$

This follows from the simple inequality $\log(1 + z) \leq z$ for $z \geq 0$.

References

- [Aei73] J. Aein. Power balancing in systems employing frequency reuse. *COMSAT Technical Review*, 3(2), 1973.
- [AN82] H. Alavi and R. Nettleton. Downstream power control for a spread spectrum cellular mobile radio system. In *Proceedings of Globecom*, 1982.
- [Ari92] S. Ariyavisitakul. Sir based power control in a cdma system. In *Proceedings of Globecom*, 1992.
- [ARZ96] M. Andersin, Z. Rosberg, and J. Zander. Gradual removals in cellular pcs with constrained power control and noise. *Wireless Networks*, 2(1):27–43, 1996.
- [ARZ97] M. Andersin, Z. Rosberg, and J. Zander. Gradual removals in cellular pcs with constrained power control and noise. *IEEE/ACM Trans. in Networking*, 5(2):255–265, 1997.
- [Bam98] N. Bambos. Toward power-sensitive network architectures in wireless communications: Concepts, issues and design aspects. *IEEE Personal Communications Magazine*, 5(3):50–59, 1998.
- [BCP95] N. Bambos, S. Chen, and G. Pottie. Radio link admission algorithms for wireless networks with power control and active link quality protection. In *Proceedings of the IEEE Infocom*, 1995.
- [BV99] S. Boyd and L. Vandenberghe. Introduction to convex optimization with engineering applications. Course Notes, 1999. <http://www.stanford.edu/class/ee364/>.
- [DPZ67] R. J. Duffin, E. L. Peterson, and C. Zener. *Geometric Programming — Theory and Applications*. Wiley, 1967.
- [FD85] J. P. Fishburn and A. E. Dunlop. TILOS: a posynomial programming approach to transistor sizing. In *Proceedings ICCAD’85*, pages 326–328, 1985.
- [FM93] G. Foschini and Z. Miljanic. A simple distributed autonomous power control algorithm and its convergence. *IEEE Trans. Vehicular Tech.*, 42(4), 1993.
- [FM95] G. Foschini and Z. Miljanic. Distributed autonomous wireless channel assignment with power control. *itvt*, 44(3):420–429, 1995.
- [GGF94] A. Goldsmith, L. Greenstein, and G. Foschini. Error statistics of real-time power measurements in cellular channels with multipath and shadowing. *IEEE Trans. Vehicular Tech.*, 43(3), August 1994.
- [HBL98] M. Hershenson, S. Boyd, and T. H. Lee. GPCAD: A tool for CMOS op-amp synthesis. In *Proceedings of the IEEE/ACM International Conference on Computer Aided Design*, San Jose, CA, November 1998.

- [KXY96] K. O. Kortanek, X. Xu, and Y. Ye. An infeasible interior-point algorithm for solving primal and dual geometric programs. *Mathematical Programming*, 76:155–181, 1996.
- [Lin92] J. Linnartz. Exact analysis of the outage probability in multiple user mobile radio. *IEEE Trans. Communications*, 40(1), January 1992.
- [Mit93] D. Mitra. An asynchronous distributed algorithm for power control in cellular radio systems. In *Proc. of 4th WINLAB Workshop, Rutgers University, NJ*, 1993.
- [NA83] R. Nettleton and H. Alavi. Power control for a spread-spectrum cellular mobile radio system. In *Proceedings of Vehicular Technology Conference*, 1983.
- [NN94] Yu. Nesterov and A. Nemirovsky. *Interior-point polynomial methods in convex programming*, volume 13 of *Studies in Applied Mathematics*. SIAM, Philadelphia, PA, 1994.
- [Stu97] G. Stuber. *Principles of Mobile Communication*. Kluwer Academic, 1997.
- [Van96] R. J. Vanderbei. *Linear Programming: Foundations and extensions*. Kluwer’s International Series, 1996.
- [Wri97] S. J. Wright. *Primal-Dual Interior-Point Methods*. SIAM, Philadelphia, 1997.
- [Yat95] R. Yates. A framework for uplink power control in cellular radio systems. *IEEE J. Sel. Areas in Comm.*, 13(7):1341–1347, 1995.
- [YH95] R. Yates and C. Y. Huang. Integrated power control and base station assignment. *IEEE Trans. Vehicular Tech.*, 44(3):638–644, 1995.
- [YM94] L. C. Yun and D. G. Messerschmitt. Power control for variable qos on a cdma channel. In *Proceedings of IEEE MILCOM*, volume 1, pages 178–182, 1994.
- [YM95] L. C. Yun and D. G. Messerschmitt. Variable quality of service in cdma systems by statistical power control. In *Proceedings of IEEE International Conf. on Communications*, volume 2, pages 713–719, 1995.
- [YS84] Y.-S. Yeh and S. C. Schwartz. Outage probability in mobile telephony due to multiple log-normal interferers. *IEEE Trans. Communications*, COM. 32(4), April 1984.
- [YS90] Y.-D. Yao and A. Sheikh. Outage probability analysis for microcell mobile radio systems with cochannel interferers in rician/rayleigh fading environment. *Electronics Letters*, 26:864–866, June 1990.
- [YS92] Y.-D. Yao and A. Sheikh. Investigations into cochannel interference in microcellular mobile radio systems. *IEEE Trans. Vehicular Tech.*, 41(2), May 1992.