A CHARACTERISATION THEOREM OF THE LOGARITHMIC FUNCTION MODULO 1

P. ERDŐS, I. JOÓ, AND L. Á. KÓCZY

The original idea of such characterisations is that of [2]. He proved that if a real-valued additive function is non-decreasing, or satisfies $f(n+1) - f(n) \to 0$ (as $n \to \infty$), then it must have the form $c \log n$ for some constant c. He had a separate argument for each case.

In [1] Elliott solves the problem in a generalised form: Let $a > 0$, b, $A > 0$ and B be integers with $\Delta = aB - Ab$ non-zero. If G is a real additive arithmetic function and for some constant C satisfies $G(an+b) - G(An+B) \to C$ as $n \to \infty$, then there is a further constant c such that $G(x) = c \log x$ for every $x \in N$ which are prime to $aA\Delta$.

The problem can be generalised further, there is a great variety of conditions that can be introduced giving numerous different results. This paper gives proof of the following rather interesting theorem:

Theorem 1. *Assume that $a \in \mathbb{N}$ and $G(x) : \mathbb{R} \to \mathbb{R}$ is non-decreasing modulo 1 by which we mean non-decreasing over $(0,1)$ and $[n, n+1)$ $\forall n \in \mathbb{N}$ and additive on the set Σ, where Σ is defined as*

$$\Sigma = \left\{ an + \theta : n \in \mathbb{N}, \theta < a, \theta = \frac{p}{q} \text{ s. t.} p, q \in \mathbb{N}, (p,q) = (a,p) = 1 \right\},$$

then

$$G(x) = c_0 \log x \text{ for a constant } c_0 \text{ and all } x \in \mathbb{R}.$$

In order to prove this theorem we need to give some lemmas. First some properties of the set Σ:

Lemma 2. *Σ is closed under multiplication.*

Proof. Take any two elements of Σ, $an_1 + \frac{p_1}{q_1}$ and $an_2 + \frac{p_2}{q_2}$. Their product is

$$\left(an_1 + \frac{p_1}{q_1}\right)\left(an_2 + \frac{p_2}{q_2}\right) = \frac{an_1q_1 + p_1}{q_1}\frac{an_2q_2 + p_2}{q_2} = \frac{a^2n_1n_2q_1q_2 + an_1p_2q_1 + an_2p_1q_2 + p_1p_2}{q_1q_2}$$

If $(a^2n_1n_2q_1q_2 + an_1p_2q_1 + an_2p_1q_2 + p_1p_2, a) \neq 1$ then either $(an_1q_1 + p_1, a) \neq 1$ or $(an_2q_2 + p_2, a) \neq 1$, so $(a^2n_1n_2q_1q_2 + an_1p_2q_1 + an_2p_1q_2 + p_1p_2, a) = 1$ and hence

$$\frac{a^2n_1n_2q_1q_2 + an_1p_2q_1 + an_2p_1q_2 + p_1p_2}{q_1q_2} = ak + \frac{p}{q_1q_2}$$

for some $p \in \mathbb{N}$. Lemma is proved. ☐

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Before giving the other property, for any $\rho \in \mathbb{R}$ we define

$$p_i = \left\lceil \frac{(a+1)^i}{\rho} \right\rceil \text{ and } q_i = \left\lfloor \frac{(a+1)^i}{\rho} \right\rfloor$$

where $\lceil x \rceil$ and $\lfloor x \rfloor$ mean the nearest integer that is not less and not greater than x, respectively ($\lfloor x \rfloor = \sup(n < x, n \in \mathbb{N})$, $\lceil x \rceil = \inf(n > x, n \in \mathbb{N})$), and let

$$\pi_i = \frac{(a+1)^i}{p_i} < \rho < \frac{(a+1)^i}{q_i} = \theta_i$$

π_i and θ_i are defined such that as $i \to \infty$, $\pi_i \to \rho$ and $\theta_i \to \rho$.

Lemma 3. *Σ has an element between any two distinct real numbers.*

Proof. Take real numbers $x \neq y$ and $\rho \in (x, y)$ such that $\rho \notin \mathbb{Q}$. Such ρ exists. π_i and θ_i are approximators of ρ, while $\pi_i, \theta_i \in \Sigma$ and so $\exists j$such that$\pi_j \in (x, y)$ (of course, for some j' $\theta_{j'} \in (x, y)$ is true, too, but this is not our current interest) □

Lemma 4. *Assume that $a \in \mathbb{N}$ and G is additive on the set Σ, and*

$$\lim_{n\to\infty} (G(a(n+m)+\rho) - G(an+\sigma)) = 0 \textit{ if } n, m \in \mathbb{N}, \rho, \sigma \in \Sigma, \rho < a, \sigma < a$$

Then we have

$$G(an+\rho) = c_0 \log(an+\rho) \textit{ for some } c_0 \in \mathbb{R} \tag{1}$$

Proof. Assume that $\exists h_1$ and $h_2 (h_1 \neq h_2)$ such that $h_1 \equiv \theta (\mathrm{mod}\ a)$, $h_2 \equiv \tau (\mathrm{mod}\ a)$ (where $\theta, \tau \in \Sigma$), $\frac{G(h_2)}{\log h_2} \neq \frac{G(h_1)}{\log h_1}$. Let e.g. $\frac{G(h_2)}{\log h_2} > \frac{G(h_1)}{\log h_1}$. Let x_0 be an arbitrary but fixed number, for which

$$\frac{G(h_2)}{\log h_2} > x_0 > \frac{G(h_1)}{\log h_1}$$

Denote $G_0 := G - x_0 \log$. Then G_0 is additive on Σ and

$$\lim_{n\to\infty} (G_0(a(n+m)+\rho) - G_0(an+\sigma)) = 0 \text{ if } a, n, m, \rho, \sigma \text{ are as before.} \tag{1'}$$

Further

$$c_2 := \frac{G(h_2)}{\log h_2} > \frac{G(h_1)}{\log h_1} =: c_1 \text{ and } G_0(h_2) > 0 > G_0(h_1). \tag{2}$$

Denote $d_{h_2}(n) := G_0(an+\sigma) - (1-\varepsilon)c_2 \log n$, where $0 < \varepsilon < 1$, we will choose later. We show that $d_{h_2}(n)$ is bounded above, i.e. we show that if $n > n_0(c_1, c_2, h_1, h_2, \varepsilon)$ then there exists $m < n$ for which

$$G_0(an+\sigma) - (1-\varepsilon)c_2 \log n < G_0(am+\rho) - (1-\varepsilon)c_2 \log m. \tag{3}$$

We are looking for such m for which

$$h_1(am+\rho) > an+\sigma \tag{4}$$

In the following we extend congruences to reals by defining $\equiv^*$ as follows:

$$a \equiv^* b (\mathrm{mod}\ c) \text{ for } a, b, c \in R \Leftrightarrow a = b + kc, \text{ where } k \in \mathbb{Z}$$

We are looking for m, such that

$$am + \rho \equiv^* \rho (\text{mod } h_1) \tag{5}$$

This implies $am \equiv^* 0(\text{mod } h_1)$, i.e. $am = h_1 k$ for some $k \in \mathbb{N}$. Also $a, m \in \mathbb{N}$, so $h_1 = \frac{an+p}{q}$ where $p, q \in \mathbb{N}$, and as $h_1 \in \Sigma$, $(p, a) = 1$. So $am \equiv^* 0(\text{mod } h_1) \Leftrightarrow amq \equiv^* 0(\text{mod } an+p)$. But $(a, an+p) = 1$ so there exists a unique solution $(\text{mod } an+p)$, hence also $\left(\text{mod } \frac{an+p}{q}\right)$, i.e. $(\text{mod } h_1)$, it is m_0. All solutions of (5) are $m_0 + kh_1 (k \in \mathbb{N})$. Choose the smallest k for which (4) is fulfilled, i.e. $m = m_0 + kh_1 > \frac{n}{h_1} + \frac{\sigma}{ah_1} - \frac{\rho}{a}$, hence

$$m = \frac{n}{h_1} + O(1) \text{ as before.} \tag{6}$$

Thus $m < n$ is satisfied. Since $h_1 \equiv^* \theta(\text{mod } a)$ therefore as Σ is closed under multiplication, for some $\sigma \in \Sigma, h_1(am + \rho) \equiv^* \sigma(\text{mod } a)$ with $ak + \sigma \in \Sigma$. Further, using (6), $h_1(am + \rho) - (an + \sigma) = O(1)$. Thus we can use $(1')$,

$$G_0(an + \sigma) = G_0(h_1(am + \rho)) + o_n(1) \tag{7}$$

where o_n means $o_n(1) \to 0$ as $n \to \infty$. Since G_0 is additive on Σ,

$$G_0(an + \sigma) = G_0(h_1) + G_0(am + \rho) + o_n(1),$$

where we used that h_1 and $am + \rho$ are both in Σ. From this

$$G_0(an+\sigma) - (1-\varepsilon)c_2 \log n = G_0(h_1) - (1-\varepsilon)c_2 \log \frac{n}{m} + (G_0(am+\rho) - (1-\varepsilon)c_2 \log m) + on(1)$$

From (8) $\frac{n}{m} = h_1 + O\left(\frac{1}{m}\right)$, thus $\log \frac{n}{m} = \log h_1 + O\left(\frac{1}{m}\right)$, and

$$G_0(an + \sigma) - (1 - \varepsilon)c_2 \log n < G_0(am + \rho) - (1 - \varepsilon)c_2 \log m,$$

where we chose ε such that $0 < \varepsilon < \varepsilon_0$, $\left(\frac{G(h_1)}{\log h_1} - (1-\varepsilon)\frac{G(h_2)}{\log h_2}\right) < 0$ and $n > n_0$. Hence $d_{h_2}(n)$ is bounded above.

We can prove similarly that $d_{h_2}(n)$ is bounded below. We are looking for such $m < n$ for which

$$\begin{aligned}
&(3') \quad G_0(an + \sigma) - (1 - \varepsilon)c_2 \log n &&> G_0(am + \rho) - (1 - \varepsilon)c_2 \log m.\\
&(4') \quad h_2(am + \rho) &&> an + \sigma\\
&(5') \quad am + \rho &&\equiv^* \rho(\text{mod } h_2)
\end{aligned}$$

Then we have

$$m = \frac{n}{h_2} + O(1). \tag{6'}$$

Thus $m < n$ is satisfied. Hence

$$G_0(an+\sigma) - (1-\varepsilon)c_2 \log n = G_0(h_2) - (1-\varepsilon)c_2 \log \frac{n}{m} + (G_0(am+\rho) - (1-\varepsilon)c_2 \log m) + o_n(1)$$

Here $\frac{n}{m} = h_2\left(1 + O\left(\frac{1}{m}\right)\right)$, thus $\log \frac{n}{m} = \log h_2 + O\left(\frac{1}{m}\right)$. Using this,

$$G_0(an + \sigma)(1 - \varepsilon)c_2 \log n > G_0(am + \rho) - (1 - \varepsilon)c_2 \log m,$$

where we used $G_0(h_2) > 0$ and $n > n_0$. Hence $d_{h_2}(n)$ is bounded below. Consequently $d_{h_2}(n)$ is bounded.

Now we show that $d_{h_1}(n)$ is bounded, where $d_{h_1}(n) := G_0(an+\theta) - (1-\varepsilon)c_1 \log n$.

By similar calculations, first we obtain

$$G_0(an+\sigma) - (1-\varepsilon)c_1 \log n < G_0(am+\rho) - (1-\varepsilon)c_1 \log m,$$

where we used $G_0(h_1) < 0$ and $n > n_0$. Hence $d_{h_1}(n)$ is bounded above.

Also

$$G_0(an+\sigma) - (1-\varepsilon)c_1 \log n > G_0(am+\rho) - (1-\varepsilon)c_1 \log m,$$

where we chose ε such that $0 < \varepsilon < \varepsilon_0$, $\left(\frac{G(h_2)}{\log h_2} - (1-\varepsilon)\frac{G(h_1)}{\log h_1}\right) > 0$ and $n > n_0$. Hence $d_{h_1}(n)$ is bounded below. Consequently $d_{h_1}(n)$ is bounded. Since $d_{h_1}(n)$ and $d_{h_2}(n)$ are both bounded

$$k(h_1, h_2, \varepsilon) \le d_{h_2}(n) - d_{h_1}(n) \le K(h_1, h_2, \varepsilon).$$

On the other hand

$$d_{h_2}(n) - d_{h_1}(n) = (1-\varepsilon)\left(\frac{G(h_2)}{\log h_2} - \frac{G(h_1)}{\log h_1}\right)\log n$$

for every $n \in \mathbb{N}$, consequently

$$\frac{G(h_2)}{\log h_2} = \frac{G(h_1)}{\log h_1},$$

i.e. if $h \in \Sigma$ then $G_0(h) = c \log h$. Using the definition of G_0 we obtain $G(h) = c_0 \log h$, where c_0 is a constant. Lemma is proved. □

Lemma 5. *Assume that G is additive on the set Σ, and is non-decreasing. Then G satisfies*

$$\lim_{n\to\infty} (G(a(n+m)+r) - G(an+s)) = 0 \text{ if } n, m \in \mathbb{N}, r, s \in \Sigma, r < a, s < a$$

Proof. Observe that all $\frac{p}{q}$ are in Σ if $(p,q) = 1$. We use this property repeatedly in the following.

As G is monotone increasing, by additivity, G must satisfy

$$G(\theta) \to 0 \text{ for } \theta \to 1, \theta \in \Sigma$$

Hence

$$\begin{aligned}
& G(a(n+m)+r) - G(an+s) = \\
= \; & G(an+(am+r)) - G(an+r) + G(an+r) - G(an+s) = \\
= \; & G(an+r) + G\left(\frac{an+(am+r)}{an+r}\right) - G(an+r) + G(an+s) + G\left(\frac{an+r}{an+s}\right) - G(an+s) = \\
= \; & G\left(\frac{an+(am+r)}{an+r}\right) + G\left(\frac{an+r}{an+s}\right),
\end{aligned}$$

where both $\frac{an+(am+r)}{an+r} \to 1$ and $\frac{an+r}{an+s} \to 1$ as $n \to \infty$, and so

$$G\left(an+(am+r)an+r\right) \to 0 \text{ and } G\left(\frac{an+r}{an+s}\right) \to 0 \text{ as } n \to \infty,$$

hence

$$\lim_{n\to\infty} G(a(n+m)+r) - G(an+s) = 0$$

as required. Lemma is proved. □

Lemma 6. *Assume that $a \in \mathbb{N}$ and $G(x) : \mathbb{R}^+ \to \mathbb{R}$ is non-decreasing and additive on the set Σ. Furthermore*

$$\lim_{n\to\infty} (G(a(n+m)+r) - G(an+s)) = 0 \text{ if } n, m \in \mathbb{N}, r, s \in \Sigma, r < a, s < a$$

Then

$$G(x) = c_0 \log x \text{ for a constant } c_0 \text{ and all } x \in \mathbb{R}.$$

Proof. $\Sigma \subset \mathbb{R}$, and for any $x \in \Sigma$, $G(x) = c \log x$ for some constant c by Lemma 3. Let $z \in \mathbb{R}$, $G(z) = c' \log z$, we prove that $c' = c$, and so $c = c_0$. Assume also, that $z = am + r$, where $0 < r < 2a$. Then let p_i and q_i as before. As G is non-decreasing.

$$a \le b \Rightarrow G(a) \le G(b)\ \forall a, b \in \mathbb{R}. \tag{8}$$

The series q_i, and p_i satisfy $\pi_i < \pi_{i+1}$ and $\theta_{i+1} < \theta_i$, hence

$$P_i := an + \pi_i \to an + \rho = z \text{ as } i \to \infty \text{ and } Q_i := an + \theta_i \to an + \rho = z \text{ as } i \to \infty, \tag{9}$$

thus $P_i \le z \le Q_i$ for all i, and therefore by (8)

$$G(P_i) \le G(z) \le G(Q_i) \tag{10}$$

that is

$$c \log P_i \le c_0 \log z \le c \log Q_i, \tag{11}$$

but P_i and Q_i were constructed to satisfy $P_i \in \Sigma$, and $Q_i \in \Sigma$, i.e. by lemma 5 $G(P_i) = c \log P_i$, $G(Q_i) = c \log Q_i$. The function $\log x$ is continuous, so $\log P_i \to \log z$ and $\log Q_i \to \log z$ as $i \to \infty$. Hence $c' = c$ as required. Thus $c_0 = c$, and so for all z$\in \mathbb{R}$, $G(z) = c_0 \log z$. By this we proved Lemma 5. □

Lemma 7. *Assume that $a \in \mathbb{N}$ and $G(x) : \mathbb{R}^+ \to \mathbb{R}$ is non-decreasing and additive on the set Σ. Then*

$$G(x) = c_0 \log x \text{ for a constant } c_0 \text{ and all } x \in \mathbb{R}.$$

Proof. By means of Lemma 4.,

$$\lim_{n\to\infty} (G(a(n+m)+r) - G(an+s)) = 0 \text{ if } n, m \in \mathbb{N}, r, s \in \Sigma, r < a, s < a$$

is true, too. Then the conditions of Lemma 5. are all satisfied and hence Lemma follows. □

Lemma 8. *$G(x) : \mathbb{R}^+$ to $\mathbb{R}$, additive over the set $\Sigma = \{an + \theta : n \in \mathbb{N}, \theta < a, \theta = \frac{q_1}{q_2}$ s. t. $(q_1, q_2) = (a, q_1) = 1\}$, non-decreasing modulo 1 (by which we mean non-increasing over $(0, 1)$ and $[n, n+1)\ \forall n \in \mathbb{N}$), then G is non-decreasing.*

Proof. The proof is induction. We show that the result is true for the interval $(0, 2)$ and that provided it is true in the interval $(0, n)$ it is true for $(0, n + 1)$, too.

By assumption G is non-decreasing over the intervals $I_1 = (0, 1)$ and $I_2 = [1, 2)$. Let $i_1, j_1 \in I_1, j_2 \in I_2$ and let $i_2 \in I_2$ such that $i_2 j_2 \in I_2$. We also require $i_1, i_2, j_1, j_2 \in \Sigma$.

By additivity,

$$G(i_1 j_1) = G(i_1) + G(j_1) \tag{12}$$

$j_1 < 1$, so $i_1 j_1 < i_1$ hence, as G is non-decreasing over I_1,

$$G(i_1 j_1) = G(i_1) + G(j_1) \leq G(i_1) \tag{11'}$$

hence

$$G(j_1) \leq 0 \ \forall j_1 \in I_1 \tag{12}$$

Similarly by additivity over I_2:

$$G(i_2 j_2) = G(i_2) + G(j_2) \tag{13}$$

$i_2 j_2 \geq i_2$, so as G is non-decreasing in I_2

$$G(i_2 j_2) = G(i_2) + G(j_2) \geq G(i_2) \tag{13'}$$

Hence

$$G(j_2) \geq 0 \forall j_2 \in I_2$$

Therefore, putting (12) and (14) together, we get

$$G(j_2) \geq 0 \geq G(j_1) \quad \forall j_1 \in I_1 \text{ and } j_2 \in I_2.$$

Hence G is non-decreasing over $I_1 \cup I_2 = (0, 2)$. Let $J_2 = (0, 2)$.

The inductive step is proved by contradiction.

We want to prove that if G is non-decreasing over $J_k = (0, k)$ for $k \in \mathbb{N}$, then G is non-decreasing over $J_{k+1} = (0, k + 1)$.

Assume that this is not true, that is, $\exists \alpha \in \Sigma \cap I_k$ such that $G(\beta) > G(\alpha)$ where $\beta \in J_{k+1}$ and hence $\beta < \alpha$. As $\alpha \in \Sigma$, $\alpha \notin \mathbb{N}$ and so $k - 1$ and α are distinct.

Take $\rho = \frac{\alpha}{\beta} \in \mathbb{R}$ and find the corresponding approximators π_i, θ_i in the interval $\left(\frac{n-1}{\beta}, \frac{k}{\beta}\right)$, as before. For some $j \in \mathbb{N}$,

$$1 < \frac{k-1}{\beta} < \pi_j < \frac{\alpha}{\beta} = \rho$$

Hence

$$k - 1 < \pi_j \beta < \alpha$$

where $\pi_j \beta \in I_k \cap \Sigma$ and so by the non-decreasing property within I_k,

$$G(\pi_j \beta) < G(\alpha).$$

By additivity over Σ,

$$G(\pi_j \beta) = G(\pi_j) + G(\beta)$$

Hence, putting (18) and (19) together

$$G(\beta) < G(\pi_j) + G(\beta) = G(\pi_j \beta) < G(\alpha)$$

Contradiction, so the result is true.

The non-decreasing property has been proved for J_2, so lemma is proved. □

Theorem 1. By Lemma 7. G is non-decreasing, hence the conditions of Lemma 6. are satisfied. Hence

$$G(x) = c_0 \log x \text{ for a constant } c_0 \text{ and all } x \in \mathbb{R}.$$

and so theorem is proved. □

References

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