MULTIPLE-VALUED STATIONARY STATE AND ITS INSTABILITY OF THE TRANSMITTED LIGHT BY A RING CAVITY SYSTEM

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In the stationary situation the transmitted light by a ring cavity containing a homogeneously broadened two level absorber exhibits a multiple-valued response to a constant incident light. The stability of the stationary state is investigated in the fast limit of the atomic relaxation. The stationary state is not always stable even when it belongs to the branch with a positive differential gain. In some cases all the stationary states becomes unstable and the transmitted light exhibits a "chaotic" behavior.

The transmitted light by a Fabry-Pérot cavity containing a two level absorber exhibits a bistable behavior [1,2]. This phenomena is usually called by the name optical bistability, and it originates from the saturation of the light absorption by the two level absorber (the absorptive bistability) [1] or from the cooperation between the cavity mistuning and the nonlinear dispersion of the absorber (the dispersive bistability) [2]. From the theoretical point of view the optical bistability is interesting since it can be considered as a typical example of a first order like phase transition in a system far from thermal equilibrium.

In a Fabry-Pérot cavity there exist two electric fields which propagate in the counter directions and interact with each other via the absorber, and moreover the electric field varies depending on the spatial coordinate, so that the theoretical analysis of the Fabry-Pérot cavity system is very complicated even when we confine ourselves to the stationary situation [1,3]. To avoid these complexities almost all of the theoretical investigations have used the mean field model in which the spatial variation of the electric and the polarization fields is avergaed [4–6]. In the mean field model, therefore, the important effect of propagation is usually not taken into account. Recently, Bonifacio and Lugiato have proposed a considerably simple system in which the problem of the absorptive bistability can be treated analytically with taking full account of the propagation effect [7,8]. In this system the ring cavity is used in place of the Fabry-Pérot cavity as the feed back mechanism of the light.

The aim of the present letter is to report some novel features of the ring cavity system as follows. (1) If the detuning of the incident light with the absorber is introduced, the ordinary bistable behavior is drastically modified, and in the stationary situation the transmitted field becomes a multiple-valued function of the incident field. (2) The stationary solution is not always stable even when it belongs to the branch with a positive differential gain. In some cases all the stationary solutions become unstable, and the transmitted field exhibits a "chaotic" behavior.

To show the above features we first prove that under some appropriate conditions the dynamics of the transmitted light can be described by a set of difference-differential equations which do not involve the spatial coordinate. The device of the ring cavity system is illustrated in fig. 1. E_I is the incident field, and E_T and E_R are the transmitted field and reflected one, respectively. L is the length of the sample cell containing the two level absorber, and $\mathcal{L}$ the total length of the optical circuit in the ring cavity. We assume that the mirror 1 and 2 have reflectivity R while the mirror 3 and 4 have 100% reflectivity. Let $E(t,z)$ be the complex envelope of the electric field. Then the following boundary conditions are obtained:

$$E(t,0) = \sqrt{T}\,E_I(t) + R\exp(ik\mathcal{L})\,E(t - l/c, L), \qquad (1a)$$

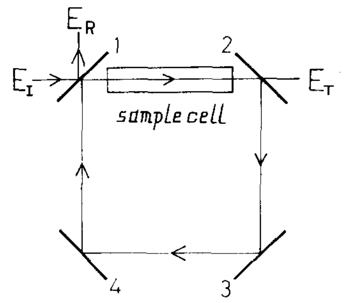

Fig. 1. The ring cavity system. See text.

$$E_T(t) = \sqrt{T}\, E(t, L) \exp(ikL), \tag{1b}$$

where $T \equiv 1 - R$ is the transmittivity coefficient of the mirror 1 and 2, $k \equiv \omega/c$ is the wave number of the electric field in the vacuum, and $l \equiv \mathcal{L} - L$. For the sake of simplicity we assume that the two level system in the sample cell is homogeneously broadened. Then the propagation of the electric field in the absorber is described by the Maxwell-Bloch equations given as follows:

$$\partial E/\partial z = 4\pi i N \mu k \rho, \tag{2a}$$

$$\partial \rho/\partial \tau = (i\Delta\omega - \gamma_\perp)\rho - i\mu w E, \tag{2b}$$

$$\partial w/\partial \tau = -\gamma_\parallel (w + 1/z) + i\mu(\rho^* E - \rho E^*)/z, \tag{2c}$$

where $\tau \equiv t - z/c$ is the retarded time, ρ and w the dimensionless polarization and half the population difference of the two level atom, respectively, μ the transition dipole moment, and $\Delta\omega \equiv \omega - \Omega$ (Ω: the transition frequency of the two level atom) the detuning frequency. $\gamma_\perp$ and $\gamma_\parallel$ are the transversal and longitudinal relaxation rates, respectively, and N the density of the atoms.

Hereafter we confine our consideration to the fast limit of the transversal realxation. Under this condition the polarization follows the electric field adiabatically:

$$\rho = i\mu w E/(i\Delta\omega - \gamma_\perp). \tag{3}$$

Substituting eq. (3) into eq. (2a), we can write the electric field $E(t, z)$ in the following integral form

$$E(\tau + z/c, z) = E(\tau, 0)$$
$$\times \exp[2\theta W(\tau, z)(i\Delta\omega + \gamma_\perp)/(\Delta\omega^2 + \gamma_\perp^2)], \tag{4}$$

where $\theta \equiv 2\pi N k \mu^2$ and the function $W(\tau, z)$ is defined by

$$W(\tau, z) = \int_0^z dz'\, w(\tau + z'/c, z'). \tag{5}$$

By substituting eqs. (4), (5) into eq. (2c) and integrating it over z, eq. (2c) leads to

$$\partial W(\tau, z)/\partial \tau = -\gamma_\parallel (W + z/2) - \mu^2 |E(\tau, 0)|^2$$
$$\times \{\exp[4\theta\gamma_\perp W/(\Delta\omega^2 + \gamma_\perp^2)] - 1\}/4\theta. \tag{6}$$

Now we introduce the following dimensionless quantities

$$\epsilon(t, z) \equiv \mu E(t, z)/z\sqrt{\gamma_\perp\gamma_\parallel(1 + \Delta^2)},$$
$$x \equiv t\gamma_\parallel, \quad \phi(t) \equiv W(t - \mathcal{L}/c, L)/L, \tag{7}$$

where $\Delta \equiv \Delta\omega/\gamma_\perp$. Combining eqs. (4) and (6) with the boundary conditions (1) and using the dimensionless quantities defined by eq. (7), we finally obtain the following set of equations which do not involve the spatial coordinate:

$$\epsilon(x, 0) = \sqrt{T}\epsilon_I(x) + R\epsilon(x - \kappa, 0)\exp(\alpha L \phi(x))$$
$$\times \exp\{i(\alpha L \Delta(\phi(x) + \tfrac{1}{2}) - \delta_0\}, \tag{8a}$$

$$d\phi(x)/dx = -(\phi(x) + 1/2) - 2|\epsilon(x - \kappa, 0)|^2$$
$$\times [\exp(2\alpha L \phi(x)) - 1]/\alpha L, \tag{8b}$$

and

$$\epsilon_T(x) = \sqrt{T}\epsilon(x - \kappa, 0)\exp(\alpha L \phi(x))$$
$$\times \exp\{i(\alpha L \Delta(\phi(x) + \tfrac{1}{2}) - (\delta_0 + kl))\}, \tag{9}$$

where $\epsilon_T(x)$ and $\epsilon_I(x)$ are defined by $\mu E_T(t - l/c)/2\{\gamma_\perp\gamma_\parallel(1 + \Delta^2)\}^{1/2}$ and $\mu E_I(t)/2\{\gamma_\perp\gamma_\parallel(1 + \Delta^2)\}^{1/2}$, respectively, $\alpha \equiv 2\theta\gamma_\perp/(\Delta\omega^2 + \gamma_\perp^2)$ is the effective absorption coefficient, and $\delta_0 = -k(\sqrt{\tilde{\epsilon}_\ell}\, L + l) + 2\pi M$ ($\tilde{\epsilon}$: the linear dielectric constant $1 - 4\pi N \mu^2 \Delta\omega/(\Delta\omega^2 + \gamma_\perp^2)$, $2\pi M$: the multiple of 2π nearest to $k(\sqrt{\tilde{\epsilon}_\ell} L + l)$) is the mistuning parameter of the ring cavity containing the linear absorber. The parameter $\kappa \equiv \gamma_\parallel \mathcal{L}/c$ denotes the dimensionless round trip time. We notice that eqs. (8a), (8b) and (9) can be interpreted as difference-differential equations whose solutions are uniquely determined if both the initial value $\phi(0)$ and the boundary condition $\epsilon(x,0)$ in the time range $-k \leqslant x < 0$ are given.

In the stationary situation we can set $d\phi(x)/dx = 0$ and $\epsilon(x,0)$ = constant. By eliminating $\epsilon(x,0)$ from eqs. (8a), (8b) and (9), the stationary solution of the transmitted field intensity is related with the incident field intensity by the following equation:

$$|\epsilon_I|^2 = |\hat{\epsilon}_T|^2 \{[\exp(-\alpha L\hat{\phi}) - R]^2 + 4R\exp(-\alpha L\hat{\phi}) \times \sin^2[\delta(|\hat{\epsilon}_T|^2)/2]\, T^{-2}, \quad (10)$$

with

$$\delta(|\hat{\epsilon}_T|^2) \equiv \delta_0 - \alpha L\Delta(\hat{\phi} + 1/2), \quad (11)$$

where ˆ denotes the stationary solution, and $\hat{\phi}$ is related with $|\hat{\epsilon}_T|^2$ by

$$(\hat{\phi} + 1/2)/[\exp(-2\alpha L\hat{\phi}) - 1] = 2|\hat{\epsilon}_T|^2/T\alpha L. \quad (12)$$

$\hat{\phi}$ is a monotone increasing function of $|\hat{\epsilon}_T|^2$ which varies from $-1/2$ to zero. $\delta(|\hat{\epsilon}_T|^2)$ denotes the intensity dependent mistuning parameter, which is originated from the nonlinear shift of the wave number in the absorber. Since $\delta(|\hat{\epsilon}_T|^2)$ varies from δ_0 to $\delta_0 - \alpha L\Delta/2$, the factor $\Phi \equiv \sin^2\delta(|\hat{\epsilon}_T|^2)/2$ oscillates as a function of $|\hat{\epsilon}_T|^2$ provided that the parameter $\alpha L\Delta$ is large enough.

In the limit of $\alpha L\Delta \to 0$ and $\delta_0 \to 0$ the relations (10) and (12) reduces to nothing but the absorptive bistability which is obtained by Bonifacio and Lugiato [7]. On the other hand, if the conditions $\alpha L\Delta \ll 1$, $\alpha L \ll 1$, $|\delta_0| \ll 1$ and $|\hat{\epsilon}_T|^2/T \ll 1$ are satisfied, eq. (10) reduces to $|\epsilon_I|^2 = |\hat{\epsilon}_T|^2[1 + 4RT^{-2}(\alpha L\Delta T^{-1}|\hat{\epsilon}_T|^2 - \delta_0/2)^2]$ by the approximation $\delta(|\hat{\epsilon}_T|^2) \approx \delta_0 - 2\alpha L\Delta|\hat{\epsilon}_T|^2 T^{-1}$, which agrees with the relation exhibiting the dispersive bistability experimentally observed by Gibbs et al. [2]. However, if the magnitude of the parameter $\alpha L\Delta$ is sufficiently large, the transmitted field intensity oscillates as a function of the incident field intensity owing to the factor $\Phi = \sin^2[\delta(|\hat{\epsilon}_T|^2)/2]$, and the ordinary bistable behavior will be drastically modified. In fig. 2 the relations between the transmitted field and the incident field are shown for various values of the parameter $\alpha L\Delta$. The parameter αL and the reflectivity R are kept fixed to 4.0 and 0.95, respectively. As $\alpha L\Delta$ increases, the ordinary bistable relation (a) obtained for $\alpha L = 0$ is drastically modified as is indicated typically by (c) and (d), that is, new branches appear in the lower intensity side of $|\hat{\epsilon}_T|$ and their number increases with $\alpha L\Delta$. Provided that the magnitude of the parameter αL is sufficiently small, a pair of new branches with negative and positive differential gains is generated whenever $\alpha L\Delta$ is increased by 2π. Such a multiple-valued behavior is due to the intensity dependent mistuning of the cavity with the incident light. The possibility of the multiple-valued response of the transmitted light has been discussed also by Felber and Marburger [9] for a Fabry-Pérot cavity system containing a Kerr medium.

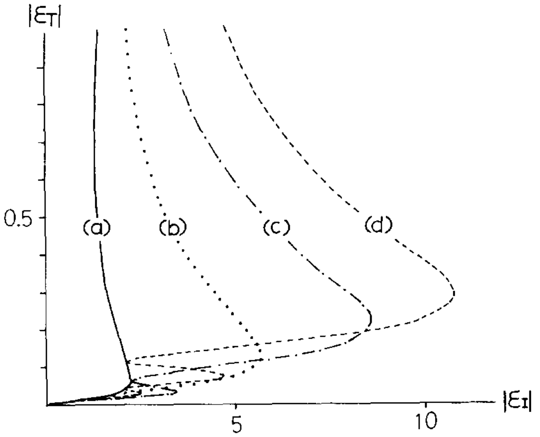

Fig. 2. Relations between the transmitted field and the incident field for various values of the parameter $\alpha L\Delta$; (a) $\alpha L = 0.0$, (b) $\alpha L\Delta = 2\pi$, (c) $\alpha L\Delta = 4\pi$ and (d) $\alpha L\Delta = 6\pi$. The parameters αL, δ_0 and R are kept fixed to 4.0, 0.0 and 0.95 respectively.

Now we discuss the stability of the multiple-valued stationary state. In this letter we confine our consideration to the limiting case $\kappa \ll 1$ i.e. the fast limit of the longitudinal relaxation. In this limit we may set $d\phi(x)/dx$ in eq. (8b) equal to zero, and eqs. (8a), (8b) and (9) reduce to the following difference equations:

$$\epsilon_{0n} = \sqrt{T}\,\epsilon_{In} + R\epsilon_{0n-1}\exp(\alpha L\phi_n) \times \exp\{i(\alpha L\Delta(\phi_n + 1/2) - \delta_0)\}, \quad (13a)$$

$$\epsilon_{Tn} = \sqrt{T}\,\epsilon_{0n-1}\exp(\alpha L\phi_n) \times \exp\{i(\alpha L\Delta(\phi_n + 1/2) - (\delta_0 + kl))\}, \quad (13b)$$

where ϵ_{0n}, ϵ_{Tn} and ϵ_{In} denotes $\epsilon(x_0 + n\kappa, 0)$, $\epsilon_T(x_0 + n\kappa)$ and $\epsilon_I(x_0 + n\kappa)$, respectively, and ϕ_n is related with ϵ_{0n-1} by

$$(\phi_n + 1/2)/[1 - \exp(\alpha L\phi_n)] = 2|\epsilon_{0n-1}|^2/\alpha L. \quad (14)$$

The stability problem is investigated by linearizing eq.

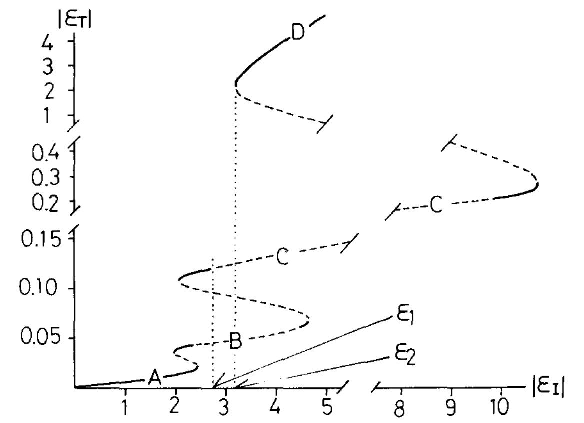

Fig. 3. Stability of the multiple-valued stationary state in case of fig. 2(d). ——— stable, ------ unstable. The branches A and D are both stable. Note that all the stationary solutions are unstable in the region $\epsilon_1 < |\epsilon_I| < \epsilon_2$.

(13a) around the stationary solution ‡. The linear motion of ϵ_{0n} around its stationary solution is characterized by two eigenvalues of a 2 X 2 evolution matrix, and the stationary solution is stable only when each of the two eigenvalues has an absolute value less than the unity.

The stability of the multiple-valued stationary state has been studied numerically. As is naturally expected, the branches with the negative differential gain $\mathrm{d}|\epsilon_T|/\mathrm{d}|\epsilon_I| < 0$ are always unstable. An interesting fact is that even the stationary solutions in the branch with a positive differential gain are not always stable. The stationary solution becomes stable when the incident field intensity $|\epsilon_I|$ is set in the vicinity of the supremum and infimum of the branch (see fig. 3).

An interesting example is the case of (d) in fig. 2. The stability of the stationary solutions in this case is displayed in fig. 3. A remarkable feature is that in the region $\epsilon_1 < |\epsilon_I| < \epsilon_2$ all stationary solutions are unstable. What happens in this region? In fig. 4(a) a typical behavior of the transmitted field in this region is displayed. This result has been obtained by iterating eqs. (13a, b) from an appropriate initial value ϵ_{00}. As the time steps are advanced in the unit κ the transmitted field varies in an apparently erattic manner. To make the situation

‡ In the special case $\alpha L = \delta_0 = 0$ we can choose ϵ_{0n} as a real quantity, so that eq. (13a) can be solved by a graphical method. By this method it is easily proven that in case of the pure absorptive bistability the branch with a positive differential gain is always stable.

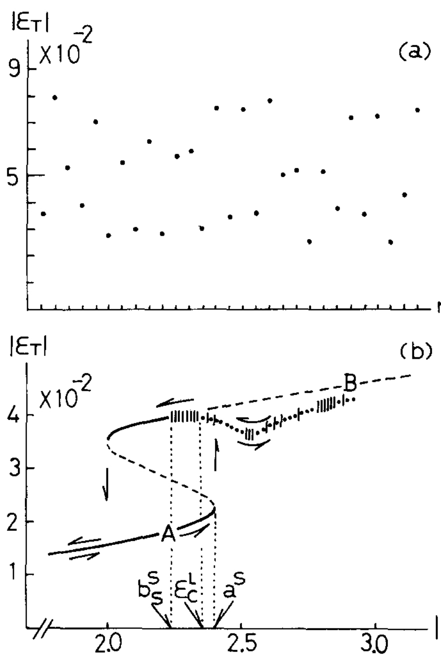

Fig. 4. "Chaotic" behavior of the transmitted field in case of fig. 2(d). (a) A typical example of the "chaotic" behavior obtained for $|\epsilon_I| = 0.3074$. (b) The hysteresis loop involving the chaotic behavior. ——— stationary, ● ● ● chaotic, | | | | | | periodic. In the chaotic or periodic region the magnitude of $|\epsilon_T|$ is expediently chosen as the mean square average of ϵ_{Tn} over a sufficiently long iteration steps.

of such a "chaotic" behavior clear we have plotted the complex sequence ϵ_{Tn} obtained by a successive iteration of eq. (13a,b) on a complex plane. It has been found that as the iterated step is advanced the plotted point tends to be attracted into a figure which appears to consist of an infinite set of one dimensional curves. Almost identical figures have been obtained when the initial value ϵ_{00} is changed over a considerably wide range. This fact strongly suggests that the figure represents the "strange attractor" of the difference equation (13a,b) [10]. Except for initial few steps the complex point ϵ_{Tn} is trapped into the strange attractor and the trajectory of ϵ_{Tn} wanders over it. Thus the "chaotic" behavior of the transmitted field is due to the wandering motion inside the strange attractor ‡‡.

‡‡ Strictly speaking the motion of the transmitted field is not always "chaotic" in the region $\epsilon_1 < |\epsilon_I| < \epsilon_2$. There exist some narrow regions in which the transmitted field exhibits periodic behaviors.

The strange attractor exists when $|\epsilon_I|$ is set outside the region $\epsilon_1 < |\epsilon_I| < \epsilon_2$, and it continues to the stable stationary solution (the stable fixed point) of the branch B as $|\epsilon_I|$ is decreased below ϵ_1. When $|\epsilon_I|$ is set below supremum a^S of the branch A, the strange attractor coexists with the stable fixed point (the stable stationary solution) corresponding to the branch A. From these facts, as is illustrated in fig. 4(b), we can expect a hysteresis loop involving the "chaotic" behavior as follows: Let us assume that $|\epsilon_I|$ is varied very slowly between zero and some value slightly larger than a^S. In the increasing process of $|\epsilon_I|$ the transmitted field is stationary in the first place and it increases along the branch A. However, as soon as $|\epsilon_I|$ exceeds a^S, the transmitted field suddenly becomes "chaotic". On the other hand, in the decreasing process the "chaotic" behavior lasts until $|\epsilon_I|$ reaches to some value ϵ_C^L which lies between a^S and the supremum b_S^S of the stable region of the branch B. As $|\epsilon_I|$ is decreased further the transmitted field flips to the branch A via the stable portion of the branch B. The transmitted field exhibits periodic motions with the period 2^m (m: a positive integer) when $|\epsilon_I|$ is set in the region bounded by b_S^S and ϵ_C^L. The value of m increases through successive bifurcations as $|\epsilon_I|$ increases from b_S^S, and it appears to become infinity as $|\epsilon_I|$ approaches ϵ_C^L [11]. As $|\epsilon_I|$ exceeds ϵ_C^L, the "chaotic" motion finally take the place of the periodic motion. We note that, as is shown in fig. 4(b), the periodic motion appears also in narrow regions sandwiched between the "chaotic" regions (the window structure [11]). In these regions the period of the motion appears to be given by $l \times 2^m$ (l: a positive integer).

In conclusion we have shown that the multiple-valued stationary state of the transmitted light by a ring cavity system is not always stable, exhibiting a new kind of instability. In some cases the transmitted light exhibits a "chaotic" behavior. In the present report we have demonstrated such a "chaotic" behavior for specific values of the parameters αL, $\alpha L\Delta$, δ_0 and R etc. The "chaotic" behavior, however, appears over a wide range of these parameters. A more detailed report will be presented in forthcoming publications.

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