A short constructive proof of the Erdős–Gallai characterization of graphic lists

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Abstract

Erdős and Gallai proved that a nonincreasing list $(d_1, \ldots, d_n)$ of nonnegative integers is the list of degrees of a graph (with no loops or multi-edges) if and only if the sum is even and the list satisfies $\sum_{i=1}^{k} d_i \le k(k-1) + \sum_{i=k+1}^{n} \min\{k, d_i\}$ for $1 \le k \le n$. We give a short constructive proof of the characterization.

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A list of nonnegative integers is *graphic* if it is the list of vertex degrees of a graph, where our model of *graph* forbids loops and repeated edges. Historically, such lists have also been called *graphic sequences*. A graph with degree list d is a *realization* of d.

Many characterizations of graphic lists are known; Sierksma and Hoogeveen [10] list seven criteria involving inequalities on the list elements. With additional results, these also appear in [9]. Additional characterizations are due to Havel [7] and Hakimi [5], Koren [8], and probably others.

The best-known explicit characterization is that by Erdős and Gallai [4]. Many proofs of it have been given, including by Berge [2] (using network flow or Tutte's f-Factor Theorem), Harary [6] (a lengthy induction), Choudum [3], Aigner–Triesch [1] (using ideals in the dominance order), Tripathi–Tyagi [11] (indirect proof), etc. The purpose of this note is to give a short direct proof that constructs a graph whose degree list is the given list.

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Theorem 1 (Erdős–Gallai [4]) *A list $(d_1,\ldots,d_n)$ of nonnegative integers in nonincreasing order is graphic if and only if its sum is even and, for each integer k with $1\le k\le n$,*

$$\sum_{i=1}^{k} d_i \le k(k-1)+\sum_{i=k+1}^{n}\min\{k,d_i\},\quad \text{for } 1\le k\le n. \tag{1}$$

Proof. Necessity is immediate and standard: each edge is counted twice to yield even sum, and the right side is the maximum contribution to the sum of the first k degrees from edges induced by the corresponding vertices and edges to the remaining vertices.

For sufficiency, let a *subrealization* of a nonincreasing list $(d_1,\ldots,d_n)$ be a graph with vertices $v_1,\ldots,v_n$ such that $d(v_i)\le d_i$ for $1\le i\le n$, where $d(v_i)$ denotes the degree of v_i. Given a list $(d_1,\ldots,d_n)$ with even sum that satisfies (1), we construct a realization through successive subrealizations. The initial subrealization has n vertices and no edges.

In a subrealization, the *critical index* r is the largest index such that $d(v_i)=d_i$ for $1\le i<r$. Initially, $r=1$ unless the list is all 0, in which case the process is complete. While $r\le n$, we obtain a new subrealization with smaller deficiency $d_r-d(v_r)$ at vertex v_r while not changing the degree of any vertex v_i with $i<r$ (the degree list increases lexicographically). The process can only stop when the subrealization is a realization of d.

Let $S=\{v_{r+1},\ldots,v_n\}$. We maintain the condition that S is an independent set, which certainly holds initially. Write $v_i\leftrightarrow v_j$ when $v_iv_j\in E(G)$; otherwise, $v_i\not\leftrightarrow v_j$.

Case 0) *$v_r\not\leftrightarrow v_i$ for some vertex v_i such that $d(v_i)<d_i$.* Add the edge v_rv_i.

Case 1) *$v_r\not\leftrightarrow v_i$ for some i with $i<r$.* Since $d(v_i)=d_i\ge d_r>d(v_r)$, there exists $u\in N(v_i)-N(v_r)$, where $N(z)=\{y\colon z\leftrightarrow y\}$. If $d_r-d(v_r)\ge 2$, then replace uv_i with $\{uv_r,v_iv_r\}$. If $d_r-d(v_r)=1$, then since $\sum d_i-\sum d(v_i)$ is even there is an index k with $k>r$ such that $d(v_k)<d_k$. Case 0 applies unless $v_r\leftrightarrow v_k$; replace $\{v_rv_k,uv_i\}$ with $\{uv_r,v_iv_r\}$.

Case 2) *$v_1,\ldots,v_{r-1}\in N(v_r)$, and $d(v_k)\ne\min\{r,d_k\}$ for some k with $k>r$.* In a subrealization, $d(v_k)\le d_k$. Since S is independent, $d(v_k)\le r$. Hence $d(v_k)<\min\{r,d_k\}$, and Case 0 applies unless $v_k\leftrightarrow v_r$. Since $d(v_k)<r$, there exists i with $i<r$ such that $v_k\not\leftrightarrow v_i$. Since $d(v_i)>d(v_r)$, there exists $u\in N(v_i)-N(v_r)$. Replace uv_i with $\{uv_r,v_iv_k\}$.

Case 3) *$v_1,\ldots,v_{r-1}\in N(v_r)$, and $v_i\not\leftrightarrow v_j$ for some i and j with $i<j<r$.* Case 1 applies unless $v_i,v_j\in N(v_r)$. Since $d(v_i)\ge d(v_j)>d(v_r)$, there exist $u\in N(v_i)-N(v_r)$ and $w\in N(v_j)-N(v_r)$ (possibly $u=w$). Since $u,w\notin N(v_r)$, Case 1 applies unless $u,w\in S$. Replace $\{uv_i,wv_j\}$ with $\{v_iv_j,uv_r\}$.

If none of these Cases apply, then $v_1,\ldots,v_r$ are pairwise adjacent, and $d(v_k)=\min\{r,d_k\}$ for $k>r$. Since S is independent, $\sum_{i=1}^{r} d(v_i)=r(r-1)+\sum_{k=r+1}^{n}\min\{r,d_k\}$. By (1), $\sum_{i=1}^{r} d_i$ is bounded by the right side. Hence we have already eliminated the deficiency at vertex r. Augment r and continue. $\square$

The proof can be implemented as an algorithm to construct a realization of the degree list. Since the subrealization improves lexicographically with each step, the number of steps is at most $\sum d_i$. To bound the time for each step, we maintain the graph as lists of neighbors and non-neighbors for each vertex. We look through the non-neighbors of v_r to see if Case 0 or Case 1 applies. To apply Case 1 we access lists twice to find u and possibly check degrees of the high-indexed vertices to find k. The implementation of Cases 2 and 3 involve similar operations. Each step is implemented using a constant number of set-membership queries. Thus the running time is naively at most $O(n \sum d_i)$. With clever maintenance of sets using sophisticated data structures involving trees, the factor of n can be reduced.

References

[1] M. Aigner and E. Triesch, Realizability and uniqueness in graphs, *Discrete Math. 136* (1994), 3–20.

[2] C. Berge, *Graphs and Hypergraphs.* (North-Holland, 1973).

[3] S. A. Choudum, A simple proof of the Erdős–Gallai theorem on graph sequences, *Bull. Austral. Math. Soc.* 33 (1986), 67–70.

[4] P. Erdős and T. Gallai, Graphs with prescribed degrees of vertices (Hungarian), *Mat. Lapok* 11 (1960), 264–274.

[5] S. L. Hakimi, On the realizability of a set of integers as degrees of the vertices of a graph, *SIAM J. Applied Math.* 10 (1962), 496–506.

[6] F. Harary, *Graph theory*, (Addison-Wesley, 1969).

[7] V. Havel, A remark on the existence of finite graphs (Czech.) *Časopis Pěst. Mat* 80 (1955), 477–480.

[8] M. Koren, Extreme degree sequences of simple graphs, *J. Combin. Theory B* 15 (1973), 213–224.

[9] N. V. R. Mahadev, and U. N. Peled, *Threshold graphs and related topics*, Annals of Discrete Mathematics 56, (North-Holland, 1995).

[10] G. Sierksma, and H. Hoogeveen, Seven criteria for integer sequences being graphic, *J. Graph Theory* 15 (1991), 223–231.

[11] A. Tripathi and H. Tyagi, A simple criterion on degree sequences of graphs, *Discrete Appl. Math.* 156 (2008), 3513–3517.