

# Towards a Logical Reconstruction of CF-Induction

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**Abstract.** CF-induction is a sound and complete hypothesis finding procedure for full clausal logic which uses the principle of inverse entailment to compute a hypothesis that logically explains a set of examples with respect to a prior background theory. Currently, CF-induction computes hypotheses by applying combinations of several complex generalisation operators to an intermediate theory called a bridge formula. In this paper we propose an alternative approach whereby hypotheses are derived from a bridge formula using a single deductive operator and a single inductive operator. We show that our simplified procedure preserves the soundness and completeness of CF-induction.

**Keywords:** inverse entailment, CF-induction, generalisation operator.

## 1 Introduction

Given a background theory  $B$  and observations  $E$ , the task of explanatory induction is to find a hypothesis  $H$  such that  $B \wedge H \models E$  and  $B \wedge H$  is consistent [5]. By the principle of *inverse entailment* (IE) [9] this is logically equivalent to find a consistent hypothesis  $H$  such that  $B \wedge \neg E \models \neg H$ . This equivalence means that the inductive hypothesis  $H$  can be computed by deducing its negation  $\neg H$  from  $B$  and  $\neg E$ . This allows highly developed *deductive* reasoning techniques to be used in the implementation of *inductive* learning systems.

IE is the basis of many successful Horn clause systems such as Progol [9], which have been used in real application domains. Recently, IE methods have been developed for full clausal theories to enable the solution of more complex problems in richer knowledge representation formalisms. One such method is CF-induction [5], which has two important benefits: unlike some related systems, such as FC-HAIL [15], CF-induction is complete for finding full clausal hypotheses; and unlike other related systems, such as the Residue Procedure

[17], CF-induction can exploit language bias to focus the procedure on some relevant part of the search space specified by the user.

CF-induction computes a hypothesis in two steps: by first constructing an intermediate bridge formula, denoted  $CC$ , such that  $B \wedge \neg E \models CC$  and then generalising the negation of this bridge formula to obtain a hypothesis  $H$  such that  $\neg CC \models H$ , where  $\models$  denotes the inverse of the classical logical entailment relation  $\models$ . Hereafter, we call  $\models$  a *generalisation relation* and we refer to its associated operators as *generalisation operators* or simply *generalisers*.

The bridge formula  $CC$  is a finite set of instances of so-called *characteristic clauses*: which are the consequences of  $B \wedge \neg E$  that satisfy a given language bias and are minimal with respect to subsumption. This bridge formula is generalised using a combination of several well-known generalisers, such as *least generalisation*, *inverse resolution* and *anti-subsumption*, to obtain  $H$  from  $\neg CC$ . In order to derive a particular hypothesis  $H$  it may be necessary to apply several different generalisers in some specific order, i.e.,

$$\neg CC \models U_1 \models U_2 \models \dots \models U_{n-1} \models U_n \models H. \quad (1)$$

The fact that these generalisers can be sequenced in many different ways, results in a large search space. Our long-term research objective is to reduce the non-determinism of CF-induction in order to make the procedure easier to apply in real-world applications.

In this paper, we concentrate on simplifying the process of obtaining hypotheses from a given bridge formula. In particular, we propose an alternative approach for CF-induction based on the observation that the negated hypothesis  $\neg H$  can be computed deductively from  $CC$ , i.e.,

$$CC \models V_1 \models V_2 \models \dots \models V_{m-1} \models V_m \models \neg H. \quad (2)$$

In the interests of computational efficiency, it is convenient to work in clausal form logic. In this case,  $\neg H$  will be represented by a clausal theory  $G$  in which any existentially quantified variables in  $\neg H$  are substituted by ground (Skolem) terms. Therefore, our new approach works in three steps: first  $G$  is deduced from  $CC$ ; then it is negated; and finally anti-instantiation is used (to replace any Skolem terms by variables) to result in the final hypothesis  $H$ .

To derive  $G$  from  $CC$ , we introduce a new deductive operator, which can be regarded as simplifying the existing operators of *subsumption*, *resolution* and *weakening*. This new operator, called the  $\gamma$ -operator, warrants the insertion of literals into the clauses of  $CC$ . We show that this single deductive operator (followed by negation and anti-instantiation) is sufficient to preserve the soundness and completeness of CF-induction. This new approach, called *CF-induction with  $\gamma$ -operator*, simplifies the original CF-induction in the sense that it only requires one deductive operator ( $\gamma$ -operator) and one generalisation operator (anti-instantiation) to be used in the construction of  $H$  from  $CC$ .

The rest of this paper is organized as follows. Section 2 introduces the theoretical background in this paper and reviews the previous procedure of CF-induction. Section 3 introduces three standard deductive operators of subsumption, resolution and weakening, and provides several results concerning the order of applying

these operators. Section 4 describes the  $\gamma$ -operator and provides the basic idea for simplifying the CF-induction procedure. Section 5 discusses some related work, and Section 6 concludes.

## 2 Background

### 2.1 Notation and Terminology

Here, we review the notation and terminology used in the paper. A *clause* is a finite disjunction of literals which is often identified with the set of its literals. A clause  $\{A_1, \dots, A_n, \neg B_1, \dots, \neg B_m\}$ , where each  $A_i, B_j$  is an atom, is also written as  $B_1 \wedge \dots \wedge B_m \supset A_1 \vee \dots \vee A_n$ . A *definite clause* is a clause which contains only one positive literal. A *positive (negative) clause* is a clause whose disjuncts are all positive (negative) literals. A *Horn clause* is a definite clause or negative clause. A *unit clause* is a clause with exactly one literal. A *clausal theory* is a finite set of clauses. A clausal theory is *full* if it contains at least one non-Horn clause. A clausal theory  $\Sigma$  is often identified with the conjunction of its clauses and is said to be in *Conjunctive Normal Form* (CNF). The *complement* of  $\Sigma$ , denoted  $\overline{\Sigma}$ , is defined as a clausal theory obtained by translating  $\neg\Sigma$  into CNF using a standard translation procedure [12]. (In brief,  $\overline{\Sigma}$  is obtained by converting  $\neg\Sigma$  into prenex conjunctive normal form with standard equivalence-preserving operations and skolemising it.) Two clausal theories  $\Sigma_1$  and  $\Sigma_2$  are said to be equivalent, denoted  $\Sigma_1 \equiv \Sigma_2$  if  $\Sigma_1 \models \Sigma_2$  and  $\Sigma_2 \models \Sigma_1$ . Let  $C$  and  $D$  be two clauses.  $C$  *subsumes*  $D$ , denoted  $C \succeq D$ , if there is a substitution  $\theta$  such that  $C\theta \subseteq D$ .  $C$  *properly subsumes*  $D$  if  $C \succeq D$  but  $D \not\succeq C$ . For a clausal theory  $\Sigma$ ,  $\mu\Sigma$  denotes the set of clauses in  $\Sigma$  not properly subsumed by any clause in  $\Sigma$ . Let  $\Sigma$  be a clausal theory and  $C$  be a clause. A *derivation* of  $C$  from  $\Sigma$  is a finite sequence of clauses  $R_1, \dots, R_k = C$  such that each  $R_i$  is either in  $\Sigma$ , or is a resolvent of two clauses in  $\{R_1, \dots, R_{i-1}\}$ . We recall the following result, called the *Subsumption Theorem*.

**Theorem 1.** [7,12] *Let  $\Sigma$  be a clausal theory and  $C$  be a clause. Then  $\Sigma \models C$  iff  $C$  is a tautology or there exists a derivation of a clause  $D$  that subsumes  $C$ .*

For a clausal theory  $\Sigma$ , a *consequence* of  $\Sigma$  is a clause entailed by  $\Sigma$ . We denote by  $Th(\Sigma)$  the set of all consequences of  $\Sigma$ . The Subsumption Theorem states the completeness of the resolution principle for finding any non-tautological consequences in  $\mu Th(\Sigma)$ .

We give the definition of *hypothesis*  $H$  in the logical setting of learning from positive examples.

**Definition 1 (Hypothesis).** *Let  $B$  and  $E$  be clausal theories, representing a background theory and (positive) examples/observations, respectively. Then  $H$  is a hypothesis wrt  $B$  and  $E$  iff  $H$  is a clausal theory such that  $B \wedge H \models E$  and  $B \wedge H$  is consistent.*

If no confusion arises, then we refer simply to a “hypothesis” instead of a “hypothesis wrt  $B$  and  $E$ ”. Note that, for a treatment of negative examples in CF-induction, we refer the reader to [5].

## 2.2 CF-Induction

CF-induction is a sound and complete method for IE. It is based on the notion of *characteristic clauses* which represent “interesting” consequences of a given problem for users [4]. Each characteristic clause is constructed over a sub-vocabulary of the representation language called a *production field*. A production field  $\mathcal{P}$  is defined as a pair,  $\langle \mathbf{L}, Cond \rangle$ , where  $\mathbf{L}$  is a set of literals closed under instantiation, and  $Cond$  is a certain condition to be satisfied, e.g., the maximum length of clauses, the maximum depth of terms, etc. When  $Cond$  is not specified,  $\mathcal{P}$  is simply denoted as  $\langle \mathbf{L} \rangle$ . A clause  $C$  belongs to  $\mathcal{P} = \langle \mathbf{L}, Cond \rangle$  if every literal in  $C$  belongs to  $\mathbf{L}$  and  $C$  satisfies  $Cond$ . For a set  $\Sigma$  of clauses, the set of consequences of  $\Sigma$  belonging to  $\mathcal{P}$  is denoted  $Th_{\mathcal{P}}(\Sigma)$ . Then, the characteristic clauses of  $\Sigma$  wrt  $\mathcal{P}$  are defined as:

$$Carc(\Sigma, \mathcal{P}) = \mu Th_{\mathcal{P}}(\Sigma)$$

Note that  $Carc(\Sigma, \mathcal{P})$  can, in general, include tautological clauses [4].

When a new clause  $F$  is added to a clausal theory, some consequences are newly derived with this additional information. The set of such clauses that belong to the production field are called *new characteristic clauses*. Formally, the *new characteristic clauses* of  $F$  wrt  $\Sigma$  and  $\mathcal{P}$  are defined as:

$$NewCarc(\Sigma, F, \mathcal{P}) = Carc(\Sigma \cup \{F\}, \mathcal{P}) - Carc(\Sigma, \mathcal{P})$$

In the following, we assume the production field  $\mathcal{P} = \langle \mathbf{L} \rangle$  where  $\mathbf{L}$  is a set of literals reflecting an *inductive bias* whose literals are the negations of those literals we wish to allow in hypothesis clauses. When no inductive bias is considered,  $\mathcal{P}$  is just set to  $\langle \mathcal{L} \rangle$ , where  $\mathcal{L}$  is the set of all literals in the first-order language. We say  $H$  is a *hypothesis wrt  $B, E$  and  $\mathcal{P}$*  iff  $H$  is a hypothesis wrt  $B$  and  $E$ , and, for every literal  $L$  appearing in  $H$ , its complement  $\bar{L}$  is in  $\mathbf{L}$ . Then it holds that for any hypothesis  $H$  wrt  $B, E$  and  $\mathcal{P}$ ,

$$B \wedge \bar{E} \models Carc(B \wedge \bar{E}, \mathcal{P}) \models \neg H; \tag{3}$$

$$B \models Carc(B, \mathcal{P}) \not\models \neg H. \tag{4}$$

The two formulae above follow from the principle of IE and the definition of characteristic clauses. In particular, Formula (3) implies we can use characteristic clauses to construct intermediate bridge formulae for IE. Formula (4) ensures the consistency of the hypothesis and background theory. As explained in [5], this can always be ensured by including at least one clause from  $NewCarc(B, \bar{E}, \mathcal{P})$  in an intermediate bridge formula. The definition of a bridge formula in CF-induction is as follows:

**Definition 2 (Bridge formula [5]).** For given  $B, E$  and  $\mathcal{P}$ , a clausal theory  $CC$  is a bridge formula wrt  $B, E$  and  $\mathcal{P}$  iff  $CC$  satisfies the following conditions:

1. Each clause  $C_i \in CC$  is an instance of a clause in  $Carc(B \wedge \bar{E}, \mathcal{P})$ ;
2. At least one  $C_i \in CC$  is an instance of a clause from  $NewCarc(B, \bar{E}, \mathcal{P})$ .

If no confusion arises, a “bridge formula wrt  $B$ ,  $E$  and  $\mathcal{P}$ ” will simply be called a “bridge formula”.

**Theorem 2.** [5] *Let  $B$ ,  $E$  be clausal theories and  $\mathcal{P}$  be a production field. Then, for any hypothesis  $H$  wrt  $B$ ,  $E$  and  $\mathcal{P}$ , there exists a bridge formula  $CC$  wrt  $B$ ,  $E$  and  $\mathcal{P}$  such that  $H \models \neg CC$ .*

Theorem 2 shows that any hypothesis can be computed by constructing and generalising the negation  $\neg CC$  of a set of characteristic clauses  $CC$ . In the original CF-induction, a bridge formula  $CC$  is first selected. Then a clausal theory  $F$  is obtained by Skolemising  $\neg CC$  and translating it to CNF. Finally,  $H$  is obtained by applying a series of generalisers to  $F$  under the constraint that  $B \wedge H$  is consistent. Many such generalisers have been proposed, such as, *reverse skolemisation* [2] (converting Skolem constants/functions to existentially quantified variables), *anti-instantiation* (replacing ground subterms with variables), *anti-weakening* (adding some clauses), *anti-subsumption* (dropping some literals from a clause), *inverse resolution* [10] (applying the inverse of the resolution principle) and *Plotkin’s least generalisation* [13].

*Example 1.* Define a background theory  $B$ , an example  $E$  and a production field  $\mathcal{P}$  as follows:

$$\begin{aligned} B &= \text{natural}(0) \vee \text{even}(0), \\ E &= \text{natural}(s(0)), \\ \mathcal{P} &= \{\{\text{natural}(X), \neg \text{natural}(X), \text{even}(X), \neg \text{even}(X)\}\}. \end{aligned}$$

Then,  $\text{NewCarc}(B, \overline{E}, \mathcal{P})$  and  $\text{Carc}(B \wedge \overline{E}, \mathcal{P})$  are as follows:

$$\begin{aligned} \text{NewCarc}(B, \overline{E}, \mathcal{P}) &= \neg \text{natural}(s(0)), \\ \text{Carc}(B \wedge \overline{E}, \mathcal{P}) &= (\text{natural}(0) \vee \text{even}(0)) \wedge \neg \text{natural}(s(0)) \wedge \text{Taut}. \end{aligned}$$

where  $\text{Taut}$  denotes the tautological clauses in  $\text{Carc}(B \wedge \overline{E}, \mathcal{P})$ .

Let  $CC$  be a clausal theory  $(\text{natural}(0) \vee \text{even}(0)) \wedge \neg \text{natural}(s(0))$ . Since each clause in  $CC$  is a clause in  $\text{Carc}(B \wedge \overline{E}, \mathcal{P})$  and the unit clause  $\neg \text{natural}(s(0))$  in  $CC$  is a clause in  $\text{NewCarc}(B, \overline{E}, \mathcal{P})$ ,  $CC$  is a bridge formula wrt  $B$ ,  $E$  and  $\mathcal{P}$  by Definition 2. The clausal theory  $F$  is obtained by translating  $\neg CC$  into CNF as follows:

$$F = (\text{natural}(0) \supset \text{natural}(s(0))) \wedge (\text{even}(0) \supset \text{natural}(s(0))).$$

Since  $B \wedge F \models E$  and  $B \wedge F$  is consistent,  $F$  is a hypothesis wrt  $B$ ,  $E$  and  $\mathcal{P}$ .

Assume that an inverse resolution generaliser is applied to  $F$  in such a way that the clause  $C_1 = \text{natural}(0) \supset \text{natural}(s(0))$  in  $F$  is replaced with the clause  $D_1 = \text{natural}(0) \supset \text{even}(0)$ , which is treated as a parent clause of  $C_1$ . This means  $C_1$  is the resolvent of  $D_1$  and  $C_2 = \text{even}(0) \supset \text{natural}(s(0))$  in  $F$ . Then the following clausal theory  $F'_1$  is constructed:

$$F'_1 = (\text{natural}(0) \supset \text{even}(0)) \wedge (\text{even}(0) \supset \text{natural}(s(0))).$$

Since  $B \wedge F'_1 \models E$  and  $B \wedge F'_1$  is consistent,  $F'_1$  is a hypothesis wrt  $B$ ,  $E$  and  $\mathcal{P}$ .

Next, assume that another inverse resolution generaliser is applied to  $F$  in such a way that the clause  $C_2$  in  $F$  is replaced with the clause  $D_2 = \text{even}(0) \supset \text{natural}(0)$ . Then the following clausal theory  $F'_2$  is constructed.

$$F'_2 = (\text{natural}(0) \supset \text{natural}(s(0))) \wedge (\text{even}(0) \supset \text{natural}(0))$$

Since  $B \wedge F'_2 \models E$  and  $B \wedge F'_2$  is consistent,  $F'_2$  is also a hypothesis wrt  $B$ ,  $E$  and  $\mathcal{P}$ . In addition to the above generaliser, if we apply an anti-instantiation generaliser to  $F'_2$  in such a way that the ground term  $0$  occurring in  $F'_2$  is replaced with the variable  $X$ , then the following theory is obtained:

$$F'_3 = (\text{natural}(X) \supset \text{natural}(s(X))) \wedge (\text{even}(X) \supset \text{natural}(X)).$$

Since  $B \wedge F'_3 \models E$  and  $B \wedge F'_3$  is consistent,  $F'_3$  is also a hypothesis wrt  $B$ ,  $E$  and  $\mathcal{P}$ .

### 3 Ordering Deductive Operators

Our motivation in this section is to develop a simplified generalisation procedure for CF-induction that uses fewer operators while preserving its soundness and completeness for finding hypotheses. We present one way to simplify the generalisation process by computing generalisations deductively. Our approach is motivated by considering the following deductive operators, which are based on [17].

**Definition 3 (Deductive operators).** *Let  $S$  and  $T$  be clausal theories. Then  $T$  is directly-derivable from  $S$  iff  $T$  is obtained from  $S$  by one of the following three operators:*

1. (resolution)  $T = S \cup \{C\}$ , where  $C$  is a resolvent of two clauses  $D_1, D_2 \in S$ .
2. (subsumption)  $T = S \cup \{C\}$ , where  $C$  is subsumed by some clause  $D \in S$ .
3. (weakening)  $T = S - \{D\}$  for some clause  $D \in S$ .

Two special cases of the subsumption operator can be further distinguished by the following two operators.

- 2a (instantiation)  $T = S \cup \{D\sigma\}$  for some clause  $D \in S$  and substitution  $\sigma$ .
- 2b (expansion)  $T = S \cup \{C\}$ , where  $C$  is a superset of some clause  $D \in S$ .

We write  $S \vdash_r T$ ,  $S \vdash_s T$ ,  $S \vdash_w T$ ,  $S \vdash_\alpha T$  and  $S \vdash_\beta T$  to denote that  $T$  is directly derivable from  $S$  by resolution, subsumption, weakening, instantiation and expansion, respectively.  $\vdash_X^*$  is the reflexive and transitive closure of  $\vdash_X$ , where  $X$  is one of the symbols  $r, s, w, \alpha, \beta$ .

We now show that entailment between theories can be established by applying these operators in a particular order.

**Theorem 3.** *Let  $S$  be a clausal theory and  $T$  be a clausal theory without any tautological clauses. If  $S \models T$ , then there are two clausal theories  $U, V$  such that*

$$S \vdash_r^* U \vdash_s^* V \vdash_w^* T$$

*Proof.* Let  $T = \{C_1, \dots, C_n\}$ . Then  $S \models C_i$  for each clause  $C_i$  in  $T$ . By the Subsumption Theorem there is a derivation  $R_1^i, \dots, R_{m_i}^i$  from  $S$  of a clause  $R_{m_i}^i$  that subsumes  $C_i$ . Hence, it is sufficient to let  $U = S \cup \{R_j^i : 1 \leq i \leq n, 1 \leq j \leq m_i\}$  and  $V = U \cup T$ .  $\square$

We also mention the following two properties concerned with reordering of deductive operators.

**Proposition 1.** *Let  $U_1$  and  $U_2$  be clausal theories. If  $U_1 \vdash_s^* U_2$ , then there exist two clausal theories  $V_1$  and  $V_2$  such that  $U_1 \vdash_\alpha^* V_1 \vdash_\beta^* V_2 \vdash_w^* U_2$ .*

*Proof.* By Definition 3, there exist a substitution  $\sigma_i$  and a clause  $D_i \in U_1$  such that  $D_i \sigma_i \subseteq C_i$ , for each clause  $C_i \in U_2 - U_1$  ( $1 \leq i \leq n$ ). Let  $T$  be a finite set  $\bigcup_{i=1}^n \{D_i \sigma_i\}$ . Let  $V_1$  be  $U_1 \cup T$  and  $V_2$  be  $V_1 \cup \{C_1, \dots, C_n\}$ . Then it holds that  $U_1 \vdash_\alpha^* V_1$  and  $V_1 \vdash_\beta^* V_2$ . Moreover, since it holds that  $U_2 \subseteq V_2$ , it holds that  $V_2 \vdash_w^* U_2$ .  $\square$

**Proposition 2.** *Let  $S, U$  and  $V$  be clausal theories. If  $S \vdash_w^* U \vdash_s^* V$ , then there exists a clausal theory  $U'$  such that  $S \vdash_s^* U' \vdash_w^* V$ .*

*Proof.* Let  $U'$  be the clausal theory  $S \cup (V - U)$ . Since  $U \vdash_s^* V$ , there exists a clause  $D_i \in U$  such that  $D_i \supseteq C_i$  for each clause  $C_i \in V - U$ . Since  $S \vdash_w^* U$ ,  $U \subseteq S$  holds. Since  $D_i \in U$ ,  $D_i \in S$  holds. Therefore  $U'$  is obtained from  $S$  using the subsumption operator, that is,  $S \vdash_s^* U'$ . Next we show that  $V \subseteq U'$  holds. Since  $U' = S \cup (V - U)$ ,  $V - U \subseteq U'$  holds. And also,  $S \subseteq U'$  holds. Since  $S \vdash_w^* U$ ,  $U \subseteq S$  holds. Accordingly,  $V \cap U \subseteq U'$  holds, since  $V \cap U \subseteq U$ . Since  $V - U \subseteq U'$  and  $V \cap U \subseteq U'$ ,  $V \subseteq U'$  holds. Hence  $U' \vdash_w^* V$  holds.  $\square$

## 4 Logical Reconstruction of CF-Induction

In this section, we use the above ordering results to show how the number of generalisation operators used in CF-induction can be reduced. Section 4.1 shows this result in the case of ground hypotheses and Section 4.2 shows it in the general case.

### 4.1 Logical Relation between Bridge Formulae and Ground Hypotheses

First, we show that resolution and instantiation can be incorporated into the selection of  $CC$ . We show this with the following two lemmas.

**Lemma 1.** *Let  $CC$  be a bridge formula wrt  $B, E$  and  $\mathcal{P}$ , and  $U$  be a clausal theory. If  $CC \vdash_r^* U$ , then there exist a clausal theory  $V$  and a bridge formula  $CC'$  wrt  $B, E$  and  $\mathcal{P}$  such that  $CC' \vdash_s^* V \vdash_w^* U$ .*

*Proof.* The proof is in two parts.

(a) First we prove that, for each clause  $C_i \in U$  ( $1 \leq i \leq n$ ), there exists a clause  $D_i \in \text{Carc}(B \wedge \overline{E}, \mathcal{P})$  such that  $D_i \supseteq C_i$ .

By the definition of characteristic clauses, for each clause  $K \in Th_{\mathcal{P}}(B \wedge \overline{E})$  there exists a clause  $M \in Carc(B \wedge \overline{E}, \mathcal{P})$  such that  $M \succeq K$ . Hence it is sufficient to show that every clause  $C_i \in U$  is included in  $Th_{\mathcal{P}}(B \wedge \overline{E})$ . Since  $CC \models U$  and  $B \wedge \overline{E} \models CC$ , it holds that every clause  $C_i \in U$  is a consequence of  $B \wedge \overline{E}$ . Then it remains to show that every clause  $C_i \in U$  belongs to  $\mathcal{P}$ , which is done by mathematical induction on the number  $n$  of the applications of  $\vdash_r$  for deriving  $U$  from  $CC$ . In the following, we write  $CC \vdash_r^n U$  to denote that  $U$  is derived from  $CC$  by  $n$  applications of  $\vdash_r$ .

*Base step:* If  $n = 0$  then  $U = CC$  and it trivially follows that every clause in  $U$  belongs to  $\mathcal{P}$ .

*Induction step:* If  $n = k + 1$  for some  $k \geq 0$ , then it holds that  $CC \vdash_r^k U' \vdash_r U$  where  $U'$  is a clausal theory. By the induction hypothesis, it holds that every clause in  $U'$  belongs to  $\mathcal{P}$ . Moreover, it follows that  $U = U' \cup \{R\}$  for some resolvent  $R$  of two clauses in  $U'$ . Since every clause in  $U'$  belongs to  $\mathcal{P}$  and  $\mathcal{P}$  is closed under instantiation, the resolvent  $R$  also belongs to  $\mathcal{P}$ . Thus every clause in  $U$  belongs to  $\mathcal{P}$  and so part (a) holds.

(b) Now we show how to construct the theories  $CC'$  and  $V$ . Start by defining the theory  $T = \bigcup_{i=1}^n \{D_i\}$  using the clauses  $D_i \in Carc(B \wedge \overline{E}, \mathcal{P})$  constructed above. Now define the bridge formula  $CC' = CC \cup T$  and the theory  $V = CC' \cup U$ . Since for each clause  $C_i \in U$  there exists  $D_i \in T$  such that  $D_i \succeq C_i$ ,  $CC' \vdash_s^* V$  holds. Since  $U \subseteq V$ ,  $V \vdash_w^* U$  holds. Hence  $CC' \vdash_s^* V \vdash_w^* U$  holds. Each clause in  $CC'$  is an instance of a clause in  $Carc(B \wedge \overline{E}, \mathcal{P})$ . Since  $CC \subseteq CC'$  and  $CC$  is a bridge formula, there exists a clause in  $CC'$  which is an instance of a clause in  $NewCarc(B, \overline{E}, \mathcal{P})$ . Then  $CC'$  is a bridge formula. Therefore there exist a bridge formula  $CC'$  and a clausal theory  $V$  such that  $CC' \vdash_s^* V \vdash_w^* U$ .  $\square$

**Lemma 2.** *Let  $CC$  be a bridge formula wrt  $B$ ,  $E$  and  $\mathcal{P}$ , and  $U$  be a clausal theory. If  $CC \vdash_{\alpha}^* U$ , then  $U$  is a bridge formula wrt  $B$ ,  $E$  and  $\mathcal{P}$ .*

*Proof.* Every clause in  $U$  is an instance of a clause in  $Carc(B \wedge \overline{E}, \mathcal{P})$ . Since  $U \supseteq CC$ , there exists a clause  $C_i \in U$  such that  $C_i$  is an instance of a clause from  $NewCarc(B, \overline{E}, \mathcal{P})$ . Therefore,  $U$  is a bridge formula.  $\square$

Then, using Lemmas 1 and 2, we can show the following theorem, which establishes the logical relation between bridge formulae and ground hypotheses.

**Theorem 4.** *Let  $H$  be a ground hypothesis wrt  $B$ ,  $E$  and  $\mathcal{P}$ . Then there exist a bridge formula  $CC$  wrt  $B$ ,  $E$  and  $\mathcal{P}$  and a clausal theory  $V$  such that  $CC \vdash_{\beta}^* V \vdash_w^* \overline{H}$ .*

*Proof.* First, we consider the case that  $\overline{H}$  has no tautological clauses. Then, by Theorem 2, there exists a bridge formula  $CC$  such that  $CC \models \neg H$ . Since  $H$  is ground,  $\overline{H} \equiv \neg H$  holds. Thus  $CC \models \overline{H}$  holds. By Theorem 3, there exist clausal theories  $V_1$  and  $V_2$  such that  $CC \vdash_r^* V_1 \vdash_s^* V_2 \vdash_w^* \overline{H}$ . By Lemma 1, there exist a clausal theory  $V_3$  and a bridge formula  $CC''$  such that  $CC'' \vdash_s^* V_3 \vdash_w^* V_1 \vdash_s^* V_2 \vdash_w^* \overline{H}$ . By Proposition 2, there exists a clausal theory  $V_4$  such that

$CC'' \vdash_s^* V_3 \vdash_s^* V_4 \vdash_w^* V_2 \vdash_w^* \overline{H}$ . Thus  $CC'' \vdash_s^* V_4 \vdash_w^* \overline{H}$ . By Proposition 1, there exist clausal theories  $V_5$  and  $V_6$  such that  $CC'' \vdash_\alpha^* V_5 \vdash_\beta^* V_6 \vdash_w^* V_4 \vdash_w^* \overline{H}$ . By Lemma 2,  $V_5$  is a bridge formula, and by letting  $CC' = V_5$ , it holds that  $CC' \vdash_\beta^* V_6 \vdash_w^* \overline{H}$ .

Next, in the case that  $\overline{H}$  contains tautological clauses, we let  $T = \{D_1, \dots, D_n\}$  denote the set of the tautological clauses in  $\overline{H}$ . Then, since every clause  $D_i \in T$  ( $1 \leq i \leq n$ ) is a consequence of  $B \wedge \overline{E}$  and  $D_i$  belongs to  $\mathcal{P}$ , for each  $D_i \in T$  there exists a clause  $C_i \in \text{Carc}(B \wedge \overline{E}, \mathcal{P})$  such that  $C_i \succeq D_i$ . Let  $S$  be the clausal theory  $\bigcup_{i=1}^n \{C_i\}$ . Then, it holds that  $S \vdash_s^* S_1 \vdash_w^* T$  where  $S_1 = S \cup T$ . Now, since the clausal theory  $\overline{H} - T$  has no tautological clauses, there exist a bridge formula  $CC$  and a clausal theory  $V_7$  such that  $CC \vdash_s^* V_7 \vdash_w^* \overline{H} - T$ . Then, it holds that  $CC \cup S \vdash_s^* V_7 \cup S_1 \vdash_w^* \overline{H}$ . Since the clausal theory  $CC \cup S$  satisfies Definition 2, the theorem holds.  $\square$

Next, we introduce a new operator, which can be regarded as concatenating weakening and expansion.

**Definition 4 ( $\gamma$ -operator).** *Let  $S$  and  $T$  be clausal theories. Then  $T$  is directly  $\gamma$ -derivable from  $S$  iff  $T$  is obtained from  $S$  under the following condition:*

$T = (S - \{D\}) \cup \{C_1, \dots, C_n\}$  for some  $n \geq 0$  where  $C_i \supseteq D$  for all  $1 \leq i \leq n$ .

Analogously to Definition 3, we write  $S \vdash_\gamma T$  iff  $T$  is directly  $\gamma$ -derivable from  $S$  and  $\vdash_\gamma^*$  is a reflexive and transitive closure of  $\vdash_\gamma$ .

**Theorem 5.** *Let  $H$  be a ground hypothesis wrt  $B$ ,  $E$  and  $\mathcal{P}$ . Then, there exists a bridge formula  $CC$  wrt  $B$ ,  $E$  and  $\mathcal{P}$  such that  $CC \vdash_\gamma^* \overline{H}$ .*

*Proof.* Since  $H$  is a ground hypothesis, by Theorem 4, there exist a bridge formula  $CC = \{C_1, \dots, C_n\}$  and a clausal theory  $U$  such that  $CC \vdash_\beta^* U \vdash_w^* \overline{H}$ . Let  $F_{C_i}$  be the clausal theory  $\{C \mid C \in \overline{H} \text{ and } C_i \subseteq C\}$ , for each clause  $C_i \in CC$  ( $1 \leq i \leq n$ ). Then, by Definition 4, for each clause  $C_i \in CC$ ,  $\{C_i\} \vdash_\gamma^* F_{C_i}$  holds. Accordingly,  $CC \vdash_\gamma^* \bigcup_{i=1}^n F_{C_i}$  holds.

Hence, it is sufficient to show that  $\overline{H} = \bigcup_{i=1}^n F_{C_i}$ . Since  $F_{C_i} \subseteq \overline{H}$  for every clause  $C_i \in CC$ , it holds that  $\bigcup_{i=1}^n F_{C_i} \subseteq \overline{H}$ . Conversely, since  $CC \vdash_\beta^* U$ , for every clause  $D \in U$ , there exists a clause  $C_i \in CC$  such that  $C_i \subseteq D$ . Also since  $U \vdash_w^* \overline{H}$ , it holds that  $\overline{H} \subseteq U$ . Then, for every clause  $D \in \overline{H}$ , there exists a clause  $C_i \in CC$  such that  $C_i \subseteq D$ , that is,  $D \in F_{C_i}$ . This means that  $\overline{H} \subseteq \bigcup_{i=1}^n F_{C_i}$ . Hence, it holds that  $\overline{H} = \bigcup_{i=1}^n F_{C_i}$ . Therefore, it holds that  $CC \vdash_\gamma^* \overline{H}$ .  $\square$

## 4.2 Deriving Non-ground Hypotheses

We generalise the result of the previous section to non-ground hypotheses. We show that any hypothesis can be obtained from a bridge formula by applying the  $\gamma$ -operator followed by anti-instantiation.

**Definition 5.** Let  $B$  and  $E$  be clausal theories and  $\mathcal{P} = \langle \mathbf{L} \rangle$  be a production field. A clausal theory  $H$  is derived by CF-induction with  $\gamma$ -operator from  $B$ ,  $E$  and  $\mathcal{P}$  iff  $H$  is constructed as follows:

- Step 1. Construct a bridge formula  $CC$  wrt  $B$ ,  $E$  and  $\mathcal{P}$ .
- Step 2. Construct a clausal theory  $G$  such that  $CC \vdash_{\gamma}^* G$ .
- Step 3. Compute the complement  $\overline{G}$  of  $G$ .
- Step 4.  $H$  is obtained by applying anti-instantiation to  $\overline{G}$ , such that  $B \wedge H$  is consistent,  $H$  contains no Skolem constants, and for every literal  $L$  in  $H$ ,  $\neg L$  belongs to  $\mathcal{P}$ .

Several remarks are necessary for Definition 5.

1. Even if  $G$  satisfies  $CC \vdash_{\gamma}^* G$  for some bridge formula  $CC$  at Step 2, any output  $H$  obtained from  $G$  cannot satisfy the conditions of Definition 1 unless  $\overline{G} \wedge B$  is consistent and  $G$  belongs to  $\mathcal{P}$ . In this respect, the constraints of  $H$  at Step 4 are introduced to guarantee the soundness of  $H$ .
2. At Step 3, the complement of  $G$  can theoretically include redundant clauses such as tautological clauses and clauses properly subsumed by other clauses. Accordingly it might be necessary to remove such clauses from the complement of  $G$  computed in Step 3<sup>1</sup>.
3. At Step 4, anti-instantiation allows us to replace subterms in  $\overline{G}$  with variables. For example, for the clause  $p(a) \vee q(a)$ , it is possible to construct  $p(X) \vee q(Y)$  obtained by replacing the constant  $a$  in  $p(a)$  and  $q(a)$  with two variables  $X$  and  $Y$ , respectively. In this way there are many possibilities to apply anti-instantiation for clauses.

We now give soundness and completeness results for CF-induction with  $\gamma$ -operator.

**Theorem 6 (Soundness).** Let  $B$ ,  $E$  and  $H$  be clausal theories, and  $\mathcal{P}$  be a production field. If  $H$  is derived by CF-induction with  $\gamma$ -operator from  $B$ ,  $E$  and  $\mathcal{P}$ , then  $H$  is a hypothesis wrt  $B$ ,  $E$  and  $\mathcal{P}$ .

*Proof.* Suppose a clausal theory  $H$  is derived by CF-induction with  $\gamma$ -operator from  $B$ ,  $E$  and  $\mathcal{P}$ . Then, by Definition 5, there exist a bridge formula  $CC$  and a clausal theory  $G$  such that  $H$  is derived by applying anti-instantiation to  $\overline{G}$  and  $CC \vdash_{\gamma}^* G$ . By Definition 2, it holds that  $B \wedge \overline{E} \models CC$ . By Definition 4, it holds that  $CC \models G$ . Accordingly, it holds that  $B \wedge \overline{E} \models G$ . Since  $H$  is derived by applying anti-instantiation to  $\overline{G}$ ,  $H \models \overline{G}$  holds. Since  $\overline{G} \models \neg G$ ,  $H \models \neg G$  follows. Equivalently,  $G \models \neg H$ . Therefore, it holds that  $B \wedge \overline{E} \models \neg H$ . Then, it holds that  $B \wedge H \models \neg \overline{E}$ . Since, from Step 4 of Definition 5,  $H$  contains no Skolem constants from  $\overline{E}$ , it holds that  $B \wedge H \models E$ . Hence, it holds that  $H$  is a hypothesis wrt  $B$ ,  $E$  and  $\mathcal{P}$ , since, from Step 4 of Definition 5,  $B \wedge H$  is consistent and for every literal  $L$  appearing in  $H$ ,  $\neg L \in \mathbf{L}$ .  $\square$

<sup>1</sup> We refer the readers to [16] concerning an efficient algorithm for computing such clauses obtained by removing redundant clauses from the complement.

**Theorem 7 (Completeness).** *Let  $B$ ,  $E$  and  $H$  be clausal theories, and  $\mathcal{P}$  be a production field. If  $H$  is a hypothesis wrt  $B$ ,  $E$  and  $\mathcal{P}$ , then there exists a theory  $H^* \equiv H$  that is derived by CF-induction with  $\gamma$ -operator from  $B$ ,  $E$  and  $\mathcal{P}$ .*

*Proof.* Suppose  $H$  is a hypothesis wrt  $B$ ,  $E$  and  $\mathcal{P}$ . By Theorem 2, there is a bridge formula  $CC$  wrt  $B$ ,  $E$  and  $\mathcal{P}$  such that  $CC \cup H$  is unsatisfiable. Using Herbrand’s theorem<sup>2</sup> [12], there are two finite sets  $CC'$  and  $H'$  such that  $CC'$  (resp.  $H'$ ) is a finite set of ground instances of  $CC$  (resp.  $H$ ) and  $CC' \cup H'$  is unsatisfiable. In this case,  $H'$  can be chosen in such a way that for every clause  $C$  in  $H$ , there is an instance  $C'$  of  $C$  such that  $C' \in H'$ , and also,  $CC'$  can be chosen in such a way that  $CC'$  contains at least one instance of a clause in  $NewCarc(B, \overline{E}, \mathcal{P})$ . Then,  $H$  can be obtained by applying an anti-instantiation generaliser to  $H'$ . We prove that  $H'$  is a ground hypothesis wrt  $B$ ,  $\overline{\overline{E}}$ , and  $\mathcal{P}$ . That is, we will show that (1)  $B \wedge H' \models \overline{\overline{E}}$ , (2)  $B \wedge H' \not\models \square$  and (3)  $\neg L \in \mathbf{L}$  for every literal  $L$  appearing in  $H'$ .

*Proof of (1):*  $CC$  is a bridge formula wrt  $B$ ,  $E$  and  $\mathcal{P}$ . Since every clause in  $CC'$  is an instance of a clause in  $CC$ ,  $CC'$  satisfies the first condition of Definition 2. Also,  $CC'$  contains at least one clause  $C'$  such that  $C'$  is an instance of a clause from  $NewCarc(B, \overline{E}, \mathcal{P})$ . Then  $CC'$  satisfies the second condition of Definition 2. Hence,  $CC'$  is also bridge formula wrt  $B$ ,  $E$  and  $\mathcal{P}$ . Thus  $B \wedge \overline{E} \models CC'$  holds. By  $CC' \models \neg H'$ ,  $B \wedge \overline{E} \models \neg H'$  holds. Then  $B \wedge H' \models \neg \overline{E}$  holds. By  $\neg \overline{E} \equiv \overline{\overline{E}}$ ,  $B \wedge H' \models \overline{\overline{E}}$  holds.

*Proof of (2):* If it holds that  $B \wedge H' \models \square$ , then it must hold that  $B \wedge H \models \square$ , by  $B \wedge H \models B \wedge H'$ . It contradicts the fact  $H$  is a hypothesis.

*Proof of (3):* Since  $H$  is a hypothesis wrt  $B$ ,  $E$  and  $\mathcal{P}$ , for every literal  $L$  appearing in  $H$ ,  $\neg L \in \mathbf{L}$  holds. Then it holds that  $\neg L \in \mathbf{L}$  for every literal  $L$  appearing in  $H'$ , since  $\mathcal{P} = \langle \mathbf{L} \rangle$  is closed under instantiation.

Now, since  $H'$  is a ground hypothesis wrt  $B$ ,  $\overline{\overline{E}}$  and  $\mathcal{P}$ , there exists a bridge formula  $CC''$  wrt  $B$ ,  $\overline{\overline{E}}$  and  $\mathcal{P}$  such that  $CC'' \vdash_{\gamma} \overline{H'}$  by Theorem 5. Since  $\overline{\overline{E}} \equiv \overline{E}$ , it holds that for every clause  $C$  in  $Carc(B \wedge \overline{\overline{E}}, \mathcal{P})$ ,  $C$  is contained in  $Carc(B \wedge \overline{E}, \mathcal{P})$ . Then, it also holds that for every clause  $C$  in  $CC''$ ,  $C$  is contained in  $Carc(B \wedge \overline{E}, \mathcal{P})$ . Hence,  $CC''$  is also bridge formula wrt  $B$ ,  $E$  and  $\mathcal{P}$ . Then, from Step 1 of Definition 5,  $CC''$  can be constructed in a CF-induction with  $\gamma$ -operator from  $B$ ,  $E$  and  $\mathcal{P}$ . Moreover,  $\overline{H'}$  can be also constructed from  $CC''$  at Step 2. Since  $H'$  is ground, it holds that  $H'$  is logically equivalent to  $\overline{H'}$  computed from  $\overline{H'}$  at Step 3. Recall that  $H$  is obtained by applying anti-instantiation to  $H'$ . Therefore a formula  $H^*$  is obtained at Step 4 with the application of anti-instantiation to  $\overline{H'}$  such that  $H^* \equiv H$ .  $\square$

*Example 2.* Recall Example 1. Let  $CC$  be the following bridge formula, which appears in Example 1.

$$CC = (natural(0) \vee even(0)) \wedge \neg natural(s(0))$$

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<sup>2</sup> A set of clauses  $\Sigma$  is unsatisfiable iff a finite set of ground instances of clauses of  $\Sigma$  is unsatisfiable.

Assume that a  $\gamma$ -operator is applied to  $CC$  so that  $\neg natural(s(0))$  is replaced with the two clauses  $\neg natural(s(0)) \vee \neg even(0)$  and  $\neg natural(s(0)) \vee natural(0)$ , then the following clausal theory  $G_1$  is constructed:

$$\begin{aligned} G_1 = & (natural(0) \vee even(0)) \\ & \wedge (\neg natural(s(0)) \vee \neg even(0)) \\ & \wedge (\neg natural(s(0)) \vee natural(0)). \end{aligned}$$

Then, we can obtain the complement  $\overline{G_1}$  of  $G_1$ , which is logically equivalent to  $F'_1$  in Example 1. Next assume that another  $\gamma$ -operator is applied to  $CC$  so that  $\neg natural(s(0))$  is replaced with the two clauses  $\neg natural(s(0)) \vee even(0)$  and  $\neg natural(s(0)) \vee \neg natural(0)$ , then the following clausal theory  $G_2$  is constructed:

$$\begin{aligned} G_2 = & (natural(0) \vee even(0)) \\ & \wedge (\neg natural(s(0)) \vee even(0)) \\ & \wedge (\neg natural(s(0)) \vee \neg natural(0)) \end{aligned}$$

Then, the complement  $\overline{G_2}$  of  $G_2$  is logically equivalent to  $F'_2$  in Example 1. Accordingly, we can obtain a clausal theory, which is logically equivalent to  $F'_3$  in Example 1 by applying an anti-instantiation generaliser to  $\overline{G_2}$ . In this way, the inverse resolution generaliser can be realised with applications of the  $\gamma$ -operator.

## 5 Related Work

The  $\gamma$ -operator can be regarded as a particular *downward refinement operator* [1,6,8,12] for the  $\vdash_\gamma^*$  order, which is closely related to the subsumption order. Let  $S$  and  $T$  be clausal theories such that  $S \vdash_\gamma^* T$ . Then  $S \succeq T$  holds. Compared with the subsumption order, one important feature of  $\gamma$ -operator lies in restraint of the operation of instantiation, which leads to a large number of choice points. There are certain desirable properties that a “good” downward refinement operator should satisfy and we intend to study which of these the  $\gamma$ -operator satisfies.

We can reduce generalisation under the entailment relation in the previous version to generalisation under the  $\gamma$ -operator. It is based on the notion that any series of processes of inductive operations on the inverse relation of entailment between the negation of a bridge formula and a hypothesis connects a certain series of processes of deductive operations on entailment between a bridge formula and the negation of hypothesis. Accordingly, there are two sides where we can grasp generalisation processes. Yamamoto and Fronhöfer [18] and Yamamoto [17] first have studied the connection between two clausal theories related by entailment and negation. It will be interesting to consider about the relation between two clausal theories ordered by the  $\gamma$ -operator instead of entailment and these negations.

CF-induction is related to several methods [9,14,15,18] developed in the field of Inductive Logic Programming (ILP) [11], and the relationship between CF-induction and such methods as Progol [9] and the residue procedure [18] has

been studied in [5]. We briefly discuss the relationship between CF-induction and the more recent HAIL/FC-HAIL [14,15] approaches.

HAIL [14] uses a combination of abduction and induction to overcome some of the limitations of Progol. In particular, HAIL learns more than one clause in response to a single example using a multi-clause generalisation of the *Bottom Clause* called a *Kernel Set*, which can be regarded as a bridge formula. The HAIL generaliser is anti-subsumption applied to this set of clauses. Recently, a full clausal generalisation of HAIL, called FC-HAIL, has been proposed in [15] that extends the mode-directed algorithm of HAIL to full clausal theories. Although FC-HAIL is not complete for hypothesis finding in full clausal logic unlike CF-induction, it can use a form of language bias called mode-declarations to restrict the hypothesis space like Progol. It will be fruitful to consider about incorporating the notion of mode-declarations into CF-induction for finding efficient hypotheses.

## 6 Conclusion and Future Work

We have studied how the generalisation procedure of CF-induction can be simplified while preserving its soundness and completeness. In Section 4.1, we introduced the  $\gamma$ -operator whose task is removing some clause  $D$  in an input clausal theory and adding a set of clauses  $C_1, \dots, C_n$  for some  $0 \leq n$  where each clause  $C_i$  is a super set of  $D$ . In Section 4.2, we showed that the  $\gamma$ -operator and anti-instantiation are sufficient to ensure the completeness of CF-induction.

The results shown in this paper will contribute toward promoting practical real-world applications of CF-induction. CF-induction is now expected to be applied in real domain such as Systems Biology [3], and we are trying to apply CF-induction with the  $\gamma$ -operator to examples shown in [3]. Some initial experiments have confirmed that these examples can be more easily solved using CF-induction with  $\gamma$ -operator, as compared with the previous version.

Future work will investigate ways of automatically finding which literals must be added to selected clauses by the  $\gamma$ -operator. In addition, we believe that studying various restrictions of the  $\gamma$ -operator may allow us to systematically compare the generalisation power of previously proposed operators. Other important future work is the selection of clauses in  $CC$ , which currently requires assistance from the user. We intend to address this issue by ranking characteristic clauses in such a way that the user is allowed to more easily select appropriate clauses for  $CC$ .

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## References

1. Badea, L.: A Refinement Operator for Theories. In: Rouveirol, C., Sebag, M. (eds.) ILP 2001. LNCS (LNAI), vol. 2157, pp. 1–14. Springer, Heidelberg (2001)
2. Cox, P.T., Pietrzykowski, T.: A Complete, Nonredundant Algorithm for Reversed Skolemization. *Theoretical Computer Science* 28(3), 239–261 (1984)

3. Doncescu, A., Inoue, K., Yamamoto, Y.: Knowledge Based Discovery in Systems Biology Using CF-Induction. In: Okuno, H.G., Ali, M. (eds.) IEA/AIE 2007. LNCS (LNAI), vol. 4570, pp. 395–404. Springer, Heidelberg (2007)
4. Inoue, K.: Linear Resolution for Consequence Finding. *Artificial Intelligence* 56(2-3), 301–353 (1992)
5. Inoue, K.: Induction as Consequence Finding. *Machine Learning* 55(2), 109–135 (2004)
6. Laird, P.D.: *Learning from Good and Bad Data*. Kluwer Academic Publishers, Dordrecht (1988)
7. Lee, C.T.: *A Completeness Theorem and Computer Program for Finding Theorems Derivable from Given Axioms*. PhD thesis, Dept. of Electronic Eng. and Computer Sci., Univ. of California, Berkeley, CA (1967)
8. Midelfart, H.: A Bounded Search Space of Clausal Theories. In: Džeroski, S., Flach, P.A. (eds.) ILP 1999. LNCS (LNAI), vol. 1634, pp. 210–221. Springer, Heidelberg (1999)
9. Muggleton, S.H.: Inverse Entailment and Progol. *New Generation Computing* 13(3-4), 245–286 (1995)
10. Muggleton, S.H., Buntine, W.L.: Machine Invention of First Order Predicates by Inverting Resolution. In: Proc. of the 5th Int. Conf. on Machine Learning, pp. 339–352 (1988)
11. Muggleton, S.H., De Raedt, L.: Inductive Logic Programming: Theory and Methods. *Logic Programming* 19(20), 629–679 (1994)
12. Nienhuys-Cheng, S., de Wolf, R.: *Foundations of Inductive Logic Programming*. LNCS, vol. 1228. Springer, Heidelberg (1997)
13. Plotkin, G.D.: A Further Note on Inductive Generalization. In: *Machine Intelligence*, vol. 6, pp. 101–124. Edinburgh University Press (1971)
14. Ray, O., Broda, K., Russo, A.M.: Hybrid Abductive Inductive Learning: a Generalisation of Progol. In: Horváth, T., Yamamoto, A. (eds.) ILP 2003. LNCS (LNAI), vol. 2835, pp. 311–328. Springer, Heidelberg (2003)
15. Ray, O., Inoue, K.: Mode Directed Inverse Entailment for Full Clausal Theories. In: Proc. of the 17th Int. Conf. on Inductive Logic Programming ILP 2007. LNCS, vol. 4894. Springer, Heidelberg (to appear 2008)
16. Satoh, K., Uno, T.: Enumerating Maximal Frequent Sets Using Irredundant Dualization. In: Grieser, G., Tanaka, Y., Yamamoto, A. (eds.) DS 2003. LNCS (LNAI), vol. 2843, pp. 256–268. Springer, Heidelberg (2003)
17. Yamamoto, A.: Hypothesis Finding Based on Upward Refinement of Residue Hypotheses. *Theoretical Computer Science* 298, 5–19 (2003)
18. Yamamoto, A., Fronhöfer, B.: Hypotheses Finding via Residue Hypotheses with the Resolution Principle. In: Arimura, H., Sharma, A.K., Jain, S. (eds.) ALT 2000. LNCS (LNAI), vol. 1968, pp. 156–165. Springer, Heidelberg (2000)