Bloch-Kato conjecture for $\mathbf{Z}/2$-coefficients and algebraic Morava K-theories.[1]

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Contents

1 Introduction.

In this paper we show that the existence of algebro-geometrical analogs of the higher Morava K-theories satisfying some basic properties would imply the Bloch-Kato conjecture with $\mathbf{Z}/2$-coefficients for fields which admit resolution of singularities (see [2] for a precise formulation of this condition).

Our approach is inspired by two different ideas. The first is the use of algebraic K-theory and norm varieties in the proof of Bloch-Kato conjecture with $\mathbf{Z}/2$-coefficients in weight three given by A. Merkurjev and A. Suslin in [4] and independently by M. Rost in [6]. The second is the "chromatic" approach to algebraic topology which was developed by Jack Morava, Mike Hopkins, Douglas Ravenel and others.

The Bloch-Kato conjecture in its original form asserts that for any field k and any prime l not equal to $char(k)$ the canonical homomorphisms

$$K_M^n(k)/l \to H_{et}^n(k, \mu_l^{\otimes n})$$

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from Milnor's K-groups of k with $\mathbf{Z}/l$-coefficients to its etale cohomology are isomorphisms. Modulo the facts we already know it can also be reformulated by saying that the etale cohomology algebras $H^n_{et}(k, \mu_l^{\otimes n})$ are quadratic, i.e. that it is generated by H^1 with all relations being in H^2. It implies at least three other major conjectures[2]:

1. Milnor's conjecture relating the Witt ring of quadratic forms over a field to its Milnor's K-groups.

2. Beilinson-Lichtenbaum conjecture which describes motivic cohomology with finite coefficients in terms of etale cohomology and, in particular, Beilinson-Soule vanishing conjecture for finite coefficients.

3. Quillen-Lichtenbaum conjecture which asserts (in its original form) that for a field of cohomological dimension n the natural homomorphism from Quillen's K-theory (with finite coefficients) to etale K-theory is an isomorphism in dimensions greater or equal to n.

Let us explain briefly the main idea of our approach. We fix a base field k (which we assume to admit resolution of singularities) and a prime number l not equal to the characteristic of k. Denote by $\mathbf{Z}_{(l)}$ the localization of $\mathbf{Z}$ in the prime ideal (l). As was shown in [11] the Bloch-Kato conjecture in weight n over k is equivalent to the fact that for any field F of finite type over k the *etale* motivic cohomology group $H^{n+1}_{et}(Spec(F), \mathbf{Z}_{(l)}(n))$ vanishes. Assuming, inductively, that the Bloch-Kato conjecture is proven for weights less than n we shown that $H^{n+1}_{et}(Spec(F), \mathbf{Z}_{(l)}(n)) = 0$ for any field F such that $K^M_n(F)$ is l-divisible. After that is done, it remains to show that for any field E over k and any symbol $\underline{a} = (a_1, \ldots, a_n)$ in $K^M_n(E)$ there exists a variety $X_{\underline{a}}$ over $Spec(E)$ such that $\underline{a}$ is divisible by l in the Milnor's K-group of the function field $E_{\underline{a}}$ of $X_{\underline{a}}$ and the homomorphism

$$H^{n+1}_{et}(Spec(E), \mathbf{Z}_{(l)}(n)) \to H^{n+1}_{et}(Spec(E_{\underline{a}}), \mathbf{Z}_{(l)}(n))$$

is injective. There are clearly many varieties $X_{\underline{a}}$ satisfying the first of these two conditions. However, to make it easier to prove the second condition it is

[2]The fact that the Bloch-Kato conjecture implies the Beilinson-Lichtenbaum and Quillen-Lichtenbaum conjectures was proven in [11] and requires the base field k to admit resolution of singularities.

important to choose $X_{\underline{a}}$ which is in some sense "maximal" among the varieties satisfying the first one. These maximal varieties are called norm varieties for $\underline{a}$ (and l). There are several possible definitions of what "maximal" can mean in this context. The most restrictive one says that $X_{\underline{a}}$ is a norm variety for $\underline{a}$ if it is smooth and projective and for any field F of finite type over k the following two conditions are equivalent:

1. $X_{\underline{a}}$ has an F-rational point.
2. $\underline{a}$ is l-divisible in $K_n^M(F)$.

Note that there still can be many norm varieties for $\underline{a}$. For example if $X_{\underline{a}}$ is a norm variety for $\underline{a}$ and Y is any smooth projective variety which has a k-rational point then $X_{\underline{a}} \times Y$ is again a norm variety for $\underline{a}$.

For any scheme X over $Spec(k)$ one may consider the corresponding *Cheh simplicial scheme* $\check{C}(X)$ such that $\check{C}(X)_m = X^m$ and faces and degeneracy morphisms are given by partial projections and diagonals respectively. Let $X_{\underline{a}}$ be a norm variety for $\underline{a}$. Denote the corresponding Cheh simplicial scheme by $\mathcal{X}_{\underline{a}}$. It is easy to see that, as an object of the homotopy category of simplicial schemes in Nisnevich topology, $\mathcal{X}_{\underline{a}}$ does not depend on the choice of $X_{\underline{a}}$. Moreover, as an object of the homotopy category of the simplicial sheaves (in Nisnevich topology on Sm/k) is it isomorphic to the sheaf of the form

$$\Phi_{\underline{a}}(U) = \begin{cases} \emptyset & \text{if } \underline{a} \neq 0 \text{ in } K_n^M(k(U))/l \\ \{\emptyset\} & \text{if } \underline{a} = 0 \text{ in } K_n^M(k(U))/l \end{cases}$$

where $k(U)$ is the function field of U and therefore has an "explicite" description purely in terms of the symbol $(a_1, \ldots, a_n)$. In this paper we try to work (when it is possible) with $\mathcal{X}_{\underline{a}}$ instead of $X_{\underline{a}}$. However, to prove the basic properties of $\mathcal{X}_{\underline{a}}$ (see 4.5,6.5,6.3) which are necessary for the proof of the Bloch-Kato conjecture we have to use some explicitly defined norm varieties and this is the point where we have to restrict our considerations to the case $l = 2$.

The problem is that, at this time, we only know how to construct norm varieties $X_{\underline{a}}$ for $l = 2$. In this case they can be chosen to be some special quadrics. The "geometry" of these quadrics in the motivic sense was studied by Markus Rost in [8]. Using his results we show that the Bloch-Kato conjecture in weight n is equivalent to the vanishing of the motivic cohomology groups $H^{n+1}(\mathcal{X}_{\underline{a}}, \mathbf{Z}_{(l)}(n))$ (which are now considered in Zariski topology!).

Another result of Markus Rost ([7]) implies that we have a vanishing result of the following form:

$$H^{2^n-1}(\mathcal{X}_{\underline{a}}, \mathbf{Z}(2^{n-1})) = 0.$$

Note that for $n = 2$ that is all we need. For $n > 2$ however we should be able to "move" the numbers of cohomology groups somehow.

This problem was solved for $n = 3$ (using a rather different language) in [4],[6]. The required connection between different motivic cohomology groups was provided in this case by the first nontrivial differential in the spectral sequence which converges from motivic cohomology to the algebraic K-theory. The reason why this differential gives enough information lies in the remarkable fact that, since for a quadric Q we have

$$\sum(-1)^i dim H^i(Q, \mathcal{O}) = 1,$$

the canonical morphism $\mathcal{X}_{\underline{a}} \to Spec(k)$ induces isomorphisms on algebraic K-groups. This approach however does not work directly for higher n since the number of possibly nontrivial differentials grows exponentially with n.

We conjecture in Section 5 that there should exist higher algebraic Morava K-theories $K(n)$ which are related to the usual algebraic K-theory and motivic cohomology in the same way as the higher Morava K-theories in topology are related to the complex K-theory and usual cohomology respectively. These theories should admit, in particular, direct image homomorphisms of certain type and, as topological case suggests, the composition of the inverse and direct image homomorphisms on $K(n-1)$ with respect to the projection $X_{\underline{a}} \to Spec(k)$ (where $X_{\underline{a}}$ is the norm quadric for a symbol $(a_1, \ldots, a_n)$) should be, roughly speaking, the identity morphism. In particular if the theories $K(n)$ exist the morphism $\mathcal{X}_{\underline{a}} \to Spec(k)$ should give isomorphisms on all $K(i)$ for $i \leq n-1$. It turns out that this is sufficient to carry out a partial computation of the motivic cohomology groups of $\mathcal{X}_{\underline{a}}$ and in particular to prove the vanishing result we need for the Bloch-Kato conjecture.

In fact, it seems that the knowledge of algebraic Morava K-theories of the simplicial schemes $\mathcal{X}_{\underline{a}}$ together with a certain "semi-periodicy" of their motivic cohomology (4.11) which follows from the results of [8] on the motivic decomposition for norm quadrics is sufficient to compute all motivic cohomology of these simplicial schemes (as modules over motivic cohomology of the base field). This computation, which will be presented in another paper, implies some results on the structure of the etale cohomology ring of fields (with $\mathbf{Z}/2$-coefficients) which are considerably stronger than the ones which follow directly from the Bloch-Kato conjecture. It suggests that there should exist some fundamental property of the etale cohomology algebras of fields underlying all these effects. Aa was already mentioned, the Bloch-Kato conjecture is equivalent to the fact that these algebras are quadratic. At least some of the other results which follows from the computations of the motivic cohomology of $\mathcal{X}_{\underline{a}}$ would become clear if the following much stronger conjecture due to Alexander Vishik and Leonid Positselski were true.

Conjecture 1 *The etale cohomology algebra $H^*(k, \mu_l^{\otimes *})$ of a field k of characteristic not equal l is a Koszul algebra.*

Finally let us make some comments on the nature of the algebraic Morava K-theories. They are (or should be) examples of generalized cohomology theories on algebraic varieties and therefore there construction is a part of the *stable homotopy theory of algebraic varieties*. In some sense this theory may be considered as the next (after "triangulated motives") step toward the Grothendieck's vision of *motivic homotopy type*. It is fairly clear at the moment how to define the basic objects of the stable homotopy theory ([14]). After that is done one can define *algebraic cobordisms* as the theory representable by the obvious algebro-geometrical analog of the Thom spectrum (we have no idea at the moment about possible "geometrical" interpretation of these cobordisms). Algebraic Morava K-theories should then be constructed out of algebraic cobordisms in the same way usual Morava K-theories are constructed out of usual complex cobordisms.

In the rest of this introduction we give a brief summary of individual sections of the paper.

In Section 2 we remind what the Bloch-Kato conjecture is and give several equivalent formulations of it. The later is important since in the proof of our main theorem (Theorem 6.1) we use induction by the weight and have to refer to different results modulo the Block-Kato conjectures for lower weights.

In Section 3 we prove a fairly simple general result which shows that Bloch-Kato conjecture can be reformulated in terms of certain properties of motivic cohomology groups of norm varieties.

In Section 4 we recall some properties of norm quadrics and translate them into the corresponding properties of the norm simplicial schemes $\mathcal{X}_{\underline{a}}$ for $l = 2$.

Algebraic analogs of Morava K-theories appear in Section 5 where we formulate Conjecture 2 describing these theories. Most of the results of this section are left without detailed proofs since there properly belong to the general stable homotopy theory of algebraic varieties and will be considered in this context elsewhere.

Finally in the last section we show that Conjecture 2 implies Bloch-Kato conjecture for $\mathbf{Z}/2$-coefficients.

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2 Motivic cohomology and Bloch-Kato conjecture.

In this section we recall the definition of motivic cohomology used in [11] and give several equivalent reformulations of the Bloch-Kato conjecture.

Motivic cohomology are defined as certain groups of morphisms in a tensor triangulated category $DM_{gm}^{eff}(k)$ which is called the triangulated category of mixed motives and which was constructed in [15]. We recall here its definition and basic properties.

For any field k denote by $SmCor(k)$ the category of finite correspondences over k. It is an additive category with the same objects as the category Sm/k of smooth schemes over k and morphisms from X to Y being cycles on $X \times Y$ which are formal linear combinations of closed subsets which are finite over X and surjective over a connected component of X. This category has a tensor structure given by direct product of schemes on objects and by external multiplication of finite cycles on morphisms.

Let $\mathcal{H}^b(SmCor(k))$ be the homotopy category of bounded complexes over $SmCor(k)$. Let further T be the thick subcategory in it generated by complexes of the form $X \times \mathbf{A}^1 \to X$ and $U \cap V \to U \oplus V \to X$ for all smooth schemes X over k and all Zariski open coverings $X = U \cup V$. One defines $DM_{gm}^{eff}(k)$ as the localization of $\mathcal{H}^b(SmCor(k))$ with respect to morphisms with cones being in T. It is again a tensor triangulated category.

The category $DM_{gm}^{eff}(k)$ can be embedded into a bigger tensor triangulated category which we denote by $DM_{-}^{eff}(k)$. To define this category let us recall first that a *presheaf with transfers* on Sm/k is an additive contravariant functor from $SmCor(k)$ to the category of abelian groups. If F is such a functor we may consider it as a presheaf on Sm/k and in particular may take the associated Nisnevich sheaf F_{Nis}. It was shown in [15] that F_{Nis} also has a canonical structure of a presheaf with transfers. It gives us an abelian category $Shv_{Nis}(SmCor(k))$ of Nisnevich sheaves with transfers. A presheaf F (resp. sheaf, presheaf with transfers, sheaf with transfers) is called homotopy invariant if for any smooth scheme X the homomorphism $F(X) \to F(X \times \mathbf{A}^1)$ given by the projection is an isomorphism. The category $DM_-^{eff}(k)$ is defined as the full subcategory in the derived category of complexes over $Shv_{Nis}(SmCor(k))$ which consists of complexes bounded from above with homotopy invariant cohomology sheaves. This category has a structure of a tensor triangulated category.

For any variety X let $L(X)$ be the presheaf with transfers representable by X on $SmCor(k)$. It is easily seen to be a sheaf in Nisnevich topology. For any presheaf F and any smooth scheme U over k denote by $\underline{C} * (F)(U)$ the normalization of the simplicial abelian group $F(U \times \Delta^\bullet)$ where $\Delta^\bullet$ is the standard cosimplicial object in Sm/k. It gives us a complex of presheaves $\underline{C}*$ (F) and a simple exersise shows that it has homotopy invariant cohomology presheaves for any F. If F is a presheaf with transfers then $\underline{C}_*(F)$ has an obvious canonical structure of a complex of presheaves with transfers. In particular the correspondence

$$X \mapsto \underline{C}_*(L(X))$$

is a functor from $SmCor(k)$ to $DM_-^{eff}(k)$. As was shown in [15] this functor extends to a full embedding $DM_{gm}^{eff}(k) \to DM_-^{eff}(k)$ and one has the following result.

Theorem 2.1 *For any smooth X over k and any complex of Nisnevich sheaves with transfers K with homotopy invariant cohomology sheaves one has canonical isomorphisms*

$$Hom_{D(Shv_{Nis}(SmCor(k)))}(\underline{C}_*(L(X)), K[i]) =$$

$$= Hom_{D(Shv_{Nis}(SmCor(k)))}(L(X), K[i]) = \mathbf{H}^i_{Nis}(X, K).$$

Remark: Strictly speaking, Theorem 2.1 as well as most other properties of the categories DM are proven in [15] for *perfect* base fields. However, simple transfer arguments show that all the properties we use hold for any base field k if we consider $\mathbf{Z}[1/char(k)]$-coefficients. Since everywhere in this paper we work with $\mathbf{Z}_{(l)}$-coefficients for some l prime to the characteristic of k we will disregard this problem.

Let $\mathbf{Z}(1)$ be the object of $DM_{gm}^{eff}(k)$ given by the complex $\mathbf{P}^1 \to Spec(k)$ with $\mathbf{P}^1$ placed in (cohomological) degree two. We set $\mathbf{Z}(n) = \mathbf{Z}(1)^{\otimes n}$. Considered as an object in $DM_{-}^{eff}(k)$, $\mathbf{Z}(n)$ is a certain complex of Nisnevich sheaves with transfers with homotopy invariant cohomology presheaves which is called the *motivic complex* of weight n on Sm/k. Note that by definition $\mathbf{Z}(n)$ is a direct summand of the complex $\underline{C}_*((\mathbf{P}^1)^n)$. We have the following result (see [15]):

Proposition 2.2 *The complex $\mathbf{Z}(n)$ has no cohomology sheaves in (cohomological) dimension greater than n.*

Together with cohomological dimension theorem in Nisnevich topology Proposition 2.2 and Theorem 2.1 imply the following result which will be widely used below.

Lemma 2.3 *Let X be a smooth variety of dimension d and Y be any smooth variety over k. Then for any $n > d$ one has*

$$Hom_{DM^{eff}(k)}(M(X), M(Y)(n)[2n]) = 0.$$

One defines motivic cohomology of any smooth scheme X over k setting

$$H^j_{\mathcal{M}}(X, \mathbf{Z}(i)) = Hom_{DM_{gm}^{eff}}(M(X), \mathbf{Z}(i)[j]).$$

The following results are direct corollaries of Theorem 2.1 and Proposition 2.2 respectively.

Corollary 2.4 *For any smooth scheme X over k there are canonical isomorphisms*

$$H^j_{\mathcal{M}}(X, \mathbf{Z}(i)) = Hom_{D(Shv_{Nis}(SmCor(k)))}(L(X), \mathbf{Z}(i)[j]) =$$

$$= \mathbf{H}^j_{Nis}(X, \mathbf{Z}(i)).$$

Corollary 2.5 *For any smooth scheme X one has $H^j_{\mathcal{M}}(X, \mathbf{Z}(i)) = 0$ for $j - i > dim(X)$.*

The following proposition presents a similar vanishing result which will be also useful below.

Proposition 2.6 *Let k be a field which admits resolution of singularities and X be a smooth scheme over k. Then for any $i \geq 0$ and any $j > 2i$ one has*

$$H^j(X, \mathbf{Z}(i)) = 0.$$

Using the motivic complex for finite coefficients $\mathbf{Z}/l(n) = \mathbf{Z}(n) \otimes \mathbf{Z}/l$ one defines the corresponding motivic cohomology with finite coefficients. We will also use the notations $H^j_{et}(X, \mathbf{Z}(i))$ (resp. $H^j_{et}(X, \mathbf{Z}/l(i))$) for motivic cohomology in the etale topology, i.e. for hypercohomology in the etale topology with coefficients in the complex of etale sheaves associated with $\mathbf{Z}(i)$ (resp. $\mathbf{Z}/l(i)$).

Below we usually denote the groups $H^j_{\mathcal{M}}(X, \mathbf{Z}(i))$ simply by $H^j(X, \mathbf{Z}(i))$.

We recall the original form of the Bloch-Kato conjecture. Let k be a field and $K^M_n(k)$ be the n-th Milnor K-group of k. Then for any l prime to $char(k)$ there is a well defined homomorphism

$$K^M_n(k) \to H^n_{et}(k, \mu_l^{\otimes n}).$$

The Bloch-Kato conjecture asserts that the induced homomorphism

$$K^M_n(k)/l \to H^n_{et}(k, \mu_l^{\otimes n})$$

is an isomorphism for any k, l and n. This conjecture is now known to be true for $n \leq 2$ ([3],[9]) and for $n = 3, l = 2$ ([4],[6]).

Milnor K-groups are related to motivic cohomology by the following result (see [11, Th.]).

Proposition 2.7 *For any field k there is a canonical isomorphism*

$$H^n(Spec(k), \mathbf{Z}(n)) = K^M_n(k).$$

Since $H^m(Spec(k), \mathbf{Z}(n)) = 0$ for $m > n$ we also have

$$H^n(Spec(k), \mathbf{Z}/l(n)) = K^M_n/l(k).$$

This relation between motivic cohomology and Milnor K-theory can be pushed a little further. Let us recall that for a discrete valuation ring R with the residue field k and the function field K there is a residue homomorphism (see [])

$$K_n^M(K) \to K_{n-1}^M(k).$$

This allows us to define a presheaf $\underline{K}_n^M$ on Sm/k such that for a smooth scheme U over k the group $\underline{K}_n^M(U)$ is the subgroup in $K_n^M(k(U))$ which is the intersection of kernels of the residue homomorphisms associated to all points of codimension 1 on U. One can verify that $\underline{K}_n^M$ is, in fact, a homotopy invariant Nisnevich sheaf with transfers. Proposition 2.7 implies than easily the following more precise result.

Proposition 2.8 *The sheaf $\underline{K}_n^M$ on Sm/k is canonically isomorphic to the cohomology sheaf $\underline{H}^n(\mathbf{Z}(n))$ of the complex $\mathbf{Z}(n)$.*

Corollary 2.9 *Let k be a field and X be a smooth scheme over k. Then there are canonical homomorphisms*

$$H^j(X, \mathbf{Z}(i)) \to H^{j-i}(X, \underline{K}_i^M)$$

which are isomorphisms for $j - i \geq dim(X)$.

Proof: There is a standard hypercohomology spectral sequence which converges to motivic cohomology and whose E_2-term consists of cohomology with coefficients in the cohomology sheaves of the complex $\mathbf{Z}(i)$. By Proposition 2.2 the first nontrivial cohomology sheaf of the complex $\mathbf{Z}(i)$ lies in (cohomological) dimension i. By Proposition 2.8 it is isomorphic to $\underline{K}_i^M$. For $j - i \geq dim(X)$ the cohomological dimension theorem in Nisnevich topology implies that the only group which contributes to $H^j(X, \mathbf{Z}(i))$ is $H^{j-i}(X, \underline{K}_i^M)$.

Remark: It can be shown that if k admits resolution of singularities the homomorphisms considered in Corollary 2.9 are also isomorphisms for $j \geq 2(i-1)$.

Proposition 2.7 gives an obvious way to reformulate the Bloch-Kato conjecture in terms of motivic cohomology and etale cohomology. To make this reformulation more natural we need the following fact.

Lemma 2.10 *For any field k and any l which is prime to $char(k)$ there is a quasi-isomorphism in the etale topology on Sm/k of the form*

$$\mathbf{Z}/l(n)_{et} \cong \mu_l^{\otimes n}.$$

Proof: This follows from the results of [10] if k admits resolution of singularities and from [15] in the general case.

Note that Lemma 2.10 implies that the motivic cohomology with finite coefficients in the etale topology are isomorphic to the usual etale cohomology

$$H^j_{et}(X, \mathbf{Z}/l(i)) = H^j_{et}(X, \mu_l^{\otimes i})$$

and modulo this fact the Bloch-Kato conjecture asserts that motivic cohomology of fields in Nisnevich and etale topology are the same in certain dimensions.

Let $\pi : (Sm/k)_{et} \to (Sm/k)_{Nis}$ be the obvious morphism of sites. Since $\mathbf{Z}/l(n)$ has no cohomology sheaves in dimension greater than n there is a canonical morphism of complexes:

$$BL_{n,l} : \mathbf{Z}/l(n) \to \tau^{\leq n}\mathbf{R}\pi_*\pi^*(\mu_l^{\otimes n})$$

where $\tau^{\leq n}\mathbf{R}\pi_*\pi^*(\mu_l^{\otimes n})$ is the part of the canonical filtration on $\mathbf{R}\pi_*\pi^*(\mu_l^{\otimes n})$ which consists of cohomology sheaves of degree less or equal to n. Note that for any X over k one has

$$\mathbf{H}^j_{Nis}(X, \tau^{\leq n}\mathbf{R}\pi_*\pi^*(\mu_l^{\otimes n})) = H^j_{et}(X, \mu_l^{\otimes n})$$

for $j \leq n$ and the canonical homomorphism

$$\mathbf{H}^{n+1}_{Nis}(X, \tau^{\leq n}\mathbf{R}\pi_*\pi^*(\mu_l^{\otimes n})) \to H^{n+1}_{et}(X, \mu_l^{\otimes n})$$

is a monomorphism.

A stronger version of the Bloch-Kato conjecture which is called the Beilinson-Lichtenbaum conjecture asserts that for any k and any l prime to $char(k)$ the morphism $BL_{n,l}$ is a quasi-isomorphism of complexes of sheaves in Nisnevich topology on Sm/k. In view of the isomorphisms given above the Beilinson-Lichtenbaum conjecture implies the original form of Bloch-Kato conjecture. The following theorem based on the main result of [11] shows that at least if k admits resolution of singularities the Bloch-Kato and Beilinson-Lichtenbaum conjectures are in fact equivalent.

Theorem 2.11 *Let k be a field which admits resolution of singularities and l be a prime not equal to $char(k)$. Denote by $\mathbf{Z}_{(l)}$ the localization of $\mathbf{Z}$ in (l) (i.e. all the primes but l are invertible in $\mathbf{Z}_{(l)}$). Then the following conditions are equivalent:*

1. *For any field F over k the canonical homomorphism*

$$K_n^M(F) \to H_{et}^n(Spec(F), \mu_l^{\otimes n})$$

   is surjective.

2. *The Beilinson-Lichtenbaum conjecture holds over k in weight n with $\mathbf{Z}/l$ coefficients, i.e. the morphism $BL_{n,l}$ is a quasi-isomorphism.*

3. *The canonical morphism*

$$\mathbf{Z}_{(l)}(n) \to \tau^{\leq n+1}\mathbf{R}\pi_*\pi^*(\mathbf{Z}_{(l)}(n))$$

   is a quasi-isomorphism in Nisnevich topology.

4. *For any field F over k one has*

$$H_{et}^{n+1}(Spec(F), \mathbf{Z}_{(l)}(n)) = 0.$$

Proof: **(1=>2)** It is the main theorem of [11].
(2=>3) Due to the fact that both complexes are complexes of presheaves with transfers with homotopy invariant cohomology sheaves it is sufficient to verify that for a field F over k the obvious homomorphisms

$$H^i(Spec(F), \mathbf{Z}_{(l)}(n)) \to H_{et}^i(Spec(F), \mathbf{Z}_{(l)}(n))$$

are isomorphisms for $i \leq n+1$. It holds for $\mathbf{Q}_l$-coefficients by the comparison result [13]. By our assumption it also holds for $\mathbf{Z}/l$ (and, therefore for $\mathbf{Q}_l/\mathbf{Z}_l$) coefficients for $i \leq n$.

It remains to show that $H_{et}^{n+1}(Spec(F), \mathbf{Z}_{(l)}(n)) = 0$. Any element a satisfies the condition $l^k a = 0$ for some k since the corresponding group is zero with rational coefficients. We may assume that $la = 0$, i.e. a belongs to the image of the last homomorphism in the exact sequence

$$H_{et}^n(Spec(F), \mathbf{Z}_{(l)}(n)) \to H_{et}^n(Spec(F), \mathbf{Z}/l) \to H_{et}^{n+1}(Spec(F), \mathbf{Z}_{(l)}(n)).$$

By our assumption the left hand side group is isomorphic to $K_n^M(F)/l$ and therefore the homomorphism

$$H^n_{et}(Spec(F), \mathbf{Z}_{(l)}(n)) \to H^n_{et}(Spec(F), \mathbf{Z}/l)$$

is surjective which implies that $a = 0$.
(3=>4) It follows immediately from Proposition 2.2.

(4=>1): The condition (4) implies in particular that all the Bokstein homomorphisms

$$H^n_{et}(Spec(F), \mathbf{Z}/l(n)) \to H^{n+1}_{et}(Spec(F), \mathbf{Z}/l^k(n))$$

are zero. It implies the Beilinson-Lichtenbaum conjecture by 2.10 and [11].

We will further refer to the equivalent conditions of Theorem 2.11 as to $BK(n, l)$ over k. The main goal of this paper is to prove that Conjecture 2 implies that $BK(n, 2)$ holds for all fields k which admit resolution of singularities.

The following lemma is a formal corollary of the condition (3) of Theorem 2.11.

Lemma 2.12 *Assume that $BK(n, l)$ holds and let $\underline{K}$ be a complex of Nisnevich sheaves with transfers concentrated in negative cohomological degrees. Then for any $j \leq n + 1$ the homomorphism*

$$Hom(\underline{K}, \mathbf{Z}_{(l)}(n)[j]) \to Hom(\underline{K}_{et}, \mathbf{Z}_{(l)}(n)[j])$$

is an isomorphism and for $j = n+2$ it is a monomorphism. (Here the Hom-groups on the left hand side are morphisms in the derived category of sheaves with transfers in Nisnevich topology and the Hom-groups on the right hand side are morphisms in the derived category of etale sheaves with transfers)

Lemma 2.13 *Assume that $BK(n, l)$ holds over k. Then Theorem Hilbert 90 holds for K_n^M over k, i.e. for any field F over k which contains a primitive root of unit of degree l and a cyclic extension E/F of degree l the sequence*

$$K_n^M(E) \stackrel{1-\sigma}{\to} K_n^M(E) \stackrel{N_{E/k}}{\to} K_n^M(F)$$

(where σ is a generator of $Gal(E/k)$) is exact.

Proof: We may clearly assume that $F = k$ and replace $\mathbf{Z}$ by $\mathbf{Z}_{(l)}$. For any finite separable field extension $p : Spec(k') \to Spec(k)$ the presheaf with transfers $L(Spec(k'))$ on Sm/k is canonically isomorphic to the direct image $p_*(\mathbf{Z})$. In particular there is a complex of presheaves with transfers of the form

$$0 \to L(k) \to L(E) \stackrel{1-\sigma}{\to} L(E) \to L(k) \to 0$$

Denote this complex with the last $L(k)$ placed in degree zero by $\underline{K}$. It is clearly exact in the etale topology and therefore Lemma 2.12 implies that

$$Hom(\underline{K}, \mathbf{Z}_{(l)}(n)[n+2]) = 0$$

in the derived category of Nisnevich sheaves with transfers. The standard spectral sequence which converges to this group from morphisms

$$Hom(L(-), \mathbf{Z}_{(l)}(n)[j]) = H^j(-, \mathbf{Z}_{(l)}(n))$$

together with Proposition 2.7 implies the result we need.

We will often use below the following very simple fact.

Proposition 2.14 *Let $a \in K_n^M(k), b \in K_m^M(k)$ be two elements of Milnors K-theory and suppose that there is a finite field extension E/k such that a is divisible by l in $K_n^M(E)$ and b belongs to the image of the norm homomorphism $K_m^M(E) \to K_m^M(k)$. Then the product $a \wedge b$ is divisible by l in $K_{n+m}(k)$.*

3 The approach to Bloch-Kato conjecture based on norm varieties.

The main goal of this section is to prove Proposition 3.8 which is the basis of our approach to the Bloch-Kato problem. To do it we have to prove several corollaries of $BK(n, l)$ which seem to be of independent interest. While most of the results of this section can be proven simplier in the case of $\mathbf{Z}/2$-coefficients we give the general proofs for the possible future applications to $BK(-, l)$ with $l > 2$. We formulate most of our results for fields which admit resolution of singularities since we want to be able to use Theorem 2.11. One can easily see however that most of the results of this section do not really depend on this assumption.

Let k be a field, l be a prime not equal to $char(k)$. We fix an algebraic closure $\bar{k}$ of k. Assume that k has no extensions of degree prime to l. Then there exists a primitive root of unit ξ of degree l in k. Let E/k be a cyclic extension of k of degree l in $\bar{k}$. We have $E = k(b)$ where $b^l = a$ for an element a in k^*. We denote by σ_ξ the generator of the Galois group $G_b = Gal(E/k)$ which acts on b by multiplication on ξ and by $[a]_\xi$ the class in $H^1_{et}(k, \mathbf{Z}/l)$ which corresponds to the homomorphism $Gal(\bar{k}/k) \to G \to \mathbf{Z}/l$ which takes σ_ξ to the canonical generator of $\mathbf{Z}/l$ (one can easily see that this class is indeed determined by a and ξ and does not depend on b).

Let $p : Spec(E) \to Spec(k)$ be the projection. Consider the etale sheaf $F = p_*(\mathbf{Z}/l)$. The group G acts on F in the natural way. Denote by F_i the kernel of the homomorphism $(1-\sigma)^i : F \to F$. One can verify easily that $F_i = Im(1-\sigma)^{l-i}$ and that as a $\mathbf{Z}/l[Gal(\bar{k}/k)]$-module F_i has dimension i. In particular we have $F = F_l$. There are exact sequences of the form:

$$0 \to \mathbf{Z}/l \to F_i \stackrel{u_i}{\to} F_i \to \mathbf{Z}/l \to 0$$

where $u_i = 1 - \sigma$ and $Im(u_i) = F_{i-1}$.

These exact sequences define certain elements α_i in

$$Ext^2_{\mathbf{Z}/l[Gal(\bar{k}/k)]}(\mathbf{Z}/l, \mathbf{Z}/l) = H^2(k, \mathbf{Z}/l).$$

Lemma 3.1 *The element α_l equals to $c[a]_\xi \wedge [\xi]_\xi$ where c is an invertible element in $\mathbf{Z}/l$ and $\alpha_i = 0$ for $i < l$.*

Proof: The fact that $\alpha_i = 0$ for $i < l$ follows from the commutativity of the diagram

$$\begin{array}{ccccccccc} 0 \to & \mathbf{Z}/l & \to & F_{i+1} & \to & F_{i+1} & \to & \mathbf{Z}/l & \to 0 \\ & 0 \downarrow & & \downarrow & & \downarrow & & \downarrow Id & \\ 0 \to & \mathbf{Z}/l & \to & F_i & \to & F_i & \to & \mathbf{Z}/l & \to 0 \end{array}$$

To compute α_l note first that since the action of $Gal(\bar{k}/k)$ on $F = F_l$ factors through $G = Gal(E/k)$ it comes from a well defined element in $H^2(G, \mathbf{Z}/l)$. This element is not zero for trivial reasons. On the other hand the group

$$H^2(G, \mathbf{Z}/l) = H^2(\mathbf{Z}/l, \mathbf{Z}/l) = \mathbf{Z}/l$$

is generated by the element $\beta(\gamma)$ where γ is the canonical generator of $H^1(G, \mathbf{Z}/l)$ and β is the Bokstein homomorphism. Thus we conclude that up to multiplication by an invertible element of $\mathbf{Z}/l$ our class α_l equals $\beta([a]_\xi)$. It remains to show that

$$\beta([a]_\xi) = c[a]_\xi \wedge [\xi]_\xi.$$

It follows by simple explicite computations from the fact that $[a]_\xi$ has a lifting to an element of $H^1_{et}(Spec(k), \mu_{l^2})$.

Lemma 3.2 *Let k be a field which admits resolution of singularities and assume that $BK(n,l)$ holds over k. Then for all $m \leq n$ and all $i = 1, \ldots, l-1$ one has:*

1. *The sequence*

$$H^m_{et}(k, \mathbf{Z}/l) \oplus H^m_{et}(k, F_{i+1}) \to H^m_{et}(k, F_{i+1}) \to H^m_{et}(k, \mathbf{Z}/l)$$

where the first homomorphism is given on the second summand by $1-\sigma$ is exact.

2. *The homomorphisms*

$$\nu_{m,i} : H^m_{et}(k, \mathbf{Z}/l) \oplus H^m_{et}(k, F_{i+1}) \to H^m_{et}(k, F_i)$$

given by the canonical morphisms $\mathbf{Z}/l \to F_i$, $F_{i+1} \to F_i$ are surjective.

Proof: We proceed by induction on i. Consider first the case $i = l-1$. The first statement follows immediately from Lemma 2.13 and our assumption that $BK(n,l)$ holds.

Let us prove the second one. The image of $H^m_{et}(k, F_{l-1})$ in $H^m_{et}(k, F_l) = H^m_{et}(E, \mathbf{Z}/l)$ coincides with the kernel of the norm homomorphism

$$H^m_{et}(E, \mathbf{Z}/l) \to H^m_{et}(k, \mathbf{Z}/l).$$

The first statement implies then that $H^m_{et}(k, \mathbf{Z}/l) \oplus H^m_{et}(k, F_{i+1})$ maps surjectively to this image. It is sufficient therefore to show that an element $\gamma \in H^m_{et}(k, F_{l-1})$ which goes to zero in $H^m_{et}(k, F_l)$ belongs to the image of $\nu_{m,l-1}$. Any such element is a composition of a cohomology class in $H^{m-1}_{et}(k, \mathbf{Z}/l)$ with the canonical extension

$$0 \to F_{l-1} \to F_l \to \mathbf{Z}/l \to 0$$

Thus we may assume that $m = 1$ and γ is the element which corresponds to this extension. Let further δ be the image of $c[\xi]_\xi$ (where c is as in Lemma 3.1) under the homomorphism $H^1_{et}(k, \mathbf{Z}/l) \to H^1_{et}(k, F_{l-1})$. The composition

$$H^1_{et}(k, \mathbf{Z}/l) \to H^1_{et}(k, F_{l-1}) \to H^2_{et}(k, \mathbf{Z}/l)$$

where the later homomorphism corresponds to the extension

$$0 \to \mathbf{Z}/l \to F_l \to F_{l-1} \to 0$$

equals to multiplication by $[a]_\xi$. We conclude now by Lemma 3.1 that the image of $\gamma - \delta$ in $H^2(k, \mathbf{Z}/l)$ is zero. Then it lifts to $H^1_{et}(k, F_l)$ which proves our Lemma in the case $i = l - 1$.

Suppose that the lemma is proven for all $i > j$. Let us show that it holds for $i = j$. The first statement follows immediately from the inductive assumption and the commutativity of the diagram

$$\begin{array}{ccc} F_{j+2} & \stackrel{1-\sigma}{\to} & F_{j+2} \\ \downarrow & & \downarrow \\ F_{j+1} & \stackrel{1-\sigma}{\to} & F_{j+1} \end{array}$$

The proof of the second one is now similar to the case $i = l - 1$ with a simplification due to the fact that $\alpha_i = 0$ for $i < l$ (Lemma 3.1).

Lemma 3.3 *Let k be a field which admits resolution of singularities and has no extensions of degree prime to l. Assume that $BK(n-1, l)$ holds over k. Then for any cyclic extension E/k of k of degree l there is an exact sequence of the form*

$$H^i_{et}(E, \mathbf{Z}/l) \stackrel{N_{E/k}}{\to} H^i_{et}(k, \mathbf{Z}/l) \stackrel{-\wedge[a]}{\to} H^{i+1}_{et}(k, \mathbf{Z}/l) \to H^{i+1}_{et}(E, \mathbf{Z}/l)$$

where $[a] \in H^1_{et}(k, \mathbf{Z}/l)$ is the class which corresponds to E/k.

Proof: It follows trivially from Lemma 3.2.

Lemma 3.4 *Let k be a field which admits resolution of singularities and l be a prime not equal to $char(k)$. Assume that $BK(n,l)$ holds over k for some $n > 0$. Assume further that k has no extensions of degree prime to l and E/k is a cyclic extension of degree l such that the norm homomorphism $K_n^M(E) \to K_n^M(k)$ is surjective. Then the sequence*

$$K_{n+1}^M(E) \stackrel{1-\sigma}{\to} K_{n+1}^M(E) \stackrel{N_{E/k}}{\to} K_{n+1}^M(k)$$

where σ is a generator for $Gal(E/k)$ is exact.

Proof: It is essentially a version of the proof given in [9] for $n = 2$ and in [4] for $p = 3$. Let us define a homomorphism

$$\phi : K_{n+1}^M(k) \to K_{n+1}^M(E)/(Im(1-\sigma))$$

as follows. Let a be an element in $K_{n+1}^M(k)$ of the form $(a_0, \ldots, a_n)$ and let b be an element in $K_{n-1}^M(E)$ such that

$$N_{E/k}(b) = (a_0, \ldots, a_{n-1}).$$

We set $\phi(a) = ba_n$. By Lemma 2.13 the element $\phi(a)$ in $K_{n+1}^M(E)/(Im(1-\sigma))$ does not depend on the choice of b and one can easily see that ϕ is a homomorphism from $(k^*)^{\otimes n+1}$ to $K_{n+1}^M(E)/(Im(1-\sigma))$. To show that it is a homomorphism from $K_n^M(k)$ it is sufficient to verify that ϕ takes an element of the form $(a_0, \ldots, a_n)$ such that say $a_0 + a_n = 1$ to zero. Let b be a preimage of $(a_0, \ldots, a_{n-1})$ in $K_n^M(k)$. We have to show that $(b, a_n) \in (1-\sigma)K_{n+1}^M(E)$. Assume first that a_0 is not in $(k^*)^l$ and let c be an element in $\bar{k}^*$ such that $c^l = a_0$. Let further $F = k(c)$. Then one has:

$$ba_n = b(1-a_0) = N_{EF/E}(b_{EL}(1-c)) =$$

$$= N_{EF/E}((b - (c, a_1, \ldots, a_n - 1))(1-c)) \in (1-\sigma)K_{n+1}^M(E)$$

since $N_{EF/F}(b - (c, a_1, \ldots, a_n - 1)) = 0$ we have by Lemma 2.13 and the assumption that $b - (c, a_1, \ldots, a_n - 1) \in (1-\sigma)K_n^M(EF)$ which implies that $(b - (c, a_1, \ldots, a_n - 1))(1-c) \in (1-\sigma)K_{n+1}^M(E)$. The proof for the case when $a_0 \in (k^*)^l$ is similar.

Clearly ϕ is a section for the obvious morphism $K_{n+1}^M(E)/(Im(1-\sigma)) \to K_{n+1}^M(k)$. It remains to show that it is surjective. It follows immediately from the fact that under our assumption on k the group $K_{n+1}^M(E)$ is generated by symbols of the form $(b, a_1, \ldots, a_n)$ where $b \in E^*$ and $a_1, \ldots, a_n \in k^*$ (see [1]).

Lemma 3.5 *Let k be a field which admits resolution of singularities and has no extensions of degree prime to l. Suppose that $BK(n-1,l)$ holds over k. Then $K_n^M(k) = lK_n^M(k)$ if and only if for any cyclic extension E/k the norm homomorphism $K_{n-1}^M(E) \to K_{n-1}^M(k)$ is surjective.*

Proof: The if part is trivial. Since $BK(n-1,l)$ holds we conclude that there is a commutative square with surjective horizontal arrows of the form

$$\begin{array}{ccc} K_{n-1}^M(E) & \to & H^{n-1}(E, \mathbf{Z}/l) \\ \downarrow & & \downarrow \\ K_{n-1}^M(k) & \to & H^{n-1}(k, \mathbf{Z}/l) \end{array}$$

and thus the cokernel of the left vertical arrow is the same as the cokernel of the right one. By Lemma 3.3 it gives us an exact sequence

$$K_{n-1}^M(E) \to K_{n-1}^M(k) \to H^n(k, \mathbf{Z}/l)$$

and since the last arrow clearly factors through $K_n^M(k)/l$ it is zero. Lemma is proven.

Lemma 3.6 *Let k be a field which admits resolution of singularities, has no extensions of degree prime to l and such that $K_n^M(k) = lK_n^M(k)$. Assume further that $BK(n-1,l)$ holds over k. Then for any finite extension E/k one has $K_n^M(E) = lK_n^M(E)$.*

Proof: It is a variant of the proof given in [9] for $n = 2$. By Lemma 3.3 and Lemma 3.4 we have an exact sequence

$$K_n^M(E) \stackrel{1-\sigma}{\to} K_n^M(E) \stackrel{N_{E/k}}{\to} K_n^M(k).$$

Let α be an element in $K_n^M(E)$ and let $\beta \in K_n^M(k)$ be an element such that $N_{E/k}(\alpha) = l\beta$. Then $N_{E/k}(\alpha - \beta_E) = 0$ and we conclude that the endomorphism

$$1 - \sigma : K_n^M(E)/l \to K_n^M(E)/l$$

is surjective. Since $(1-\sigma)^l = 0$ it implies that $K_n^M(E)/l = 0$.

Theorem 3.7 *Let k be as in Lemma 3.6. Then $cd_l(k) \leq n-1$.*

Proof: Let α be an element of $H^n_{et}(k, \mathbf{Z}/l)$. We have to show that $\alpha = 0$. By Lemma 3.6 and obvious induction we may assume that α vanishes on a cyclic extension of k. Then by Lemma 3.3 it is of the form $\alpha_0 a$ where $a \in H^1_{et}(k, \mathbf{Z}/l)$ is the element which represents our cyclic extension. Thus since $BK(n-1, l)$ holds it belongs to the image of the homomorphism $K^M_n(k)/l \to H^n_{et}(k, \mathbf{Z}/l)$ and therefore is zero.

Theorem 3.7 suggests the following approach to the Bloch-Kato problem.

Proposition 3.8 *Let k be a field which admits resolution of singularities and l be a prime not equal to $char(k)$. Suppose that $BK(n-1, l)$ holds over k and that for any field F over k and any sequence of elements $\underline{a} = (a_1, \ldots, a_n)$ in F^* there exists a variety $X_{\underline{a}}$ over F satisfying the following conditions:*

1. *$\underline{a}$ is divisible by l in $K^M_n(F(X_{\underline{a}}))$.*
2. *The homomorphism*

$$H^{n+1}_{et}(Spec(F), \mathbf{Z}_{(l)}(n)) \to H^{n+1}_{et}(Spec(F(X_{\underline{a}})), \mathbf{Z}_{(l)}(n))$$

   is injective.

Then $BK(n, l)$ holds over k.

Proof: In view of Theorem 2.11 it is sufficient to show that for any field F over k the group $H^{n+1}_{et}(Spec(F), \mathbf{Z}_{(l)}(n))$ is zero. Let α be an element of this group. In view of Lemma 2.10 $H^{n+1}_{et}(Spec(L), \mathbf{Z}_{(l)}(n)) = 0$ for any field L over k of cohomological dimension $\leq n-1$. Thus by Theorem 3.7 there exists a sequence of pairs $(F_i, \underline{a}_i)_{i=0,\ldots,k}$ where $F_0 = F$ and $\underline{a}_i$ is a sequence $(a_{1i}, \ldots, a_{ni})$ of elements in F_i^* such that

1. F_{i+1} is a finite extension of $F_i(X_{\underline{a}_i})$ of degree prime to l.
2. α is zero in $H^{n+1}_{et}(Spec(F_k), \mathbf{Z}_{(l)}(n))$.

which immediately implies the result we need.

We will use further the following simple fact.

Lemma 3.9 *Let k be a field which admits resolution of singularities and suppose that $BK(n-1,l)$ holds over k. Let further X be a smooth variety over k. Then an element $\alpha \in H_{et}^{n+1}(Spec(k), \mathbf{Z}_{(l)}(n))$ goes to zero in $k(X_a)$ if and only if its image in $H_{et}^{n+1}(X_a, \mathbf{Z}_{(l)}(n))$ comes from $H^{n+1}(X_a, \mathbf{Z}_{(l)}(n))$.*

Proof: The "if" part is trivial (by Corollary 2.5). To prove the "only if" part we may assume using obvious induction that there is a smooth closed subscheme Z in X such that the restriction of α to $X - Z$ lies in the image of $H^{n+1}(U, \mathbf{Z}_{(l)}(n))$. Then our statement follows from the assumption that $BK(n-1,l)$ holds and the Gysin exact sequence associated with the smooth pair (Z,X) (see [11]).

4 Pfister quadrics and their motives.

In this section we describe explicitly norm varieties for $l = 2$. We assume everywhere that k is a field of characteristic not equal 2 which admits resolution of singularities.

Let k be such a field. For any elements $a_1, \ldots, a_n$ in k^* let $< a_1, \ldots, a_n >$ be the quadratic form $\sum a_i x_i^2$. Let further

$$<< a_1, \ldots, a_n >>=< 1, -a_1 > \otimes \ldots \otimes < 1, -a_n > .$$

The quadratic forms $<< a_1, \ldots, a_n >>$ are called Pfister forms. We denote by $Q_{\underline{a}} = Q_{a_1,\ldots,a_n}$ the projective quadric of dimension $2^{n-1} - 1$ given by the equation $<< a_1, \ldots, a_{n-1} >>= a_n t^2$. This quadric is called the norm quadric associated with the sequence $(a_1, \ldots, a_n)$. The most important property of Pfister forms is given by the following theorem (see []).

Theorem 4.1 *Let $<< a_1, \ldots, a_n >>$ be a Pfister form which represents zero. Then it is hyperbolic.*

We are going to show now that if $Q_{\underline{a}}$ has a rational point over k then $\underline{a}$ is divisible by 2 in $K_n^M(k)$. In particular it implies that $\underline{a}$ is divisible by 2 in the generic point of $Q_{\underline{a}}$. We need first the following result.

Lemma 4.2 *For any $\underline{a} = (a_1, \ldots, a_n)$ the following two conditions are equivalent*

1. *$Q_{\underline{a}}$ has a k-rational point (i.e. the form*

$$<< a_1, \ldots, a_{n-1} >> \oplus < -a_n >$$

represents zero over k).

2. *The form $<< a_1, \ldots, a_n >>$ represents zero over k.*

Proof: Since $<< a_1, \ldots, a_{n-1} >> \oplus < -a_n >$ is a subform in $<< a_1, \ldots, a_n >>$ the first condition implies the second one for obvious reasons. Let us show that if $<< a_1, \ldots, a_n >>$ represents zero over k then so does $<< a_1, \ldots, a_{n-1} >> \oplus < -a_n >$. Denote the quadric given by the equation $<< a_1, \ldots, a_n >>= 0$ by $P_{\underline{a}}$. By Theorem 4.1 we may assume that $<< a_1, \ldots, a_n >>$ is hyperbolic. The quadric $Q_{\underline{a}}$ is a section of $P_{\underline{a}}$ by a linear subspace L of codimension $2^{n-1} - 1$ in $\mathbf{P}^{2^n - 1}$. On the other hand since $P_{\underline{a}}$ is hyperbolic for any rational point p of $P_{\underline{a}}$ there exists a linear subspace H of dimension $dim(P_{\underline{a}})/2 = 2^{n-1} - 1$ which lies on $P_{\underline{a}}$ and passes through p. The intersection of H and L is a point on $Q_{\underline{a}}$.

Proposition 4.3 *Let $\underline{a} = (a_1, \ldots, a_n)$ be a symbol such that the quadric $Q_{\underline{a}}$ has a rational point. Then $\underline{a}$ is divisible by 2 in $K_n^M(k)$.*

Proof: We proceed by induction on n. Consider first the case $n = 2$. Then $Q_{\underline{a}}$ is given by the equation $x^2 - a_1 y^2 = a_2 z^2$. We may assume that it has a point of the form $(x_0, y_0, 1)$ (otherwise a_1 is a square root in k and the statement is obvious). Then a_2 is the norm of the element $x_0 + a_1^{1/2} y_0$ from $k(a_1^{1/2})$ and thus the symbol (a_1, a_2) is divisible by 2 by Proposition 2.14.

Suppose that the lemma is proven for weights smaller than n. The quadric $Q_{\underline{a}}$ is given by the equation $<< a_1, \ldots, a_{n-1} >>= a_n t^2$. The form $<< a_1, \ldots, a_{n-1} >>$ is of the form $< 1 > \oplus \mu_{\underline{a}}$. By induction we may assume that our point $q \in Q_{\underline{a}}(k)$ belongs to the affine part $t \neq 0$. Consider the plane L generated by points $(1, 0, \ldots, 0)$ and q. The restriction of the quadratic form $<< a_1, \ldots, a_{n-1} >>$ to L is of the form $<< b >>$ for some b (the idea is that L is a "subfield" in the vector space where $<< a_1, \ldots, a_{n-1} >>$ lives). Consider the field extension $k(b^{1/2})$. The form $<< b >>$ and therefore the form $<< a_1, \ldots, a_{n-1} >>$ represents zero over $k(b^{1/2})$ and thus by the inductive assumption $(a_1, \ldots, a_{n-1}) = 0$ in $K_{n-1}^M(k(b^{1/2}))/2$. On the other hand by the construction $<< b >>$ represents a_n and therefore we have

$a_n \in ImN_{k(b^{1/2})/k} \subset k^*$ which proves the proposition in view of Proposition 2.14.

In fact the opposite assertion to the one proven in Proposition 4.3 also holds. We give this fact here without proof since we do not use it in this paper.

Proposition 4.4 *Let k be a field and $\underline{a} = (a_1, \ldots, a_n)$ be a sequence of invertible elements of k such that $\underline{a}$ is divisible by 2 in $K_n^M(k)$. Then the quadric $Q_{\underline{a}}$ has a k-rational point.*

For any scheme X over k denote by $\check{C}(X)$ the simplicial scheme such that $\check{C}(X)_m = X^m$ and faces and degeneracy morphisms are given by partial projections and diagonals respectively. For any symbol $\underline{a} = (a_1, \ldots, a_n)$ denote $\check{Q}_{\underline{a}}$ by $\mathcal{X}_{\underline{a}}$. The main result of this section is the following theorem.

Theorem 4.5 *For any $n > 1$ there exists a motivic cohomology class*

$$\mu^{\underline{a}} \in H^{2^n-1}(\mathcal{X}_{\underline{a}}, \mathbf{Z}(2^{n-1}-1))$$

such that the object $M_{\underline{a}}$ of the category $DM_-^{eff}(k)$ defined by the distinguished triangle

$$M(\mathcal{X}_{\underline{a}})(2^{n-1}-1)[2^n-2] \to M_{\underline{a}} \to M(\mathcal{X}_{\underline{a}}) \stackrel{Id\wedge\mu^{\underline{a}}}{\to} M(\mathcal{X}_{\underline{a}})(2^{n-1}-1)[2^n-1]$$

is a direct summand of $M(Q_{\underline{a}})$.

Before we proceed to the proof of this theorem let us note that the main "philosophical" reason why this result is important is that unlike the quadric $Q_{\underline{a}}$ itself the object $\mathcal{X}_{\underline{a}}$ has a direct description in terms of the symbol $\underline{a}$. More precisely one has the following result (which is left without proof since we do not make any use of it below).

Proposition 4.6 *Considered as an object of the homotopy category of simplicial sheaves in Nisnevich topology on Sm/k, the simplicial scheme $\mathcal{X}_{\underline{a}}$ is canonically isomorphic to the sheaf of sets $\Phi_{\underline{a}}$ of the form*

$$\Phi_{\underline{a}}(U) = \begin{cases} \emptyset & \text{if } \underline{a} \neq 0 \text{ in } K_n^M(k(U))/2 \\ \{\emptyset\} & \text{if } \underline{a} = 0 \text{ in } K_n^M(k(U))/2 \end{cases}$$

where $k(U)$ is the function field of a smooth connected scheme U.

We will show below (see Proposition 6.5) that the element $\mu^{\underline{a}}$ which together with $\mathcal{X}_{\underline{a}}$ determines the motive $M_{\underline{a}}$ also has an abstract definition (at least modulo the existence of the hypothetical algebraic Morava K-theories).

Our proof of Theorem 4.5 is based on the following important result which is due to M. Rost (see [8]).

Theorem 4.7 (M. Rost) *There exists a direct summand $M_{\underline{a}}$ of the motive $M(Q_{\underline{a}})$ together with two morphisms*

$$\psi^* : \mathbf{Z}(2^{n-1}-1)[2^n-2] \to M_{\underline{a}}$$

$$\psi_* : M_{\underline{a}} \to \mathbf{Z}$$

such that for any field F over k where $Q_{\underline{a}}$ has a point the sequence

$$\mathbf{Z}(2^{n-1}-1)[2^n-2] \to (M_{\underline{a}})_F \to \mathbf{Z}$$

is a splitting distinguished triangle in $DM^{eff}(F)$.

The compositions

$$\mathbf{Z}(2^{n-1}-1)[2^n-2] \to M_{\underline{a}} \to M(Q_{\underline{a}})$$

$$M(Q_{\underline{a}}) \to M_{\underline{a}} \to \mathbf{Z}$$

are equal to the fundamental class and the canonical morphism $M(Q_{\underline{a}}) \to \mathbf{Z} = M(Spec(k)$ respectively.

Remark: Theorem 4.7 is formulated in [8] in the context of Chow motives. It implies the same result in the category $DM^{eff}(k)$ by [15].

Theorem 4.5 is, in fact, a rather formal corollary of Theorem 4.7. We start with the following lemmas.

Lemma 4.8 *The sequence of morphisms*

$$M(Q_{\underline{a}})(2^{n-1}-1)[2^n-2] \stackrel{Id\otimes\psi^*}{\to} M(Q_{\underline{a}}) \otimes M_{\underline{a}} \stackrel{Id\otimes\psi_*}{\to} M(Q_{\underline{a}})$$

is a splitting distinguished triangle.

Proof: Let *cone* be a cone of the morphism $\psi^* : \mathbf{Z}(2^{n-1}-1)[2^n-2] \to M_{\underline{a}}$. The morphism $\psi_* : M_{\underline{a}} \to \mathbf{Z}$ factors through a well defined morphism $\phi : cone \to \mathbf{Z}$. By Theorem 4.7 ϕ is an isomorphism in the generic point of $Q_{\underline{a}}$. It implies immediately by general properties of category $DM^{eff}(k)$ that $\phi \otimes Id_{M(Q_{\underline{a}})}$ is an isomorphism. Thus the sequence

$$M(X)(2^{n-1}-1)[2^n-2] \to M(X) \otimes M_{\underline{a}} \to M(X)$$

can be extended to a distinguished triangle. This triangle splits since

$$Hom(M(X), M(X)(2^{n-1}-1)[2^n-1]) = 0$$

by Proposition 2.6.

Lemma 4.9 *Consider the canonical morphism $M(\mathcal{X}_{\underline{a}}) \to \mathbf{Z}$. Let further N be an object of the thick (resp. localizing) subcategory in $DM^{eff}(k)$ generated by objects of the form $M(Q_{\underline{a}}) \otimes L$. Then this morphism induces isomorphisms*

$$Hom(\mathbf{Z}, N(i)[j]) \to Hom(M(\mathcal{X}_{\underline{a}}), N(i)[j])$$

(resp. isomorphisms

$$Hom(N(i)[j], M(\mathcal{X}_{\underline{a}})) \to Hom(N(i)[j], \mathbf{Z})$$

) for all $i, j \in \mathbf{Z}$[3] *.*

Proof: It is clearly sufficient to consider the case $N = M(Q_{\underline{a}}) \otimes L$. Then it follows from duality and the obvious fact that

$$M(\mathcal{X}_{\underline{a}}) \otimes M(Q_{\underline{a}}) \to M(Q_{\underline{a}})$$

is an isomorphism.

Lemma 4.9 implies in particular that

$$Hom(M(\mathcal{X}_{\underline{a}}), M(\mathcal{X}_{\underline{a}})(i)[j]) = Hom(M(\mathcal{X}_{\underline{a}}), \mathbf{Z}(i)[j]).$$

[3]The thick (resp. localizing) subcategory in a triangulated category $\mathcal{D}$ generated by a class of objects A is the minimal triangulated subcategory in $\mathcal{D}$ which is closed under direct summands (resp. direct summands and arbitrary direct sums) and contains A.

In particular any morphism

$$M(\mathcal{X}_{\underline{a}}) \to M(\mathcal{X}_{\underline{a}})(2^{n-1}-1)[2^n-1]$$

is of the form $Id \wedge \mu^{\underline{a}}$ for some motivic cohomology class $\mu^{\underline{a}}$ of $\mathcal{X}_{\underline{a}}$. Therefore to prove Theorem 4.5 it is sufficient to verify that the Rost motive $M_{\underline{a}}$ can be included into a distinguished triangle of the form

$$M(\mathcal{X}_{\underline{a}})(2^{n-1}-1)[2^n-2] \to M_{\underline{a}} \to M(\mathcal{X}_{\underline{a}}) \to M(\mathcal{X}_{\underline{a}})(2^{n-1}-1)[2^n-1]$$

Using again Lemma 4.9 we see that there is a sequence of morphisms

$$M(\mathcal{X}_{\underline{a}})(2^{n-1}-1)[2^n-2] \to M_{\underline{a}} \to M(\mathcal{X}_{\underline{a}})$$

whose composition equals zero. Let *cone* be a cone of the first morphism. Then the morphism

$$M_{\underline{a}} \to M(\mathcal{X}_{\underline{a}})$$

factors through a unique morphism

$$\phi : cone \to M(\mathcal{X}_{\underline{a}})$$

and we only have to show that ϕ is an isomorphism. The category of objects N such that $cone \otimes Id_N$ is an isomorphism is a localizing subcategory. Since $M(\mathcal{X}_{\underline{a}}) \otimes M(Q_{\underline{a}}) = M(Q_{\underline{a}})$ we conclude from Lemma 4.8 that $M(Q_{\underline{a}})$ belongs to this subcategory. Then $M(\mathcal{X}_{\underline{a}})$ also belongs to this subcategory. On the other hand we have a commutative diagram

$$\begin{array}{ccc} cone \otimes M(\mathcal{X}_{\underline{a}}) & \to & M(\mathcal{X}_{\underline{a}}) \otimes M(\mathcal{X}_{\underline{a}}) \\ \downarrow & & \downarrow \\ cone & \to & M(\mathcal{X}_{\underline{a}}) \end{array}$$

with both vertical arrows being isomorphisms. This finishes the proof of Theorem 4.5.

Corollary 4.10 *In the notations of Theorem 4.5 one has*

$$H^{2^n-1}(\mathcal{X}_{\underline{a}}, \mathbf{Z}(2^{n-1}-1)) = \begin{cases} 0 & \text{if } Q_{\underline{a}} \text{has a zero cycle sof degree 1} \\ \mathbf{Z}/2 & \text{otherwise} \end{cases}$$

In particular a class $\mu^{\underline{a}}$ satisfying the conditions of Theorem 4.5 is unique.

Proof: By Lemma 2.6 and Theorem 4.5 we have

$$Hom(M_{\underline{a}}, \mathbf{Z}(2^{n-1}-1)[2^n-1]) = 0.$$

Therefore the homomorphism

$$-\wedge \mu^{\underline{a}} : H^0(\mathcal{X}_{\underline{a}}, \mathbf{Z}) \to H^{2^n-1}(\mathcal{X}_{\underline{a}}, \mathbf{Z}(2^{n-1}-1))$$

is surjective. The left hand side group is $\mathbf{Z}$ and the right hand side group is of order 2 for transfers reasons since it is clearly zero if $\mathbf{Q}_{\underline{a}}$ has a point. It implies easily that it is zero if $Q_{\underline{a}}$ has a 0-cycle of degree 1. It remains to show that it is not zero otherwise. If it were zero we would have $\mu^{\underline{a}} = 0$ and thus the morphism $M_{\underline{a}} \to \mathcal{X}_{\underline{a}}$ would split. By Lemma 4.9 it would imply that there is a morphism $\mathbf{Z} \to M_{\underline{a}}$ whose composition with the canonical morphism $M_{\underline{a}} \to \mathbf{Z}$ is identity. But since $M_{\mathbf{Q}a}$ is a direct summand of $M(Q_{\underline{a}})$ it means that $Q_{\underline{a}}$ has a zero cycle of degree 1.

Corollary 4.11 *The multiplication homomorphisms*

$$-\wedge \mu : H^j(\mathcal{X}^{\underline{a}}, \mathbf{Z}(i)) \to H^{j+2^n-1}(\mathcal{X}_{\underline{a}}, \mathbf{Z}(i+2^{n-1}-1))$$

are epimorphisms for $j \geq i$ and isomorphisms for $j \geq i+1$.

Proof: This follows immediately from Theorem 4.5 and Corollary 2.5.

Corollary 4.12 *We have*

$$H^{2^n-2}(\mathcal{X}_{\underline{a}}, \mathbf{Z}(2^{n-1}-1)) = 0.$$

Proof: By Theorem 4.5 it is sufficient to show that the homomorphism

$$H^{2^n-2}(M_{\underline{a}}, \mathbf{Z}(2^{n-1}-1)) \to H^0(\mathcal{X}_{\underline{a}}, \mathbf{Z}) = \mathbf{Z}$$

is a monomorphism. It follows from the fact that $M_{\underline{a}}$ is a direct summand of $M(Q_{\underline{a}})$ of the form described in Theorem 4.7 since $H^{2^n-2}(M(Q_{\underline{a}}), \mathbf{Z}(2^{n-1}-1))$ is isomorphic to the group of zero cycles on $Q_{\underline{a}}$ and our homomorphism is the degree homomorphism which is injective for any quadric by [].

Corollary 4.13 *The kernel of the composition*

$$k^* = H^1(\mathcal{X}_{\underline{a}}, \mathbf{Z}(1)) \stackrel{\wedge\mu}{\to} H^{2^n}(\mathcal{X}_{\underline{a}}, \mathbf{Z}(2^{n-1}))$$

is the subgroup of elements generated by the images of norm homomorphisms from those finite extensions of k where $Q_{\underline{a}}$ has a rational point. In particular if $b \wedge \mu^{\underline{a}} = 0$ for some b then the symbol $(b, a_1, \ldots, a_n)$ is divisible by 2 in $K_n^M(k)$.

Proof: The last assertion follows from the first one and Proposition 2.14. Since $Hom(\mathcal{X}_{\underline{a}}, \mathbf{Z}(1)[1]) = k^*$, Theorem 4.5 implies that this kernel coincides with the image of the homomorphism

$$H^{2^n-1}(M_{\underline{a}}, \mathbf{Z}(2^{n-1})) \to k^*$$

given by the composition with the canonical morphism

$$\mathbf{Z}(2^{n-1} - 1)[2^n - 2] \to M_{\underline{a}}.$$

Thus, by Theorem 4.7, it coincides with the image of the homomorphism

$$H^{2^n-1}(M_{\underline{a}}, \mathbf{Z}(2^{n-1})) \to k^*$$

given by the restriction to the fundamental class

$$\mathbf{Z}(2^{n-1} - 1)[2^n - 2] \to M(Q_{\underline{a}})$$

of $Q_{\underline{a}}$. By duality it implies that it coincides further by the image of the homomorphism

$$Hom(\mathbf{Z}, M(Q_{\underline{a}})(1)[1]) \to k^*$$

defined by the canonical projection $Q_{\underline{a}} \to Spec(k)$. The group on the right hand side can be described explicitly as a certain quotient of the group of zero cycles on $Q_{\underline{a}} \times (\mathbf{A}^1 - \{0\})$. Any such cycle is a formal sum of pairs of the form (x, t) where x is a closed point on $Q_{\underline{a}}$ and t is an invertible element of its residue field k_x. Our homomorphism takes (x, t) to $N_{k_x/k}(t)$ which finishes the proof in view of Proposition 4.3.

Remark: Proposition 4.4 implies that the image of the homomorphism from Corollary 4.13 is, in fact, equal to the sum of images of norm homomorphisms from the field extensions E/k where $\underline{a}$ becomes 2-divisible in K_n^M. We are not going to use this fact below.

We will need the following non-vanishing result about $\mu^{\underline{a}}$. By definition motivic cohomology of $\mathcal{X}_{\underline{a}}$ are morphisms in the category $DM_-^{eff}(k)$ from the complex of sheaves with transfers of the form

$$\ldots \to L(Q^3_{\underline{a}}) \to L(Q^2_{\underline{a}}) \to L(Q_{\underline{a}})$$

(where the diffrencial is given by the alternated susm of morphisms induced by the projections) to the corresponding Tate motive. Since

$$Hom(L(Q_{\underline{a}}), \mathbf{Z}(2^{n-1}-1)[2^n-1]) = H^{2^n-1}(Q_{\underline{a}}, \mathbf{Z}(2^{n-1}-1)) = 0$$

by Proposition 2.6 the element $\mu^{\underline{a}}$ defines a a class $\tau^{\underline{a}}$ in the homology group of the complex

$$H^{2^n-2}(Q_{\underline{a}}, \mathbf{Z}(2^{n-1}-1)) \to H^{2^n-2}(Q^2_{\underline{a}}, \mathbf{Z}(2^{n-1}-1)) \to H^{2^n-2}(Q^3_{\underline{a}}, \mathbf{Z}(2^{n-1}-1))$$

Proposition 4.14 *The class $\tau^{\underline{a}}$ is zero if and only if $Q_{\underline{a}}$ has a zero cycle of degree 1.*

Proof: The "if" part is trivial since in this case $\mu^{\underline{a}} = 0$. Suppose that $\tau^{\underline{a}}$ is zero. It implies easily by Lemma 4.9 that there exists a morphism $M(Q_{\underline{a}}) \to M_{\underline{a}}$ such that its composition with the canonical morphism $M_{\underline{a}} \to \mathbf{Z}$ coincides with the canonical morphism $M(Q_{\underline{a}}) \to M(Spec(k)) = \mathbf{Z}$ and its composition with the difference of two projections $M(Q_{\underline{a}})^2 \to M(Q_{\underline{a}})$ is zero. Theorem 4.7 implies that then there is a morphism $M(Q_{\underline{a}}) \to M(Q_{\underline{a}})$ with the same property. Since the group of zero cycles on a smooth projective variety X is canonically isomorphic to $Hom(\mathbf{Z}, M(X))$ it remains to show that the sequence

$$Hom(M(Spec(k)), M(Q_{\underline{a}})) \to Hom(M(Q_{\underline{a}}), M(Q_{\underline{a}})) \to$$

$$\to Hom(M(Q_{\underline{a}})^2, M(Q_{\underline{a}}))$$

is exact. This follows immediately from duality and the fact that the sequence

$$M(Q_{\underline{a}})^3 \stackrel{(pr-1-pr_2)\otimes pr_3}{\to} M(Q_{\underline{a}})^2 \stackrel{pr_2}{\to} M(Q_{\underline{a}})$$

splits by the morphisms

$$Id \otimes M(\Delta) : M(Q_{\underline{a}})^2 \to M(Q_{\underline{a}})^3$$
$$M(\Delta) : M(Q_{\underline{a}}) \to M(Q_{\underline{a}})^2$$

Theorem 4.5 enables us to make the final step in our reduction of the Bloch-Kato problem to a question about motivic cohomology groups.

Proposition 4.15 *Let k be a field of characteristic not equal to 2 which admits resolution of singularities and assume that $BK(n-1,2)$ holds over k. Then $BK(n,2)$ holds over k is and only if for any field F over k and any sequence $\underline{a} = (a_1, \ldots, a_n)$ of invertible elements of F of length n the homomorphism*

$$H^{n+1}(\mathcal{X}_{\underline{a}}, \mathbf{Z}_{(2)}(n)) \to H^{n+1}_{et}(\mathcal{X}_{\underline{a}}, \mathbf{Z}_{(2)}(n)) = H^{n+1}_{et}(Spec(k), \mathbf{Z}_{(2)}(n))$$

is zero.

Proof: The "only if" part is obvious. In fact, if $BK(n,2)$ holds over k the group $H^{n+1}(\mathcal{X}_{\underline{a}}, \mathbf{Z}_{(2)}(n))$ is isomorphic to $H^{n+1}_{et}(Spec(k), \mathbf{Z}_{(2)}(n))$ by Lemma 2.12 which is zero by Theorem 2.11. To prove the "if" part it is sufficient to show that if this homomorphism is zero then the conditions of Proposition 3.8 are verified for $X_{\underline{a}} = Q_{\underline{a}}$. The first condition holds by Proposition 4.4 since $Q_{\underline{a}}$ obviously has a point over its function field. Let α be an element in $H^{n+1}_{et}(Spec(k), \mathbf{Z}_{(2)}(n))$ which goes to zero in the generic point of $Q_{\underline{a}}$. We claim that its image in $H^{n+1}_{et}(\mathcal{X}_{\underline{a}}, \mathbf{Z}_{(2)}(n))$ belongs to the image of the natural homomorphism

$$H^{n+1}(\mathcal{X}_{\underline{a}}, \mathbf{Z}_{(2)}(n)) \to H^{n+1}_{et}(\mathcal{X}_{\underline{a}}, \mathbf{Z}_{(2)}(n))$$

which is sufficient since this homomorphism is zero by the condition of the proposition. By Lemma 3.9 the image of α in $H^{n+1}_{et}(Q_{\underline{a}}, \mathbf{Z}_{(2)}(n))$ comes from $H^{n+1}(Q_{\underline{a}}, \mathbf{Z}_{(2)}(n))$. Since $M_{\underline{a}}$ is a direct summand of $M(Q_{\underline{a}})$ such that the canonical morphism $M(Q_{\underline{a}}) \to \mathbf{Z}$ factors through $M_{\underline{a}}$ the same holds for $M_{\underline{a}}$. Our statement follows now from Theorem 4.5 and $BK(n-1,2)$.

Theorem 4.16 (M. Rost) *For $\underline{a} = (a_1, \ldots, a_n)$ the natural homomorphism*

$$H^{2^n-1}(Q_{\underline{a}}, \mathbf{Z}(2^{n-1})) \to k^*$$

is a monomorphism.

Proof: Since the dimension of $Q_{\underline{a}}$ equals $2^{n-1}-1$ the left hand side group is isomorphic to the group $H^{2^{n-1}-1}(Q_{\underline{a}}, \underline{K}^M_{2^{n-1}})$ by Corollary 2.9 and our result follows from [7].

Corollary 4.17 *For* $\underline{a} = (a_1, \ldots, a_n)$ *we have*

$$H^{2^n-1}(\mathcal{X}_{\underline{a}}, \mathbf{Z}(2^{n-1})) = 0.$$

Proof: It follows immediately from Theorems 4.5 and 4.16.

5 Algebraic Morava K-theories.

In this section we describe hypothetical theories on the category of algebraic varieties over a field k which are called algebraic Morava K-theories. The construction of these theories will be considered in future papers.

Let $\Delta^{op}(Sm/k)$ be the category of smooth simplicial schemes. For any smooth scheme X we denote by the same symbol the simplicial scheme whose terms are X in all dimensions and faces and degeneracy morphisms are identity morphisms. Let further $\Delta^{op}_{\bullet}(Sm/k)$ be the category of pointed smooth simplicial schemes. For a smooth simplicial scheme $\mathcal{X}$ we denote by the same symbol the object of $\Delta^{op}_{\bullet}(Sm/k)$ which corresponds to the simplicial scheme $\mathcal{X} \coprod Spec(k)$ pointed by the canonical embedding $Spec(k) \to \mathcal{X} \coprod Spec(k)$. Since everywhere below we are going to work only with pointed objects this should not cause any confusion.

For any morphism $f : (\mathcal{X}, x) \to (\mathcal{Y}, y)$ we denote by $cone(f)$ the pointed simplicial cone of f. Note that the standard construction of cones in the category of simplicial sets works for simplicial objects in any category with finite coproducts. Considering the cone of the canonical morphism $(\mathcal{X}, x) \to (Spec(k), Id)$ we get the suspension functor

$$\Sigma_s : \Delta^{op}_{\bullet}(Sm/k) \to \Delta^{op}_{\bullet}(Sm/k)$$

and exactly in the same way as for simplicial sets, any morphism $f : (\mathcal{X}, x) \to (\mathcal{Y}, y)$ can be extended into the cofibration sequence

$$(\mathcal{X}, x) \xrightarrow{f} (\mathcal{Y}, y) \to cone(f) \to \Sigma_s(\mathcal{X}, x)$$

The reason, why we denote the suspension functor on our category by Σ_s instead of the usual Σ is that there is another "suspension" $\Sigma_t(\mathcal{X}, x)$ called the t-suspension. Morally we would like it to be the smash product with $(\mathbf{A}^1 - \{0\}, 1)$. However, since the smash products are not in general defined in $\Delta^{op}_{\bullet}(Sm/k)$ we set $\Sigma_t(\mathcal{X}, x)$ to be the cone of the obvious morphism

$$(\mathbf{A}^1 - \{0\}, 1) \to cone(Id \times \{1\})$$

where $cone(Id \times \{1\})$ is the cone of the morphism

$$Id \times \{1\} : (\mathcal{X}, x) \to (\mathcal{X}, x) \times_{Spec(k)} (\mathbf{A}^1 - \{0\}, 1).$$

A morphism $f : (\mathcal{X}, x) \to (\mathcal{Y}, y)$ in $\Delta^{op}_{\bullet}(Sm/k)$ is called a weak equivalence in the Nisnevich topology if for any (smooth) henselian local scheme S the corresponding morphism of the pointed simplicial sets

$$Hom(S, (\mathcal{X}, x)) \to Hom(S, (\mathcal{Y}, y))$$

is a weak equivalence in the usual sense.

A cohomology theory over k is a family of contravariant functors

$$H^{*,*} : \Delta^{op}_{\bullet}(Sm/k) \to Ab$$

together with two natural "suspension isomorphisms":

$$H^{*,*}(\Sigma_s(\mathcal{X}, x)) = H^{*-1,*}((\mathcal{X}, x))$$

$$H^{*,*}(\Sigma_t(\mathcal{X}, x)) = H^{*-1,*-1}((\mathcal{X}, x))$$

such that the following conditions hold:

Exactness For a cofiber sequence

$$(\mathcal{X}, x) \xrightarrow{f} (\mathcal{Y}, y) \to cone(f) \to \Sigma_s(\mathcal{X}, x)$$

the sequence

$$H^{*,*}(\Sigma_s(\mathcal{X}, x)) \to H^{*,*}(cone(f)) \to H^{*,*}((\mathcal{Y}, y)) \to H^{*,*}((\mathcal{X}, x))$$

is exact.

Nisnevich descent For a morphism $f : (\mathcal{X}, x) \to (\mathcal{Y}, y)$ which is a weak equivalence in the Nisnevich topology the homomorphisms $H^{*,*}(f)$ are isomorphisms.

Homotopy invariance I For any $(\mathcal{X}, x)$ the projection

$$(\mathcal{X}, x) \times I^1 \to (\mathcal{X}, x)$$

(where I^1 is the simplicial interval pointed by one of the vertices) induces isomorphisms on $H^{*,*}$.

Homotopy invariance II For any $(\mathcal{X}, x)$ the projection

$$(\mathcal{X}, x) \times (\mathbf{A}^1, \{0\}) \to (\mathcal{X}, x)$$

induces isomorphisms on $H^{*,*}$.

Conjecture 2 *For any prime l and any $n \geq 1$ there exist a cohomology theory $K(n)^{*,*}(-)$ over k with $\mathbf{Z}_{(l)}$-coefficients which has the following properties:*

Periodicity. *There are canonical isomorphisms*

$$K(n)^{i,j}(-) = K(n)^{i+2(l^n-1),j+l^n-1}(-).$$

Motivic spectral sequence. *There are canonical spectral sequences with*

$$E_1^{r,s} = H^r(X, \mathbf{Z}_{(l)}((l^n-1)s + j))$$

$$d_m^{r,s} : E_m^{r,s} \to E_m^{r+1+2(l^n-1)m,s+m}$$

which converge to $K(n)^{r-2(l^n-1)s,j}(X)$. The first nontrivial differentials in these spectral sequences are diffrenciations on the motivic cohomology ring, i.e.

$$d_1(a \wedge b) = d_1(a) \wedge b \pm a \wedge d_1(b).$$

Direct images. *For a smooth projective morphism $p : X \to Y$ of relative dimension d of smooth schemes over k there are homomorphisms*

$$p_* : K(n)^{*,*}(X) \to K(n)^{*-2d,*-d}(Y)$$

such that for any morphism $f : Y' \to Y$ the diagram

$$\begin{array}{ccc} K(n)^{*,*}(X) & \to & K(n)^{*-2d,*-d}(Y) \\ \downarrow & & \downarrow \\ K(n)^{*,*}(X \times_Y Y') & \to & K(n)^{*-2d,*-d}(Y') \end{array}$$

commutes.

"Normalization". *Let $p_X : X \to Spec(k)$ be a smooth hypersurface of degree l in $\mathbf{P}^{l^n}$. Then for any smooth scheme Y the composition*

$$(p_X \times Id_Y)_*(p_X \times Id_Y)^* : K(n)^{*,*}(Y) \to K(n)^{*-2(l^n-1),*-l^n+1}(Y)$$

coincides with the inverse to the periodicity isomorphism.

Example 5.1 *The Adams operations give a decomposition of usual algebraic K-theory with $\mathbf{Z}_{(l)}$-coefficients into a direct sum of the form*

$$(K \otimes \mathbf{Z}_{(l)})^{-i}(-) = K(1)^{i,0}(-) \oplus K(1)^{i+2,1}(-) \oplus \ldots \oplus K(1)^{i+2(l-1),l-1}(-)$$

and the first algebraic Morava K-theory can be defined as a direct summand of the usual algebraic K-theory.

In the rest of this section we consider some simple constructions related to the skeletons of simplicial schemes which will be used in the proof of Theorem 6.1. Note that for a smooth simplicial scheme $\mathcal{X}$ its i-th skeleton $Sk_i(\mathcal{X})$ does not have to be smooth since for $n > i$ the scheme $Sk_i(\mathcal{X})_n$ is the union of the images of the degeneracy morphisms from $\mathcal{X}_{n-1}$ to $\mathcal{X}_n$. However, it is clearly possible to extend any theory given on smooth simplicial schemes to their skeletons. Since it requires some unpleasant formalities we will refer below to the fact that if k admits resolution of singularities any cohomology theory $H^{*,*}(k)$ can be canonically extended to a family of functors on the category $\Delta^{op}_{\bullet}(Sch/k)$ of all pointed simplicial schemes (of finite type) over k (see [14]).

For any simplicial scheme $\mathcal{X}$ denote by $Aug(\mathcal{X})$ the corresponding "augmented" simplicial scheme, i.e. the cone of the canonical morphism $\mathcal{X} \to Spec(k)$ (note that it is not the same as $\Sigma_s(\mathcal{X})$ since $Spec(k)$ as a pointed scheme is $(Spec(k) \coprod Spec(k), i)$). We will use the following simple fact.

Lemma 5.2 *Let $\mathcal{X}$ be a simplicial scheme and $H^{*,*}$ be a cohomology theory over k such that for any $p, q \in \mathbf{Z}$ the sequence*

$$0 \to H^{p,q}(Spec(k)) \to H^{p,q}(\mathcal{X}_1) \to H^{p,q}(\mathcal{X}_2) \to \dots$$

is exact (here the first homomorphism is induced by the canonical projection and the other ones are the alternated sums of homomorphisms induced by the face morphisms). Then the restrictions homomorphisms

$$H^{*,*}(Aug(Sk_i(\mathcal{X}))) \to H^{*,*}(Aug(Sk_{i-1}(\mathcal{X})))$$

are zero and in particular $H^{*,*}(Aug(\mathcal{X})) = 0$.

6 Bloch-Kato conjecture for $\mathbf{Z}/2$-coefficients.

The goal of this section is to prove the following theorem.

Theorem 6.1 *Let k be a field of characteristic not equal to 2 which admits resolution of singularities and assume that Conjecture 2 holds over k. Then the Bloch-Kato conjecture with $\mathbf{Z}/2$-coefficients holds over k.*

We assume below that k satisfies the conditions of Theorem 6.1.

By Proposition 4.15 in order to prove the Bloch-Kato conjecture we have to show that, assuming that $BK(n-1,2)$ holds over k, the homomorphism

$$H^{n+1}(\mathcal{X}_{\underline{a}}, \mathbf{Z}_{(2)}(n)) \to H^{n+1}_{et}(\mathcal{X}_{\underline{a}}, \mathbf{Z}_{(2)}(n)) = H^{n+1}_{et}(Spec(k), \mathbf{Z}_{(2)}(n))$$

is zero. We are going to do it using the motivic spectral sequences for the algebraic Morava K-theories on $\mathcal{X}_{\underline{a}}$ and $Aug(\mathcal{X}_{\underline{a}})$.

Lemma 6.2 *Let $\mathcal{X}$ be a smooth simplicial scheme over k. Then for any $p, q, m \geq 0$ and any $n > dim(\mathcal{X}_m + m)$ we have*

$$H^{p+n}(Sk_m(\mathcal{X}), \mathbf{Z}(q)) = 0$$

$$H^{p+n+1}(Aug(Sk_m(\mathcal{X})), \mathbf{Z}(q)) = 0$$

Proof: We will only show that the first equality holds. The second one follows then immediately from the cofibration sequence

$$Sk_m(\mathcal{X}) \to Spec(k) \to Aug(Sk_m(\mathcal{X})).$$

By definition the motivic cohomology groups of a simplicial scheme $\mathcal{Y}$ are morphisms in the category $DM^{eff}(k)$ from the complex of sheaves which is the normalization of the sheaf of simplicial abelian groups $L(\mathcal{Y})$ to the corresponding Tate object. For the m-th skeleton of $\mathcal{X}$ this complex is of the form

$$0 \to F_m \to \ldots \to F_0 \to 0$$

where F_i is a certain direct summand of $L(\mathcal{X}_m)$. By Corollary 2.5 we have $Hom(F_i, \mathbf{Z}(p)[q]) = 0$ for $q - p > dim(\mathcal{X}_i)$. Since degeneracy morphisms are embeddings we have $dim(\mathcal{X}_{i+1}) \geq dim(\mathcal{X}_i)$ and the standard spectral sequence implies now the result we need.

Proposition 6.3 *Let $(a_1, \ldots, a_n)$ be a sequence of invertible elements of k of length n. The all the restriction homomorphisms*

$$K(n-1)^{*,*}(Aug(Sk_i(\mathcal{X}_{\underline{a}}))) \to K(n-1)^{*,*}(Aug(Sk_{i-1}(\mathcal{X}_{\underline{a}})))$$

are zero. In particular, we have

$$K(n-1)^{*,*}(Aug(\mathcal{X}_{\underline{a}})) = 0.$$

Proof: By Lemma 5.2 it is sufficient to verify that for any p, q the complex

$$0 \to K^{p,q}(Spec(k)) \to K^{p,q}(X) \to K^{p,q}(X^2) \to \ldots \to K^{p,q}(X^i) \to \ldots$$

with the differential given by the alternated sum of morphisms induced by the partial projections on $K^{p,q}(X^j)$ for $j > 0$ and by the restriction homomorphism for $j = 0$ is exact. We will show that this complex is in fact homotopic to zero. To construct this homotopy consider the compositions of the direct image homomorphisms

$$K^{p,q}(X^i) \to K^{p-2^n+2, q-2^{n-1}+1}(X^{i-1})$$

(with respect to the first projection) with the periodicity isomorphisms (see Conjecture 2). It is easy to see (using the properties of direct image homomorphisms together with the "normalization" property asserted by Conjecture

2) that these composition form a homotopy from the identity morphism of our complex to zero.

In order to analyze the motivic spectral sequences for the higher Morava K-theories on $\mathcal{X}_{\underline{a}}$ we will need an alternative description of the element $\mu^{\underline{a}}$ given in Proposition 6.5 below. We start with the following lemma.

Lemma 6.4 *Let $\underline{a} = (a_1, \dots, a_n)$ be a symbol of length n and assume that $BK(n-1, 2)$ holds over k. Then the group $H^{n+1}(\mathcal{X}_{\underline{a}}, \mathbf{Z}_{(2)}(n-1))$ is canonically isomorphic to the subgroup in $H^n_{et}(k, \mathbf{Z}/2(n-1))$ which consists of elements vanishing in the generic point of $Q_{\underline{a}}$.*

Proof: Note first that $Q_{\underline{a}} \to Spec(k)$ is a covering in the etale topology and therefore the morphism $\mathcal{X}_{\underline{a}} \to Spec(k)$ induces isomorphisms on all theories with etale descent. In particular

$$H^n_{et}(k, \mathbf{Z}/2(n-1)) = H^n_{et}(\mathcal{X}_{\underline{a}}, \mathbf{Z}/2(n-1)).$$

By $BK(n-1, 2)$ the morphism

$$H^n(\mathcal{X}_{\underline{a}}, \mathbf{Z}/2(n-1)) \to H^n_{et}(\mathcal{X}_{\underline{a}}, \mathbf{Z}/2(n-1))$$

is a monomorphism and its image consists exactly of elements which vanish in the generic point of $Q_{\underline{a}}$. We have further an exact sequence

$$H^n(\mathcal{X}_{\underline{a}}, \mathbf{Z}_{(2)}(n-1)) \to H^n(\mathcal{X}_{\underline{a}}, \mathbf{Z}/2(n-1)) \to H^{n+1}(\mathcal{X}_{\underline{a}}, \mathbf{Z}_{(2)}(n-1)) \to$$
$$\to H^{n+1}(\mathcal{X}_{\underline{a}}, \mathbf{Z}_{(2)}(n-1))$$

where the last arrow is multiplication by 2. It remains to observe that $H^{n+1}(\mathcal{X}_{\underline{a}}, \mathbf{Z}_{(2)}(n-1))$ is a 2-torsion group (since it becomes zero after any field extension of degree two where $Q_{\underline{a}}$ has a point) and that by $BK(n-1, 2)$ we have

$$H^n(\mathcal{X}_{\underline{a}}, \mathbf{Z}_{(2)}(n-1)) = H^n_{et}(\mathcal{X}_{\underline{a}}, \mathbf{Z}_{(2)}(n-1)) = H^n_{et}(Spec(k), \mathbf{Z}_{(2)}(n-1)) = 0.$$

Lemma 6.4 implies in particular that there is a distinguished element in $H^{n+1}(\mathcal{X}_{\underline{a}}, \mathbf{Z}_{(2)}(n-1))$ which corresponds to the class of $\underline{a}$ in

$$H^n_{et}(Spec(k), \mathbf{Z}/2(n-1)) = H^n_{et}(Spec(k), \mathbf{Z}/2(n)).$$

Denote this element by $\eta_1^{\underline{a}}$.

Let $d^{(n)}$ be the first nontrivial differential in the motivic spectral sequence for $K(n)$. Note that it is of the form

$$d^{(n)} : H^*(-, \mathbf{Z}_{(2)}(*)) \to H^{*+2^{n+1}-1}(-, \mathbf{Z}_{(2)}(* + 2^n - 1)).$$

We set:

$$\eta_2^{\underline{a}} = d^{(1)}(\eta_1^{\underline{a}})$$

$$\dots$$

$$\eta_{k+1}^{\underline{a}} = d^{(k)}(\eta_k^{\underline{a}}).$$

The element $\eta_{n-1}^{\underline{a}}$ belongs to the same motivic cohomology group of $\mathcal{X}_{\underline{a}}$ as $\mu^{\underline{a}}$.

Proposition 6.5 *We have $\mu^{\underline{a}} = \eta_{n-1}^{\underline{a}}$.*

Proof: By Corollary 4.10 we have only to show that if $Q_{\underline{a}}$ has no zero cycle of degree 1 then $\eta_{n-1}^{\underline{a}}$ is not zero. For any $m \leq n$ denote by $\underline{a}^m$ the sequence $(a_1, \dots, a_m)$. We have a sequence of morphisms

$$\mathcal{X}_{\underline{a}^2} \to \mathcal{X}_{\underline{a}^3} \to \dots \to \mathcal{X}_{\underline{a}^n} = \mathcal{X}_{\underline{a}}.$$

Note that $\eta_1^{\underline{a}^2}$ is not zero by Lemma 6.4. Thus by induction we may assume that $\eta_{n-2}^{\underline{a}^{n-1}}$ is not zero and $n \geq 3$.

Lemma 6.6 *Under the inductive assumption we have*

$$\eta_{n-1}^{\underline{a}^{n-1}} = \mu^{\underline{a}^{n-1}} \wedge \mu^{\underline{a}^{n-1}} \neq 0.$$

Proof: In view of Corollaries 4.10,4.11 it is sufficient to show that $\eta_{n-1}^{\underline{a}^{n-1}}$ is not zero. By the inductive assumption we have

$$\eta_{n-1}^{\underline{a}^{n-1}} = d^{(n-2)}(\mu^{\underline{a}^{n-1}}).$$

For an element $a \in H^p(\mathcal{X}_{\underline{a}^{n-1}}, \mathbf{Z}_{(2)}(q))$ denote by $Aug(a)$ its image under the homomorphism

$$H^p(\mathcal{X}_{\underline{a}^{n-1}}, \mathbf{Z}_{(2)}(q)) \to H^{p+1}(Aug(\mathcal{X}_{\underline{a}^{n-1}}), \mathbf{Z}_{(2)}(q))$$

arising from the cofibration long exact sequence. These homomorphisms commute with all the differentials in the motivic spectral sequences and are isomorphisms for all $p > q$. Suppose that $Aug(\eta_{n-1}^{\underline{a}^{n-1}}) = 0$. The value of the second differential in the motivic spectral sequence for $K(n-2)$ on $Aug(\mu^{\underline{a}^{n-1}})$ belongs to the motivic cohomology group which is zero by Corollaries 4.11 and 4.12. The third differential belongs to the group

$$H^{2^{n+1}-5}(Aug(\mathcal{X}_{\underline{a}^{n-1}}), \mathbf{Z}_{(2)}(2^n - 4)).$$

Consider the simplicial scheme $Aug(Sk_3(\mathcal{X}_{\underline{a}^{n-1}}))$. Since

$$dim(Sk_3(\mathcal{X}_{\underline{a}^{n-1}})) = 2^{n-2} + 2^{n-1} - 1$$

we conclude, by Lemma 6.2, that all the differentials in the motivic spectral sequence for $K(n-2)$ vanish on the restriction of $Aug(\mu^{\underline{a}^{n-1}})$ to $Aug(Sk_3(\mathcal{X}_{\underline{a}^{n-1}}))$. Thus by Proposition 6.3 the restriction of $Aug(\mu^{\underline{a}^{n-1}})$ to $Aug(Sk_2(\mathcal{X}_{\underline{a}^{n-1}}))$ belongs to the image of an incoming differential in the motivic spectral sequence for $K(n-2)$. On the other hand all these differentials are zero for dimension reasons and therefore the restriction of $Aug(\mu^{\underline{a}^{n-1}})$ to this skeleton should be zero. But then the same should hold for the restriction of $\mu^{\underline{a}^{n-1}}$ to $Sk_2(\mathcal{X}_{\underline{a}^{n-1}})$ which is not the case by Proposition 4.14 and our assumption that $Q_{\underline{a}}$ (and, therefore, $Q_{\underline{a}^{n-1}}$) has no zero cycles of degree 1.

To finish the proof of Proposition 6.5 it remains to notice that

$$(\eta_{n-1}^{\underline{a}})_{|\mathcal{X}_{\underline{a}^{n-1}}} = \eta_{n-1}^{\underline{a}^{n-1}} \wedge a_n$$

(where a_n is considered as an element of $H^1(\mathcal{X}_{\underline{a}^{n-1}}, \mathbf{Z}_{(2)}(1))$) which implies that this element is not zero by Lemma 6.6 and Corollaries 4.11, 4.13.

Remark: As was mentioned before (see 4.6) the object of the homotopy category of simplicial schemes in Nisnevich topology which corresponds to the simplicial scheme $\mathcal{X}_{\underline{a}}$ can be constructed in a very simple way directly from the symbol $\underline{a}$ without any use of the theory of Pfister quadrics. Proposition 6.5 shows that the element $\mu^{\underline{a}}$ which defines the motive $M_{\underline{a}}$ also has a purely formal description. Thus we have a construction of the Rost motive $M_{\underline{a}}$ which does not rely on the theory of Pfister forms. Unfortunately using only this formal construction we are unable to show that $M_{\underline{a}}$ is indeed a pure motive.

Proposition 6.5 has the following very important corollary.

Proposition 6.7 *Let $(a_1, \ldots, a_n)$ be a sequence of invertible elements of k^* of length $n \geq 3$ and suppose that $BK(n-1,2)$ holds over k. Denote by $E^{i,j}$ the cohomology group of the complex*

$$H^{j-2^{n-1}+1}(\mathcal{X}_{\underline{a}}, \mathbf{Z}_{(2)}(i-2^{n-2}+1)) \stackrel{d^{(n-2)}}{\to} H^j(\mathcal{X}_{\underline{a}}, \mathbf{Z}_{(2)}(i)) \to$$

$$\stackrel{d^{(n-2)}}{\to} H^{j+2^{n-1}-1}(\mathcal{X}_{\underline{a}}, \mathbf{Z}_{(2)}(i+2^{n-2}-1)).$$

Then for any $j > i + 2^{n-1}$ we have $E^{i,j} = 0$.

Proof: Let a be an element of $H^j(\mathcal{X}_{\underline{a}}, \mathbf{Z}_{(2)}(i))$. By Lemma 4.11 our condition on j, i implies that a is of the form $a_0 \wedge \mu^{\underline{a}}$ for some element a_0. By Proposition 6.5 we have $\mu^{\underline{a}} = d^{(n-2)}(\eta^{\underline{a}}_{n-2})$. Therefore, $d^{(n-2)}(\mu^{\underline{a}}) = 0$ and

$$d^{(n-2)}(a) = d^{(n-2)}(a_0) \wedge \mu^{\underline{a}}.$$

If this element is zero, Lemma 4.11 implies that $d^{(n-2)}(a_0) = 0$, but then

$$a = d^{(n-2)}(a_0 \wedge \eta^{\underline{a}}_{n-2})$$

which proves the proposition.

Corollary 6.8 *Suppose that $BK(m,2)$ holds over k. Then the differential*

$$d^{(n-2)} : H^j(\mathcal{X}_{\underline{a}}, \mathbf{Z}_{(2)}(i)) \to H^{j+2^{n-1}-1}(\mathcal{X}_{\underline{a}}, \mathbf{Z}_{(2)}(i+2^{n-2}-1))$$

is a monomorphism for all i, j satisfying the conditions

$$i \leq m + 2^{n-2} - 1$$

$$1 \leq j - i \leq 2^{n-2} + 1.$$

Proof: Let a be an element in $H^j(\mathcal{X}_{\underline{a}}, \mathbf{Z}_{(2)}(i))$ which belongs to the kernel of $d^{(n-2)}$. Then by Proposition 6.7 all the higher differentials in the motivic spectral sequence for $K(n-2)$ vanish on a as well. It is sufficient to show that the image $Aug(a)$ of a in $H^{j+1}(Aug(\mathcal{X}_{\underline{a}}), \mathbf{Z}_{(2)}(i))$ is zero. By Proposition 6.3 we have

$$K(n-2)^{*,*}(Aug(\mathcal{X}_{\underline{a}})) = 0.$$

Therefore, $Aug(a)$ should lie in the image of the incoming differentials in the motivic spectral sequence for $K^{(n-2)}$ on $Aug(\mathcal{X}_{\underline{a}})$. Note that by our condition on i and j all this differentials are coming from motivic cohomology groups $H^p(-.\mathbf{Z}_{(2)}(q))$ with $p \leq q+2$. and $q \leq m$. By $BK(m,1)$ these groups map monomorphically to the corresponding etale motivic cohomology groups. On the other hand any theory with etale descent is zero on $Aug(\mathcal{X}_{\underline{a}})$ which proves our corollary.

Proof of Theorem 6.1: By induction we may assume that $BK(n-1,2)$ holds over k and that $n > 2$. By Proposition 4.15 all we have to do is to show that the homomorphism

$$H^{n+1}(\mathcal{X}_{\underline{a}}, \mathbf{Z}_2(n)) \to H^{n+1}_{et}(\mathcal{X}_{\underline{a}}, \mathbf{Z}_2(n)) = H^{n+1}_{et}(Spec(k), \mathbf{Z}_2(n))$$

is zero. Since we have

$$H^*_{et}(Spec(k), \mathbf{Z}_{(2)}(*)) = H^*_{et}(\mathcal{X}_{\underline{a}^2}, \mathbf{Z}_{(2)}(*)) = \ldots = H^*_{et}(\mathcal{X}_{\underline{a}}, \mathbf{Z}_{(2)}(*))$$

it is sufficient to show that the restriction of any class $\alpha \in H^{n+1}(\mathcal{X}_{\underline{a}}, \mathbf{Z}_2(n))$ to $\mathcal{X}_{\underline{a}^2}$ is zero. We set:

$$\alpha_1 = d^{(1)}(\alpha)$$

$$\ldots$$

$$\alpha_i = d^{(i)}(\alpha_{i-1}).$$

By Corollary 4.17 we have $\alpha_{n-2} = 0$ on $\mathcal{X}_{\underline{a}}$. Thus we may assume by induction that α_{n-2-i} is zero on $\mathcal{X}_{\underline{a}^{n-i}}$. Then by Corollary 6.8 we have that α_{n-3-i} is zero on $\mathcal{X}_{\underline{a}^{n-i-1}}$ which proves the theorem.

References

[1] H. Bass and J. Tate. The Milnor ring of a global field. In *K-theory II*, volume 342 of *Lecture Notes in Math.*, pages 349–446. Springer-Verlag, 1973.

[2] E. Friedlander and V. Voevodsky. Bivariant cycle cohomology. *Cycles, transfers and motivic homology theories*, page ..., 1995.

[3] A. Merkurjev and A. Suslin. K-cohomology of Severy-Brauer varieties and the norm residue homomorphism. *Math. USSR Izvestiya*, 21:307–340, 1983.

[4] A. Merkurjev and A. Suslin. The norm residue homomorphism of degree three. *Math. USSR Izvestiya*, 36(2):349–367, 1991.

[5] D. C. Ravenel. *Nilpotence and periodicity in stable homotopy theory.* Ann. of Math. Studies 128. Princeton, 1992.

[6] M. Rost. Hilbert 90 for K_3 for degree-two extensions. *Preprint*, 1986.

[7] M. Rost. On the spinornorm and $A_0(X, K_1)$ for quadrics. *Preprint*, 1988.

[8] M. Rost. Some new results on the Chowgroups of quadrics. *Preprint*, 1990.

[9] A. Suslin. Algebraic K-theory and the norm residue homomorphism. *J. Soviet Math.*, 30:2556–2611, 1985.

[10] A. Suslin and V. Voevodsky. Singular homology of abstract algebraic varieties. *Preprint.*, 1992.

[11] A. Suslin and V. Voevodsky. Motivic cohomology with finite coefficients and Bloch-Kato conjecture. *Preprint*, Spring 1995.

[12] R. Thomason. Algebraic K-theory and etale cohomology. *Ann. Sci. ENS*, 18:437–552, 1985.

[13] V. Voevodsky. Cohomological theory of presheaves with transfers. *Cycles, transfers and motivic homology theories*, page ..., 1995.

[14] V. Voevodsky. The Spanier-Whitehead category of algebraic varieties. *In preparation*, 1995.

[15] V. Voevodsky. Triangulated categories of motives over a field. *Cycles, transfers and motivic homology theories*, page ..., 1995.

[16] V. Voevodsky, E. Friedlander, and A. Suslin. *Cycles, transfers and motivic homology theories.* .. .., .., ..