A variant on the compressed sensing of Emmanuel Candès

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1 Introduction

This paper is motivated by the outsanding achievements of Emmanuel Candès and Terence Tao on what is now called "compressed sensing". Let us begin with a theorem by Terence Tao. Let p be a prime number and $\mathbb{F}_p$ be the finite field with p elements. We denote by $\#E$ the cardinality of $E \subset \mathbb{F}_p$. The Fourier transform of a complex valued function f defined on $\mathbb{F}_p$ is denoted by $\hat{f}$. Let M_q be the collection of all $f : \mathbb{F}_p \mapsto \mathbb{C}$ such that the cardinality of the support of f does not exceed q. Then Terence Tao proved that for $q < p/2$ and for any set Ω of frequencies such that $\#\Omega \geq 2q$, the mapping $\Phi : M_q \mapsto l^2(\Omega)$ defined by $f \mapsto \hat{f}$ is injective. We want to generalize this fact to functions defined on the unit square with applications to image processing. In a forthcoming work the hypothesis that f is supported by the unit square will be removed. Here and in what follows the action takes place on $[0,1]^2$ identified to $(\mathbb{R}/\mathbb{Z})^2$. Since the unit square $[0,1]^2$ has been identified to $(\mathbb{R}/\mathbb{Z})^2$, the Fourier transform of $f \in L^1([0,1]^2)$ is the sequence of its Fourier coefficients defined by

$$\hat{f}(k) = \int_0^1 \int_0^1 \exp(-2\pi i k \cdot x) f(x) dx, \;\; k \in \mathbb{Z}^2. \tag{1.1}$$

To generalize Tao's theorem to the continuous setting we begin with a parameter $\beta \in (0, 1/2)$ which will play the role of q and define a collection M_β of images $f \in L^2([0,1]^2)$ as follows: we write $f \in M_\beta$ if f is supported by a compact set $K \subset [0,1]^2$ whose measure $|K|$ does not exceed β. This compact set K depends on f and M_β is not a vector space. If f, g belong to M_β, then $f + g$ belongs to $M_{2\beta}$, a situation which is classical in nonlinear approximation. As it will be proved below, for every $\alpha \in (0, 1/2)$ there exists a set $\Lambda_\alpha \subset \mathbb{Z}^2$ with the following properties: (a) density $\Lambda_\alpha = 2\alpha$ and (b) the mapping $\Phi : M_\beta \mapsto \ell^2(\Lambda_\alpha)$ defined by $\Phi(f) = (\hat{f}(k))_{k \in \Lambda_\alpha}$ is injective when $0 < \beta < \alpha$. This set Λ_α plays the role of Ω in Tao's work and the density of Λ_α is then playing the role of the cardinality of Ω. Any $f \in M_\beta$ can be retrieved from the information given by the "irregular sampling" $\hat{f}(k) = a(k), \; k \in \Lambda_\alpha$, and one would like to do it by some fast algorithm. If we a priori know that the data $a(k), k \in \Lambda_\alpha$, are the Fourier coefficients of some *nonnegative* $f \in M_\beta$, then it will be proved that f is the unique solution of the following problem

$$\inf\{\|u\|_1 \,; u \in L^1(\mathbb{T}^2), \; \hat{u}(k) = a(k), \;\; k \in \Lambda_\alpha\}. \tag{1.2}$$

We do not impose any condition on the support of u in (1.2). The uniqueness of the solution of the problem (1.2) is coming from the peculiar structure of the data $a(k), \; k \in \Lambda_\alpha$. This is no longer true when f both takes positive and negative values (see Lemma 4.2). We now construct the sparse set Λ_α.

Definition 1.1. *If $\alpha \in (0, 1/2)$ we define $\Lambda_\alpha \subset \mathbb{Z}^2$ by*

$$\Lambda_\alpha = \{(m,n) \in \mathbb{Z}^2; \; \exists r \in \mathbb{Z} \text{ such that } |m\sqrt{2} + n\sqrt{3} - r| \leq \alpha\}. \tag{1.3}$$

The choice of $\sqrt{2}$ and $\sqrt{3}$ is irrelevant and other irrational numbers γ_1 and γ_2 could be used as long as γ_1, γ_2 and 1 are linearly independent over $\mathbb{Q}$. We know from the theory of "model sets" [2] that the density of $\Lambda_\alpha \subset \mathbb{Z}^2$ is uniform and equals 2α. It means that for every $\varepsilon > 0$ there exists a $R(\varepsilon)$ such that for $R \geq R(\varepsilon)$ and uniformly in $x_0 \in \mathbb{Z}^2$

$$(2\alpha - \varepsilon)\pi R^2 \leq \#\{\Lambda_\alpha \cap B(x_0, R)\} \leq (2\alpha + \varepsilon)\pi R^2. \tag{1.4}$$

Here $B(x_0, R)$ is the disc centered at x_0 with radius R. As above we write $f \in M_\beta$ if f is supported by a compact set $K \subset [0,1]^2$ whose measure $|K|$ does not exceed β.

Theorem 1.1. *The mapping $\Phi : M_\beta \mapsto l^2(\Lambda_\alpha)$ defined by $f \mapsto (\hat{f}(k))_{k\in\Lambda_\alpha}$ is injective if $\beta < \alpha$. If $f \in M_\beta$ is nonnegative, then f is the unique solution u of the following variational problem*

$$\inf\{\|u\|_1\,; u \in L^1(\mathbb{T}^2),\ \hat{u}(k) = a(k),\ k \in \Lambda_\alpha\}. \tag{1.5}$$

Moreover if $u \in L^1(\mathbb{T}^2)$, $u \geq 0$ and $\hat{u}(k) = a(k),\ k \in \Lambda_\alpha$, then $u = f$.

Let us stress that we do not assume anything on the support of u in (1.5). Theorem 1.1 is sharp since the hypothesis $\beta < \alpha$ cannot be replaced by $\beta > \alpha$. If one does not assume $f \geq 0$ then f is not in general the argument of the variational problem (1.5). These two remarks will be proved later on (see Section 4). We also want to investigate the stability in Theorem 1.1 which is given by the following statement where the L^∞ norm measuring the error term is the supremum over $\mathbb{T}^2$.

Theorem 1.2. *If $\beta < \alpha$, let us assume we are given a noisy sampling $a(k) = \hat{f}(k) + \hat{R}(k),\ k \in \Lambda_\alpha$, where the unknown function f is nonnegative and belongs to M_β and where the error term R fulfils $\|R\|_\infty \leq \varepsilon$. Then for any nonnegative u, the property $\hat{u}(k) = a(k),\ k \in \Lambda_\alpha$, implies $\|u - f\|_\infty \leq C\varepsilon$ where $C = C(f)$ only depends on the geometry of the closed support of f.*

Let us say a few words on $C(f)$. One might hope that $C(f)$ be a function of α and β. This is not the case. Indeed $C(f)$ depends on a function $\beta(\varepsilon)$ which is the measure of the sum $K + B(0, \varepsilon)$ between K and the disc centered at 0 with radius ε. The behavior of $\beta(\varepsilon)$ near 0 depends on the geometry of K. This will be detailed in Section 3 (see Proposition 3.1). The following section is devoted to $(L^2(K), l^2(\Lambda))$ estimates. Theorems 1.1 and 1.2 will be proved in Section 3 and the last section will be devoted to some counter-examples.

2 L^2 and L^p estimates

The theorem by Tao can be split into two pieces. We return to the finite field $\mathbb{F}_p$ and assume that we are given two sets $T \subset \mathbb{F}_p$ and $\Omega \subset \mathbb{F}_p$ with the same cardinality. We denote by l^2_T the collection of all $f \in l^2_p$ which are supported by T (i.e. vanish outside T). Similarly $l^2(\Omega)$ denotes the restrictions to Ω of signals $f \in l^2_p$. This looks pedantic but will be adapted to functional spaces in a continuous setting. Then the mapping $\mathcal{F} : l^2_T \mapsto l^2(\Omega)$ defined by $f \mapsto \hat{f}$ is an isomorphism. The second piece in Tao's theorem concerns the situation where T is not given. If $\#\Omega \geq 2q$, then the mapping $\mathcal{F} : M_q \mapsto l^2(\Omega)$ is injective. We aim at generalizing these two facts to the continuous setting. Let us begin with the case where T is given. Let $\Omega \subset \mathbb{T}^2$ be a Borel set and $\Lambda \subset \mathbb{Z}^2$ be a set of frequencies. Let L^2_Ω be the Hilbert space of all square integrable functions supported by Ω. In other words we write $f \in L^2_\Omega$ if f belongs to $L^2(\mathbb{T}^2)$ and vanishes almost everywhere on $\mathbb{T}^2 \setminus \Omega$. We define L^p_Ω by the same condition if $p \in [1, \infty]$. The space $L^p(\Omega)$ will consist of all restrictions to Ω of functions in L^p. We will use similar notations Y_K and $Y(\Omega)$ for a functional Banach space Y. A functional space is a Banach space which contains all sufficiently regular functions on $\mathbb{T}^2$ and is continuously embedded in the space of distributions on $\mathbb{T}^2$. These remarks lead to the following definition

Definition 2.1. *If $K \subset \mathbb{T}^2$ is a compact set, $f \in Y_K$ means that the support of f is contained in K. If Ω is an open set, $Y(\Omega)$ denotes the Banach space of all restrictions to Ω of functions in Y equipped with the quotient norm.*

The first problem will be raised in the simplest case where $Y = L^2(\mathbb{T}^2)$. We will discuss the following estimate

$$\sum_{k\in\Lambda} |a(k)|^2 \le C \int_\Omega |\sum_{k\in\Lambda} a(k)\exp(2\pi i k\cdot x)|^2\, dx \tag{2.1}$$

This problem led H. J. Landau to introduce the definition of a set of *stable interpolation* [1]

Definition 2.2. $\Lambda \subset \mathbb{T}^2$ *is a set of stable interpolation for* E^2_Ω *if for every* $(a(\lambda))_{\lambda\in\Lambda} \in l^2(\Lambda)$, *there exists a* $f \in E^2_\Omega$ *such that* $f(\lambda) = a(\lambda)$, $\lambda \in \Lambda$.

Landau observed that Λ is a *set of stable interpolation* for E^2_Ω if and only if the condition (2.1) is satisfied. In the same paper H. J. Landau defined a *set of stable sampling* by the following condition

Definition 2.3. *Let $\Omega \subset \mathbb{T}^2$ be a Borel set and $\Lambda \subset \mathbb{Z}^2$ a set of frequencies. We say that Λ is a set of stable sampling for E^2_Ω if a constant C exists such that for every sequence $f \in E^2_\Omega$ one has*

$$\sum_{k\in\mathbb{Z}^2} |f(k)|^2 \le C \sum_{k\in\Lambda} |f(k)|^2 \tag{2.2}$$

In other words Λ is a set of stable sampling for E^2_Ω if and only if the functions $\exp(2\pi i k\cdot x)$, $k \in \Lambda$, are a frame of $L^2(\Omega)$. The familiar Shannon's identity tells us that any "band-limited" $f \in E^2_\Omega$ can be reconstructed from its sampling $f(k)$ $k \in \Lambda$.

Lemma 2.1. *If $\Lambda \subset \mathbb{Z}^2$ is a set of stable sampling for E^2_Ω then there exists a collection $\phi_k \in E^2_\Omega$, $k \in \Lambda$, such that for every $f \in E^2_\Omega$ one has $f = \sum_{k\in\Lambda} f(k)\phi_k$.*

If $\Lambda \subset \mathbb{Z}^2$ is a set of stable sampling and at the same time a set of stable interpolation for E^2_Ω then the functions $\exp(2\pi i k\cdot x)$, $k \in \Lambda$, are a Riesz basis for $L^2(\Omega)$.

Sampling and interpolation are the same problem as it will be proved now.

Proposition 2.1. *If K is the complement of Ω in $\mathbb{T}^2$ and if M is the complement of Λ in $\mathbb{Z}^2$, then M is a set of stable sampling for E^2_K if and only if Λ is a set of stable interpolation for E^2_Ω.*

For proving (2.1) $\Rightarrow$ (2.2) we start with a sequence $f \in E^2_K$. Then $f(k)$, $k \in \mathbb{Z}^2$, are the Fourier coefficients of a function $F \in L^2(\mathbb{T}^2)$ which is supported by K. We split f into the sum $g + h$ where $g = f\mathbf{1}_M$ and $h = f\mathbf{1}_\Lambda$. Here and in what follows $\mathbf{1}_E$ denotes the indicator function of the set E. Then $g(k)$, $k \in \mathbb{Z}^2$, are the Fourier coefficients of $G \in L^2(\mathbb{T}^2)$ and $h(k)$, $k \in \mathbb{Z}^2$, are the Fourier coefficients of $H \in L^2(\mathbb{T}^2)$. We have $F = G + H$. From (2.1) we know that

$$\|h\|_2 \le C\Big(\int_\Omega |H(x)|^2 dx\Big)^{1/2}. \tag{2.3}$$

But $G + H = 0$ on Ω and

$$\int_\Omega |H(x)|^2 dx = \int_\Omega |G(x)|^2 dx \le \|G\|_2^2 = \|g\|_2^2 = \sum_{k\in M} |f(k)|^2. \tag{2.4}$$

Therefore

$$\|h\|_2 \le C\Big(\sum_{k\in M} |f(k)|^2\Big)^{1/2}. \tag{2.5}$$

and

$$\|f\|_2 = \Big(\|g\|_2^2 + \|h\|_2^2\Big)^{1/2} \le \sqrt{1+C^2}\Big(\sum_{k\in M} |f(k)|^2\Big)^{1/2} \tag{2.6}$$

which ends the proof.

The converse implication is just as easy. We assume that $f(k)$ vanishes outside Λ and we consider $F(x) = \sum_{k\in\Lambda} f(k)\exp(2\pi ik\cdot x)$. We want to prove (2.1). We split $F(x)$ into the sum $G(x) + H(x)$ where G is the product between F and the indicator function of K and $H(x) = \mathbf{1}_\Omega F(x)$. Let $g(k)$ and $h(k)$, $k \in \mathbb{Z}^2$, be the Fourier coefficients of G and H. We know that $f = g + h$ and we have $f(k) = 0$ whenever $k \in M$. Then (2.2) yields $\|G\|_2^2 \le C\sum_{k\in M} |g(k)|^2$. But $g(k) = -h(k)$, $k \in M$, and $\|G\|_2^2 \le C\sum_{k\in M} |h(k)|^2 \le C\|H\|_2^2 = C\int_\Omega |F|^2\,dx$ which ends the proof. Indeed $\|F\|_2^2 = \|G\|_2^2 + \|H\|_2^2$ implies $\|F\|_2 \le \sqrt{1+C^2}\|H\|_2$.

We now extend these definitions to $p \ne 2$. Let E^p be the Banach space of Fourier coefficients of functions in $L^p(\mathbb{T}^2)$, a Banach space that authors denote by $\mathcal{F}L^p$.

Definition 2.4. *When $K \subset \mathbb{T}^2$ is a Borel set, let E^p_K be the Banach space of all sequences $f(k)$, $k \in \mathbb{Z}^2$, which are the Fourier coefficients of a function $F \in L^p_K$. In other words $f \in E^p_K$ if and only if $F(x) = \sum_{k\in\mathbb{Z}^2} f(k)\exp(2\pi ik\cdot x)$ belongs to $L^p(\mathbb{T}^2)$ and vanishes almost everywhere on $\mathbb{T}^2 \setminus K$.*

If $\Lambda \subset \mathbb{Z}^2$ we define the Banach space $E^p(\Lambda)$ as the space of restrictions to Λ of $f \in E^p$ and we equip $E^p(\Lambda)$ with the quotient norm. We now define a *set of stable sampling* for E^p_K.

Definition 2.5. *We say that Λ is a set of stable sampling for E^p_K if one has $\|f\|_{E^p} \le C\|f\|_{E^p(\Lambda)}$ for every $f \in E^p_K$.*

In other words $F \in L^p(\mathbb{T}^2)$, $F(x) = 0$ almost everywhere on $\mathbb{T}^2 \setminus K$, and $\hat{F}(k) = \hat{G}(k)$, $k \in \Lambda$, imply $\|F\|_p \le C\|G\|_p$. This is the most convenient formulation. As we did above let us denote by $L^p(K)$ the Banach space of all restrictions to K of functions in $L^p(\mathbb{T}^2)$. This looks pedantic since $L^p(K)$ and L^p_K are identical. But function in L^p_K needs to be viewed as a function which vanishes outside K. If $1 \le p < \infty$ and $1/p + 1/q = 1$, then Definition 2.5 is equivalent to the following condition

Lemma 2.2. *A set of frequencies Λ is a set of stable sampling for E^p_K if and only if every $F \in L^q(K)$ is the restriction to K of a function $G(x) = \sum_{k\in\Lambda} a(k)\exp(2\pi ik\cdot x)$ which belongs to $L^q(\mathbb{T}^2)$.*

We now replace $L^q(\mathbb{T}^2)$ by a functional Banach space Y. If $\Omega \subset \mathbb{T}^2$ is an open set, Y_Ω denotes the Banach space of all restrictions to Ω of functions in Y and Y_Ω is equipped with the quotient norm. A set of stable sampling can be defined using a "direct" or a "dual" point of view. We begin with the "dual definition" which is Lemma 2.2 when $Y = L^q$.

Definition 2.6. *We denote by Y^Λ the Banach space of all functions in Y whose Fourier coefficients vanish outside Λ. Then Λ is a set of stable sampling for $Y(\Omega)$ if every function (or distribution) $f \in Y(\Omega)$ is the restriction to Ω of a function $g \in Y^\Lambda$.*

We now try to mimic the proof of Lemma 2.2. Let us assume that Y is the dual space of some Banach space B and let $\mathcal{B}(\Lambda)$ denote the space of restrictions to Λ of the Fourier transforms (i.e. Fourier coefficients) of functions in B. Then $\mathcal{B}(\Lambda)$ is equipped with the quotient norm. Then the direct definition reads as follows

Definition 2.7. *If $K \subset \mathbb{T}^2$ is a compact set, Λ is a set of stable sampling for Y_K if one has $\|f\|_B \leq C\|\hat{f}\|_{\mathcal{B}(\Lambda)}$ for every f in B_K.*

In other words if F is supported by K, if G belongs to the Banach space B and if the Fourier transforms of F and G coincide on Λ, we shall have $\|F\|_B \leq C\|G\|_B$. These two definitions may differ since we did not relate K to Ω and if Ω is the interior of K, then $Y(\Omega)$ is not the dual space of B_K.

We now define a set of *stable interpolation* for E^p_K. Let us denote by $\mathcal{C}_\Lambda$ the space of all continuous functions F on $\mathbb{T}^2$ such that $\hat{F}(k) = 0$ if $k \notin \Lambda$. We notice that (2.1) can be written $F \in \mathcal{C}_\Lambda \Rightarrow \|F\|_2 \leq \|F\|_{L^2(\Omega)}$. This observation leads to the following definition

Definition 2.8. *Let $K \subset \mathbb{T}^2$ be a Borel set and $p \in [1, \infty]$. We say that Λ is a set of stable interpolation for E^p_K if*

$$F \in \mathcal{C}_\Lambda \Rightarrow \|F\|_p \leq \|F\|_{L^p(K)} \tag{2.7}$$

When $p = 2$ this new definition is the one we gave above. If $1/p + 1/q = 1$ a duality argument yields the following lemma

Lemma 2.3. *Λ is a set of stable interpolation for E^p_K if and only if each sequence $(a(k))_{k\in\Lambda}$ belonging to $E^q(\Lambda)$ is the restriction to Λ of a sequence $f(k), k \in \mathbb{Z}^2,\ f \in E^q_K$.*

If $p = 1$ or $p = \infty$ this statement needs to be modified accordingly. Keeping these definitions in mind we have

Theorem 2.1. *As in Proposition 2.1, let $M \subset \mathbb{Z}^2$, let Λ be the complement of M in $\mathbb{Z}^2$ and let $K \subset \mathbb{T}^2$ be a Borel set. Let Ω be the compement of K in $\mathbb{T}^2$. If M is a set of stable sampling for E^p_K, then Λ is a set of stable interpolation for E^p_Ω. Conversely if Λ is a set of stable interpolation for E^p_Ω and if the indicator function of Λ is a multiplier for $\mathcal{F}L^p$, then M is a set of stable sampling for E^p_K.*

Before proving this theorem, let us observe that the set Λ which will be used below has the required property. This follows from the fact that the indicator function of an interval is a multiplier for $\mathcal{F}L^p$ and from the transference methods of R. Coifman and G. Weiss. The proof of Theorem 2.1 is similar to the one we gave for Proposition 2.1. We want to prove (2.7) when $F \in \mathcal{C}_\Lambda$. We have $F(x) = \sum_{k\in\Lambda} f(k)\exp(2\pi i k\cdot x)$. We then split $F(x)$ into the sum $G(x) + H(x)$ where G is the product between F and the indicator function of K and $H(x) = \mathbf{1}_\Omega F(x)$. Let $g(k)$ and $h(k),\ k \in \mathbb{Z}^2$, be the Fourier coefficients of G and H. We know that $f = g + h$ and we have $f(k) = 0$ whenever $k \in M$. Then $g = -h$ on M and the equivalent definition of a set of stable sampling yields $\|G\|_p \leq C\|H\|_p = C\|F\|_{L^p(\Omega)}$. The proof of (2.7) ends with $\|F\|_p \leq \|G\|_p + \|H\|_p \leq (1 + C)\|F\|_{L^p(\Omega)}$.

The converse implication is as easy. We start with two functions F and G such that F is supported by K and $\hat{F} = \hat{G}$ on M. We want to prove that $\|F\|_p \leq C\|G\|_p$. For proving it we denote by $f(k)$ the Fourier coefficients of F and by $g(k)$ the Fourier coefficients of G. We write $F = F_1 + F_2$ where $F_1(x) = \sum_{k\in M} f(k)\exp(ik\cdot x) = \sum_{k\in M} g(k)\exp(ik\cdot x)$ and $F_2(x) = \sum_{k\in\Lambda} f(k)\exp(ik\cdot x)$. Since Λ is a set of stable interpolation for E^p_Ω we obtain $\|F_2\|_p \leq C\|F_2\|_{L^p(\Omega)}$. But $F_1 + F_2 = 0$ on Ω which implies $\|F_2\|_{L^p(\Omega)} = \|F_1\|_{L^p(\Omega)} \leq \|F_1\|_p$. Finally the Fourier coefficients of F_1 are given by $f_1(k) = f(k) = g(k)$, $k \in M$, and 0 elsewhere. If $\mathbf{1}_M$ is a multiplier of $\mathcal{F}L^p$, we have $\|F_1\|_p \leq C\|G\|_p$ which ends the proof.

Proposition 2.2. *Let $\Omega \subset \mathbb{T}^2$ be an open set containing a compact set K. Let us assume that $q \geq p$ and that $\Lambda \subset \mathbb{Z}^2$ is a set of stable interpolation for E^q_K. Then Λ is a set of stable interpolation for E^p_Ω.*

The proof is not difficult and will be detailed if $q = \infty$ and $p = 2$. Let one denote by $B(0, \varepsilon)$ the ball centered at 0 with radius ε where ε is fixed such that $K + B(0, \varepsilon) \subset \Omega$. Next one denotes by $g \in L^2(B(0, \varepsilon))$ an arbitrary function satisfying $\|g\|_2 \leq 1$. Then if $\|F\|_{L^2(\Omega)} \leq 1$, one has $\|F * g\|_{L^\infty(E)} \leq 1$ and the spectrum of $F * g$ is included in that of F. Then (2.7) implies $\|F * g\|_\infty \leq C$, i.e. $|\int F(x)g(x_0 - x)dx| \leq C$, for every x_0. Optimizing in g one obtains

$$\Big(\int_{B(x_0,\varepsilon)} |F(x)|^2 dx\Big)^{1/2} \leq C. \tag{2.8}$$

It suffices to cover $\mathbb{T}^2$ with ε^{-2} such discs to obtain $\|F\|_2 \leq C\varepsilon^{-1}$. The same proof shows that a set of stable interpolation for E^q_K is a set of stable interpolation for E^p_Ω when $p \leq q$.

A final remark concerns the extension to l^p-norms, $p \in [1, \infty]$, of the definition of a set of stable sampling. Let us assume that $K \subset \mathbb{T}^2$ is a compact set. Given an exponent p we would like to know whether or not there exists a constant $C = C(K, M, p)$ such that

$$(\sum_{k \in \mathbb{Z}^2} |f(k)|^p)^{1/p} \leq C(\sum_{k \in M} |f(k)|^p)^{1/p} \tag{2.9}$$

whenever $f(k)$ are the of Fourier coefficients of a function or a distribution F supported by K. Let us stress that in general F is no longer a function when $p > 2$. Therefore the support of F is the closed support of a distribution. That is why K is closed in (2.9). The estimate (2.9) says that M is a set of stable sampling for a space that we define now. We let Y^p be the Banach space consisting of the functions or distributions (when $p > 2$) whose Fourier coefficients belong to l^p. Then the left-hand side of (2.9) is the norm in Y^p of $F(x) = \sum_{k \in \mathbb{Z}^2} f(k) \exp(2\pi i k \cdot x)$. This function or distribution is supported by K.

Definition 2.9. *Let Y^p_K denote the space of all functions or distributions in Y^p which are supported by K. If (2.9) holds for every $F \in Y^p_K$ we say that M is a set of stable sampling for the space Y^p_K.*

Let now $\Omega \subset \mathbb{T}^2$ be an open set. We define the Banach space $Y^p(\Omega)$ as the space of restrictions to Ω of all generalized functions $F \in Y^p$, the norm being the obvious quotient norm.

Definition 2.10. *A set Λ of stable interpolation for $Y^p(\Omega)$ is defined by $F \in \mathcal{C}_\Lambda \Rightarrow \|F\|_p \leq \|F\|_{Y^p(\Omega)}$.*

We then have

Proposition 2.3. *Let $1 \leq p \leq \infty$, $\Omega \subset \mathbb{T}^2$ be an open set and K a compact set contained in Ω. If we have $F \in \mathcal{C}_\Lambda \Rightarrow \|F\|_\infty \leq \|F\|_{L^\infty(K)}$, then $F \in \mathcal{C}_\Lambda \Rightarrow \|F\|_p \leq \|F\|_{Y^p(\Omega)}$. Therefore the complement M of Λ in $\mathbb{T}^2$ is a set of stable sampling for Y^p_R where R is the complement of Ω in $\mathbb{T}^2$.*

The proof is similar to the one we gave in the L^2 setting. It relies on a generalization of Proposition 2.1. This generalization only concerns the trivial implication (2.1) $\Rightarrow$ (2.2). The proof is identical to the one we gave when $p = 2$. We now prove Proposition 2.3. First we observe that Y^p is the dual of $Y^{p'}$ when p and p' are conjugate exponents (with an obvious modification if $p = 1$). Moreover a function on $\mathbb{T}^2$ belongs to Y^p if and only if it locally belongs to Y^p. We assume that ε is small enough to ensure $K + B(0, \varepsilon) \subset \Omega$. We then pick a test function g supported in $B(0, \varepsilon)$ and belonging to the unit ball of $Y^{p'}$. If $\|F\|_{Y^p(\Omega)} \leq 1$, then $\|F * g\|_{L^\infty(K)} \leq 1$. We then use the hypothesis to obtain $\|F * g\|_\infty \leq C$. We optimize in g as we did before. Then all local Y^p norms of F are controlled. This implies the required estimate on the full Y^p norm of F.

Finding necessary and sufficient conditions for (2.1) or (2.2) is out of reach and only necessary conditions are known when Ω and Λ are arbitrary. These necessary conditions were obtained by H. J. Landau in [1]. The upper or lower density of Λ are compared to the measure of Ω or K.

Definition 2.11. *The upper density* $D^+(\Lambda)$ *of* Λ *is defined as* $\limsup_{R\to\infty}\sup_x \frac{\#\{B(x,R)\cap\Lambda\}}{\pi R^2}$ *and the lower density* $D_-(\Lambda)$ *is defined by* $\liminf_{R\to\infty}\inf_x \frac{\#\{B(x,R)\cap\Lambda\}}{\pi R^2}$.

H. J. Landau proved the implications (2.1) $\Rightarrow D^+(\Lambda)\le|\Omega|$ and (2.2) $\Rightarrow D_-(\Lambda)\ge|K|$. These necessary conditions are not sufficient. Some sufficient conditions will be given below in Theorem 2.1. A sketch of the proof of Theorem 2.1 can be found in [2] (pages 39 to 50). For the reader's convenience a detailed proof is given now. Let us define $\Pi:\mathbb{R}\mapsto\mathbb{T}=\mathbb{R}/\mathbb{Z}$ by $\Pi(t)=t \pmod 1$. Then $\mathbb{Z}^2$ can be embedded in $\mathbb{T}$ by $\gamma^*:\mathbb{Z}^2\mapsto\mathbb{T}$ which is defined by

$$\gamma^*(m,n)=\Pi(m\sqrt{2}+n\sqrt{3}) \tag{2.10}$$

This mapping γ^* is injective with a dense range. With an obvious abuse of notations we still denote by Π the canonical mapping from $\mathbb{R}^2$ to $\mathbb{T}^2$. Then the dual mapping $\gamma:\mathbb{Z}\mapsto\mathbb{T}^2$ is given by $\gamma(k)=\Pi(k\sqrt{2},k\sqrt{3})$ and the range of γ will be denoted by Γ. Then Γ is dense in $\mathbb{T}^2$.

One denotes by $I\subset\mathbb{T}$ an arbitrary interval (or arc) of the circle. This arc is not necessarily centered in 0 and the complement of I in $\mathbb{T}$ is also an interval. Then the subset $\Lambda_I\subset\mathbb{Z}^2$ will be defined by

$$\Lambda_I=\{(m,n)\in\mathbb{Z}^2;\gamma^*(m,n)\in I\}. \tag{2.11}$$

If $I=[a,b]$ where $0<a<b<1$, $\Lambda_I=\{(m,n)\in\mathbb{Z}^2;\ \exists r\in\mathbb{Z}$ such that $a\le m\sqrt{2}+n\sqrt{3}-r\le b\}$. The density of Λ_I is uniform and equals $|I|$. An compact set $K\subset\mathbb{T}^2$ is Riemann integrable if the measure of the boundary of K is 0. Let us then define M_K as the set of all $k\in\mathbb{Z}$ such as $\gamma(k)\in K$. Then the density of M_K is uniform and equals the Lebesgue measure $|K|$ of K as it is proved in [2]. A first step leading to L^2 estimates is the following result.

Proposition 2.4. *Let us assume that* Ω *is an open set of measure* $|\Omega|>|I|$. *Then* Λ_I *is a set of stable interpolation for* E^∞_Ω.

In other words there exists a constant $C=C_{\Omega,I}$ such that for any function $F\in\mathcal{C}(\mathbb{T}^2)$ whose Fourier coefficients vanish outside Λ_I one has

$$\|F\|_\infty\le C\sup_{x\in\Omega}|F(x)|. \tag{2.12}$$

Before proving it, let us observe that one cannot replace the open set Ω by a compact set K in this Proposition 2.4. Indeed $|K|>|I|$ does not suffice to obtain (2.12). For studying the dependence in Ω of the constant $C_{\Omega,I}$, we introduce a new definition

Definition 2.12. *Let us consider a sequence* W *of positive numbers* w_j, $j\in\mathbb{N}$. *We say that an open set* $\Omega\subset\mathbb{T}^2$ *is* W*-thick if one can find a sequence of pairwise disjoint discs* $Q_j\subset\Omega$ *s.t.* $|Q_j|>w_j>0$, $j\in\mathbb{N}$.

For a given Ω one can always find a sequence W such that Ω is $W-$thick. When W is given, if an open set Ω is $W-$thick, then Ω contains "W-large discs".

Proposition 2.5. *We have* $C=C_{\Omega,I}\le C_{W,I}$ *in Proposition 2.4 if there exists a sequence* $W=(w_j)_{j\in\mathbb{N}}$ *with the following properties: (a)* $\sum_0^\infty w_j>|I|$ *and (b)* Ω *is* W*-thick.*

We return to the proof of Proposition 2.4. For proving (2.12) it suffices to assume that F is a finite trigonometrical sum. Then one has

$$F(x_1,x_2)=\sum_{(m,n)\in\Lambda_I}a_{(m,n)}e^{2\pi i(mx_1+nx_2)}. \tag{2.13}$$

It implies

$$F(k\sqrt{2}, k\sqrt{3}) = \sum_{(m,n)\in\Lambda_I} a_{(m,n)} e^{2\pi i k(m\sqrt{2}+n\sqrt{3})} = \hat{\mu}(-k) \tag{2.14}$$

where μ is the measure on $\mathbb{T}$ which is the sum of the Dirac masses $a_{(m,n)}$ at the points $\Pi(m\sqrt{2}+n\sqrt{3})$ belonging to I. If $\Pi(k\sqrt{2}, k\sqrt{3}) \in K$ then $k \in M_K$. Since $\Gamma = \gamma(\mathbb{Z})$ is dense in $\mathbb{T}^2$ we have

$$\sup_{\Omega} |F| = \sup_{k\in M_\Omega} |\hat{\mu}(-k)|. \tag{2.15}$$

Before stating our next lemma we return to the definition of the lower density.

Definition 2.13. *The lower density $D_-(M)$ of $M \subset \mathbb{Z}$ is the upper bound of the set of nonnegative numbers d such that for every $\varepsilon > 0$ there exists a $R(\varepsilon)$ such that for $R \geq R(\varepsilon)$ we have, uniformly in $m \in \mathbb{Z}$,*

$$d(R-\varepsilon) \leq \#\{M \cap [m, m+R]\} \tag{2.16}$$

We now use the following lemma

Lemma 2.4. *Then if $|I|$ denotes the length of the arc $I \subset \mathbb{T}$ and if $d > |I|$ where d is the lower density of M, then there exists a constant $C = C(M, I)$ such that*

$$\sup_{k\in\mathbb{Z}} |\hat{\mu}(k)| \leq C \sup_{k\in M} |\hat{\mu}(k)| \tag{2.17}$$

for any measure μ carried by I. If we are given a positive number d and a sequence M_j of sets of integers for which (2.16) holds uniformly in j, and if $d > |I|$, then we have $C(M_j, I) \leq C$.

A proof of Lemma 2.2 can be found in [2]. Let us sketch the argument for the reader's convenience. The proof relies on the following observations. If $M_j \subset \mathbb{Z}$, $j \in \mathbb{N}$, is a sequence of sets of integers, we say that M_j weakly converges to M if for each integer R, we have $M \cap [-R, R] = M_j \cap [-R, R]$ when $j \geq j_R$. The limit set may be the empty set and this weak convergence is the weak convergence in the weak-star topology $\sigma(L^\infty, L^1)$ of the indicator functions of M_j. If for a given d the sets M_j satisfy (2.16) uniformly in j then $d \leq D_-(M)$. We now denote by d the lower density of $M \subset \mathbb{Z}$. Then from any sequence n_j, $j \in \mathbb{N}$, one can extract a subsequence n'_j such that the sequence of sets $M_j = M - n'_j$ weakly converges to a limit set M'. The lower density of this set M' is still d. We then argue by contradiction. If (2.17) does not hold, one can find a sequence μ_j of measures carried by I and a sequence n_j of integers such that $|\hat{\mu}_j(n_j)| = 1$ while $\|\hat{\mu}_j\|_{l^\infty(M)} \leq 1/j$. Multiplying μ_j by a suitable constant we can assume $\hat{\mu}_j(n_j) = 1$. Let $\hat{\nu}_j(x) = \hat{\mu}_j(x + n_j)$ and let ν be a weak limit of a subsequence of these ν_j. Then ν is supported by I, $\hat{\nu} = 0$ on M' while $\hat{\nu}(0) = 1$. This contradicts the classical results on the density of zeros of entire functions of exponential type. The same proof yields the second statement in Lemma 2.2.

We now return to Proposition 2.4. Once again we use the fact that Γ is dense in $\mathbb{T}^2$ and we have

$$\sup_{k\in\mathbb{Z}} |\hat{\mu}(-k)| = \sup_{k\in\mathbb{Z}} |F(k\sqrt{2}, k\sqrt{3})| = \|F\|_\infty. \tag{2.18}$$

Then (2.15), (2.17), and (2.18) yield the required estimate (2.12). We now check Proposition 2.5. We set $E = Q_1 \cup \ldots \cup Q_N$ where $w_1 + \ldots + w_N > |I|$. Now N is fixed as everything else but the centers of the discs Q_j. The arguments used in Proposition 2.4 apply here with E replacing Ω and the proof of Proposition 2.5 ends with the following lemma

Lemma 2.5. *If $r > 0$ is given, $x \in \mathbb{T}^2$ is arbitrary, then the set of integers defined by $M = \{k \in \mathbb{Z},\ \gamma(k) \in B(x,r)\}$ has a uniform density given by πr^2. Moreover the estimates (2.16) are uniform in x.*

We turn to L^2 estimates.

Proposition 2.6. *We still assume that $\Omega \subset \mathbb{T}^2$ is an open set whose measure satisfies $|\Omega| > |I|$. Then there exists a constant $C = C(\Omega, I)$ such that for any continuous function F on $\mathbb{T}^2$ whose spectrum is included in Λ_I, one has*

$$\|F\|_2 \le C\|F\|_{L^2(\Omega)} \tag{2.19}$$

Moreover $C(\Omega, I) \le C(W, I)$ if there exists a sequence $W = (w_j)_{j\in\mathbb{N}}$ with the following properties: (a) $\sum_0^\infty w_j > |I|$ and (b) Ω is W-thick.

Proposition 2.2 and Proposition 2.4 imply Proposition 2.6. We now state our main theorem.

Theorem 2.2. *With the preceding notations, let $K \subset \mathbb{Z}^2$ be a compact set such that $|K| < |I|$. Then Λ_I is a set of stable sampling for Y_K^p for $1 \le p \le \infty$. In other words for any sequence $f \in l^p(\mathbb{Z}^2)$ of Fourier coefficients of a function F is supported by K, one has*

$$\Big(\sum_{k\in\mathbb{Z}^2} |f(k)|^p\Big)^{1/p} \le C\Big(\sum_{\lambda\in\Lambda_I} |f(\lambda)|^p\Big)^{1/p}. \tag{2.20}$$

Moreover $C = C_{K,I} \le C_{W,I}$ when the complement Ω of K satisfies :(a) $\sum_0^\infty w_j > 1 - |I|$ and (b) Ω is W-thick.

For proving Theorem 2.2 we denote by $J = I^c$ the complement of I in $\mathbb{T}$. We observe that $J \subset \mathbb{T}$ is still an arc. It suffices now to observe that the complement of Λ_I in $\mathbb{Z}^2$ is $M = \Lambda_J$ and to apply Proposition 2.3. The last assertion in Theorem 2.2 is following from the corresponding statement in Lemma 2.4. One cannot hope for a uniform estimate where $C = C(a_1, a_2)$ would only depend on the positive numbers $a_1 = |K|$ and $a_2 = |I|$. A counter-example will be given in Section 4. We now return to the mapping $\Phi : M_\beta \mapsto l^2(\Lambda_\alpha)$. We want to prove that this mapping is injective. If F_1 and F_2 are two images supported by two compact sets K_1 and K_2 whose measures do not exceed β, then $F = F_1 - F_2$ is carried by the compact set $K = K_1 \cup K_2$ whose measure does not exceed 2β. It suffices to apply Theorem 2.2 to F to conclude.

Independently A. Olevskii and A. Ulanovskii constructed sets of stable sampling and sets of stable interpolation. For functions of one real variable, they proved the following

Theorem 2.3. *For every positive d and ε there exists a sequence $\Lambda \subset \mathbb{R}$ satisfying the following conditions*

(a) $\|\Lambda - (1/d)\mathbb{Z}\|_\infty < \varepsilon$

(b) the family $\exp(i\lambda x)$, $\lambda \in \Lambda$, *is a frame in $L^2(S)$ for every compact set S of measure $< 2\pi d$.*

(c) for every open set S of measure $> 2\pi d$, any square summable sequence $a(\lambda)$, $\lambda \in \Lambda$, is the restriction to Λ of the Fourier transform of a function $f \in L^2(S)$.

3 Nonnegative images

The proof of Theorem 1.1 relies on the following theorem

Theorem 3.1. *Let $K \subset \mathbb{T}^2$ be a compact set such that $|K| < \alpha$. If $x_0 = \Pi(k_0\sqrt{2}, k_0\sqrt{3})$ does not belong to K, there exists an atomic measure σ on $\mathbb{T}^2$ with the following properties*

$$\sigma \geq 0 \text{ and } \sigma(\{x_0\}) = 1 \tag{3.1}$$

$$\sigma(K) = 0 \tag{3.2}$$

$$\text{the Fourier transform } \hat{\sigma} \text{ of } \sigma \text{ is supported by } \Lambda_\alpha. \tag{3.3}$$

A bound of the total mass $\|\sigma\|$ of σ will be given below. We postpone the proof of Theorem 3.1 and return to the second assertion in Theorem 1.1. Let K be the closed support of f. We know that $f \geq 0$ and $|K| < \alpha$. We want to compare f to a competitor u which verifies $u \geq 0$ and $\hat{u} = \hat{f}$ on Λ_α. Our first claim is that the proof reduces to the case where f and u are continous functions. For proving this remark let us consider $\varphi_j(x) = j^2\varphi(jx)$ where $\varphi \in \mathcal{C}_0^\infty, \varphi \geq 0$ and $\int \varphi(x)dx = 1$. Replacing f and u by $f_j = f * \varphi_j,\ u_j = u * \varphi_j$, we have $\hat{f}_j(k) = \hat{u}_j(k),\ k \in \Lambda_\alpha$. Moreover the support of f_j is contained in $K_j = \{x; \operatorname{dist}(x, K) \leq \frac{C}{j}\}$ where C depends on the support of φ. If we can prove $f_j = u_j$ for $j \geq j_0$ then we can conclude. We have $\lim |K_j| = |K| < \alpha$ which implies $|K_j| < \alpha$ for $j \geq j_1$. This shows that we can restrict our attention to f_j, u_j and K_j. We forget the subscript j and assume that f and u are smooth. Replacing K by a slightly larger set we can assume that K is Riemann integrable.

We have $\Lambda_\alpha = -\Lambda_\alpha$. Then if $x_0 \notin K$ we use Theorem 3.1 and write

$$0 \leq \int_{\mathbb{T}^2} u d\sigma = \sum_{k\in\mathbb{Z}^2} \hat{u}(k)\hat{\sigma}(-k) = \sum_{k\in\Lambda_\alpha} \hat{u}(k)\hat{\sigma}(-k) = \sum_{k\in\Lambda_\alpha} \hat{f}(k)\hat{\sigma}(-k) = \int_{\mathbb{T}^2} f d\sigma \tag{3.4}$$

But f vanishes on $\mathbb{T}^2 \setminus K$ and $\sigma(K) = 0$. Therefore $\int_{\mathbb{T}^2} f d\sigma = 0$ which together with (3.4) implies $\int_{\mathbb{T}^2} u d\sigma = 0$. Finally we use again the fact that u and σ are nonnegative. We have

$$0 \leq u(x_0) = \sigma(\{x_0\})u(x_0) \leq \int_{\mathbb{T}^2} u d\sigma = 0 \tag{3.5}$$

and $u(x_0) = 0$. Since the subgroup $\Gamma = \{\Pi(k\sqrt{2}, k\sqrt{3}),\ k \in \mathbb{Z}\}$ is dense in $\mathbb{T}^2$, we obtain $u = 0$ on $\mathbb{T}^2 \setminus K$. Therefore u is supported by K and Theorem 2.2 yields the required result.

The proof of the first assertion in Theorem 1.1 is almost trivial. Indeed let us assume that a competitor u exists with $\|u\|_1 \leq \|f\|_1$. We decompose u into a sum $u = u_1 - u_2 + iu_3$ where u_1 and u_2 are nonnegative functions or measures with disjoint supports and u_3 is real valued. Since $0 \in \Lambda_\alpha$ and f is nonnegative we have

$$\hat{u}_1(0) - \hat{u}_2(0) + i\hat{u}_3(0) = \hat{u}(0) = \hat{f}(0) = \|f\|_1 \geq \|u\|_1 \geq \|u_1 - u_2\|_1 = \hat{u}_1(0) + \hat{u}_2(0). \tag{3.6}$$

Therefore $\hat{u}_3(0) = \hat{u}_2(0) = 0$ which implies $u_2 = 0$ since u_2 is nonnegative. Finally the first and the last term in (3.6) are equal. Therefore all terms in (3.6) are equal and $\|u\|_1 = \|f\|_1$. Then (3.6) reduces to $\|u_1\|_1 = \hat{u}_1(0) = \hat{u}(0) = \|f\|_1 = \|u\|_1$ which implies $u_3 = 0$. Finally u is nonnegative and it now suffices to use the second assertion in Theorem 1.1.

We now prove Theorem 3.1. Once again $\gamma : \mathbb{Z} \mapsto \mathbb{T}^2$ is defined by $\gamma(k) = \Pi(k\sqrt{2}, k\sqrt{3})$. As above we write $\Gamma = \gamma(\mathbb{Z})$ and $x_0 = \gamma(k_0)$. Let $S \subset \mathbb{Z}$ be defined by $\gamma(k) \in K$. Since K is Riemann integrable, then S has a uniform density d which is given by $d = |K|$. We forget $\mathbb{T}^2$ and focus on $\mathbb{T}$ and $\mathbb{Z}$. The Fourier coefficients of a function $f \in L^1(\mathbb{T})$ are defined by $c(k) = \int_{\mathbb{T}} f(x)\exp(-2\pi ikx)\,dx$. The proof relies on the following lemmas.

Lemma 3.1. *Let us assume that $S \subset \mathbb{Z}$ has a uniform density $d \in (0,1)$. Let $J \subset \mathbb{T}$ be an interval centered at 0 with length $|J| > d$. Then there exists a constant C such that if $k_0 \notin S$ there exists a function $h \in L^2(J)$ such that $\hat{h}(k) = 0$ for every $k \in S$, $\hat{h}(k_0) = 1$ and $\|h\|_{L^2(J)} \leq C$.*

We then have

Lemma 3.2. *Let us assume that $S \subset \mathbb{Z}$ has a uniform density $d \in (0, 1/2)$. Let $I \subset \mathbb{T}$ be an interval centered at 0 with length $|I| > 2d$. Then there exists a constant C such that if $k_0 \notin S$ there exists a continuous function ϕ supported by I such that $\hat{\phi}(k) \geq 0$, $k \in \mathbb{Z}$, $\hat{\phi}(k) = 0$ for every $k \in S$, $\sum_{-\infty}^{+\infty} \hat{\phi}(k) \leq C$, and $\hat{\phi}(k_0) = 1$.*

The proof of Lemma 3.2 is obvious if Lemma 3.1 is accepted. It suffices to define ϕ by $\phi = h * \tilde{h}$ where $\tilde{h}(x) = \bar{h}(-x)$. We now return to Lemma 3.1. The proof is based on the following estimate.

Lemma 3.3. *Let us assume that $S \subset \mathbb{Z}$ has a uniform density $d \in (0,1)$. Let $J \subset \mathbb{T}$ be any interval of length larger than d. Then there exists a positive constant β such that for any $l \notin S$ and for any sequence $c(k) \in l^2(S)$, we have*

$$\| \exp(2\pi i l x) - \sum_{k \in S} c(k) \exp(2\pi i k x) \|_{L^2(J)} \geq \beta. \tag{3.7}$$

This estimate implies Lemma 3.1 with $C = 1/\beta$. We now prove Lemma 3.3 using the simplest form of Beurling's theorem [1]. Here is the statement

Theorem 3.2. *Let $\Lambda \subset \mathbb{Z}$ and let $D^+(\Lambda) = \limsup_{R \to +\infty} R^{-1} \sup_{k \in \mathbb{Z}} \#\{\Lambda \cap [k, k+R]\}$ be the upper density of Λ. Then if the length $|J|$ of an interval $J \subset \mathbb{T}$ satisfies $|J| > D^+(\Lambda)$ there exists a constant C such that for every sequence $c(k)$, $k \in \Lambda$*

$$\sum_{k \in \Lambda} |c(k)|^2 \leq C \int_J |\sum_{k \in \Lambda} c(k) \exp(2\pi i k x)|^2 \, dx. \tag{3.8}$$

For proving Lemma 3.3 we mimic the proof of Lemma 2.4 and argue by contradiction. Let us denote by H the Hilbert space $L^2(J)$. Let us assume that one can find a sequence $l_j \notin S$ and some coefficients $c(k,j)$ such that

$$\| \exp(2\pi i l_j x) - \sum_{k \in S} c(k,j) \exp(2\pi i k x) \|_H \leq 1/j. \tag{3.9}$$

The triangle inequality gives

$$\| \sum_{k \in S} c(k,j) \exp(2\pi i k x) \|_H \leq 2 \tag{3.10}$$

Then Beurling's theorem yields

$$(\sum_{k \in S} |c(k,j)|^2)^{1/2} \leq 2C. \tag{3.11}$$

This being said, we rewrite (3.9) as

$$\|1 - \sum_{k \in S_j} c(k + l_j, j) \exp(2\pi i k x)\|_H \leq 1/j. \tag{3.12}$$

with $S_j = S - l_j$.

We now use the fundamental assumption that S has a uniform density. Therefore we can replace the sequence S_j by a subsequence such that $S_j \rightharpoonup S'$. It means that for each $R \geq 1$ and $j \geq j(R)$

we have $S_j \cap [-R, R] = S' \cap [-R, R]$. Similarly we set $c_j(k) = c(k + l_j, j)$ and we can also assume that $c_j(k)$ weakly converges to $c'(k)$, $k \in \mathbb{Z}$. These two convergences imply the weak convergence $f_j(x) = \sum_{k\in S_j} c(k + l_j, j) \exp(2\pi ikx) \rightharpoonup f(x) = \sum_{k\in S'} c'(k) \exp(2ikx)$ as j tends to infinity. This weak convergence refers to the weak topology in the Hilbert space H. But $0 \notin S'$ since $0 \notin S_j$. Finally (3.12) yields $1 = \sum_{k\in S'} c'(k) \exp(ikx)$ in $L^2(J)$ which contradicts Beurling's theorem applied to $S' \cup \{0\}$.

We now return to Theorem 3.1. We have $|K| < \alpha$. This compact set K is replaced by a slightly larger compact set L which is Riemann integrable of measure $|L| < \alpha$. Then the set $S = \{k \in \mathbb{Z}\,; \gamma(k) \in L\}$ has a uniform density $d = |L| < \alpha$. Lemma 3.2 is applied to $I = [-\alpha, \alpha]$ when k_0 is defined by $\gamma(k_0) = x_0$. The atomic measure σ is defined by

$$\sigma = \sum_{-\infty}^{\infty} \hat{\phi}(k)\delta_{\gamma(k)} \tag{3.13}$$

where δ_a is the Dirac mass at a. Then σ is nonnegative. We have $\hat{\phi}(k) = 0$ whenever $\gamma(k) \in L$. This implies $\sigma(L) = 0$. We also have $\sigma \geq \delta_{x_0}$ since $\hat{\phi}(k_0) = 1$. Finally

$$\hat{\sigma}(-p, -q) = \sum_{-\infty}^{\infty} \hat{\phi}(k) \exp\left[2\pi i(p\sqrt{2} + q\sqrt{3})k\right] = \phi(p\sqrt{2} + q\sqrt{3}) = 0 \text{ when } (p, q) \notin \Lambda_\alpha. \tag{3.14}$$

This concludes the proof of Theorem 3.1.

We now prove Theorem 1.2. This proof relies on an estimate of the total mass of the measure σ in Theorem 3.1. This estimate will depend on the growth of the function $\beta(\varepsilon)$ of $\varepsilon > 0$ defined by $\beta(\varepsilon) = |K + B(0, \varepsilon)|$. We begin with a few remarks.

Lemma 3.4. *For every positive number η there are finitely many Riemann integrable compact sets $L \in \mathcal{L}$ such that for every compact set $K \subset \mathbb{T}^2$ one can find a $L \in \mathcal{L}$ such that $K \subset L \subset K + B(0, \eta)$.*

The proof of Lemma 3.4 is trivial. One uses a "fine grid" on $\mathbb{T}^2$ with step size $\eta/2$ and $\mathcal{L}$ is simply the collection of all finite unions of squares delimited by this grid.

Proposition 3.1. *With the preceding notations, we let η be small enough so that the measure $\beta(2\eta)$ of $K + B(0, 2\eta)$ is less than α. We also assume that η is smaller than the distance from x_0 to K. Then in Theorem 3.1 the total mass of σ does not exceed $C(\alpha, \beta(\eta))$.*

The value of η depends on the geometrical structure of the compact set K and not only on the measure of K. The proof of Proposition 3.1 is not difficult. We first use Lemma 3.4 and enlarge K into $L \in \mathcal{L}$ with $K \subset L \subset K + B(0, \eta)$ where η is small enough to ensure $\beta(\eta) = |K + B(0, 2\eta)| < \alpha$. We can assume that $x_0 \notin L$. Finally it suffices to rewrite the proof of Theorem 3.1 and to keep track of the constants which come out. As η tends to 0, the cardinality of $\mathcal{L}$ blows up and so does the mass of σ.

We now prove Theorem 1.2. As in the proof of Theorem 1.1 we can assume that u and f are continuous. Let K be the closed support of f. We define L by Lemma 3.4 and let $x_0 \notin L$. The total mass of the measure σ provided by Theorem 3.1 does not exceed $C(\alpha, \beta)$ which is defined by Proposition 3.1. Keeping notations as simple as possible, we write Λ for Λ_α and $u \geq 0$ implies

$$0 \leq u(x_0) \leq \int u\, d\sigma = \sum_{\Lambda} \hat{u}(\lambda)\hat{\sigma}(-\lambda) = \sum_{\Lambda} \hat{f}(\lambda)\hat{\sigma}(-\lambda) + \sum_{\Lambda} \hat{R}(\lambda)\hat{\sigma}(-\lambda) = \int f\, d\sigma + \int R\, d\sigma = I_1 + I_2.$$

Then $I_1 = 0$ since f is supported by L and $\sigma(L) = 0$. Moreover $\|R\|_\infty \le \varepsilon$ and $\|\sigma\| \le C$ imply $|I_2| \le C\varepsilon$. We obtain $0 \le u(x_0) \le C'\varepsilon$. This estimate is uniform in $x_0 \notin L$. We now write $u = u_1 + u_2$ where u_2 is the product between u and the indicator function of L. We then have $\|u_1\|_\infty \le C'\varepsilon$. This implies $\hat{u}_2(\lambda) = \hat{f}(\lambda) + \hat{r}_2(\lambda),\ \lambda \in \Lambda$, where $\|r_2\|_\infty \le C\varepsilon$. Theorem 1.2 results from the following lemma applied to $v = u_2 - f$ and $v = r_2$

Lemma 3.5. *If K is a compact set of measure $|K| < 2\alpha$, then there exists a constant C such that for any function u supported by K we have*

$$\|u\|_\infty \le C_K \inf\{\|v\|_\infty;\ \hat{u}(\lambda) = \hat{v}(\lambda),\ \lambda \in \Lambda_\alpha\} \tag{3.15}$$

where the infimum runs over all competitors v without any restriction on their supports. In other words Λ is a set of stable sampling for E_K^∞.

By duality Lemma 3.5 implies the following. Every measure μ on K is the restriction to K of a measure $\nu = \sum_{k\in\Lambda_\alpha} a(k)\exp(2\pi k \cdot x)$. This is almost Theorem 3.1. In this application, K is replaced by $K \cup \{x_0\}$. But Theorem 3.1 says more since we need to prove that ν is nonnegative. That explains why another proof is used for proving Theorem 3.1. Let us observe that in Theorem 3.1 $|K|$ is less than α while here it suffices to assume $|K| < 2\alpha$.

We now prove Lemma 3.5. We split u into a sum $u_1 + u_2$ where $\hat{u}_1 = \hat{u}\mathbf{1}_\Lambda$ and $\mathbf{1}_\Lambda$ is the indicator function of Λ. We know that $\hat{v}\mathbf{1}_\Lambda = \hat{v}\mathbf{1}_\Lambda$. This crude definition of u_1 will be modified at the end of the proof. Let us denote by M the complement of Λ is $\mathbb{Z}^2$ and by Ω the complement of K in $\mathbb{T}^2$. Then $\hat{u}_2$ is supported by M. The measure of Ω exceeds the density of M and Proposition 2.4 yields $\|u_2\|_\infty \le C\|u_2\|_{L^\infty(\Omega)}$. But $u_1 + u_2 = 0$ on Ω which implies $\|u_2\|_\infty \le C\|u_1\|_{L^\infty(\Omega)} \le C\|u_1\|_\infty$. The proof would finish if we could believe that $\hat{u}_1 = \hat{v}\mathbf{1}_\Lambda$ implies $\|u_1\|_\infty \le C\|v\|_\infty$. This cannot be true since the indicator function of Λ is not the Fourier-Stieljes transform of a measure on $\mathbb{T}^2$. For facing this issue we introduce a function β on $\mathbb{T}$ which is 1 on $[-\alpha+\varepsilon, \alpha-\varepsilon]$, is smooth and is supported by $[-\alpha, \alpha]$. We then define $B(p,q) = \beta(p\sqrt{2} + q\sqrt{3}),\ (p,q) \in \mathbb{Z}^2$, on $\mathbb{Z}^2$. Then $B(p,q),\ (p,q) \in \mathbb{Z}^2$, are the Fourier coefficients of an atomic measure ν. Finally we define u_1 by $\hat{u}_1(p,q) = B(p,q)\hat{u}(p,q)$ and proceed as above. The support of $u_2 = u - u_1$ is contained in M' which is defined as the set of all pairs $(p,q) \in \mathbb{Z}^2$ such that there exists a $r \in \mathbb{Z}$ with $|p\sqrt{2} + q\sqrt{3} - r| > \alpha - \varepsilon$. Therefore the density of M' is given by $1 - 2\alpha + 2\varepsilon$ which is smaller than the measure of Ω when ε is sufficiently small. This being said, the argument used in the "wrong proof" is valid and yields Lemma 3.5.

4 Counter-examples

We now construct some contre-examples.

Lemma 4.1. *There exist two nonnegative continuous functions u and v on $\mathbb{T}^2$ such that $u \ne v$ while $\hat{u}(k) = \hat{v}(k), k \in \Lambda_\alpha$.*

This lemma says that we cannot have uniqueness in Theorem 1.1 if the information concerning the measure of the support of f is dropped. The proof is simple. Let $\theta(t)$ be the triangle function on $\mathbb{T} = \mathbb{R}/\mathbb{Z}$ defined by $\theta(1/2) = 1$, $\theta(\alpha) = \theta(-\alpha) = 0$, θ being affine on $[\alpha, 1/2]$ and on $[1/2, 1-\alpha]$. Then

$$\theta(t) = \sum_{-\infty}^{\infty} (-1)^k \tau_k \exp(2\pi i k t),$$

where $\tau_k > 0$. We now consider the atomic measure $\tau = \sum_{-\infty}^{\infty} (-1)^k \tau_k \delta_{\gamma(k)}$ and we have, as above, $\hat{\tau}(p,q) = 0$ on Λ_α. The atomic measure τ can be written as the difference $\sigma - \rho$ where $\sigma = \sum_{-\infty}^{\infty} \tau_{2k} \delta_{\gamma(2k)}$. Then we have $\sigma > 0$, $\rho > 0$. To prove Lemma 4.1 it suffices to use the same approximation to the identity as in Section 3 and to define u_j and v_j by $u_j = \sigma * \varphi_j$, $v_j = \rho * \varphi_j$ where $\varphi \geq 0$. We have $u_j \neq v_j$ if j is large enough.

In the same spirit we have

Lemma 4.2. *For every positive ε there exists a compact set K of measure not exceeding ε and a continuous function f supported by K such that f is not the argument of the problem*

$$\inf\{\|u\|_1\,; \hat{u}(\lambda) = \hat{f}(\lambda),\ \lambda \in \Lambda_\alpha\}. \tag{4.1}$$

This lemma says that $f \geq 0$ is playing a key role in Theorem 1.1. We use the same atomic measure τ as before and split it into $\tau_N + \rho_N$ where $\tau_N = \sum_{|k|\leq N} (-1)^k \tau_k \delta_{\gamma(k)}$. Next we set $f_N = \tau_N * \phi_N$, $r_N = -\rho_N * \phi_N$ where $\phi_N = N^2 \phi(Nx)$. The function f we are looking for is f_N. We let $K = K_N$ be the closed support of f_N. Then the measure of K_N does not exceed $2/N$. Moreover

$$\hat{f}_N(\lambda) = \hat{r}_N(\lambda),\ \lambda \in \Lambda_\alpha. \tag{4.2}$$

But $\|f_N\|_1 > \|r_N\|_1$ since the latter tends to 0 as N tends to infinity. Therefore the challenger r_N is winning against f_N. This ends the proof of Lemma 4.2.

Lemma 4.3. *If $\beta > \alpha$, there exists a real valued continuous function $f \in M_{2\beta}$ which is not identically 0 and whose Fourier transform vanishes on Λ_α.*

We have $\alpha < 1/2$ and $\beta > \alpha$. We start with the rectangle $K \subset \mathbb{T}^2$ defined by $-1/2 \leq x_1 \leq 1/2$ and $-\beta \leq x_2 \leq \beta$. We let $Z \subset \mathbb{Z}$ be the set of all $k \in \mathbb{Z}$ such that $\gamma(k) \notin K$. Then $Z = -Z$ and the uniform density of Z is $1 - 2\beta < 1 - 2\alpha$. Lemma 3.1 yields a function $\theta \in L^2([\alpha, 1-\alpha])$ which is not identically 0, is supported by $[\alpha, 1-\alpha]$ and such that $\hat{\theta}(k) = 0$ if $k \in Z$. It is easy to see that one can impose to $\hat{\theta}$ to be real valued. By regularization θ can be assumed to belong to $\mathcal{C}_0^\infty([\alpha, 1-\alpha])$. Indeed the above construction can be applied to an interval $[-\alpha', \alpha']$ where $\alpha' > \alpha$ is close to α. This is letting enough room for a convolution with a smooth approximation of the identity. We now consider the atomic measure

$$\tau = \sum_{-\infty}^{\infty} \hat{\theta}(k) \delta_{\gamma(k)}. \tag{4.3}$$

Then τ is supported by K. Indeed $\hat{\theta}(k) = 0$ if $\gamma(k) \notin K$. Finally $\hat{\tau}(p,q) = 0$ when $(p,q) \in \Lambda_\alpha$. We the define $T_j \in \mathcal{C}^\infty(\mathbb{T}^2)$ by $T_j = \tau * \varphi_j$ where φ_j is defined as above. Then T_j is supported by $K_j = \{-1/2 \leq x_1 \leq 1/2; -\beta - 1/j \leq x_2 \leq \beta + 1/j\}$. The Fourier transform of T_j vanishes on Λ_α and T_j is not identically 0 if j is large enough. At the end of the proof, β has been replaced by $\beta + 1/j$ where j is large and this tells us that we needed to start with a β' such that $\beta > \beta' > \alpha$. Then we end with $\beta' + 1/j < \beta$ if j is large enough.

Lemma 4.4. *If $\beta > \alpha$ the mapping $\Phi:\ M_\beta \mapsto \Lambda_\alpha$ is not injective.*

We return to the proof of Lemma 4.3 and we decompose the rectangle K into two closed halves K' and K" of measure β. These adjacent rectangles have disjoint interiors. Then $f = T_j$ can be split into $f' - f$" where f' is the restriction of f to K' and is supported by K' and f" is the restriction of $-f$ to

K". We have $\Phi(f') = \Phi(f")$ but $f' \neq f"$.

We now turn to the issue discussed in Theorem 2.1. We wanted to know if the contant C in (2.20) depends only on $|I| - |K|$. A counter-example is given by the following theorem where $I = [-\alpha, \alpha]$ and $\Lambda = \Lambda_\alpha$ as in Theorem 2.1.

Theorem 4.1. *For every $\eta > 0$ and every integer N there exist a compact set K whose measure does not exceed η and a function f supported by K such that $\|f\|_2 = 1$ while $\sum_{\lambda\in\Lambda} |\hat{f}(\lambda)|^2 \leq N^{-2}$.*

Let M be the complement of Λ in $\mathbb{Z}^2$. The proof of Theorem 4.1 begins with the following lemma

Lemma 4.5. *Keeping the same notations as above, there exist a compact set K of measure not exceeding η and a function g such that the Fourier transform of g is supported by M, $\|g\|_2 = 1$ and $\int_{K^c} |g|^2\, dx \leq N^{-2}$.*

Here $K^c = \mathbb{T}^2 \setminus K$. We first accept this lemma and prove Theorem 4.1. We let f be the product between g and the indicator function of K. Then $\|f - g\|_2 \leq N^{-1}$ which implies $\sum_{\lambda\in\Lambda} |\hat{f}(\lambda) - \hat{g}(\lambda)|^2 \leq N^{-2}$. But $\hat{g}(\lambda) = 0$ if $\lambda \in \Lambda$. Therefore f is enjoying the properties listed in Theorem 4.1.

We now prove Lemma 4.4. Let θ and the atomic measure τ be defined as in Lemma 4.1. We consider the atomic measure $\tau = \sum_{-\infty}^{\infty} (-1)^k \tau_k \delta_{\gamma(k)}$ and we have, as above, $\hat{\tau}(p, q) = 0$ on Λ. As we did in the proof of Lemma 4.2, we split τ into $\tau_N + \rho_N$. We now consider $g = g_\varepsilon = \tau * \phi_\varepsilon = u_{N,\varepsilon} + v_{N,\varepsilon}$ where $u = u_{N,\varepsilon} = \tau_N * \phi_\varepsilon$ and $v = v_{N,\varepsilon} = \rho_N * \phi_\varepsilon$ with $\phi_\varepsilon(x) = \varepsilon^{-1}\phi(x/\varepsilon)$, ϕ being supported by $|x| \leq 1$ and normalized in L^2. Then $u_{N,\varepsilon}$ is supported by the union K of $2N + 1$ discs of measure $\pi\varepsilon^2$ Therefore the measure of K does not exceed $\pi(2N + 1)\varepsilon^2$. We then observe that the total mass of ρ_N is less than C/N. It implies $\|v\|_2 \leq C/N$ uniformy on ε. Once N is fixed, ε can be chosen small enough so that the supports of the $2N + 1$ terms in the expansion of u have disjoint supports. Then $C \leq \|u\|_2 \leq C'$ where C and C' are two positive constants. The triangle inequality implies the same conclusion for g. Finally the norm in $L^2(K^c)$ of g coincides with that of v since u is supported by K. But $\|v\|_2 \leq C/N$ which ends the proof.

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