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Galois Groupoids and Covering Morphisms in Topos Theory

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Abstract. The goals of this paper are (1) to compare the Galois groupoid that appears naturally in the construction of the fundamental groupoid of a topos $\mathscr{E}$ bounded over an arbitrary base topos $\mathscr{S}$ given by Bunge (1992), with the formal Galois groupoid defined by Janelidze (1990) in a very general setting given by a pair of adjoint functors, and (2) to discuss a good notion of covering morphism of a topos $\mathscr{E}$ over $\mathscr{S}$ which is general enough to include, in addition to the covering projections determined by the locally constant objects, also the unramified morphisms of topos theory given by those local homeomorphisms which are at the same time complete spreads in the sense of Bunge-Funk (1996, 1998).

In connection with these matters we also (3) introduce and study a notion of a Galois topos over $\mathscr{S}$ which generalizes that of Grothendieck (1971) and Moerdijk (1989), (4) explain the role of stack completions in distinguishing Galois groupoids from fundamental groupoids when the base topos $\mathscr{S}$ is arbitrary, (5) extend to the case of an arbitrary base topos $\mathscr{S}$ results of Bunge-Moerdijk (1997) concerning the comparison between the coverings and the paths fundamental group toposes, and (6) discuss pseudofunctorialy of the fundamental groupoid constructions and apply it to give a simple version (for locally paths simply connected toposes) of the (pushout version of the) van Kampen theorem for toposes given in Bunge-Lack (2001).

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1 Introduction

One of the purposes of this paper is to compare, given a locally connected and locally simply connected topos $\mathscr{E}$ over a base topos $\mathscr{S}$, the Galois groupoid of automorphisms of the canonical "point" (in effect, a "bag of points") of the coverings fundamental group topos of $\mathscr{E}$ as defined in [8], with the formal Galois groupoid of Janelidze [21] in this setting.

The notion of a locally constant object is central to these considerations; we begin then by investigating in section 2 (Locally constant objects in toposes) the connection between the notion of a locally constant object given in [12] (inspired by [8]), with that of Janelidze [21, 22] and, in passing, also with that of Barr and Diaconescu [3], with which both agree if the topos $\mathscr{E}$ is connected and the base topos $\mathscr{S}$ is Set, but not in general.

In section 3 (Stack completions and the fundamental groupoid of a topos) we revisit the construction of the coverings fundamental groupoid of a locally connected topos $\mathscr{E}$ over a base topos $\mathscr{S}$ given in [8], an account of which is briefly given in [13] under the implicit assumption that the base topos $\mathscr{S}$ is Set. A new ingredient is the observation, not previously made explicit in either [8] or [13], that there are *two* groupoids involved, to wit, for each cover U in $\mathscr{E}$, there is the Galois groupoid G_U of automorphisms of the universal cover of the topos of U-split objects in $\mathscr{E}$ on the one hand, and its stack completion π_U which classifies U-split torsors on the other. When the base topos is Set, this distinction dissapears and the two groupoids are usually identified in practice [19, 13]. This brings into consideration the desirability of assuming that the base (elementary) topos $\mathscr{S}$ satisfies an "axiom of (small) stack completions", which was suggested by Lawvere in 1974 and which is known to hold at least of all Grothendieck toposes $\mathscr{S}$, as shown in [16, 23].

We prove in section 4 (Galois groupoids and Galois toposes) that there is indeed an equivalence (not just a Morita equivalence) between the Galois groupoid of automorphisms of a universal cover of a locally connected and locally simply connected topos $e : \mathscr{E} \to \mathscr{S}$, and the Galois groupoid of the pure theory associated with the first pair of adjoints in the 3-tuple $e_! \dashv e^* \dashv e_*$ given by the (locally connected) geometric morphism e. We also show that, unlike the pure Galois theory of [21, 5] in the general case, the Galois theory for toposes is *implicit* in the construction of the Galois groupoid, and that this is so on account of the presence of the third adjoint in the sequel of three determined by e.

Also in section 4 we introduce and study the notion of a Galois topos over an arbitrary base topos $\mathscr{S}$. Although these relative Galois toposes are not assumed to be either connected or pointed, they come naturally equipped with a bag of points indexed by the connected components of a (non-connected) universal cover; this is in line with the view advocated by Grothendieck [20] that rather than fixing a single base point, one ought work with a suitable bag of points ("paquet des points"). Pointed connected

Galois toposes over $\mathscr{S}$ have been investigated by Moerdijk [27] following Grothendieck [18] (see also [23]). We obtain here characterization theorems in a manner parallel to [27]; in particular we show that Galois toposes in our sense (and which correspond, modulo the intervention of locales in the subject, to the multi-Galois toposes of Grothendieck [19]) are precisely the classifying toposes of prodiscrete (localic) groupoids in $\mathscr{S}$.

In section 5 (Locally paths simply connected toposes over an arbitrary base) we recall the paths version of the fundamental group topos [29, 30], give a constructive version of the existence of a comparison map from the paths to the coverings fundamental group toposes, and then we prove the equivalence of the comparison map under an assumption of the locally paths simply connected type. We also investigate pseudonaturality of the fundamental group topos constructions and their universal properties, and apply this to give a simplified version of the van Kampen theorem for locally paths simply connected toposes, in connection with [12].

In section 6 (Generalized covering morphisms and a van Kampen theorem) we turn to the topic of covering morphisms, also in connection with the Galois and fundamental groupoids. The notion we give here has good enough properties so that (a pushout version of) the van Kampen theorem from [12] holds for these coverings and is also general enough to include, not only the familiar covering projections determined by the locally constant objects, but also the unramified coverings introduced and studied in [10, 11, 17], namely those local homeomorphisms which are also complete spreads.

2 Locally constant objects in toposes

Let $\mathscr{E}$ be a topos bounded over an elementary topos $\mathscr{S}$ by means of a geometric morphism $e : \mathscr{E} \to \mathscr{S}$. Assume that $\mathscr{E}$ is *locally connected* over $\mathscr{S}$ by means of e in the sense of [4], that is, that there is given an $\mathscr{S}$-indexed left adjoint $e_! \dashv e^*$, where e is the geometric morphism given by the (necessarily $\mathscr{S}$-indexed) adjoint pair $e^* \dashv e_*$.

We begin by recalling a *fundamental pushout* construction in the definition of the fundamental groupoid of $\mathscr{E}$ as given in [8]. For an epimorphism $U \to 1$ in $\mathscr{E}$ (a *cover* in $\mathscr{E}$) denote by $\mathscr{G}_U$ the topos defined by the following pushout in $\mathbf{Top}_{\mathscr{S}}$

$$\begin{tikzcd} \mathscr{E}/U \arrow[r, "\varphi_U"] \arrow[d, "\rho_U"] & \mathscr{E} \arrow[d, "\sigma_U"] \\ \mathscr{S}/e_!U \arrow[r, "p_U"'] & \mathscr{G}_U \end{tikzcd} \qquad (*)$$

where $\varphi_U : \mathscr{E}/U \to \mathscr{E}$ is the canonical local homeomorphism and where ρ_U is the connected locally connected part in the (unique) factorization of

the composite $e\ \varphi_U : \mathscr{E}/U \to \mathscr{S}$, which is the composite of two locally connected hence locally connected, into a connected locally connected morphism followed by a (surjective) local homeomorphism.

The fundamental pushout construction $(*)$ was the motivation behind the "family" definition of a locally constant object formally introduced in [12] and given below in the special context of a locally connected topos $\mathscr{E}$ over $\mathscr{S}$. It is, in fact, the natural notion to consider in topology in the case of a non-connected cover; in the context of Grothendieck toposes it also appears in [18]. There is a non-constructively given equivalence with a "single object" description when the topos $\mathscr{E}$ is connected and defined over *Set*, as shown by Barr and Diaconescu [3], but it is not clear whether this equivalence can be made constructive.

Definition 2.1 An object A of (a locally connected topos) $\mathscr{E}$ (with structure map $e : \mathscr{E} \to \mathscr{S}$) is said to be *locally constant* if there exists a cover U in $\mathscr{E}$ which splits A in the sense that there is a morphism $\alpha : S \to e_!U$ in $\mathscr{S}$ and an isomorphism $\theta : e^*S \times_{e^*e_!U} U \to A \times U$ over U, where U is equipped with the morphism $\eta_U : U \to e^*e_!U$ given by the unit of the adjointness $e_! \dashv e^*$ evaluated at U.

The following result is proven in [12].

Proposition 2.2 *If $e : \mathscr{E} \to \mathscr{S}$ is locally connected, A an object and U a cover in $\mathscr{E}$, the following are equivalent:*

1. *A is locally constant split by U in the sense of Definition 2.1.*
2. *A is U-locally trivial with respect to the adjoint pair $e_! \dashv e^*$ in the sense of* [21]*, that is, there is a morphism $\zeta : A \times U \to e^*S$ such that the square*

$$\begin{tikzcd} A \times U \arrow[r, "\pi_2"] \arrow[d, "\zeta"] & U \arrow[d, "\eta"] \\ e^*S \arrow[r, "e^*\alpha"'] & e^*e_!U \end{tikzcd}$$

   is a pullback.
3. *There is a morphism $\alpha : J \to I$ in $\mathscr{S}$, a morphism $\eta : U \to e^*I$ in $\mathscr{E}$, and an isomorphism $\theta : e^*J \times_{e^*I} U \to A \times U$ over U.*

Remark 2.3 The equivalence between (1) and (2) in Proposition 2.2 shows that an object A of a locally connected topos $\mathscr{E}$ is locally constant (relative to its structure map $e : \mathscr{E} \to \mathscr{S}$) in the sense of Definition 2.1 if and only if the unique morphism $A \to 1$ is locally trivial with respect to the adjoint pair $e_! \dashv e^*$ in the sense of Janelidze [21]. The equivalence between (1) and (3) in Proposition 2.2 strips the notion to its barest terms and in particular has the desirable consequence of its *stability* under pullback along arbitrary geometric morphisms. Note that (3) is meaningful in an arbitrary topos $\mathscr{E}$ bounded over $\mathscr{S}$ and that it agrees with (1) (and so also with (2)) if

$\mathscr{E}$ is locally connected over $\mathscr{S}$. In [12] we gave (3) as the notion of a locally constant object in a topos $\mathscr{E}$ defined over $\mathscr{S}$.

Corollary 2.4 *Let $\varphi : \mathscr{F} \to \mathscr{E}$ be a geometric morphism in* $\mathbf{Top}_{\mathscr{S}}$. *If A is a locally constant object in $\mathscr{E}$ split by the cover U in $\mathscr{E}$ in the sense of* (2) *in Proposition 2.2, then $\varphi^* A$ is a locally constant object in $\mathscr{F}$, split by the cover $\varphi^* U$ in $\mathscr{F}$.*

Proof The conclusion is easily checked. □

Denote by $\mathrm{Spl}(U)$ the full subcategory of $\mathscr{E}$ determined by the U-split objects of $\mathscr{E}$ in the sense of Definition 2.1.

Lemma 2.5 [8] *Let*

$$\begin{tikzcd} \mathscr{C} \arrow[r, "q"] \arrow[d, "p"] & \mathscr{B} \arrow[d, "h"] \\ \mathscr{A} \arrow[r, "k"'] & \mathscr{D} \end{tikzcd}$$

be a pushout in $\mathbf{Top}_{\mathscr{S}}$ *where p and q are both locally connected. Then, each of h and k is locally connected. Furthermore, if p (respectively, q) is connected, then so is h (respectiveky k).*

Proposition 2.6 *Let $\mathscr{G}_U$ be the topos in the fundamental pushout $(*)$ applied to $\mathscr{E}$ and U. Then there exists an equivalence functor $\Phi : \mathscr{G}_U \to \mathrm{Spl}(U)$. Under this equivalence, the fully faithful functor ${\sigma_U}^* : \mathscr{G}_U \to \mathscr{E}$ corresponds to the inclusion of* $\mathrm{Spl}(U)$ *into $\mathscr{E}$.*

Proof An object of $\mathscr{G}_U$ in the fundamental pushout $(*)$ of toposes is precisely a locally constant object split by U in the sense of Definition 2.1. This is a consequence of the well-known fact that pushouts in $\mathbf{Top}_{\mathscr{S}}$ are calculated as pullbacks in **CAT** of the inverse image parts. A morphism from $\langle X\ , S \to e_! U\ , \theta\rangle$ to $\langle X'\ , S' \to e_! U\ , \theta'\rangle$, both objects in $\mathscr{G}_U$, is a pair of morphisms $f : X \to X'$ and $\alpha : S \to S'$, the latter over $e_! U$, compatible with the isomorphisms θ and θ'.

It follows from Lemma 2.5 that σ_U is connected and locally connected. Since the forgetful functor ${\sigma_U}^* : \mathscr{G}_U \to \mathscr{E}$ is then fully faithful and factors through the inclusion $\mathrm{Spl}(U) \hookrightarrow \mathscr{E}$, it follows from the equivalence of (1) and (3) in Proposition 2.2 that there is an equivalence $\Phi : \mathscr{G}_U \to \mathrm{Spl}(U)$. □

Remark 2.7 The equivalence functor $\Phi : \mathscr{G}_U \to \mathrm{Spl}(U)$ forgets the splitting of the U-split objects in the sense of Definition 2.1, as well as the compatibility with the splittings of any morphism between objects in $\mathscr{G}_U$. In the absence of Choice there is no inverse to Φ. The importance of this remark will emerge from the developments in the next section.

We next compare explicitly our notion of locally constant object with that of Janelidze [21, 5]. Therein it is given a pair of adjoint functors $I \dashv H$ with $I : \mathscr{E} \to \mathscr{S}$ and $H : \mathscr{S} \to \mathscr{E}$ (here $\mathscr{E}$ and $\mathscr{S}$ are *categories*) such that

for every object $E \in \mathscr{E}$ the counit $I^E H^E \to \mathrm{Id}_{\mathscr{S}/IE}$ of the induced adjoint pair $I^E \dashv H^E$ in the diagram

$$\begin{tikzcd} \mathscr{E}/E \arrow[r, shift right, "I^E"'] & \mathscr{S}/IE \arrow[l, shift right, "H^E"'] \end{tikzcd}$$

is an iso. In this context, a morphism $\alpha : A \to B$ in $\mathscr{E}$ is said to be *trivial* if the diagram

$$\begin{tikzcd} A \arrow[r, "\alpha"] \arrow[d, "\eta_A"] & B \arrow[d, "\eta_B"] \\ HIA \arrow[r, "HI\alpha"'] & HIB \end{tikzcd}$$

is a pullback; $\alpha : A \to B$ is a *locally trivial* morphism relative to the adjoint pair $I \dashv H$ if there exists $p : E \to B$ of effective descent such that $p^*(A, \alpha)$ is trivial in $\mathscr{E}/E$.

The category $\mathrm{Spl}(p)$ of locally trivial morphisms split by p is easily seen to be given as the pullback

$$\begin{tikzcd} \mathscr{E}/E & \mathscr{E}/B \arrow[l, "p^*"] \\ \mathscr{S}/IE \arrow[u, "H^E"] & \mathrm{Spl}(p) \arrow[l] \arrow[u] \end{tikzcd}$$

.

It is shown in [21] that if in the pure setting one lets $\mathrm{Gal}(p)$ be given by the canonical diagram

$$I((E \times_B E) \times_E (E \times_B E)) \begin{array}{c} \longrightarrow \\ \longrightarrow \\ \longrightarrow \end{array} I(E \times_B E) \begin{array}{c} \longrightarrow \\ \longleftarrow \\ \longrightarrow \end{array} I(E)$$

then, although $\mathrm{Gal}(p)$ is not in general an internal category in $\mathscr{S}$, it makes sense to consider its actions and to form a category $\mathscr{S}^{\mathrm{Gal}(p)}$. Furthermore, there is an equivalence

$$\mathrm{Spl}(p) \simeq \mathscr{S}^{\mathrm{Gal}(p)}$$

to be interpreted as the *fundamental theorem of Galois theory.*

In order to get an actual groupoid, it is assumed of the cover $p : E \to B$ considered in [21] that it is a normal (or regular) cover. As we show in the next section, the existence of a normal cover (actually of the *universal cover*) in the topos setting is a consequence of properties of the fundamental pushout construction.

If $p : E \to B$ is an I-*normal* object of $\mathscr{E}$ in the sense that $p \in \mathrm{Spl}(p)$, then as shown in [21], the canonical morphism

$$I((E \times_B E) \times_E (E \times_B E)) \to I(E \times_B E) \times_{I(E)} I(E \times_B E)$$

is an isomorphism, from which the desired groupoid structure can be extracted. The reader ought to consult [21, 5] for details.

In preparation for later work we prove next that there is also a Galois groupoid in the topos context and associated with a cover $p : U \to 1$, extracted from the pushout topos $\mathscr{G}_U$ by means of the Joyal-Tierney theorem [24] and additional topos-theoretic considerations [8].

Proposition 2.8 *Let $\mathscr{E}$ be a locally connected topos over $\mathscr{S}$, and let U be a cover in $\mathscr{E}$. Then, the topos $\mathscr{G}_U$ in the fundamental pushout $(*)$ is the classifying topos $\mathscr{B}G_U$ of an etale complete discrete localic groupoid G_U in $\mathscr{S}$, by an equivalence*

$$\mathscr{G}_U \simeq \mathscr{B}G_U$$

which identifies p_U with the canonical bag of points of $\mathscr{B}G_U$.

Proof It follows from properties of the fundamental pushout $(*)$ that the geometric morphism $p_U : \mathscr{S}/e_!U \to \mathscr{G}_U$ is a local homeomorphism. Indeed, since both φ_U and ρ_U are locally connected, so is p_U by Lemma 2.5, and since p_U is also totally disconnected in the sense of [8], it must be a local homeomorphism; it is surjective since $U \to 1$ is an epi, hence φ_U is surjective. From general facts about descent toposes [24, 29] it is derived that there is determined an (etale complete) discrete localic groupoid G_U whose discrete locale of objects is $e_!U$ and is such that $\mathscr{G}_U$ is equivalent to the classifying topos $\mathscr{B}G_U$. □

Recall that one of our goals is to compare, in the case of a *locally simply connected* topos $\mathscr{E}$ over $\mathscr{S}$, the Galois groupoid of Janelidze [21] associated with the universal cover, with the Galois groupoid G_U that arises as in Proposition 2.8 above. In section 4 we shall prove their equivalence as groupoids, but prior to that we will, in section 3, make explicit its connection with the fundamental groupoid $\pi_1^c(\mathscr{E})$ of $\mathscr{E}$ as constructed in [8].

3 Stack completions and the fundamental groupoid of a topos

In this section we update the definition of the coverings fundamental groupoid $\pi_1^c(\mathscr{E})$ given in [8] (see also [13]), where $\mathscr{E}$ is a locally connected topos over $\mathscr{S}$ by means of a geometric morphism $e : \mathscr{E} \to \mathscr{S}$. A new ingredient here is making explicit what was only implicit in those sources, namely, the role of stacks (for the topology of regular epimorphisms) of groupoids in $\mathscr{S}$, both in the definition of the fundamental groupoid, as in the nature of the coefficients for cohomology of $\mathscr{E}$. These developments require that the (elementary) topos $\mathscr{S}$ satisfy an additional assumption, to wit, roughly speaking, that the stack completions of groupoids in $\mathscr{S}$ be again groupoids in $\mathscr{S}$.

We begin by making some remarks about stack completions in the case of groupoids in $\mathscr{S}$. Stacks of category objects (for the regular epimorpisms topology of $\mathscr{S}$) are discussed in [14], and stack completions shown to exist (and described) in [15]. We are here interested only in groupoids.

For G a groupoid in $\mathscr{S}$ (regarded as a category object), its stack completion is given (up to equivalence) by any pair $< \mathscr{A}, F >$, with $\mathscr{A}$ an

$\mathscr{S}$-indexed category, $F : [G] \to \mathscr{A}$ an $\mathscr{S}$-indexed functor, and where $\mathscr{A}$ is a stack (of groupoids) and $F : [G] \to \mathscr{A}$ a weak equivalence. The stack completion of a groupoid G in $\mathscr{S}$, as a category object in which idempotents split, is identified in [15] with the $\mathscr{S}$-indexed category $\mathrm{Pointess}_{\mathscr{S}}(\mathscr{B}G)$ of $\mathscr{S}$-essential points of $\mathscr{B}G$. Also the $\mathscr{S}$-indexed category $\mathrm{Tors}(G)$, given by assigning to an object I of $\mathscr{S}$ the category $\mathrm{Tors}(G)^I$ of G-torsors in $\mathscr{S}/I$, and letting transition morphisms given by pullback, is a stack completion of G. Diaconescu's theorem [23], stating that for a groupoid G the topos $\mathscr{B}G$ classifies G-torsors, suggests yet another identification of the stack completion of $[G]$, as in the following theorem.

Theorem 3.1 *Let G be a groupoid in $\mathscr{S}$ given by a diagram*

$$G_1 \times_{G_0} G_1 \begin{array}{c}\longrightarrow\\ \longrightarrow\\ \longrightarrow\end{array} G_1 \begin{array}{c}\longrightarrow\\ \longleftarrow\\ \longrightarrow\end{array} G_0 \ .$$

Then, the stack completion of G is given by the $\mathscr{S}$-indexed category of points of $\mathscr{B}G$.

Proof There is a canonical bag of points $p_G : \mathscr{S}/G_0 \to \mathscr{B}G$ such that composing with it defines an $\mathscr{S}$-indexed functor

$$F : [G] \to \mathrm{Points}_{\mathscr{S}}(\mathscr{B}G).$$

Explicitly, given an object I of $\mathscr{S}$, $F^I : [I, G] \to \mathbf{Top}_{\mathscr{S}}(\mathscr{S}/I, \mathscr{B}G)$ assigns, to an object $x : I \to G_0$ of the category $[I, G]$, the composite $\mathscr{S}/I \to \mathscr{S}/G_0 \to \mathscr{B}G$ of the geometric morphism induced by x with p_G, and to a morphism $f : x \to y$ in $[I, G]$, given by a morphism $f : I \to G_1$ in $\mathscr{S}$ such that $d_0 f = x$ and $d_1 f = y$, the natural transformation $f : p_G x \to p_G y$. This assignment is easily checked to be $\mathscr{S}$-indexed.

It follows just as in [7] (Proposition 3.1) that $\mathrm{Points}_{\mathscr{S}}(\mathscr{B}G)$ is a stack, using for this now that the $\mathscr{S}$-essential surjections (a particular case of open surjections) are of effective descent in $\mathbf{Top}_{\mathscr{S}}$ [24].

It remains to verify that F is a weak equivalence. That for each I the functor F^I is fully faithful is a direct consequence of the etale completeness of G, which means that the square in $\mathbf{Top}_{\mathscr{S}}$ given below

$$\begin{tikzcd} \mathscr{S}/G_1 \arrow[r, "d_1"] \arrow[d, "d_0"] & \mathscr{S}/G_0 \arrow[d, "p_G"] \\ \mathscr{S}/G_0 \arrow[r, "p_G"'] & \mathscr{B}G \end{tikzcd} \tag{1}$$

is a pullback, and which holds for any discrete (localic) groupoid G. We show next that F is essentially surjective. Let $q : \mathscr{S}/I \to \mathscr{B}G$ be a geometric morphism, that is, an object of $\mathrm{Points}_{\mathscr{S}}(\mathscr{B}G)^I$. Consider the pullback of q

along p_G, which we claim looks like this:

$$\begin{tikzcd} \mathscr{S}/J \arrow[r, "r"] \arrow[d, "\alpha"] & \mathscr{S}/G_0 \arrow[d, "p_G"] \\ \mathscr{S}/I \arrow[r, "q"'] & \mathscr{B}G \end{tikzcd} \tag{2}$$

where $\alpha : J \to I$ is an epimorphism in $\mathscr{S}$ and where $r : \mathscr{S}/J \to \mathscr{S}/G_0$ is induced by some $y : J \to G_0$, so that $F^J(y) \cong \alpha^*(q)$, or F is indeed essentially surjective. The claim is true because the canonical p_G is a surjective local homeomorphism, a fact already noticed in [8] and used in the proof of Proposition 2.8 in the previous section, and that therefore so is the geometric morphism opposite it in the pullback; it follows that the latter is induced by an epimorphism $\alpha : J \to I$ in $\mathscr{S}$, as claimed. Furthermore, the geometric morphism $r : \mathscr{S}/J \to \mathscr{S}/G_0$ opposite q in the pullback is necessarily induced by a morphism $y : J \to G_0$ in $\mathscr{S}$, hence the result. □

Corollary 3.2 *Let G be a groupoid in $\mathscr{S}$ and let $\mathscr{G} = \mathscr{B}G$ be its classifying topos. Then, the following hold:*

1. *Any geometric morphism $p : \mathscr{S}/I \to \mathscr{G}$ of $\mathscr{G}$ is $\mathscr{S}$-essential.*
2. *With p as above, the inverse image part $p^* : \mathscr{G} \to \mathscr{S}/I$ is naturally equivalent to a functor*

$$\mathrm{Hom}_{\mathscr{G}/g^*(I)}(A \to g^*(I), (-) \times g^*(I)) : \mathscr{G} \to \mathscr{S}/I$$

where $A \to g^*(I)$ has the structure of a G-torsor in $\mathscr{G}/g^*(I)$.

Proof 1. Since the $\mathscr{S}$-indexed category $\mathrm{Points}_{\mathscr{S}}(\mathscr{B}G)$ and its subindexed category $\mathrm{Pointess}_{\mathscr{S}}(\mathscr{B}G)$ are both stack completions of (the externalization $[G]$ of) G, they must be equivalent.

2. The $\mathscr{S}$-indexed category $\mathrm{Tors}(G)$ is also a stack completion of $[G]$ and a full subcategory of $\mathscr{B}G = \mathscr{S}^{G^{op}}$, hence equivalent to $\mathrm{Pointess}_{\mathscr{S}}(\mathscr{B}G)$ by an equivalence which gives the desired representability of inverse image parts of $\mathscr{S}$-essential (bags of) points by torsors.

□

Definition 3.3 A topos $\mathscr{S}$ is said to satisfy the *(ASC)* (or the axiom of stack completions) if for any groupoid G in $\mathscr{S}$ there exists (i) a groupoid $\tilde{G}$ in $\mathscr{S}$ whose externalization $[\tilde{G}]$ is a stack, and (ii) a weak equivalence internal functor (or homomorphism of groupoids) $G \to \tilde{G}$.

It is known that any Grothendieck topos $\mathscr{S}$ satisfies this axiom on account of the existence of a small generating class [16, 23].

The *Morita equivalence theorem* from [15] takes on the following form in the case of groupoids.

Theorem 3.4 *Let $\mathscr{S}$ be a topos which satisfies the axiom of stack completions in the sense of Definition 3.3. Let G and H be groupoids in $\mathscr{S}$.*

Then there exists an equivalence of categories

$$\mathrm{Hom}_{\mathscr{S}}(\tilde{G},\tilde{H}) \simeq \mathbf{Top}_{\mathscr{S}}(\mathscr{B}G,\mathscr{B}H)$$

where $\tilde{G}$ and $\tilde{H}$ are the stack completions of G and H.

We now turn to *topos cohomology* of a locally connected topos $\mathscr{E}$ defined over a base topos $\mathscr{S}$, of which we shall furthermore assume satisfies (ASC) in the sense of Definition 3.3. Unlike previous accounts [23, 8, 13], we shall assume that the coefficients are (not just groupoids but)*stacks of groupoids* in $\mathscr{S}$. Denote by Gpstacks($\mathscr{S}$) (respectively Gpdstacks($\mathscr{S}$)) the category of groups K (respectively groupoids G) in $\mathscr{S}$ which are stacks (or whose externalizations, as $\mathscr{S}$-indexed categories, are stacks). Notice that classically this is not an issue, as when $\mathscr{S}$ is Set, the axiom of choice guarantees that every weak equivalence is a strong equivalence.

For each object K in Gpstacks($\mathscr{S}$) as defined above, the cohomology of $\mathscr{E}$ with coefficients in K is denoted by $H^1(\mathscr{E};K)$ and defined as the object in $\mathscr{S}$ of connected components of the category (which is a groupoid and a stack!) $\mathrm{Tors}(\mathscr{E};K)$, that is, as the "set" of isomorphism classes of K-torsors in $\mathscr{E}$. (We are not asserting, by the way, that the group $e^*(K)$ in $\mathscr{E}$ is a stack in $\mathscr{E}$.)

Consider now the equivalence

$$H^1(\mathscr{E};K) \simeq \Pi_0(\mathrm{colim}\{\mathrm{Tors}(\mathscr{E};K)^U\})$$

where the indexing is taken over a (cofinal set in a) generating poset of covers U in $\mathscr{E}$, and where $\mathrm{Tors}(\mathscr{E};K)^U$ is the groupoid of U-split K-torsors in $\mathscr{E}$. It is now *a priori* clear that the question of representing the functor $H^1(\mathscr{E};-)$ by some suitable (prodiscrete localic) groupoid $\pi_1^c(\mathscr{E})$ reduces to that of obtaining discrete groupoids π_U representing the functors $\mathrm{Tors}(\mathscr{E};-)^U$: Gpstacks($\mathscr{S}$)] $\to$ [Gpdstacks($\mathscr{S}$)], since then the desired groupoid $\pi_1^c(\mathscr{E})$ would be definable as a limit of a filtered system of the π_U.

The central idea in definining the fundamental groupoid of $\mathscr{E}$ is contained in the pushouts $(*)$ of the previous section and in the Galois groupoids G_U associated with them as in Proposition 2.8. The G_U are etale complete groupoids (hence Galois groupoids), and the toposes $\mathscr{B}(G_U)$ their classifying toposes.

Let $U \leq U$, where U and V are covers in $\mathscr{E}$, and let $\alpha : U \to V$ be a given morphism in $\mathscr{E}$ witnessing the relation. Then there is an induced geometric morphism $\varphi_\alpha : \mathscr{G}_U \to \mathscr{G}_V$ over $\mathscr{S}$ which commutes with the bags of points p_U and p_V, and which therefore may be interpreted as an object of $\mathbf{Top}_{\mathscr{S}}[\mathscr{B}G_U,\mathscr{B}G_V]_+$, the full subcategory of $\mathbf{Top}_{\mathscr{S}}[\mathscr{B}G_U,\mathscr{B}G_V]$ whose objects are geometric morpisms commuting with the canonical (bags of) points. The square brackets in both cases indicate that the morphisms in those Hom-categories are to be taken to be iso 2-cells in $\mathbf{Top}_{\mathscr{S}}$.

This is clear from the cube below, using the pushout property $(*)$ of the front face:

$$\begin{tikzcd}
 & \mathscr{E}/V \arrow[r] \arrow[d] & \mathscr{E} \arrow[d] \arrow[dl, phantom, "//"] \\
\mathscr{E}/U \arrow[ur] \arrow[r] \arrow[d] & \mathscr{E} \arrow[d] & \\
 & \mathscr{S}/e_!V \arrow[r] & \mathscr{G}_V \\
\mathscr{S}/e_!U \arrow[ur] \arrow[r] & \mathscr{G}_U \arrow[ur] &
\end{tikzcd}$$

As shown in [7] there is a strong equivalence of categories:

$$\mathrm{Hom}(G_U, G_V) \simeq \mathbf{Top}_{\mathscr{S}}[\mathscr{B}(G_U), \mathscr{B}(G_V)]_+.$$

We derive from it that there are induced groupoid homomorphisms $g_\alpha : G_U \to G_V$, for each α. Furthermore, since the geometric morphisms φ_α are connected (and locally connected), the homomorphisms g_α are *full and essentially surjective* by [28].

From the above diagram follows, by virtue of the connectedness of $\sigma_U : \mathscr{E} \to \mathscr{G}_U$, that the geometric morphisms φ_α depend on α up to unique iso 2-cell. In particular, the groupoid homomorphisms g_α, which are determined by the geometric morphisms φ_α up to iso 2-cells, also depend on the α up to unique iso 2-cells. This and the cofinality of $\mathrm{Cov}(\mathscr{E})$ implies that the 2-system is bifiltered (and biordered) in the sense of the following definition [25].

Definition 3.5 A 2-system $\{G_i\}$ of discrete groupoids, groupoid homomorphisms and 2-cells between them, indexed by a category $\mathscr{C}$, is said to be *bifiltered and biordered* if

1. For any two groupoids G_i , G_j in the system, there is $k \in \mathscr{C}$ and morphisms $\alpha : i \to k$ and $\beta : j \to k$, inducing homomorphisms $g_\alpha : G_i \to G_k$, $g_\beta : G_j \to G_k$.
2. For any other morphism $\alpha^{'} : i \to j$, and corresponding $g^{'}_\alpha : G_i \to G_j$, there exists a unique (iso) 2-cell $\lambda_{i,i'} : g_\alpha \to g^{'}_\alpha$.

We have also that for any group K (whether or not it is a stack) there is an equivalence

$$\mathbf{Top}_{\mathscr{S}}(\mathscr{B}(G_U), \mathscr{B}(K)) \simeq \mathrm{Tors}(\mathscr{E}; K)^U$$

natural in K. Indeed, by Diaconescu's theorem [23], the topos $\mathscr{B}(K)$ classifies K-torsors in $\mathscr{E}$, and under this correspondence, the U-split K-torsors in $\mathscr{E}$ are in bijection with those geometric morphisms $\varphi : \mathscr{E} \to \mathscr{B}(K)$ whose inverse image part φ^* factors through the inclusion of $\mathrm{Spl}(U)$ in $\mathscr{E}$ and the latter is equivalent to $\mathscr{B}(G_U)$); notice that the canonical $p_G : \mathscr{S}/G_0 \to \mathscr{B}(G_U)$ has no bearing in this equivalence.

It follows that the Galois groupoids G_U only *weakly* represent cohomology. In order to get a strong representation, we must consider their stack completions.

Lemma 3.6 *Let $\mathscr{S}$ be a topos satisfying (ASC) in the sense of Definition 3.3. Let $\mathscr{E}$ be a locally connected topos over $\mathscr{S}$, U a cover in $\mathscr{E}$, and G_U the Galois groupoid obtained in Proposition 2.8. Then, if $F_U : G_U \to \tilde{G_U}$ is the stack completion of G_U, the discrete groupoid $\pi_U = \tilde{G_U}$ represents the functor*

$$\mathrm{Tors}(\mathscr{E}; -)^U : \tilde{\mathrm{Gps}}(\mathscr{S}) \to \tilde{\mathrm{Gpds}}(\mathscr{S}).$$

Further, the weak equivalence $F_U : G_U \to \tilde{G_U}$ induces an equivalence $\tilde{F_U} : \mathscr{B}(G_U) \to \mathscr{B}(\tilde{G_U})$.

Proof It follows from Theorem 3.4 that for the groupoid G_U and any group K which is a stack in $\mathscr{S}$, there is a strong equivalence

$$\mathrm{Hom}(\tilde{G_U}, K) \simeq \mathbf{Top}_{\mathscr{S}}[\mathscr{B}(G_U), \mathscr{B}(K)]$$

and therefore, also a strong equivalence

$$\mathrm{Hom}(\tilde{G_U}, K) \to \mathrm{Tors}(\mathscr{E}; K)^U.$$

The equivalence of the classifying toposes is a consequence of the fact that $\mathscr{S}$ is a stack and therefore the functor $\mathscr{B}(-) \simeq \mathscr{S}^{(-)^{op}}$ carries weak equivalence functors into (strong) equivalence functors. □

Definition 3.7 Denote by $\Pi_1^c(\mathscr{E})$ the limit in $\mathbf{Top}_{\mathscr{S}}$ of the filtered system of toposes $\mathrm{Spl}(U)$ and connected locally connected geometric morphisms $h_{UV} : \mathrm{Spl}(U) \to \mathrm{Spl}(V)$ for each pair of covers U, V with $U \leq V$, indexed by a cofinal generating poset of covers in $\mathscr{E}$. The topos $\Pi_1^c(\mathscr{E})$ is said to be the *coverings fundamental group topos* of $\mathscr{E}$ over $\mathscr{S}$. It is equivalent, as a category, to the full subcategory of $\mathscr{E}$ generated by the locally constant covers in $\mathscr{E}$, in the sense of Definition 2.1. There is a connected locally connected geometric morphism $\sigma_{\mathscr{E}} : \mathscr{E} \to \Pi_1^c(\mathscr{E})$.

Theorem 3.8 *Let $\mathscr{E}$ be a locally connected topos over a base topos $\mathscr{S}$, where $\mathscr{S}$ is assumed to satisfy (ASC) in the sense of Definition 3.3. Then the coverings fundamental group topos $\Pi_1^c(\mathscr{E})$ defined in Definition 3.7 is the classifying topos of a prodiscrete localic groupoid $\pi_1^c(\mathscr{E})$ which represents first-degree cohomology of $\mathscr{E}$ with coefficients in discrete groups.*

Proof Let $\{\pi_U\}$ be system of groupoids obtained from the system of groupoids $\{G_U\}$ as in Lemma 3.6, that is, by taking stack completions, and where the transition homomorphisms are homomorphisms $g_{\tilde{UV}} : \pi_U \to \pi_V$ induced by the $g_\alpha : G_U \to G_V$ for *any* α witnessing $U \leq V$, as follows from the Morita equivalence theorem for discrete groupoids. Since the $g_\alpha : G_U \to G_V$ are full and essentially surjective homomorhisms, the same is true of the homomorphisms $g_{UV} : \tilde{\pi_U} \to \pi_V$. Furthermore, since the original 2-system is *bi*filtered and *bi*ordered solely on account of the dependency of the transition homomorphisms g_α on the witnessing morphisms $\alpha : U \to V$, the 1-system of the stack completions, whose transition homomorphisms $g_{\tilde{UV}}$ only depend on $U \leq V$, is simply filtered.

Let $\pi_1^c(\mathscr{E}) = \lim\{\pi_U\}$ be the limit of the filtered system of discrete groupoids, taken in the category of *localic groupoids*. We now prove that the prodiscrete groupoid $\pi_1^c(\mathscr{E})$ represents cohomology of $\mathscr{E}$ with coefficients in discrete groups which are stacks. Explicilty, the claim is that for a discrete group K which is a stack, there is an isomorphism

$$H^1(\mathscr{E}; K) \simeq [\pi_1^c(\mathscr{E}), K].$$

This follows from Lemma 3.6 and various canonical isomorphisms:

$$\begin{aligned}[\pi_1^c(\mathscr{E}), K] &\simeq \Pi_0(\mathrm{Hom}(\pi(\mathscr{E}), K)) = \Pi_0(\mathrm{Hom}(\lim\{\pi_U\}, K)) \\ &\simeq \Pi_0(\mathrm{colim}\{\mathrm{Hom}(\pi_U, K)\}) \\ &\simeq \Pi_0(\mathrm{colim}\{\mathrm{Tors}(\mathscr{E}; K)^U\}) \simeq \Pi_0(\mathrm{Tors}(\mathscr{E}; K)) \simeq H^1(\mathscr{E}; K)\end{aligned}$$

□

In the rest of the paper we shall be dealing for the most part with locally simply connected toposes. We recall the definition.

Definition 3.9 A locally connected topos $\mathscr{E}$ over $\mathscr{S}$ is said to be *locally simply connected* if there is a single cover U in $\mathscr{E}$ which splits all locally constant objects in $\mathscr{E}$ in the sense of Definition 2.1.

4 Galois groupoids and Galois toposes

In this section we compare, in the case of a (locally connected and) locally simply connected topos $\mathscr{E}$ over $\mathscr{S}$, the Galois groupoid of [21] given any $e_!$-normal cover in $\mathscr{E}$, with the fundamental groupoid $\pi_1^c(\mathscr{E})$; in particular, this applies to the universal cover in $\mathscr{E}$, easily seen to be $e_!$-normal cover in $\mathscr{E}$.

Recall (e.g., from [7]) the notion of a G-torsor in $\mathscr{E}$, for G a groupoid in $\mathscr{E}$ given by means of a diagram

$$G_1 \times_{G_0} G_1 \;\begin{array}{c}\longrightarrow\\ \longrightarrow\\ \longrightarrow\end{array}\; G_1 \;\begin{array}{c}\longrightarrow\\ \longleftarrow\\ \longrightarrow\end{array}\; G_0 \ .$$

A (right) G-object $\mathbf{T}$ is given by the data $< \gamma : T \to G_0 \ , \ \theta >$ with T an object of $\mathscr{E}$, $p : T \to 1$ is the unique morphism into the terminal object 1, and $\theta : T \times_{G_0} G_1 \to T$ an action (unitary and associative). We say that $\mathbf{T}$ is a *G-torsor* in $\mathscr{E}$ if it is a right G-object for which $p : T \to 1$ is an epimorphism and

$$< \mathrm{proj}_1 \ , \ \theta >: T \times_{G_0} G_1 \to T \times T$$

is an isomorphism.

Definition 4.1 Let $e : \mathscr{E} \to \mathscr{S}$ be a locally connected geometric morphism. By an *$\mathscr{S}$-Galois family* of $\mathscr{E}$ we mean an object X of $\mathscr{E}$ of global support, equipped with a morphism $\delta : X \to e^*I$, such that the pair $(X \ , \ \delta)$ is both connected and a $\mathrm{Aut}(\delta)$-torsor in $\mathscr{E}/I$ over $\mathscr{S}$. Explicitly, for the

the canonical action θ of $e^*(\mathrm{Aut}(\delta))$ (a discrete groupoid in $\mathscr{E}$ with object of objects e^*I) on δ, the induced

$$< \mathrm{proj}_1 \ , \ \theta >: X \times_{e^*I} e^*(Aut(\delta)) \to X \times X$$

is an isomorphism. In particular, X is a locally constant object in the sense of Definition 2.1 and a cover in $\mathscr{E}$.

We omit the subscripts U in what follows for the geometric morphisms σ_U and p_U which occur in the pushout diagram $(*)$ defining $\mathscr{G}_U$.

Denote by $g : \mathscr{G}_U \to \mathscr{S}$ the structure geometric morphism. There is a natural isomorphism $g\sigma \simeq e$. Since $\sigma : \mathscr{E} \to \mathscr{G}_U$ is connected and locally connected and since $e : \mathscr{E} \to \mathscr{S}$ is locally connected, it follows that $g : \mathscr{G} \to \mathscr{S}$ is locally connected. We will use this in what follows.

Theorem 4.2 *Let $\mathscr{E}$ be a locally connected topos bounded over $\mathscr{S}$ by means of a geometric morphism $e : \mathscr{E} \to \mathscr{S}$. Let U be a cover in $\mathscr{E}$. Let $p : \mathscr{S}/e_!U \to \mathscr{G}_U$ be the discrete point that arises in the pushout definition $(*)$. Then, The morphism $p : \mathscr{S}/e_!U \to \mathscr{G}_U$ is a surjective local homeomorphism, its inverse image part is represented (uniquely up to isomorphism) by an $\mathscr{S}$-Galois family $(\sigma^*A, \sigma^*\zeta)$ and the square*

$$\begin{tikzcd} \mathscr{E}/\sigma^*A \arrow[r, "\varphi_{\sigma^*A}"] \arrow[d, "\rho_{\sigma^*\zeta}"'] & \mathscr{E} \arrow[d, "\sigma"] \\ \mathscr{S}/e_!U \arrow[r, "p"'] & \mathscr{G}_U \end{tikzcd}$$

is a pullback. In addition, $\mathscr{G}_U$ is generated by a single $\mathscr{S}$-Galois family $(\sigma^*A, \sigma^*\zeta : \sigma^*A \to e^*e_!U)$ and is the classifying topos of the discrete localic groupoid $\mathrm{Aut}(\sigma^*\zeta)$.

Proof It follows from Lemma 2.5 that p is locally connected therefore $\mathscr{S}$-essential; it is also surjective. Therefore, by [6] Proposition 1.2, its inverse image p^* is represented by a family $\zeta\colon A \to g^*e_!U$ in the sense that there is a natural isomorphism

$$p^* \simeq \hom_{\mathscr{G}_U(\mathscr{E})/g^*e_!U}(\zeta \ , \ \Delta_{e_!(U)}(-))$$

where $\Delta_{e_!(U)}(X) = X \times g^*e_!(U)$, equipped with the appropriate projection. The morphism $\zeta\colon A \to g^*e_!U$ represents p^* which has a right adjoint; in particular, $\zeta\colon A \to g^*e_!U$ is connected in $\mathscr{G}_U/g^*e_!U$. From the remark that there is a natural isomorphism

$$\hom_{\mathscr{G}_U/g^*e_!U}(\zeta \ , \ \Delta_{e_!(U)}(-)) \simeq \hom_{\mathscr{G}_U}(A, -)$$

and the faithfulness of p follows that A has global support. An explicit description of the representing family is given in the proof of [7] Theorem 1.2; let $\zeta = (p^{e_!U})_!(\delta_{e_!U} : e_!U \to e_!U \times e_!U)$.

We shall also regard this family as one in $\mathscr{E}/e^*e_!U$ by applying the fully faithful σ^*.

That p is a local homeomorphism is proven in [8] by recourse to the notion of a totally disconnected geometric morphism. In particular, there is induced an equivalence $\mathscr{G}_U/A \simeq \mathscr{S}/e_!U$ which identifies the localic point p with the local homeomorphism $\varphi_A : \mathscr{G}_U/A \to \mathscr{G}_U$. The statement about the mentioned square being a pullback is a consequence of this.

Also a consequence of this is the claim that $(\sigma^*A, \sigma^*\zeta)$ is a torsor, as can be seen by evaluating the naturally isomorphic functors p^* and $\varphi^*\sigma^*$ at σ^*A; this gives the desired isomorphism

$$< \mathrm{proj}_1 \ , \ \theta >: \sigma^*A \times_{e^*e_!U} e^*(Aut(\sigma^*\zeta)) \to \sigma^*A \times \sigma^*A$$

as in Definition 4.1.

The family $\zeta : A \to g^*e_!U$ representing p^* is generating for $\mathscr{G}_U$ since p^* is faithful. Hence $\mathscr{G}_U$ is generated by a single $\mathscr{S}$-Galois family.

Finally, since p is a local homeomorphism hence open, and since it is surjective, it is of effective descent by [24]. It follows that there is an equivalence $\mathscr{G}_U \simeq \mathscr{B}(G_U)$ where $G_U = \mathrm{Aut}(p)$ is a localic groupoid, discrete since p is (not just open but) a local homeomorphism.

From the fact that ζ represents p^* now follows that $\mathrm{Aut}(p)$ and $\mathrm{Aut}(\sigma^*\zeta)$ are equivalent groupoids in $\mathscr{S}$. □

The following proposition states that the universal cover $\sigma^*\zeta : \sigma^*A \to e^*e_!U$ in the locally simply connected topos $\mathscr{E}$ is $e_!$-normal in the sense of [21]. This is immediate from the properties of universal covers; we give below an alternative and direct proof of the assertion in question using only part of the data supplied by the pushout square $(*)$ in $\mathbf{Top}_{\mathscr{S}}$.

Lemma 4.3 *The $\mathscr{S}$-Galois family ζ representing p^* as in Theorem 4.2 may be alternatively obtained as the morphism*

$$\sigma^*(u_{\sigma_!U}) : \sigma^*\sigma_!U \to \sigma^*g^*g_!\sigma_!U \simeq e^*e_!U,$$

where u is the unit of the adjointness $g_! \dashv g^*$.

Proof This follows readily from Theorem 4.2 and the commutativity of the pushout square $(*)$, in particular, from the natural isomorphism $p_!\rho_! \simeq \sigma_!\varphi_{U!}$. □

Proposition 4.4 *Let $\mathscr{E}$ be locally simply connected with U as a single cover splitting all locally constant objects. Let (A, ζ) be the $\mathscr{S}$-Galois family of $\mathscr{E}$ with ζ representing $p : \mathscr{S}/e_!U \to \mathscr{G}_U = \mathscr{B}(\pi_1^c(\mathscr{E}))$. Then (A, ζ), which we identify with $(\sigma^*A, \sigma^*\zeta)$ if regarded in $\mathscr{E}$, is a universal cover in the sense that σ^*A is a cover, $(\sigma^*A, \sigma^*\zeta)$ is a normal object for the adjoint pair $e_! \dashv e^*$, and σ^*A splits every locally constant object of $\mathscr{E}$.*

Proof It is shown in Theorem 4.2 that the discrete groupoid G_U is the groupoid of automorphisms of the localic point p and, since the latter is represented by an $\mathscr{S}$-Galois family $\zeta : A \to g^*e_!U$, G_U is also realized as the discrete groupoid $\mathrm{Aut}(\zeta)$. Furhermore, a concrete instance of such a family is $\sigma^*(u_{\sigma_!U}) : \sigma^*\sigma_!U \to e^*e_!U$.

We now show that $\sigma^*A = \sigma^*\sigma_!U$ has global support and is a normal object (for the adjoint pair $e_! \dashv e^*$). For the former, notice that since σ is connected and locally connected (in particular $\sigma_!1 \simeq 1$) and U has global support, $\sigma^*A = \sigma^*\sigma_!U \to \sigma^*\sigma_!1 \simeq 1$ is an epimorphism.

For the latter, observe that since σ^* is fully faithful and $A = \sigma_!U \in \mathrm{Spl}(U)$, $\sigma^*A \simeq \sigma^*\sigma_!U$ is U-split. Use now Lemma 5.3 of [12] applied to σ (since it is locally connected, and in particular the adjoint pair $\sigma_! \dashv \sigma^*$ is $\mathscr{S}$-indexed); it says in this case that for any object X of $\mathscr{G}_U$, U splits σ^*X if and only if $\sigma_!U$ splits X. In particular, since U splits $\sigma^*A = \sigma^*\sigma_!U$, it follows that $\sigma_!U$ splits $\sigma_!U$, hence (by stability) $\sigma^*A = \sigma^*\sigma_!U$ splits $\sigma^*A = \sigma^*\sigma_!U$. □

Corollary 4.5 *Let $\mathscr{E}$ be a locally connected and locally simply connected topos over $\mathscr{S}$. Let U be a single cover which splits all locally constant objects of $\mathscr{E}$. Let A be the universal cover. Let $H(\mathscr{E}) = \mathrm{Gal}(\sigma^*\zeta)$ be the Galois groupoid obtained as in* [21] *using that $\sigma^*\zeta$ is an $e_!$-normal object, and let $G = \pi_1^c(\mathscr{E})$. Then, there is an equivalence $H \simeq G$, explicitly,*

$$\mathrm{Gal}(\sigma^*\zeta) \simeq \pi_1^c(\mathscr{E}).$$

Proof This follows readily from the isomorphism

$$\sigma^*A \times_{e^*e_!U} e^*(Aut(\sigma^*\zeta)) \to \sigma^*A \times \sigma^*A$$

applying the functor $e_!$ to both sides to get

$$e_!(\sigma^*A \times_{e^*e_!U} e^*(Aut(\sigma^*\zeta))) \simeq e_!(\sigma^*A \times \sigma^*A)$$

and then using that

$$e_!\sigma^*A \simeq g_!\sigma_!\sigma^*\sigma_!U \simeq g_!\sigma_!U \simeq e_!U$$

and *Frobenius Reciprocity* for the $\mathscr{S}$-indexed adjoint pair $e_! \dashv e^*$ to obtain

$$e_!(\sigma^*A \times_{e^*e_!U} e^*(Aut(\sigma^*\zeta))) \simeq e_!U \times_{e_!U} Aut(\sigma^*\zeta) \simeq Aut(\sigma^*\zeta)$$

hence the desired isomorphism

$$Aut(\sigma^*\zeta) \simeq e_!(\sigma^*A \times \sigma^*A).$$

□

Remark 4.6 In Proposition 4.4 we have obtained a normal cover (X, δ) solely from the ($\mathscr{S}$-indexed) adjoint pair $e_! \dashv e^*$ and the pullback of inverse image parts arising from the pushout $(*)$ in the construction of $\mathscr{G} = \mathscr{G}_U$. As in the pure theory of [21], it follows that there is given a groupoid $\mathrm{Gal}(X, \delta)$ in $\mathscr{S}$ and that the "fundamental theorem of Galois theory" holds.

There is one further step which is needed in order to legitimate the claim that $\mathrm{Gal}(X, \delta)$ is indeed equivalent to the fundamental groupoid and not just Morita equivalent to it; it is precisely in that further step that the *right adjoint* e_* (and what derives from it in the pushout $(*)$) enters. This suggests that an "enriched" pure setting could perhaps give everything one would want of a Galois theory, but it is also quite possible that were the pure theory be strenghtened in the sense indicated, then some of its interesting examples [21, 5] might be lost. This needs further inverstigation.

To sum up: starting with a pure set-up given by an adjoint pair $e_! \dashv e^*$ satisfying certain conditions explicitated in [21], we have furthermore in the topos setting, *primo*, that from the $\mathscr{S}$-indexedness of the adjoint pair $e_! \dashv e^*$ follows the existence of a normal cover (X, δ), and that in consequence one gets (as in [21]) a groupoid $\mathrm{Gal}(X, \delta)$ for which the "fundamental theorem of Galois theory" holds; *secundo*, that using the further right adjoint e_* and everything that derives from it in the pushout $(*)$, one has that (X, δ) is σ^* applied to a family which represents (the inverse image part of) the surjective $\mathscr{S}$-essential point p_U of the topos $\mathrm{Spl}(U)$ and is therefore an $\mathscr{S}$-ATO in the appropriate slice category, and *tertio*, from the additional observation that p_U is a local homeomorphism follows that (X, δ) is a Galois family (a torsor, not just an $\mathscr{S}$-ATO) in $\mathscr{E}$, and that $\mathrm{Gal}(X, \delta)$ *is* equivalent to the (Galois) groupoid of automorphisms of the universal cover.

We will now consider an unpointed version of a notion of a Galois topos introduced by Moerdijk [27] following Grothendieck [1].

Definition 4.7 A topos $\mathscr{E}$ bounded over $\mathscr{S}$ will be said to be a *$\mathscr{S}$-Galois topos* if $\mathscr{E}$ is locally connected and has an internal site $\mathbf{C}$ of definition determined by objects A of $\mathscr{E}$ for which there exists a morphism $\zeta : A \to e^*I$ which is an $\mathscr{S}$-Galois family in $\mathscr{E}$ in the sense of Definition 4.1.

We now set out to *characterize* $\mathscr{S}$-Galois toposes. For this we need to observe the behaviour of $\mathscr{S}$-Galois objects under the transition morphisms in the system whose limit gives the fundamental groupoid.

Lemma 4.8 *Let $\mathscr{E}$ and $\mathscr{F}$ be locally connected toposes over $\mathscr{S}$ with corresponding structure maps e and f. Let $\varphi : \mathscr{F} \to \mathscr{E}$ be a connected and locally connected geometric morphism over $\mathscr{S}$ and let $\zeta : A \to e^*I$ be any $\mathscr{S}$-Galoisfamily in $\mathscr{E}$. Then,*

$$\varphi^*A \xrightarrow{\varphi^*\zeta} \varphi^*e^*I \cong f^*I$$

is an $\mathscr{S}$-Galois family in $\mathscr{F}$.

Proof Observe that $\varphi^*(Aut(\zeta)) \simeq Aut(\varphi^*(\zeta))$ since φ^* is a fully faithful left exact functor which has a right adjoint. Observe also that, since φ is connected and locally connected, if $\zeta : A \to e^*I$ is connected in $\mathscr{E}/e^*I$, then $\varphi^*(\zeta) : \varphi^*(A) \to \varphi^*(e^*I) = f^*I$ is connected in $\mathscr{F}/f^*I$. $\square$

Proposition 4.9 *The limit of an inversely filtered system of $\mathscr{S}$-Galois toposes of the form $\mathscr{B}(G_\alpha)$ with G_α a discrete groupoid in $\mathscr{S}$, and connected locally connected transition morphisms between them, is an $\mathscr{S}$-Galois topos.*

Proof For each $\mathscr{G}_\alpha$ there is a single generating $\mathscr{S}$-Galois family $\zeta_\alpha : A_\alpha \to e^*(I_\alpha)$ by assumption. Moreover, for each α one has the canonical internal site given by the discrete groupoid $\mathrm{Aut}(\zeta_\alpha)$. By the construction of filtered inverse limits of toposes in terms of (internal) sites [29], it follows that these families determine families in $\mathscr{G}$ which form an internal site for $\mathscr{G}$.

Since the transition morphisms are connected locally connected, it follows from Lemma 4.8 that these are $\mathscr{S}$-Galois objects of $\mathscr{G}$. □

The following theorem is the analogue of [27] Theorem 3.2 in the unpointed case. We need to assume for parts of it that the base topos is *Set*.

Theorem 4.10 *Let $\mathscr{E}$ be a locally connected bounded over $\mathscr{S}$. Then, (1) and (3) are equivalent. If furthermore the base topos $\mathscr{S}$ is assumed to satisfiy the axiom of stack completions in the sense of Definition 3.3, then all four conditions are equivalent.*

1. *$\mathscr{E}$ is an $\mathscr{S}$-Galois topos.*
2. *The canonical morphism $\sigma_{\mathscr{E}} : \mathscr{E} \to \Pi_1^c(\mathscr{E})$ is an equivalence.*
3. *$\mathscr{E}$ is generated by its locally constant covers.*
4. *$\mathscr{E}$ is equivalent to a topos $\mathscr{B}(G)$ for G a prodiscrete localic groupoid.*

Proof The implication (1) $\Rightarrow$ (3) follows from Proposition 4.4. The converse (3) $\Rightarrow$ (1) amounts to the existence of a Galois closure in this constructive setting and follows immediately from Theorem 4.2. The implication (2) $\Rightarrow$ (1) follows from the conjunction of Theorem 3.8 and Proposition 4.9. The converse (1) $\Rightarrow$ (2) holds because, from the construction of filtered inverse limits of toposes in terms of sites [29], it follows that the inverse image of $\sigma_{\mathscr{E}}$ establishes an equivalence of sites, hence of the respective toposes. The implications (2) $\Rightarrow$ (4) and (4) $\Rightarrow$ (2) are part of Theorem 3.8. □

Corollary 4.11 *Let $\mathscr{E}$ be a locally connected topos bounded over $\mathscr{S}$. Then, the following are equivalent*

1. *$\mathscr{E}$ is a locally simply connected $\mathscr{S}$-Galois topos.*
2. *$\mathscr{E}$ is locally simply connected and the canonical morphism $\sigma_{\mathscr{E}} : \mathscr{E} \to \Pi_1^c(\mathscr{E})$ is an equivalence.*
3. *$\mathscr{E}$ is locally connected and $\mathscr{E} \simeq \mathscr{E}_{lc}$ as a category, where $\mathscr{E}_{lc}$ is the full subcategory of $\mathscr{E}$ whose objects are the locally constant objects in $\mathscr{E}$.*
4. *$\mathscr{E}$ is the classifying topos $\mathscr{B}(G)$ for G a discrete groupoid in $\mathscr{S}$.*

Remark 4.12 Grothendieck [19] *defines* the fundamental groupoid of a (locally simply connected) Galois (Grothendieck) topos $\mathscr{G}$ as the groupoid of points of $\mathscr{G}$. This is consistent with the results of the last section, particularly Theorem 3.1 and Theorem 3.8, since Set satisfies the axiom of choice, hence trivially the axiom of stack completions. This definition is then extended in [19] to what are called therein "multi-Galois toposes" and which classify the actions of a progroup. In the case of $\mathscr{S}$-Galois toposes for any topos $\mathscr{S}$ (including Set), we are dealing with classifying toposes $\mathscr{B}(\pi)$ of prodiscrete groupoids π, which are localic but not discrete; as proven in [7], the stack completion of such a π (for the topology of open surjections of locales) is obtained by considering the *localic* points of $\mathscr{B}(\pi)$. In the case of $\pi_1^c(\mathscr{E}) = lim\{\pi_U\}$, there is a canonical localic point p of the classifying topos $\Pi_1^c(\mathscr{E}) = lim\{\mathscr{B}(\pi_U)\}$ and this localic point may be said to be a *prodiscrete localic point*, as it is obtained as a limit of (discrete) bags of points [8].

5 Locally paths simply connected toposes over an arbitrary base

Recall the definition of the paths fundamental group topos $\Pi_1^p(\mathscr{E})$ for $\mathscr{E}$ a topos bounded over $\mathscr{S}$ [27]. It is given as the colimit $\tau_{\mathscr{E}} : \mathscr{E} \to \Pi_1^p(\mathscr{E})$ in $\mathbf{Top}_{\mathscr{S}}$ for the descent diagram

$$\begin{tikzcd} \mathscr{E}^{\Delta} \arrow[r, shift left=2] \arrow[r] \arrow[r, shift right=2] & \mathscr{E}^{I} \arrow[r, shift left, "\varepsilon_0"] \arrow[r, shift right, "\varepsilon_1"'] & \mathscr{E} \end{tikzcd}$$

obtained from the cosimplicial locale by taking its exponential into $\mathscr{E}$. The assignment of $\Pi_1^p(\mathscr{E})$ to $\mathscr{E}$ is evidently pseudofunctorial and the geometric morphisms $\tau_{\mathscr{E}} : \mathscr{E} \to \Pi_1^p(\mathscr{E})$ are the components of a natural transformation. In the case of a *connected* locally connected topos, $\tau_{\mathscr{E}} : \mathscr{E} \to \Pi_1^p(\mathscr{E})$ is an open surjection, hence $\Pi_1^p(\mathscr{E})$ is the classifying topos of an open localic groupoid $\pi_1^p(\mathscr{E})$; we shall not make this assumption here.

The (unique) path-lifting property can be expressed in topos theory. Let Δ_n denote the standard n-simplex locale in the topos $\mathscr{S}$, constructed from the unit interval locale I in the usual manner.

Definition 5.1 (i) An object Y of $\mathscr{E}$ is said to have the *path-lifting property* if the induced geometric morphism

$$(\varphi_Y)^{\Delta_n} : (\mathscr{E}/Y)^{\Delta_n} \to (\mathscr{E})^{\Delta_n}$$

is an open surjection.

(ii) An object Y of $\mathscr{E}$ is said to have the *unique path-lifting property* if the commutative diagram

$$\begin{tikzcd} (\mathscr{E}/Y)^{\Delta_n} \arrow[r, "\varphi_Y^{\Delta_n}"] \arrow[d, "\varepsilon_0"] & \mathscr{E}^{\Delta_n} \arrow[d, "\varepsilon_0"] \\ \mathscr{E}/Y \arrow[r, "\varphi_Y"'] & \mathscr{E} \end{tikzcd}$$

is a pullback.

Proposition 5.2 *Any cover Y in $\mathscr{E}$ with the unique path-lifting property also satisfies the path-lifting property.*

Proof Open surjections are pullback stable. □

Proposition 5.3 *Let A be a locally constant object in $\mathscr{E}$. Then A has the unique path-lifting property.*

Proof The Lemma 6.1 of [13], established for Grothendieck toposes, is valid over an arbitrary base topos. Indeed, locally constant sheaves in the sense of Definition 2.1 (or "constructible sheaves" in the sense of Grothendieck [1]) are constant on contractible locales, the argument is just as in topology. Furthermore, the required stability holds by Lemma 4.8. □

The *comparison lemma* between the paths and the coverings fundamental group toposes was established in [13] for a *connected* locally connected

topos $\mathscr{E}$ over $\mathscr{S}$. In the non-connected case, we need to work with the locally constant objects of Definition 2.1 rather than with those of Barr and Diaconescu [3]. The proof we give here uses Galois closure and the fact that locally constant objects have the (unique) path-lifting property.

Theorem 5.4 *Let $\mathscr{E}$ be a locally connected topos bounded over $\mathscr{S}$. Then there exists a geometric morphism*

$$\kappa_{\mathscr{E}} : \Pi_1^p(\mathscr{E}) \to \Pi_1^c(\mathscr{E})$$

for which the triangle

$$\begin{tikzcd} \mathscr{E} \arrow[r, "\sigma_{\mathscr{E}}"] \arrow[d, "\tau_{\mathscr{E}}"'] & \Pi_1^c(\mathscr{E}) \\ \Pi_1^p(\mathscr{E}) \arrow[ur, "\kappa_{\mathscr{E}}"'] & \end{tikzcd}$$

commutes.

Proof Let U be a cover in $\mathscr{E}$ and let $X \in \mathrm{Spl}_U(\mathscr{E})$. We wish to show that X has a (canonical) action by paths. We resort to Galois closure in $\mathscr{G}_U(\mathscr{E}) \simeq \mathrm{Spl}_U(\mathscr{E})$. Let $\zeta : A \to e^*e_!U$ be a generating $\mathscr{S}$-Galois family for $\mathscr{G}_U(\mathscr{E})$ as obtained in Theorem 4.2. Recall from Theorem 4.2 that the diagram

$$\begin{tikzcd} \mathscr{E}/A \arrow[r, "\varphi_A"] \arrow[d] & \mathscr{E} \arrow[d] \\ \mathscr{S}/e_!U \arrow[r, "p_U"'] & \mathscr{G}_U \end{tikzcd}$$

is a pullback (in particular, it commutes). Consider now the diagram

$$\begin{tikzcd} (\mathscr{E}/A)^\Delta \arrow[r] \arrow[d, shift left, shift left, shift left] \arrow[d, shift left, shift left] \arrow[d, shift left] & (\mathscr{E})^\Delta \arrow[d, shift left, shift left, shift left] \arrow[d, shift left, shift left] \arrow[d, shift left] \\ (\mathscr{E}/A)^I \arrow[r] \arrow[d, shift left] \arrow[d] & \mathscr{E}^I \arrow[d, shift left] \arrow[d] \\ \mathscr{E}/A \arrow[r] & \mathscr{E} \end{tikzcd}$$

It follows now from Proposition 5.3 and Proposition 5.2 that the three horizontal arrows in the above descent diagram are open surjections. Therefore, it is sufficient to prove that $(\varphi_A)^*X$, as an object of $\mathscr{E}/A$ which arises as $\rho^*(\zeta)$ for some $\zeta : S \to e_!U$, has an action by paths. In turn, we claim that is enough to prove that $(\varphi_A)^*X$, as an object of $\mathscr{E}/A$ regarded as a topos over $\mathscr{S}/e_!U$, has an action by paths. This is because, since $\mathscr{S}/e_!U \to \mathscr{S}$ is a surjective local homeomorphism, the top arrow in the pullback square

below is also one:

$$\begin{tikzcd} {[(\mathscr{E}/A)^I]_U} \arrow[r] \arrow[d] & \mathscr{E}/A^I \arrow[d, "\varepsilon_0"] \\ \mathscr{S}/e_!U \arrow[r, "\varphi_{e_!U}"'] & \mathscr{S} \end{tikzcd}$$

and similarly for $\Delta = \Delta_1$ and the other simplices locales.

The next observation is that for $i = 0, 1$ the triangle

$$\begin{tikzcd} {((\mathscr{E}/A)^I)_U} \arrow[r, "\varepsilon_i"] \arrow[d, "\varepsilon"'] & \mathscr{E}/A \\ \mathscr{E}/A \times_{\mathscr{S}/e_!U} \mathscr{E}/A \arrow[ur, "\pi_i"'] & \end{tikzcd}$$

commutes up to iso 2-cell. By the theorem of Moerdijk and Wraith [30], ε is an open surjection since $\mathscr{E}/A \to \mathscr{S}/e_!U$ is connected locally connected. Therefore, since (for the same reason) the latter is the *coequalizer* of the pair π_0, π_1, it coequalizes the pair $\varepsilon_0, \varepsilon_1$ considered above. It follows easily from this that the object $(\varphi_U)^* X = \rho^* \zeta$ has an action by paths.

The preceding argument shows the existence of a canonical geometric morphism $\kappa_U : \Pi_1^p(\mathscr{E}) \to \mathscr{G}_U$, for each cover U, from which the existence of the desired morphism $\kappa_{\mathscr{E}} : \Pi_1^p(\mathscr{E}) \to \mathscr{G}(\mathscr{E})$ follows. □

An obvious problem with the definition of $\Pi_1^c(\mathscr{E})$ for a locally connected topos $\mathscr{E}$ is the lack of pseudofunctoriality, by contrast with the pseudofunctoriality of $\Pi_1^p(\mathscr{E})$, as noted in [13]. As in topology, it is primarily the *paths* fundamental groupoid that is of interest, with the *coverings* version as a means for calculating the latter for spaces where the two are equivalent.

We have seen in the previous section that there is always a comparison map

$$\kappa_{\mathscr{E}} : \Pi_1^p(\mathscr{E}) \to \Pi_1^c(\mathscr{E})$$

and that this is a connected locally connected geometric morphism. For $\mathscr{E}$ a Galois topos, $\kappa_{\mathscr{E}}$ is an equivalence, in which case (that is defined on $\mathscr{S}$-Galois toposes) the assignment of $\Pi_1^c(\mathscr{E})$ to $\mathscr{E}$ is pseudofunctorial. But this case is not too interesting if what we are seeking is somesort of reflection of a *comprehensive* 2-category of toposes including the $\mathscr{S}$-Galois toposes into the 2-category of $\mathscr{S}$-Galois toposes, both as full sub 2-categories of $\mathbf{LTop}_{\mathscr{S}}$. Since the assignment of $\mathscr{E}_{lc}$ (the full subcategory of $\mathscr{E}$ consisting of its locally constant objets) to $\mathscr{E}$ is an idempotent operation, as is more generally the assignment of the full subcategory of $\mathscr{E}$ *generated* by the locally constant objects, the reflection would immediately result from the pseudofunctoriality and from it, in turn, a *van Kampen* theorem (fundamental group topos version) in the form of the preservation of pushouts could be directly inferred.

Let $\mathbf{GTop}_{\mathscr{S}}$ the full sub 2-category of $\mathbf{LTop}_{\mathscr{S}}$ with objects the $\mathscr{S}$-Galois toposes and let $I : \mathbf{GTop}_{\mathscr{S}} \to \mathbf{LTop}_{\mathscr{S}}$ be the inclusion 2-functor.

Remark 5.5 Denote by lsc($\mathbf{LTop}_{\mathscr{S}}$) (respectively lsc($\mathbf{GTop}_{\mathscr{S}}$)) the full sub 2-category of $\mathbf{LTop}_{\mathscr{S}}$ (respectively of $\mathbf{GTop}_{\mathscr{S}}$) whose objects are locally simply connected toposes. The inclusion I above restricts to an inclusion I : lsc$\mathbf{GTop}_{\mathscr{S}} \to$ lsc$\mathbf{LTop}_{\mathscr{S}}$. The only apparent way to render pseudofunctorial this assignment is (as done in [12]) to replace lsc($\mathbf{LTop}_{\mathscr{S}}$) by the non-full sub 2-category of $\mathbf{LTop}_{\mathscr{S}}$ whose objects are pairs ($\mathscr{E}$, A) with $\mathscr{E}$ locally connected and A a *universal cover* in $\mathscr{E}$ (by which we mean a locally constant object A of $\mathscr{E}$ with global support which splits all of the locally constant objects in $\mathscr{E}$, as implicilty in Proposition 4.4), and where a morphism $(\mathscr{E}, A) \to (F, B)$ is given by a pair (φ, s) consisting of a geometric morphism $\varphi : \mathscr{E} \to \mathscr{F}$ and a morphism $s : A \to \varphi^*(B)$ in $\mathscr{E}$. However, this results in a loss of the reflection property, so that a simple-minded van Kampen theorem is not a consequence. In this non-full setting, however, a van Kampen theorem can be derived by imposing certain restrictions on the pushouts, as in [12] Theorem 6.5; the same restrictions reappear naturally in the coverings version of a van Kampen theorem, as in [12] or, more generally as in the next section.

Notice now that if $\mathscr{E}$ is a locally simply connected (and locally connected) topos with universal cover A, then $\mathscr{E}/A$ has the property that $\Pi_1^c(\mathscr{E}/A) \simeq \mathscr{S}/e_!A$, in such a way that the equivalence is compatible with the given morphisms with domain $\mathscr{E}/A$. From the existence of the comparison map follows then that to require that $\Pi_1^p(\mathscr{E}/A) \simeq \mathscr{S}/e_!A$ is a stronger condition. We thus make the following definition (see [13] for a related notion in the case of Grothendieck toposes).

Definition 5.6 A locally connected topos $\mathscr{E}$ is said to be *locally paths simply connected* if $\mathscr{E}$ is locally simply connected and the universal cover A has the property that $\Pi_1^p(\mathscr{E}/A) \simeq \mathscr{S}/e_!A$ by an equivalence which commutes with $\tau_{\mathscr{E}/A} : \mathscr{E}/A \to \Pi_1^p(\mathscr{E}/A)$ and $\rho_{\eta_A} : \mathscr{E}/A \to \mathscr{S}/e_!A$.

Theorem 5.7 *For $\mathscr{E}$ a locally connected and locally paths simply connected topos, the comparison map $\kappa_{\mathscr{E}} : \Pi_1^p(\mathscr{E}) \to \Pi_1^c(\mathscr{E})$ is an equivalence.*

Proof It is enough to prove that there exists a unique geometric morphism $\lambda_{\mathscr{E}} : \Pi_1^c(\mathscr{E}) \to \Pi_1^p(\mathscr{E})$ for which the triangle

$$\begin{tikzcd} \mathscr{E} \arrow[r, "\tau_{\mathscr{E}}"] \arrow[d, "\sigma_{\mathscr{E}}"'] & \Pi_1^p(\mathscr{E}) \\ \Pi_1^c(\mathscr{E}) \arrow[ur, "\lambda_{\mathscr{E}}"'] & \end{tikzcd}$$

commutes and is such hat $\lambda_{\mathscr{E}}\kappa_{\mathscr{E}} = \mathrm{id}_{\Pi_1^p(\mathscr{E})}$.

To this end, consider the diagram

$$\begin{tikzcd}
\mathscr{E}/A \arrow[r, "\varphi_A"] \arrow[d, "\rho_A"] \arrow[dd, bend right=60, "\tau_{\mathscr{E}/A}"'] & \mathscr{E} \arrow[d, "\sigma_{\mathscr{E}}"'] \arrow[dd, bend left=60, "\tau_{\mathscr{E}}"] \\
\mathscr{S}/e_!A \arrow[r, "p_A"] \arrow[d, "\simeq"] & \Pi_1^c(\mathscr{E}) \arrow[d, dotted, "\lambda_A"'] \\
\Pi_1^p(\mathscr{E}/A) \arrow[r, "\Pi_1^p(\varphi_A)"'] & \Pi_1^p(\mathscr{E})
\end{tikzcd}$$

where A is a universal cover of $\mathscr{E}$. Since A splits every locally constant object in $\mathscr{E}$, the top square is a pushout. The desired morphism $\lambda_{\mathscr{E}}$ arises from the pseudofunctoriality of Π_1^p and pseudonaturality of τ, using now that A is a universal cover and $\mathscr{E}$ is locally paths simply connected, so that $\Pi_1^p(\mathscr{E}/A) \simeq \mathscr{S}/e_!A$ by an equivalence which commutes with $\tau_{\mathscr{E}/A} : \mathscr{E}/A \to \Pi_1^p(\mathscr{E}/A)$. □

With obvious notations, we now have the following.

Proposition 5.8 *The inclusion* I : lpsc$\mathbf{GTop}_{\mathscr{S}} \to$ lpsc$\boldsymbol{LTop}_{\mathscr{S}}$ *has a left biadjoint given by the 2-functor* Π_1^c : lpsc$\boldsymbol{LTop}_{\mathscr{S}} \to$ lpsc$\mathbf{GTop}_{\mathscr{S}}$. *The unit of adjointness has as components the (connected locally connected) geometric morphisms* $\sigma_{\mathscr{E}} : \mathscr{E} \to \Pi_1^c(\mathscr{E})$.

Proof The pseudofunctoriality of the assigment of the topos $\Pi_1^c(\mathscr{E})$ to a locally connected and locally paths simply connected topos $\mathscr{E}$ in the sense of Definition 5.6 is easily deduced from Theorem 5.7 and the stability of locally constant objects under pullback along arbitrary geometric morphisms. That $\Pi_1^c(\mathscr{E})$ is a (locally paths simply connected) $\mathscr{S}$-Galois topos for each locally connected (and locally paths simply connected) topos $\mathscr{E}$ follows in turn (since $\Pi_1^c(\mathscr{E})$ is again locally connected) from the claim that the induced geometric morphism

$$\Pi_1^c(\sigma_{\mathscr{E}}) : \Pi_1^c(\mathscr{E}) \to \Pi_1^c(\Pi_1^c(\mathscr{E}))$$

is an equivalence. To see this it is sufficient to observe that the operation of carving out $\mathscr{E}_{lc}$ from $\mathscr{E}$ is an idempotent one. It is now obvious that Π_1^c is a reflection onto lpsc$\mathbf{GTop}_{\mathscr{S}}$ with σ the unit of the adjointness. This proves the first assertion; the second assertion is a consequence of the first and Corollary 4.11. □

It follows from Proposition 5.8 that pushouts in lpsc$\mathbf{LTop}_{\mathscr{S}}$ are taken by $\Pi_1^c(-)$ into pushouts in $\mathbf{Top}_{\mathscr{S}}$ in which all toposes in the pushout are locally paths simply connected Galois toposes, a statement which may be interpreted as a van Kampen theorem for (locally paths simply connected) toposes in terms of fundamental groupoid toposes [12]. We now derive a version of the van Kampen theorem in terms of fundamental groupoids.

Proposition 5.9 *Let*

$$\begin{tikzcd} \mathscr{E}_1 \arrow[r, "\alpha_1"] & \mathscr{E} \\ \mathscr{E}_0 \arrow[u, "\beta_1"] \arrow[r, "\beta_2"'] & \mathscr{E}_2 \arrow[u, "\alpha_2"'] \end{tikzcd}$$

be a pushout in $\mathbf{Top}_{\mathscr{S}}$ *in which all four geometric morphisms are locally connected surjections and where all four toposes are locally connected and locally paths simply connected. Then, the induced diagram*

$$\begin{tikzcd} \mathscr{B}(G_1) \arrow[r, "\alpha_1"] & \mathscr{B}(G) \\ \mathscr{B}(G_0) \arrow[u, "\beta_1"] \arrow[r, "\beta_2"'] & \mathscr{B}(G_2) \arrow[u, "\alpha_2"'] \end{tikzcd}$$

is a pushout of classifying toposes of discrete groupoids, where all four morphisms commute with the canonical localic points.

Proof It is enough to prove that if $\varphi : \mathscr{E} \to \mathscr{F}$ is a locally connected surjection, with $\mathscr{E}$ and $\mathscr{F}$ locally paths simply connected, then the induced $\Pi_1^c(\varphi) : \Pi_1^c(\mathscr{E}) \to \Pi_1^c(\mathscr{F})$, alternatively viewed as a geometric morphism $\mathscr{B}(\varphi) : \mathscr{B}(G(\mathscr{E})) \to \mathscr{B}(G(\mathscr{F}))$, is a $+$-geometric morphism of classifying toposes in the sense that it commutes with the canonical localic points.

To see this, observe that since φ is locally connected, it follows that if we let U be an object of $\mathscr{E}$ splitting all of the locally constant objects in $\mathscr{E}$ then $V = \varphi_!(U)$ has the same property in $\mathscr{F}$. Further, since φ is assumed surjective, it also follows that if U is a cover in $\mathscr{E}$ (has global support), then also V is a cover in $\mathscr{F}$.

Consider now the universal cover $\tilde{V}$ associated with V, taken together with the canonical morphisms $\delta : \tilde{V} \to f^* f_! V$ which represents (the inverse image part of) the canonical localic point p_V of $\mathscr{F}$.

We have a cube

$$\begin{tikzcd} & \mathscr{F}/\tilde{V} \arrow[rr] \arrow[dd] & & \mathscr{F} \arrow[dd] \\ \mathscr{E}/U \arrow[ur] \arrow[rr] \arrow[dd] & & \mathscr{E} \arrow[ur] \arrow[dd] & \\ & \mathscr{S}/f_!V \arrow[rr] & & \mathscr{G}_V(\mathscr{F}) \\ \mathscr{S}/e_!U \arrow[ur] \arrow[rr] & & \mathscr{G}_U(\mathscr{E}) \arrow[ur] & \end{tikzcd}$$

where the back square is a pushout and the front square is a pullback.

Since there is given a morphism $e_!U \to f_!V$ in $\mathscr{S}$, namely the identity, there is first of all a unique geometric morphism $\mathscr{E}/U \to \mathscr{F}/\tilde{V}$ making the left side and the top squares commute, using for this that the front square is a pullback (universal cover $\tilde{V}$ in $\mathscr{F}$). Since the back square is a pushout, we now obtain a unique geometric morphism $\mathscr{G}_U(\mathscr{E}) \to \mathscr{G}_V(\mathscr{F})$ which makes

the right side and the bottom squares commute. In particular, this is a morphism $\mathscr{B}(G_U(\mathscr{E})) \to \mathscr{B}(G_V(\mathscr{F}))$ which commutes with the canonical localic points p_U and p_V as claimed.

It remains to see that the above argument can applied consistently to the four geometric morphisms in the pushout square: starting with a cover U_0 in $\mathscr{E}_0$ splitting all of the locally constant objecst in $\mathscr{E}$, one now lets $U_1 = \beta_{1!}(U_0)$, $U_2 = \beta_{2!}(U_0)$, and finally $U = \alpha_{1!}(U_1) \simeq \alpha_{2!}(U_2)$. □

Denote by $\mathbf{Gpd}(\mathscr{S})$ the 2-category of (discrete localic) groupoids in $\mathscr{S}$.

Corollary 5.10 *Let*

$$\begin{tikzcd} \mathscr{E}_1 \arrow[r, "\alpha_1"] & \mathscr{E} \\ \mathscr{E}_0 \arrow[u, "\beta_1"] \arrow[r, "\beta_2"'] & \mathscr{E}_2 \arrow[u, "\alpha_2"'] \end{tikzcd}$$

be a pushout diagram in $\mathbf{Top}_{\mathscr{S}}$ *where all four toposes are locally connected and locally paths simply connected and where all four geometric morphisms are locally connected surjections. Then the diagram*

$$\begin{tikzcd} \pi_1^c(\mathscr{E}_1) \arrow[r, "\pi_1^c(\alpha_1)"] & \pi_1^c(\mathscr{E}) \\ \pi_1^c(\mathscr{E}_0) \arrow[u, "\Pi_1^c(\beta_1)"] \arrow[r, "\pi_1^c(\beta_2)"'] & \pi_1^c(\mathscr{E}_2) \arrow[u, "\pi_1^c(\alpha_2)"'] \end{tikzcd}$$

is a pushout in **Gpd**.

Proof This is an immediate consequence of Proposition 5.9 and the Morita theorem for localic groupoids from [7]. □

Remark 5.11 We haste to point out that the above result may not be too interesting from the point of view of topology, where often the van Kampen theorems are seen as means for construction a universal cover of a space in a pushout given universal covers for the other three, in a compatible fashion. This, in fact, is the point of view taken in [12]. In particular, we need not assume therein, unlike above, that the toposes in question be anything more than (locally connected and) locally simply connected.

6 Generalized covering morphisms and a van Kampen theorem

In topology, covering morphisms (in a more general sense than that of covering projections) are usually required to be local homeomorphisms which furthermore satisfy a path-lifting property. The latter should be thought of as topological coverings. Since morphisms not satisfying either condition, such as certain complete spreads in topos theory [10, 11], have also been thought of as covering morphisms "with singularities", we want to also make precise in what (generalized) sense are these still to be considered as coverings.

We shall restrict, as we have done so far in this paper, to locally connected toposes over $\mathscr{S}$, even if this assumption is not always strictly required. Denote by $\mathbf{LTop}_{\mathscr{S}}$ the full sub 2-category of $\mathbf{Top}_{\mathscr{S}}$ whose objects are the locally connected toposes.

A *notion* $\mathscr{C}$ *of covering morphism* on $\mathbf{LTop}_{\mathscr{S}}$ will be given by an assignment, for each $\mathscr{E} \in \mathbf{LTop}_{\mathscr{S}}$, of a category $\mathscr{C}(\mathscr{E}) \subseteq \mathbf{LTop}_{\mathscr{S}}/\mathscr{E}$, where the slice category $\mathbf{LTop}_{\mathscr{S}}/\mathscr{E}$ has as objects, geometric morphisms with codomain $\mathscr{E}$, with a morphism from $\varphi : \mathscr{F} \to \mathscr{E}$ to $\psi : \mathscr{G} \to \mathscr{E}$ given by a geometric morphism $\chi : \mathscr{F} \to \mathscr{G}$ for which there is a natural isomorphism $\alpha : \varphi \to \psi\chi$.

Denote by $\mathscr{L} : \mathscr{K}^{\mathrm{op}} \to \mathbf{CAT}$ the pseudofunctor which assigns to a topos $\mathscr{E}$ the slice category $\mathbf{LTop}_{\mathscr{S}}/\mathscr{E}$ and to a geometric morphism $\varphi : \mathscr{F} \to \mathscr{E}$ the functor given by pulling back along φ.

We will say (following [14]) that a subpseudofunctor $\mathscr{C}$ of $\mathscr{L} : \mathscr{K}^{\mathrm{op}} \to \mathbf{CAT}$ is a *stack* for a class Φ of morphisms of effective descent in $\mathbf{Top}_{\mathscr{S}}$, assumed to be closed under composition and pullbacks, if for any object $\mathscr{E}$ of $\mathscr{K}$, given $\alpha \in \mathscr{L}(\mathscr{E})$ such that for some $\varphi : \mathscr{F} \to \mathscr{E}$ with $\varphi \in \Phi$, it is the case that $\varphi^*(\alpha) \in \mathscr{C}(\mathscr{F})$, then it follows that $\alpha \in \mathscr{C}(\mathscr{E})$.

Definition 6.1 Let $\mathscr{K}$ be a sub 2-category of $\mathbf{LTop}_{\mathscr{S}}$. A notion $\mathscr{C}$ of covering morphisms is said to define *a fibration of covering morphisms for* $\mathscr{K}$ with respect to a class Φ as above if,

1. $\mathscr{C}$ extends to a pseudofunctor $\mathscr{C} : \mathscr{K}^{\mathrm{op}} \to \mathbf{CAT}$,
2. there is a fully faithful pseudonatural transformation $H : \mathscr{C} \to \mathscr{L}$, and
3. $\mathscr{C}$ is a stack for the class Φ.

We say that a pseudofunctor $\mathscr{C} : \mathscr{K}^{\mathrm{op}} \to \mathbf{CAT}$ *preserves binary products* if for each pair of objects E and F in $\mathscr{K}$, the canonical functors

$$\mathscr{C}(E+F) \to \mathscr{C}(E) \times \mathscr{C}(F)$$

induced by the injections $i : E \to E+F$ and $j : F \to E+F$ are equivalences. In the terminology of [26], $\mathscr{C}$ is an *intensive quantity* on $\mathscr{K}$.

Notice that the 2-extensivity of $\mathscr{K}$ (for $\mathscr{K} = \mathbf{Top}_{\mathscr{S}}$, or $\mathscr{K} = \mathbf{LTop}_{\mathscr{S}}$) says that $\mathscr{L}$ preserves binary products. Notice also that there is a fully faithful pseudonatural transformation $J : \mathscr{A} \to \mathscr{L}$. Indeed, given any composite $\zeta = \psi\varphi$, where both ζ and ψ are local homeomorphisms, so is φ.

The following is a version of the *van Kampen theorem* in terms of covering morphisms. It is an abstraction of Theorem 5.5 of [12], given therein for covering projections on toposes; the proof is analogous.

Theorem 6.2 *Let $\mathscr{C}$ be a fibration of covering morphisms on $\mathscr{K}$ with respect to a class Φ of morphisms of effective descent in $\mathscr{K}$. Assume furthermore that $\mathscr{C}$ preserves binary products and that $H : \mathscr{C} \to \mathscr{L}$ factors*

through the canonical inclusion $J : \mathscr{A} \to \mathscr{L}$. *Let*

$$\begin{tikzcd} \mathscr{E}_1 \arrow[r, "\alpha_1"] & \mathscr{E} \\ \mathscr{E}_0 \arrow[u, "\beta_1"] \arrow[r, "\beta_2"'] & \mathscr{E}_2 \arrow[u, "\alpha_2"'] \end{tikzcd}$$

be a pushout diagram in $\boldsymbol{LTop}_{\mathscr{S}}$ *in which the induced map* $\alpha : \mathscr{E}_1 + \mathscr{E}_2 \to \mathscr{E}$ *is a morphism in* Φ. *Then*

$$\begin{tikzcd} \mathscr{C}(\mathscr{E}_1) \arrow[r, "\beta_1^*"] & \mathscr{C}(\mathscr{E}_0) \\ \mathscr{C}(\mathscr{E}) \arrow[u, "\alpha_1^*"] \arrow[r, "\alpha_2^*"'] & \mathscr{C}(\mathscr{E}_2) \arrow[u, "\beta_2^*"'] \end{tikzcd}$$

is a pullback in **CAT**.

The assignment to a topos $\mathscr{E}$ of the class of *local homeomorphisms* over it constitutes a pseudofunctor $\mathscr{A} : \mathbf{LTop}_{\mathscr{S}}^{\mathrm{op}} \to \mathbf{CAT}$; given any geometric morphism $\varphi : \mathscr{F} \to \mathscr{E}$ in $\mathbf{LTop}_{\mathscr{S}}$, pulling back along φ restricts to a functor $A(\varphi) : \mathscr{A}(\mathscr{E}) \to \mathscr{A}(\mathscr{F})$. The pseudofunctor $\mathscr{A}$ is trivially a stack for any (morphism of effective descent) class Φ. That $\mathscr{A}$ preserves binary products follows from the fact that colimits in $\mathbf{Top}_{\mathscr{S}}$ (or $\mathbf{LTop}_{\mathscr{S}}$) are calculated as limits in **CAT** of the corresponding diagrams of inverse image parts [29].

Another example of a fibration of covering morphisms is given by letting $\mathscr{C}(\mathscr{E}) = \mathscr{E}_{lc}$ be the full subcategory of $\mathscr{E}$ determined by the (locally trivial) covering morphisms in terms of the adjunction $e_! \dashv e^*$ in the sense of [21]; equivalently $\mathscr{C}(\mathscr{E})$ is the category of local homeomorphisms over $\mathscr{E}$ defined by a locally constant object in the sense of Definition 2.1. The pseudofunctoriality of $\mathscr{C}$ on $\mathbf{LTop}_{\mathscr{S}}$ follows from the stability of locally constant objects under pullbacks along arbitrary geometric morphisms, as remarked in Section 2. There is a pseudonatural transformation $H : \mathscr{C} \to \mathscr{A}$ (with fully faithful components) by the very definition of $\mathscr{C}$. It is shown in [12] that H is φ-cartesian for every surjective locally connected φ. This means that the pseudonaturality square

$$\begin{tikzcd} \mathscr{A}(X) \arrow[r, "\mathscr{A}(\varphi)"] & \mathscr{A}(Y) \\ \mathscr{C}(X) \arrow[u, "HX"] \arrow[r, "\mathscr{C}(\varphi)"'] & \mathscr{C}(Y) \arrow[u, "HY"'] \end{tikzcd}$$

is a pullback in $\mathscr{K}$. This readily implies that $\mathscr{C}$ is a stack for the class Φ of locally connected surjections. It is also shown in [12] that $\mathscr{C}$ preserves binary products.

In particular, there is a *van Kampen theorem* for $\mathscr{C}$ as an application of Theorem 6.2, already shown in [12].

We shall now consider a different fibration of covering morphisms for which we need to recall some basic facts about complete spreads from [10, 11]. Let $\mathscr{E}$ be a (locally connected) topos presented over $\mathscr{S}$ by a site $\mathbf{C}$. A geometric morphism $\varphi : \mathscr{F} \to \mathscr{E}$ in $\mathbf{LTop}_{\mathscr{S}}$ is a *complete spread* if, for the discrete opfibration $D : \mathbf{D} \to \mathbf{C}$ which is associated with the functor $D : \mathbf{C} \to \mathscr{S}$ which is the composite

$$\mathbf{C} \xrightarrow{\varepsilon} \mathscr{E} \xrightarrow{\varphi^*} \mathscr{F} \xrightarrow{f_!} \mathscr{S} \ ,$$

the square

$$\begin{tikzcd} \mathscr{F} \arrow[r] \arrow[d, "\varphi"] & \mathscr{S}^{\mathbf{D}^{\mathrm{op}}} \arrow[d, "u"] \\ \mathscr{E} \arrow[r] & \mathscr{S}^{\mathbf{C}^{\mathrm{op}}} \end{tikzcd}$$

where u is induced by D, is a pullback.

Complete spreads are related to distributions in the sense of Lawvere [26]. A distribution on a (locally connected) topos $\mathscr{E}$ over $\mathscr{S}$ is any $\mathscr{S}$-cocontinuous functor $\mu : \mathscr{E} \to \mathscr{S}$. Denote by $\mathbf{Dist}(\mathscr{E})$ the category of distributions on $\mathscr{E}$. Explicitly, $\mathbf{Dist}(\mathscr{E}) = \mathrm{Coc}_{\mathscr{S}}(\mathscr{A}(\mathscr{E}) \ , \ \mathscr{S})$ and each geometric morphism $\varphi : \mathscr{F} \to \mathscr{E}$ induces, by composition with the $\mathscr{S}$-cocontinuous functor $\mathscr{A}(\varphi) = \varphi^* : \mathscr{E} \to \mathscr{F}$, a functor $\mathbf{Dist}(\varphi) : \mathbf{Dist}(\mathscr{F}) \to \mathbf{Dist}(\mathscr{E})$.

For each $\mathscr{E} \in \mathbf{Top}_{\mathscr{S}}$ there is a topos $\mathscr{M}(\mathscr{E})$ [9] (the "symmetric topos") and an $\mathscr{S}$-essential geometric morphism $\delta : \mathscr{E} \to \mathscr{M}(\mathscr{E})$ such that the pair $\langle \mathscr{M}(\mathscr{E}) \ , \ \delta \rangle$ classifies distributions on $\mathscr{E}$.

If $\varphi : \mathscr{F} \to \mathscr{E}$ is $\mathscr{S}$-essential, then composition with the $\mathscr{S}$-cocontinuous functor $\varphi_! : \mathscr{F} \to \mathscr{E}$ is a functor $\mathscr{R}_\varphi : \mathbf{Dist}(\mathscr{E}) \to \mathbf{Dist}(\mathscr{F})$ which is right adjoint to the functor $\mathbf{Dist}(\varphi) : \mathbf{Dist}(\mathscr{F}) \to \mathbf{Dist}(\mathscr{E})$, as the latter is given by composition with φ^*. Under the equivalence between distributions and complete spreads, it is the right adjoint $\mathscr{R}_\varphi$ which corresponds to pulling back complete spreads over $\mathscr{E}$ along the $\mathscr{S}$-essential morphism $\varphi : \mathscr{F} \to \mathscr{E}$ to give a complete spread over $\mathscr{F}$.

For objects $\mathscr{E}$ and $\mathscr{F}$ in $\mathbf{Top}_{\mathscr{S}}$, their coproduct $\mathscr{E} + \mathscr{F}$ exists in $\mathbf{Top}_{\mathscr{S}}$ and is given by the product category $\mathscr{E} \times \mathscr{F}$; the injections $\mathrm{i} : \mathscr{E} \to \mathscr{E} + \mathscr{F}$ and $\mathrm{j} : \mathscr{F} \to \mathscr{E} + \mathscr{F}$ have as inverse image parts the left adjoints to the projections. It is shown in [12] that $\mathbf{Top}_{\mathscr{S}}$ is an extensive 2-category; this says that for the 2-categorical versions of the slices, the coproduct pseudofunctor $F : \mathbf{Top}_{\mathscr{S}} \times \mathbf{Top}_{\mathscr{S}} X \to \mathbf{Top}_{\mathscr{S}}$ induces biequivalences

$$F_{\langle \mathscr{E}, \mathscr{F} \rangle} : \mathbf{Top}_{\mathscr{S}}/\mathscr{E} \times \mathbf{Top}_{\mathscr{S}}/\mathscr{F} \to \mathbf{Top}_{\mathscr{S}}/(\mathscr{E} + \mathscr{F})$$

for all $\mathscr{E}$ and $\mathscr{F}$ in $\mathbf{Top}_{\mathscr{S}}$.

Lemma 6.3 *For any $\mathscr{E}$ and $\mathscr{F}$ objects in $\mathbf{Top}_{\mathscr{S}}$, there exists an equivalence*

$$\mathscr{M}(\mathscr{E} + \mathscr{F}) \simeq \mathscr{M}(\mathscr{E}) \times \mathscr{M}(\mathscr{F})$$

Proof For simplicity, we will argue by considering points of $\mathscr{M}(\mathscr{E})$, but this argument can be justified if instead we take generalized points of $\mathscr{M}(\mathscr{E})$. Morphisms $\mathscr{S} \to \mathscr{M}(\mathscr{E})$ are in bijection with the $\mathscr{S}$-valued distributions on $\mathscr{E}$, that is, with $\mathscr{S}$-cocontinuous functors $\mu : \mathscr{E} \to \mathscr{S}$. Furthermore, an $\mathscr{S}$-cocontinuous functor $\mu : \mathscr{E} + \mathscr{F} \to \mathscr{S}$ corresponds to a complete spread $\varphi : \mathscr{G} \to \mathscr{E} + \mathscr{F}$. From the extensivity of $\mathbf{LTop}_{\mathscr{S}}$ follows that, as a geometric morphism, φ splits into two geometric morphisms $\varphi_1 : \mathscr{G}_1 \to \mathscr{E}$ and $\varphi_2 : \mathscr{G}_2 \to \mathscr{F}$, where $\mathscr{G} = \mathscr{G}_1 + \mathscr{G}_2$; in turn, these determine uniquely a pair of $\mathscr{S}$-cocontinuous functors $\nu : \mathscr{E} \to \mathscr{S}$ and $\zeta : \mathscr{F} \to \mathscr{S}$, such that $\mu = \nu + \zeta$. □

Denote by $\mathscr{D}(\mathscr{E})$ the category of complete spreads (with locally connected domain) over $\mathscr{E}$. There is an functor $B : \mathscr{D}(\mathscr{E}) \to \mathbf{Dist}(\mathscr{E})$ which assigns, to a complete spread (in fact to any geometric morphism) $\varphi : \mathscr{F} \to \mathscr{E}$ in $\mathbf{LTop}_{\mathscr{S}}$ the distribution $\mu : \mathscr{E} \to \mathscr{S}$ given by the composite $\mu = f_!\varphi^*$. This functor is an equivalence. If considered as a functor $B : \mathbf{LTop}_{\mathscr{S}}/\mathscr{E} \to \mathbf{Dist}(\mathscr{E})$, then the correct assertion is that it has a right adjoint which assigns, to a distribution μ on $\mathscr{E}$ a complete spread over $\mathscr{E}$; this gives rise to a (unique) factorization of any given geometric morphism $\varphi : \mathscr{F} \to \mathscr{E}$ in $\mathbf{LTop}_{\mathscr{S}}$ into a composite of a *pure* morphism $\pi : \mathscr{F} \to \mathscr{D}$ followed by a complete spread $\psi : \mathscr{D} \to \mathscr{E}$.

All of the above considerations remain true if $\mathbf{Top}_{\mathscr{S}}$ is replaced by $\mathbf{LTop}_{\mathscr{S}}$ [12]. Furthermore, the assignment of $\mathscr{D}(\mathscr{E})$ to $\mathscr{E}$ can be made into a pseudofunctor $\mathscr{D} : \mathscr{K}^{\mathrm{op}} \to \mathbf{CAT}$ letting $\mathscr{K}$ be the sub 2-category of $\mathbf{LTop}_{\mathscr{S}}$ with morphisms the $\mathscr{S}$-essential morphisms, where to an $\mathscr{S}$-essential $\varphi : \mathscr{F} \to \mathscr{E}$ there is induced the functor $\mathscr{D}(\varphi) : \mathscr{D}(\mathscr{E}) \to \mathscr{D}(\mathscr{F})$ given by pulling back complete spreads along $\mathscr{S}$-essential geometric morphisms.

Proposition 6.4 *With $\mathscr{K}$ as above, the pseudofunctor $\mathscr{D} : \mathscr{K}^{\mathrm{op}} \to \mathbf{CAT}$ is a fibration of covering morphisms for the class Φ of surjective local homoemorphisms. Furthermore $\mathscr{D}$ preserves binary products.*

Proof First of all, $\mathscr{D}$ is a subfibration of $\mathscr{L}$ as pseudofunctors on $\mathscr{K}$; more generally, given any geometric morphism $\psi : \mathscr{G} \to \mathscr{F}$ over $\mathscr{E}$ with $\zeta : \mathscr{G} \to \mathscr{E}$ and $\varphi : \mathscr{F} \to \mathscr{E}$ complete spreads, it follows from $\varphi\psi = \zeta$ that φ is a complete spread [11]. Secondly, the local character of complete spreads established in [11] (Proposition 7.1) says that $\mathscr{D}$ is a stack for the class Φ of surjective local homeomorphisms. Finally, That $\mathscr{D}$ preserves binary products is a consequence of the equivalences

$$\mathscr{D}(\mathscr{E}) \simeq \mathbf{Dist}(\mathscr{E}) \simeq \mathscr{K}(\mathscr{S}, \mathscr{M}(\mathscr{E}))$$

and Lemma 6.3. Indeed we have the equivalences

$$\mathscr{K}(\mathscr{S}, \mathscr{M}(\mathscr{E} + \mathscr{F})) \simeq \mathscr{K}(\mathscr{S}, \mathscr{M}\mathscr{E} \times \mathscr{M}\mathscr{F}) \simeq \mathscr{K}(\mathscr{S}, \mathscr{M}\mathscr{E}) \times \mathscr{K}(\mathscr{S}, \mathscr{M}\mathscr{F})$$

where $\mathscr{K} = \mathbf{Top}_{\mathscr{S}}$. □

Let $\mathscr{U}(\mathscr{E})$ be the full subcategory of $\mathscr{E}$ determined by its complete spread objects [11, 17], alternatively viewed as the "unramified morphisms" over $\mathscr{E}$, consisting of those complete spreads over $\mathscr{E}$ which are also local homeomorphisms. Denote by $\mathscr{U}$ the intersection of $\mathscr{A}$ and $\mathscr{D}$.

Proposition 6.5 *$\mathscr{U}$ is a fibration of covering morphisms on $\boldsymbol{LTop}_{\mathscr{S}}$ for the class Φ of surjective local homeomorphisms. Furthermore, $\mathscr{U}$ preserves binary products.*

Proof The stability of complete spread objects (or of the unramified morphisms they determine) under pullback in $\mathbf{LTop}_{\mathscr{S}}$ has been shown in [17]. Hence, $\mathscr{U} : \mathbf{LTop}_{\mathscr{S}}^{\mathrm{op}} \to \mathbf{CAT}$ may be regarded as a sub pseudofunctor of $\mathscr{A}$ (not just of $\mathscr{L}$) . The remaining properties are a consequence Proposition 6.4. □

Corollary 6.6 *Let*

$$\begin{tikzcd} \mathscr{E}_1 \arrow[r, "\alpha_1"] & \mathscr{E} \\ \mathscr{E}_0 \arrow[u, "\beta_1"] \arrow[r, "\beta_2"'] & \mathscr{E}_2 \arrow[u, "\alpha_2"'] \end{tikzcd}$$

be a pushout diagram in $\boldsymbol{LTop}_{\mathscr{S}}$ in which the induced map $\alpha : \mathscr{E}_1 + \mathscr{E}_2 \to \mathscr{E}$ is a surjective local homeomorphism. Then

$$\begin{tikzcd} \mathscr{U}(\mathscr{E}_1) \arrow[r, "\beta_1^*"] & \mathscr{U}(\mathscr{E}_0) \\ \mathscr{U}(\mathscr{E}) \arrow[u, "\alpha_1^*"] \arrow[r, "\alpha_2^*"'] & \mathscr{U}(\mathscr{E}_2) \arrow[u, "\beta_2^*"'] \end{tikzcd}$$

is a pullback in **CAT**.

Proof It follows directly from Theorem 6.2 and Proposition 6.5. □

A topological notion of covering morphism on $\mathbf{LTop}_{\mathscr{S}}$ ought to restrict at least to all local homeomorphisms satisfying a path-lifting property.

Definition 6.7 A fibration $\mathscr{C}$ of covering morphisms in the sense of Definition 6.1 is said to be *topological* if:

1. $\mathscr{C}$ is a subpseudofunctor of $\mathscr{A} : \mathbf{LTop}_{\mathscr{S}}^{\mathrm{op}} \to \mathbf{CAT}$, and
2. for every (locally connected) topos $\mathscr{E}$, every cover $X \in \mathscr{C}(\mathscr{E})$ has the unique path-lifting property in the sense of Definition 5.1.

Let us examine how the three examples of fibrations of covering morphisms (given so far) fare from the topological viewpoint expressed in Definition 6.7. The *second condition* in the above definition excludes $\mathscr{A}$ as a topological fibration of covering morphisms since not every cover X in an arbitrary (locally connected) topos $\mathscr{E}$ has the unique path-lifting property, or not every local homeomorphism is a (discrete) fibration. Thus, arbitrary local homeomorphisms, though covering morphisms, are not in

general topological covering morphisms. The *first condition* in the above definition excludes also $\mathscr{D}$ as a topological fibration of covering morphisms since $\mathscr{D}$ is not a subpseudofunctor of $\mathscr{A}$, that is, not all complete spreads are local homeomorphisms. By contrast, the fibration $\mathscr{C}$ of locally trivial covering morphisms satisfies the required conditions.

Proposition 6.8 *Let $\mathscr{C}(\mathscr{E}) = \mathscr{E}_{lc}$ be the full subcategory of $\mathscr{E}$ determined by the locally constant objects in the sense of Definition 2.1. Then $\mathscr{C}$ defines a topological fibration of covering morphisms in the sense of Definition 6.7.*

Proof That $\mathscr{C}$ is a subpseudofunctor of $\mathscr{A}$ is part of the definition of $\mathscr{C}$. That every object of every $\mathscr{C}(\mathscr{E})$ satisfies the unique path-lifting property was established in Proposition 5.3. □

It is natural to ask whether the topological notion expressed in Definition 6.7 includes examples other than the locally trivial covering morphisms.

The following result was shown in [11] for the locally constant objects in the sense of Barr and Diaconescu [3]; we need to establish it here for the locally constant objects in the sense of Definition 2.1.

Proposition 6.9 *Let Y be a locally constant object in a locally connected topos $\mathscr{E}$. Then, Y is a complete spread object in $\mathscr{E}$.*

Proof Let Y be a locally constant object in $\mathscr{E}$ in the sense of Definition 2.1. By Proposition 2.2 an equivalent condition (expressed therein as condition (3)) is that there is a cover U in $\mathscr{E}$, a morphism $\alpha : J \to I$ in $\mathscr{S}$, a morphism $\eta : U \to e^*I$, and a morphism $\zeta : Y \times U \to e^*J$, for which the square

$$\begin{tikzcd} Y \times U \arrow[r, "\pi_2"] \arrow[d, "\zeta"] & U \arrow[d, "\eta"] \\ e^*J \arrow[r, "e^*\alpha"'] & e^*I \end{tikzcd}$$

is a pullback. By the above and the local character of complete spreads along surjective local homeomorphisms [11](Proposition 7.1), it is sufficient to prove that any geometric morphism $\mathscr{E}/e^*J \to \mathscr{E}/e^*I$ (induced by a morphism $\alpha : J \to I$ in $\mathscr{S}$) is a complete spread. We know [11](proof of Theorem 7.3) that constant objects are complete spreads. Therefore the composite $\mathscr{E}/e^*J \to \mathscr{E}/e^*I \to \mathscr{E}$ as well as the second morphisms in the composite are both complete spreads. By [10] (Proposition 1.2.3) this implies that $\mathscr{E}/e^*J \to \mathscr{E}/e^*I$ is a complete spread. □

Theorem 6.10 *Let X be a complete spread cover in a locally connected topos $\mathscr{E}$. Then X has the unique path-lifting property in the sense of Definition 5.1.*

Proof The following is a characterization of complete spread covers [11] Proposition 7.5. Let $\mathscr{E}$ be a locally connected topos with pure inclusion

$\mathscr{E} \hookrightarrow \mathscr{S}^{\mathbf{C}^{\mathrm{op}}}$. Assume that X is a cover in $\mathscr{E}$ with corresponding (surjective) discrete opfibration $X : \mathbf{X} \to \mathbf{C}$ and associated discrete fibration $D : \mathbf{D} \to \mathbf{C}$. Then X is a complete spread object if and only if the square

$$\begin{tikzcd} \mathscr{E}/X \arrow[r] \arrow[d] & \mathscr{S}^{\mathbf{D}^{\mathrm{op}}} \arrow[d] \\ \mathscr{E} \arrow[r] & \mathscr{S}^{\mathbf{C}^{\mathrm{op}}} \end{tikzcd}$$

is a pullback, where the right vertical arrow is induced by D and therefore is a surjective discrete fibration (and opfibration); the geometric morphism that it induces has the unique paths-lifting property.

We now show that the local homeomorphism $\mathscr{E}/X \to \mathscr{E}$ has the unique path-lifting property. Consider the cube:

$$\begin{tikzcd} & \mathscr{D} \arrow[rr] \arrow[dd] & & \mathscr{S}^{\mathbf{D}^{\mathrm{op}}} \arrow[dd] \\ \mathscr{D}^{\Delta_n} \arrow[ur] \arrow[rr] \arrow[dd] & & \mathscr{S}^{\mathbf{D}^{\mathrm{op}}\Delta_n} \arrow[ur] \arrow[dd] & \\ & \mathscr{E} \arrow[rr] & & \mathscr{S}^{\mathbf{C}^{\mathrm{op}}} \\ \mathscr{E}^{\Delta_n} \arrow[ur] \arrow[rr] & & \mathscr{S}^{\mathbf{C}^{\mathrm{op}}\Delta_n} \arrow[ur] & \end{tikzcd}$$

In it, the front, back, and right side facesare pullbacks, hence so is the left side square. This gives the unique paths-lifting property for the geometric morphism $\mathscr{D} \to \mathscr{E}$, as claimed. $\square$

Proposition 6.11 *$\mathscr{U}$ is a topological fibration of covering morphisms.*

Proof By Proposition 6.5 $\mathscr{U}$ is a fibration of covering morphisms. By definition, $\mathscr{U}$ is a subfibration of $\mathscr{A}$. The path-lifting property is shown in Theorem 6.10. $\square$

We note that, as shown explicitly in [17], the inclusion $\mathscr{C} \hookrightarrow \mathscr{U}$ is proper. Hence this is a genuine example of a notion of covering morphism having analogous topological properties as that of the locally trivial covering morphisms, yet it is not (seemingly) an instance of the Janelidze notion. Although $\mathscr{U}$ is a stack and contains all trivial morphisms, it is not the stack completion of it; indeed the stack completion of the trivial morphisms is $\mathscr{C}$. Although not discussed in this paper, it can be shown, just as in topology, that in the case of locally paths simply connected toposes, the two classes $\mathscr{C}$ and $\mathscr{U}$ agree.

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