

# Parameter definability in the recursively enumerable degrees

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ABSTRACT. The biinterpretability conjecture for the r.e. degrees asks whether, for each sufficiently large  $k$ , the  $\Sigma_k^0$  relations on the r.e. degrees are uniformly definable from parameters. We solve a weaker version: for each  $k \geq 7$ , the  $\Sigma_k^0$  relations bounded from below by a nonzero degree are uniformly definable. As applications, we show that  $\text{Low}_1$  is parameter definable, and we provide a new example of a  $\emptyset$ -definable ideal. Moreover, we prove that automorphisms restricted to intervals  $[\mathbf{d}, \mathbf{1}]$ ,  $\mathbf{d} \neq \mathbf{0}$ , are  $\Sigma_7^0$ . We also show that, for each  $\mathbf{c} \neq \mathbf{0}$ ,  $(\mathbb{N}, +, \times)$  can be interpreted in  $[\mathbf{0}, \mathbf{c}]$  without parameters.

## 1. Introduction

The study of structures based on recursively enumerable sets is an important area of computability theory. Among the structures one studies are the lattice  $\mathcal{E}$  of recursively enumerable (r.e.) sets under inclusion and the degree structures  $\mathcal{R}_T$  and  $\mathcal{R}_m$  induced on the r.e. sets by Turing- and many-one reducibility. The structure  $\mathcal{R}_T$  has been in the focus of interest for 50 over years, beginning with Post's problem whether there are r.e. Turing degrees besides the degree of a computable set and the degree of the halting problem.

For all three structures, the approach of studying definability and coding with first-order formulas has been very successful. Definability with parameters was first used as a tool to prove that the theory of the structure is undecidable. For instance, the first proof that  $\text{Th}(\mathcal{R}_T)$  is undecidable, due to Harrington and Shelah [4], proceeded by coding an arbitrary  $\Delta_2^0$  partial ordering, using formulas involving four parameters. In such proofs, r.e. sets which are values for the parameters are constructed in order to encode very particular sets and relations. Here, we aim at definability results of a more general kind. Given a class  $\mathcal{C}$  of sets (or relations) on  $\mathcal{R}_T$ , we want to find a formula with parameters which enables us, as the parameters vary, to define all sets (or relations) in  $\mathcal{C}$ . Such a class is called *weakly uniformly definable*. If, in addition, we can specify a first-order condition  $\alpha$  on parameters so that the parameters satisfying  $\alpha$  give precisely the elements of  $\mathcal{C}$ , then we call  $\mathcal{C}$  *uniformly definable*.

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Uniform definability results have been proved for other structures, usually as a tool to interpret  $\text{Th}(\mathbb{N}, +, \times)$  in the theory. For instance, in  $\mathcal{R}_m$ , for sufficiently large  $k$ , the  $\Sigma_k^0$  relations which are contained in a proper initial interval are weakly uniformly definable (Nies [8]). Harrington’s Ideal Definability Lemma for  $\mathcal{E}$  (see [3]) is another example. Finally, in [10], we show that, in the structure of r.e. weak truth-table (*wtt*) degrees, the class of finite sets of low *wtt*-degrees which have pairwise have as a supremum the greatest *wtt*-degree is weakly uniformly definable.

A structure  $(\mathbf{A}, R_1, \dots, R_n)$  is called *arithmetical* if there is an onto map  $\beta : \mathbb{N} \mapsto A$  such that the preimages under  $\beta$  of the relations  $R_i$  and of equality on  $A$  are arithmetical. Furthermore, we say a given relation on  $\mathbf{A}$  is  $\Sigma_k^0$  (*arithmetical*) if its preimage is  $\Sigma_k^0$  (arithmetical). Clearly, all parameter definable relations are arithmetical.

One of our principal results is that *for all sufficiently large  $k$  (actually for all  $k \geq 7$ ), the class of  $\Sigma_k^0$  relations in an interval  $[\mathbf{d}, \mathbf{1}]$ ,  $\mathbf{d} \neq \mathbf{0}$ , is uniformly definable.* The biinterpretability (BI, for short) conjecture for an arithmetical structure  $\mathbf{A}$  asks whether this holds for all for sufficiently large  $k$  and all  $\Sigma_k^0$  relations, without further restrictions. Since  $\mathbf{A}$  is arithmetical, this property is equivalent to the usual model theoretic notion of biinterpretability (see [5]) of  $\mathbb{N}$  and the structure obtained from extending  $\mathbf{A}$  by finitely many constants. The property has interesting consequences for  $\mathbf{A}$ : the automorphisms are uniformly definable (and therefore there are only countable many), and each orbit under the action of the automorphism group on  $A^n$  is  $\emptyset$ -definable (in other words,  $\mathbf{A}$  is a prime model of its theory). With our approximation, we still obtain that the class of partial 1-1 functions  $\{\Phi \upharpoonright [\mathbf{d}, \mathbf{1}] : \Phi \in \text{Aut}(\mathcal{R}_T) \ \& \ \mathbf{d} \neq \mathbf{0}\}$  is weakly uniformly definable, and that these functions are  $\Sigma_7^0$ .

We derive from the main result that an ideal of  $\mathcal{R}_T$  which is generated by a  $\emptyset$ -definable set is itself  $\emptyset$ -definable. This leads to a new  $\emptyset$ -definable ideal, the ideal generated by the nonbounding degrees, thereby answering [12, Question 2.8]. (The recursion theoretic details will appear in [7].)

In [2], an interesting new example of a  $\Pi_5^0$ -ideal was given, namely  $\{\mathbf{x} : \forall \mathbf{y} \in \text{Low}_1 [\mathbf{x} \vee \mathbf{y} \in \text{Low}_1]\}$ . As a consequence of our result, this ideal is parameter definable. Using this together with our uniform definability result, we prove that  $\text{Low}_1$  is parameter definable. In [11] it was shown that all the jump classes except possibly  $\text{Low}_1$  are  $\emptyset$ -definable.

We also prove that for  $k \geq 7$ , the class of  $\Sigma_k^0$ -ideals of  $\mathcal{R}_T$  is uniformly definable. The lattice of all ideals of  $\mathcal{R}_T$  was studied for instance by Calhoun [1]. We show that the lattice of  $\Sigma_k^0$ -ideals,  $k \geq 7$ , with an extra predicate for being principal, is biinterpretable with  $\mathcal{R}_T$ .

To obtain our results, we introduce a more flexible construction of Slaman–Woodin (SW) sets (see [11]). We use the lower bound  $\mathbf{e}$  for SW sets as an “emergency device” to correct the  $\Delta$ -functionals of minimality requirements  $M_i$  (in [11], the lower bound was used for a different purpose). We also guess at the outcomes of those requirements in a tree construction. The greater flexibility in building SW sets makes their construction compatible with permitting below a given nonzero degree  $\mathbf{c}$ . In this way, we are able to provide an interpretation without parameters of  $(\mathbb{N}, +, \times)$  in  $[\mathbf{0}, \mathbf{c}]$ .

In [9], we announced a uniform definability result which was somewhat stronger than the one given here. It would, for instance, imply that each u.r.e. set of nonzero degrees is definable. The proof we had in mind was incorrect. However, the consequences listed above all follow as well from the present result.

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## 2. Main results

Throughout this article, “definable” means parameter definable, and “ $\emptyset$ -definable” means definable without parameters. All degrees will be r.e. Turing degrees. We recall some concepts. See [11] for more details.

**DEFINITION 2.1.** *A scheme for coding in an  $L$ -structure  $\mathbf{A}$  is given by a list of  $L$ -formulas  $\varphi_1, \dots, \varphi_n$  with a shared parameter list  $\bar{p}$ , together with a correctness condition  $\alpha(\bar{p})$ . If a scheme  $S_X$  is given,  $X, X_0, X_1, \dots$  denote objects coded via  $S_X$  by a list of parameters satisfying the correctness condition. If necessary, we write  $X(\bar{p})$  to indicate that  $X$  is coded by the list  $\bar{p}$ .*

**DEFINITION 2.2.** *A scheme  $S_M$  for coding models of Robinson arithmetic  $Q$  (in the language of arithmetic  $L(+, \times)$ ) is given by the formulas  $\varphi_{num}(x, \bar{p})$ ,  $\varphi_+(x, y, z; \bar{p})$ ,  $\varphi_\times(x, y, z; \bar{p})$  and a correctness condition  $\alpha_0(\bar{p})$  which expresses that  $\varphi_+$  and  $\varphi_\times$  define binary operations on the nonempty set  $\{x : \varphi_{num}(x; \bar{p})\}$ , and that  $\{x : \varphi_{num}(x; \bar{p})\}$  with the corresponding operations satisfies the finitely many axioms of  $Q$ .*

All models  $M$  coded by such a scheme  $S_M$  have a standard part isomorphic to  $\mathbb{N}$ . If this standard part equals  $M$ , we say that  $M$  is *standard*. Given  $k \in \mathbb{N}$ ,  $k^M$  denotes  $k$ , viewed as an element of  $M$ .

- DEFINITION 2.3.**
- (i) *A scheme  $S_g$  for coding a function  $g$  is given by a formula  $\varphi_1(x, y; \bar{p})$  defining the relation between arguments and values, and a correctness condition  $\alpha(\bar{p})$  which says (at least) that a function is defined on the intended domain.*
  - (ii) *More generally, a scheme for defining  $n$ -ary relations on  $\mathbf{A}$  is given by a formula  $\varphi(x_1, \dots, x_n; \bar{p})$  and a correctness condition  $\alpha(\bar{p})$ .*
  - (iii) *A set  $\mathcal{C}$  of  $n$ -ary relations on  $\mathbf{A}$  is uniformly definable (u.d.) in  $\mathbf{A}$  if, for some scheme  $S$  for coding relations,  $\mathcal{C}$  is the class of relations coded via  $S$  as the parameters range over tuples in  $\mathbf{A}$  which satisfy the correctness condition.*
  - (iv)  *$\mathcal{C}$  is weakly uniformly definable if  $\mathcal{C}$  is contained in a uniformly definable class.*

We are now ready to state our main result, in the language of coding schemes.

**THEOREM 2.4.** *There is a scheme  $S_M$  for coding a model of  $Q$  and a scheme  $S_h$  for coding functions with the following property. For each  $\mathbf{d} \neq \mathbf{0}$ , there is a standard  $M$  and a map  $h$  such that  $h : M \rightarrow [\mathbf{d}, \mathbf{1}]$  is onto.*

The parameter list contains a parameter  $\mathbf{e} \neq \mathbf{0}$  such that  $\mathbf{e} \leq \mathbf{n}$  for each  $\mathbf{n} \in M$ .

If  $X \subseteq \mathcal{R}_T$ , let  $\text{Ind}(X)$  denote the index set  $\{e : \text{deg}(W_e) \in X\}$ , and similarly for relations on  $\mathcal{R}_T$ . We say that  $X$  is  $\Sigma_k^0$  if its index set is  $\Sigma_k^0$ . Before phrasing the main result in the language of arithmetical sets, we need to analyze the connection between the arithmetical complexity of  $\text{Ind}(X)$  and the complexity of  $X$  as a relation on  $M$ .

LEMMA 2.5. *Suppose  $S_M$  is a scheme as in Example 2.2 and  $r$  is a number such that for every standard  $M$ , there is a function  $\alpha \leq_T \emptyset^{(r)}$  such that  $\forall i \in \mathbb{N} \ i^M = \text{deg}(W_{\alpha(i)})$ .*

*Suppose  $M$  is standard and  $X$  is an  $n$ -ary relation on  $M$ .*

- (i) *If  $k \geq \max(r+1, 4)$  and  $X$  is  $\Sigma_k^0$  as a relation on  $M$ , then  $\text{Ind}(X)$  is  $\Sigma_k^0$ .*
- (ii) *If  $k \geq \max(r+1, 5)$  then, conversely,  $\text{Ind}(X) \in \Sigma_k^0$  implies that  $X$  is  $\Sigma_k^0$  as a relation on  $M$ .*

*Proof.* To simplify notation we assume that  $n = 1$ .

(i). Suppose the  $L(+, \times)$ -formula  $\varphi(x)$  is  $\Sigma_k$  and defines  $X$  in  $M$ . Then

$$\text{Ind}(X) = \{i : \exists n[\varphi(n) (\Sigma_k^0) \ \& \ W_{\alpha(n)} \equiv_T W_i (\Sigma_{\max(4, r+1)}^0)]\}.$$

(ii). Let  $g \leq_T \emptyset^{(4)}$  be a function with range  $\{i : \forall j < i \ W_j \not\equiv_T W_i\}$ , and let  $\gamma(n) = \text{deg}(W_{g(n)})$ . Since  $k \geq 5$  and  $\text{Ind}(X)$  is  $\Sigma_k^0$ ,  $\gamma^{-1}(X)$  is  $\Sigma_k^0$ . Let  $\tilde{\gamma}$  be  $\gamma$  viewed as a function  $M \rightarrow \mathcal{R}_T$ . Clearly,  $\langle i, j \rangle \in \text{Ind}(X) \Leftrightarrow \exists n[W_{\alpha(n)} \equiv_T W_i \ \& \ W_{g(n)} \equiv_T W_j]$ , so that  $\text{Ind}(\tilde{\gamma})$  is  $\Sigma_{\max(5, r+1)}^0$ . Hence  $\tilde{\gamma}(X)$  is  $\Sigma_k^0$ , and  $X = \tilde{\gamma}^{-1}(\tilde{\gamma}(X))$  is  $\Sigma_k^0$  as a subset of  $M$ .  $\diamond$

COROLLARY 2.6. *For sufficiently large  $k$  (in fact,  $k \geq 7$ ), for each  $n$  the class  $\mathcal{D}_k^n = \{R : \text{Ind}(R) \in \Sigma_k^0 \ \& \ \exists \mathbf{d} \neq \mathbf{0} \ R \subseteq [\mathbf{d}, \mathbf{1}]^n\}$  is uniformly definable.*

*Proof.* Let  $r$  be as in Lemma 2.5, and let  $m \geq \max(r+1, 5)$  be a number such that, for each function  $h$  (coded via  $S_h$ ),  $\text{Ind}(h)$  is  $\Sigma_m^0$ . Given  $\mathbf{d} \neq \mathbf{0}$ , choose  $M, h$  as in Theorem 2.4. Suppose  $k \geq m$ . Note that, for each  $R \subseteq [\mathbf{d}, \mathbf{1}]^n$ ,

$$\text{Ind}(R) \in \Sigma_k^0 \Leftrightarrow \text{Ind}(h^{-1}(R)) \in \Sigma_k^0 \Leftrightarrow \text{Ind}(h^{-1}(R)) \Sigma_k^0 \text{ on } M$$

(for the second equivalence we used Lemma 2.5).

For notational simplicity, again assume that  $n = 1$ . We first show that each class  $\mathcal{D}_k^1$  is weakly u.d., and use this to derive a (first order) correctness condition to recognize parameters  $\bar{\mathbf{p}}$  such that  $M(\bar{\mathbf{p}})$  is standard.

The scheme  $S^k$  via which  $\mathcal{D}_k^1$  is weakly u.d. involves a list of parameters  $\bar{\mathbf{q}}$  which contains a lower bound  $\mathbf{d} \neq \mathbf{0}$  for the relation to be described, parameters to encode  $M, h$ , and also an element  $\mathbf{i}$  of  $M$ . Let  $\varphi(x, \bar{\mathbf{q}})$  express that, for some  $\mathbf{n} \in M$ ,

$$M \models \mathbf{n} \in W_{\mathbf{i}}^{\emptyset^{(k-1)}} \ \& \ h(\mathbf{n}) = \mathbf{x}.$$

As a correctness condition for  $S^k$ , firstly, require that  $\mathbf{d} \neq \mathbf{0}$  and that  $h : M \rightarrow [\mathbf{d}, \mathbf{1}]$  is onto. Clearly, each  $\Sigma_k^0$  set  $R$  can be obtained by this scheme, so that  $\mathcal{D}_k^1$  is weakly u.d. Conversely, if  $M$  is standard, only  $\Sigma_k^0$ -sets can be represented by (ii) of Lemma 2.5. Thus, to make sure that we do not include other relations, we only need to add a correctness condition to  $S_M$  which implies that each  $M$  is standard. But notice that the index set of the standard part of any  $M$  is  $\Sigma_p^0$  for a fixed  $p$ , depending only on  $S_M$ . Then, because our particular scheme  $S_M$  contains a nonzero lower bound  $\mathbf{e}$  for  $M$ , we can quantify over a class of subsets of any coded  $M$  which includes the standard part. Hence we can express that  $M$  is standard by saying that the only inductive subset coded in this way is  $M$  itself. In Section 5.3 we show that the Corollary holds for each  $k \geq 7$ , since one can choose  $r = 4$  and  $m = 7$ .  $\diamond$

An obvious modification of the proof shows:

**COROLLARY 2.7.** *The class  $\{F : F \text{ finite} \ \& \ \exists \mathbf{d} \neq \mathbf{0} \ F \subseteq [\mathbf{d}, \mathbf{1}]\}$  is uniformly definable.*

**PROPOSITION 2.8.** *The class*

$$\{\Phi \upharpoonright [\mathbf{d}, \mathbf{1}] : \Phi \in \text{Aut}(\mathcal{R}_T) \ \& \ \mathbf{d} \neq \mathbf{0}\}$$

*is weakly uniformly definable. Each such function is  $\Sigma_7^0$ .*

By Corollary 2.6, it is sufficient to show that there is a number  $k \geq 1$  such that, if  $\Phi \in \text{Aut}(\mathcal{R}_T)$  and  $\mathbf{d} \neq \mathbf{0}$ , then  $\Phi \upharpoonright [\mathbf{d}, \mathbf{1}]$  is  $\Sigma_k^0$ . Given  $\Phi$  and  $\mathbf{d}$ , choose  $M, h$  for  $[\mathbf{d}, \mathbf{1}]$  by Theorem 2.4. Let  $\widetilde{M}, \widetilde{h}$  be the standard model and map coded by the images under  $\Phi$  of the parameters coding  $M, h$ . Then  $\widetilde{h} : \widetilde{M} \mapsto [\Phi(\mathbf{d}), \mathbf{1}]$  is onto. Thus for any  $\mathbf{x} \geq \mathbf{d}$ ,  $\Phi(\mathbf{x}) = \mathbf{y} \Leftrightarrow \exists n \in \mathbb{N} [h(n^M) = \mathbf{x} \ \& \ \widetilde{h}(n^{\widetilde{M}}) = \mathbf{y}]$ , so that  $\Phi \upharpoonright [\mathbf{d}, \mathbf{1}]$  is  $\Sigma_k^0$  for some fixed  $k$ . From the remark at the end of the proof of Corollary 2.6, in fact  $k = 7$ .  $\diamond$

**THEOREM 2.9.** *Low<sub>1</sub> is parameter definable.*

*Proof.* Cholak, Slaman and Groszek [2] proved that there is a degree  $\mathbf{d} \neq \mathbf{0}$  such that  $\forall \mathbf{x} [\mathbf{x} \in \text{Low}_1 \Leftrightarrow \mathbf{x} \vee \mathbf{d} \in \text{Low}_1]$ . The class  $[\mathbf{d}, \mathbf{1}] \cap \text{Low}_1$  is  $\Sigma_4^0$ , and hence definable by Corollary 2.6.  $\diamond$

### 3. Interpreting $(\mathbb{N}, +, \times)$ in intervals $[\mathbf{0}, \mathbf{c}]$

We will develop a more flexible construction of SW-sets in the proof of the following.

**THEOREM 3.1.** *For each  $\mathbf{c} \neq \mathbf{0}$ ,  $(\mathbb{N}, +, \times)$  can be interpreted in  $[\mathbf{0}, \mathbf{c}]$  without parameters.*

First, we confine ourselves to a proof that true arithmetic, i.e.,  $\text{Th}(\mathbb{N}, +, \times)$  can be interpreted in  $\text{Th}([\mathbf{0}, \mathbf{b}])$  for low  $\mathbf{b}$  (Theorem 3.3 below). This restricted result had been announced by D. Seetapun. Thereafter we will prove the full result by an extension of the coding methods used.

To obtain the desired interpretation of  $\text{Th}(\mathbb{N}, +, \times)$ , we use a model theoretic method which is of interest by itself. The method appears to be quite versatile, and we expect it to work in arbitrary nontrivial intervals of  $\mathcal{R}_T$  as well. It is easy to adapt the proof of Theorem 3.3 to prove that the theory of nontrivial intervals  $[\mathbf{b}_0, \mathbf{b}_1]$ ,  $\mathbf{b}_1$  low, interprets true arithmetic.

**THEOREM 3.2.** *Let  $\mathbf{A}$  be a structure. Suppose schemes  $S_M$  as in Example 2.2 and  $S_g$  as in Example 2.3 are given such that some  $M$  is standard, and*

$$\forall M \forall n \in \mathbb{N} \exists M' \exists g \forall i \leq n \ g(i^M) = i^{M'}.$$

*Then  $\text{Th}(\mathbb{N}, +, \times)$  can be interpreted in  $\text{Th}(\mathbf{A})$ .*

*Proof.* This version of the proof is due to T. Slaman. By recursion, for each  $k \geq 0$ , we determine a formula  $\alpha_k(\overline{\mathbf{p}})$ . If  $\mathcal{C}_k$  is the class of models  $M(\overline{\mathbf{p}})$  defined by a parameter list  $\overline{\mathbf{p}}$  satisfying  $\alpha_k$ , then we ensure  $\forall M \in \mathcal{C}_k \ M \equiv_k \mathbb{N}$  (i.e.,  $M$  satisfies the same  $\Sigma_k$  sentences as  $\mathbb{N}$ ). We obtain the desired interpretation as follows: given a sentence  $\varphi$  in the language of arithmetic, determine  $k$  such that  $\varphi$  is  $\Sigma_k$ . Then

$$\mathbb{N} \models \varphi \Leftrightarrow \mathbf{A} \models \exists \bar{\mathbf{p}} [\alpha_k(\bar{\mathbf{p}}) \ \& \ "M(\bar{\mathbf{p}}) \models \varphi"].$$

For  $\alpha_0$  we can take the vacuous condition, since all models  $M$  satisfy  $Q$ , whence  $M \equiv_0 \mathbb{N}$ .

In the inductive step, we use the satisfaction predicate  $\text{Sat}_{\Sigma_p}(n)$  (see [6]). Thus, for each  $\Sigma_p$  sentence  $\varphi$  with code number  $n_\varphi$ , and each model  $M$  of  $Q$ ,  $M \models \varphi \Leftrightarrow M \models \text{Sat}_{\Sigma_p}(n_\varphi)$ . Inductively, suppose  $\mathcal{C}_k$  has been defined via  $\alpha_k$  such that  $\forall M \in \mathcal{C}_k \ M \equiv_k \mathbb{N}$ . Then

$$(3.1) \quad \Sigma_{k+1} - \text{Th}(\mathbb{N}) \subseteq \Sigma_{k+1} - \text{Th}(M).$$

Let  $\mathcal{C}_{k+1} = \{M \in \mathcal{C}_k : \forall x \in M \ \forall M' \in \mathcal{C}_k \ \forall g$

$$[g \text{ defines bijection } [0, x]_M \mapsto [0, g(x)]_{M'} \ \& \ M \models \text{Sat}_{\Sigma_{k+1}}(x)] \Rightarrow \\ M' \models \text{Sat}_{\Sigma_{k+1}}(g(x))].$$

Clearly there is a first-order condition  $\alpha_{k+1}(\bar{\mathbf{p}})$  describing  $\mathcal{C}_{k+1}$ . Moreover, if  $M$  is standard, then  $M \in \mathcal{C}_{k+1}$ . For suppose  $x \in M$  and  $M \models \text{Sat}_{\Sigma_{k+1}}(x)$ . Then  $x = n^M$  where  $n = n_\varphi$  for some  $\Sigma_{k+1}$  sentence  $\varphi$ . Then  $M \models \varphi$ . Suppose  $M' \in \mathcal{C}_k$ , then, by (3.1),  $M' \models \varphi$ . Hence, if  $g$  is as above,  $M' \models \text{Sat}_{\Sigma_{k+1}}(g(x))$ .

Finally, for any  $M \in \mathcal{C}_{k+1}$ ,  $\Sigma_{k+1} - \text{Th}(M) \subseteq \Sigma_{k+1} - \text{Th}(\mathbb{N})$ : by the hypothesis of the Theorem, for each  $n$ , there is a  $g$  and a *standard*  $M'$  such that  $g$  determines an isomorphism  $[0, n^M] \mapsto [0, g(n^{M'})]$ . Now suppose  $n = n_\varphi$  where  $\varphi$  a  $\Sigma_{k+1}$  sentence. By the definition of  $\mathcal{C}_{k+1}$ , if  $M \models \varphi$ , then  $M' \models \varphi$ .  $\diamond$

Using a variant of Tennenbaum's Theorem, one can show that, if  $\mathbf{A}$  is arithmetical, then in fact from some  $k$  on  $\mathcal{C}_k$  consist only of standard models.

We now turn to the recursion theoretic constructions. Throughout we use the same scheme  $S_M$ , making use of Slaman-Woodin sets above a parameter  $\mathbf{e}$  (defined in [11, Definition 2.4.]). For any degrees  $\mathbf{q}, \mathbf{p}, \mathbf{e}, \mathbf{r}$ , the Slaman-Woodin set is

$$\text{SW}(\mathbf{q}, \mathbf{p}; \mathbf{e}, \mathbf{r}) = \{\mathbf{x} \in [\mathbf{e}, \mathbf{r}] : \mathbf{x} \text{ is minimal in } [\mathbf{e}, \mathbf{r}] \text{ s.t. } \mathbf{q} \leq \mathbf{x} \vee \mathbf{p}\}.$$

To obtain copy  $M$  of  $\mathbb{N}$  encoded in  $\mathcal{R}_T$ , first we build a u.r.e. sequence  $(\mathbf{g}_n)_{n \in \mathbb{N}}$  and parameters so that

$$(3.2) \quad \{\mathbf{g}_n : n \in \mathbb{N}\} = \text{SW}(\mathbf{q}, \mathbf{p}; \mathbf{e}, \mathbf{r})$$

To define  $+$ ,  $\times$ , we use a recursive p.o.  $(\mathbb{N}, \preceq)$  encoding a copy  $L$  of  $(\mathbb{N}, +, \times)$  as described in [11, p. 246]. We may assume that  $\preceq$  is chosen in a way that  $i^L$  equals  $2i$  (as an element of  $\mathbb{N}$ ), and that these elements of  $L$  are minimal elements of  $(\mathbb{N}, \preceq)$ . We will ensure that

$$(3.3) \quad \mathbf{g}_i \leq \mathbf{g}_j \vee \mathbf{1} \Leftrightarrow i \preceq j.$$

In the following Theorem, (i) and (ii) serve to code a copy of  $\mathbb{N}$ , while (iii) is needed to satisfy the additional hypothesis of Theorem 3.2 for  $A = [\mathbf{0}, \mathbf{b}]$  when  $\mathbf{b} \neq \mathbf{0}$  is low.

**THEOREM 3.3.** (i). For each  $\mathbf{d} \neq \mathbf{0}$ , there is a u.r.e. antichain  $(\mathbf{g}_n)$  in  $[\mathbf{0}, \mathbf{d}]$  such that

$$\{\mathbf{g}_n : n \in \mathbb{N}\} = \text{SW}(\mathbf{q}, \mathbf{p}; \mathbf{e}, \mathbf{r})$$

for some  $\mathbf{e} \leq \mathbf{d}$ ,  $\mathbf{p} \leq \mathbf{q} \leq \mathbf{d}$ ,  $\mathbf{r} = \bigoplus_n \mathbf{g}_n$ . Moreover,  $\mathbf{r} \leq \mathbf{d}$ .

(ii). If, in addition, a recursive partial ordering  $\mathcal{P} = \langle \omega, \preceq \rangle$  is given, then we can also construct  $\mathbf{l}$  such that  $i \preceq j \Leftrightarrow \mathbf{g}_i \leq \mathbf{l} \vee \mathbf{g}_j$ . Moreover,  $\mathbf{r}, \mathbf{l} \leq \mathbf{d}$ .

(iii). Suppose  $\mathbf{0} < \mathbf{d} < \mathbf{b}$ ,  $\mathbf{b}$  is low and  $\mathbf{u}_0, \dots, \mathbf{u}_n \in [\mathbf{d}, \mathbf{b}]$ . Suppose further that the even numbers are minimal elements of  $\mathcal{P}$ . Then there are degrees as above such that, in addition,  $\mathbf{e} \neq \mathbf{0}$ ,  $\mathbf{l}, \mathbf{r} \leq \mathbf{b}$  and  $\mathbf{g}_{2i} \leq \mathbf{u}_j \Leftrightarrow i = j$ , for  $0 \leq i, j \leq n$ . Moreover,  $\mathbf{r}, \mathbf{l} \leq \mathbf{b}$ .

To see that (iii) is sufficient for Theorem 3.2, first add as an additional correctness condition in  $S_M$  that the lower bound  $\mathbf{e}$  be nonzero. Given  $M$ , let  $\mathbf{d}$  be the lower bound for  $M$  and let  $\mathbf{u}_i = i^M$ . Obtain  $M'$  via (iii). In the scheme  $S_g$ , use two further parameters  $\mathbf{u}, \tilde{\mathbf{u}}$  with the values  $n^M$  and  $n^{M'}$ . Then the following defines a map  $g$  as desired (where  $\leq_M, \leq_{M'}$  denote the orderings on  $M, M'$ ):

$$g(\mathbf{x}) = \mathbf{y} \Leftrightarrow [\mathbf{x} \geq \mathbf{y} \ \& \ \mathbf{x} \leq_M \mathbf{u} \ \& \ \mathbf{y} \leq_{M'} \tilde{\mathbf{u}}].$$

*Proof.* The proof is an extension of the construction of Slaman-Woodin sets given in [11, Thms 4.1 and 4.7]. We review this construction here in order to explain our extensions, but for more details, see [11, 4.2]. For each degree  $\mathbf{x}$  (given or to be constructed), the corresponding capital letter  $X$  denotes a r.e. set in this degree.

For (i), we construct r.e. sets  $E \leq_T D$  and  $P \leq_T Q$  such that  $Q \leq_T D$ . Moreover we build a u.r.e. sequence of sets  $(G_e)$  and will ensure that  $G \cap 2\mathbb{N} = E$  for each  $n$  (i.e., only odd numbers go into a set  $G_n$ , unless they also go into  $E$ ). Let  $R = \bigoplus_n G_n$ . We meet the requirements from [11, Thm 4.1]. The requirements

$$T_i : G_i \oplus P \geq_T Q$$

are met by building functionals  $\Gamma_i$  such that  $\Gamma_i(G_i \oplus P) = Q$ . To make the set of degrees  $\{\mathbf{g}_i : i \in \mathbb{N}\}$  a Slaman-Woodin set, we meet requirements

$$M_i : [\Theta_i(R) = W_i \ \& \ \Phi_i(W_i \oplus P) = Q] \Rightarrow \exists j \leq n(i) \ G_j \leq_T W_i \oplus E,$$

where  $n(i)$  is the number  $n$  such that  $T_0, \dots, T_n$  have higher priority than  $M_i$ , and  $(W_i, \Phi_i, \Theta_i)$  is an effective list of all triples consisting of a r.e. set  $W_i$  and two functionals  $\Phi_i, \Theta_i$ , such that  $W_i \cap 2\mathbb{N} = E$ .

To make the degrees  $(\mathbf{g}_i)$  pairwise incomparable, we meet requirements

$$D_{i,j,e} : \Psi_e(G_i) \neq G_j,$$

where  $(\Psi_e)$  is a list of all functionals.

The basic idea for the  $M_i$ -strategy is as before: for ascending  $j \leq n(i)$ , attempt to build a Turing reduction  $\Delta_{i,j}(W_i) = G_j$  (as an aid, in the notation  $\Delta_{i,j}$ , the index  $i$  indicates which requirement  $M_i$  builds the functional, while  $j$  indicates the set  $G_j$  being computed). If the last of them failed, we argue that we can diagonalize against  $\Phi_i(W_i \oplus P) = Q$ : there is  $y$  such that  $\Phi_i(W_i \oplus P; y) = 0$  and  $\gamma_j(y) > \varphi(y)$  for each  $j \leq n(i)$ , so that when we put  $y$  into  $Q$ , the  $\Gamma_j$  correction of  $T_j$ , which is via a  $P$ -enumeration, does not destroy our successful diagonalization. We first define a length of agreement function:

$$(3.4) \quad \begin{aligned} l(i, u) &= \mu z \{ \neg [\Phi_i(W_i \oplus P; z) \downarrow = Q(z) [u] \\ &\wedge (\forall w \leq \varphi_i(z)) (\Theta_i(R; w) \downarrow = W_i(w) [u]) ] \}. \end{aligned}$$

Unlike before, an *i-expansionary* stage is now simply a stage where this length is larger than at all previous stages.

The functionals  $\Delta_{i,j}$  are defined at stages  $s$  via numbers  $y < l(i, s)$  targeted for  $Q$ . Each  $y$  has a ‘‘chit’’, which is cancelled at a later stage if  $\Phi_i(W_i \oplus P; y)$  changes, in

which case  $y$  never goes into  $Q$ . Once  $l(i, s) > y$  and the chit for  $y$  is uncanceled and has not yet been assigned to a  $\Delta_{i,0}$  computation, define  $\Delta_{i,0}(W_i, x) = G_0(x)$  with use  $\varphi_i(y)$ , assigning the chit for  $y$  to this  $\Delta$  computation, where  $x$  is least such that  $x < y$  and  $\Delta_{i,0}(x)$  is undefined.

In the basic construction, if  $x$  is put into  $G_i$  (by some  $D$ -type requirement), then by the next  $i$ -expansionary stage we see if  $W_i \upharpoonright \delta_{i,0}(x)$  changed (note that  $W_i$  can only change in reaction to an  $R$ -change). If so we correct  $\Delta_{i,0}(x)$ . If not, we say the  $\Delta(x)$  computation *fails*. We have made progress on a possible diagonalization against  $\Phi_i(W_i \oplus P) = Q$ , since now  $\gamma_0(y) > \varphi_i(y)$ . We next assign the chit for  $y$  to a (newly defined)  $\Delta_{i,1}(W_i; x)$  computation with value  $G_1(x)$  and the same use  $\varphi_i(y)$ , etc. Once the uncanceled chit for  $y$  has been assigned to a  $\Delta_{i,n(i)}(W_i; z)$  computation which failed,  $y$  is ready as a candidate for diagonalization. Meanwhile, if one of the  $\Delta_{i,j}(x)$  computation is corrected (i.e.,  $W_i \upharpoonright \varphi_i(y)$  changes), we cancel the chit for  $y$ .

Since we want  $Q \leq_T D$ ,  $M_i$  actually appoints a list of candidates  $y_0 < y_1 < \dots$ . Each time a new candidate is appointed,  $M_i$  initializes lower priority requirements. Once  $D \upharpoonright p$  changes,  $y_p$  is enumerated into  $Q$ .

The argument above depends on the preservation of a computation  $\Phi_i(W_i \oplus P; y) = 0$ . Since we can make the  $\Delta_{i,j}(W_i; x)$  computation undefined when  $W_i \upharpoonright \varphi_i(y)$  changes (and later redefine it via a new  $y$ ), we only need to be able to restrain  $P \upharpoonright \varphi_i(y)$ . Suppose  $x$  is a candidate for a  $G_j$ -positive strategy  $U$  (say  $D_{j,l,e}$ ) and that  $M_i$  is a requirement of higher priority than  $U$  while  $j \leq n(i)$ . Suppose there are infinitely many  $i$ -expansionary stages.

(1) Till  $x$  is enumerated, it is  $U$ 's responsibility to maintain the suitability of  $x$  for enumeration into  $G_j$ , by restraining  $P \upharpoonright \delta_{i,j}(x)$ . When this restraint is violated,  $x$  is cancelled.

(2) Next we consider the  $P$ -restraint we need after an enumeration of  $x$ . If we proceeded as in [11, 4.2], we would rely on the condition that the  $P$ -restraint is not violated till the next  $i$ -expansionary stage, since only then we have new computations of  $W_i$  from  $R$  and can make progress on a possible diagonalization in case  $W_i \upharpoonright \delta_{i,j}(x)$  did not change. Thus, after the enumeration, till the next  $i$ -expansionary stage, the  $P$ -restraint is maintained by the higher priority requirement  $M_i$ . This leads to the restraint on  $P$  with finite  $\liminf$  in [11], which makes that construction incompatible with permitting by  $D$ .

Here, we enumerate into the lower bound  $E$  instead. Consider a  $P$ -positive strategy  $M_k$  (necessarily  $k > i$ , else  $M_i$  is initialized for another time), which changes  $P \upharpoonright \delta_{i,j}(x)$  before the next  $i$ -expansionary stage following the enumeration of  $x$ . Then  $M_k$  has higher priority than  $U$ . The new idea is that  $M_k$  also enumerates a  $\Delta$ -correction number  $w$  into  $E$ . This number was chosen by  $U$  right after the last time  $U$  was initialized, so that  $w < x$ . Since  $E$  is a part of  $W_i$ , we can now immediately redefine  $\Delta_{i,j}(x) = 1$  with use 0. We have to verify that enumerating  $w$  does no harm to  $M_k$  itself. But, by the initialization of lower priority requirements  $M_k$  carries out whenever it appoints a new candidate  $y$ ,  $w$  is greater than  $l(k, s)$  and hence any computation  $M_k$  is interested in. (The potential problem would be that  $E$  is a part of  $R$  and  $M_k$  needs to preserve a certain part of  $\Theta_k(R) = W_k$  in order to preserve its diagonalization.) Clearly the enumeration of  $w$  into  $E$  needs to initialize  $U$ , whose goal it was to preserve a set  $G_l$ . However, the diagonalization action of  $M_k$  is finitary, so that  $U$  is only initialized finitely often.

Note that a strategy at a single stage  $M_k$  puts numbers  $y_p$  into  $Q$ , larger numbers  $\gamma_j(y)$  into  $P$  and even larger  $\Delta$ -correction numbers into  $E$ .

The situation above is similar to the problem of building infima below  $\mathbf{d}$ . If, say,  $\mathbf{d}$  is nonbounding, a minimal pair type construction (which involves a restraint with only finite lim inf) must fail. However, there is always some  $\mathbf{e} < \mathbf{d}$  which is meet reducible within  $[\mathbf{0}, \mathbf{d}]$ . Like our  $M$ -type strategies, the infimum strategies use correction numbers targeted for  $E$ .

We also meet lowness type requirements as in [11], but use  $Q$  instead of the set  $P$ , since in a later application we need  $R \oplus Q$  to be low.

4.  $K_{e,x}$ : If there are infinitely many  $s$  such that  $\Xi_e(R \oplus Q; x) \downarrow [s]$ , then  $\Xi_e(R \oplus Q; x) \downarrow$ .

Here  $\{\Xi_e\}$  lists all partial recursive functionals and includes various ones appearing in the construction with the approximations given in the construction. Their purpose is to make the various functionals  $\Gamma$ ,  $\Delta$  converge if they are defined often enough, and to make sure there are enough numbers  $y$  with uncanceled chits. Since  $P \leq_T Q$  via permitting, we indirectly also preserve computations with oracle  $R \oplus P$ . The strategies are as before.

We now give the formal construction for (i). The same general comments apply as in [11, Section 4.2]. We order the requirements in a priority list. We describe the action at substage  $n$  of stage  $s$  according to the type of  $n$ -th requirement. Recall that  $r$  is the last stage where such a requirement was initialized (all requirements are initialized at stage 0). Comments in double brackets [...] are motivational only.

1.  $T_i : G_i \oplus P \geq_T Q$ . As before, let  $y$  be least such that  $\Gamma(G_i \oplus P; y)$  is undefined and define it with large use. Whenever later the oracle changes below the use, make the computation undefined.

2.  $D_{i,j,e} : \Psi_e(G_i) \neq G_j$  If this is the first stage at which we deal with this requirement since  $r$ , choose a large  $\Delta$ -correction number  $w$  targeted for  $E$ . [[We will do this for any  $R$ -positive strategy added in later extensions of the construction.]]

The  $D_{i,j,e}$ -strategy builds a list of followers  $x_0 < x_1 < \dots$  targeted for  $G_j$ . A follower  $x$  is *realized* when  $\Psi_e(G_i; x) = 0$ . Suppose  $D_{i,j,e}$  is not yet satisfied. If  $k > 0$  and  $x_{k-1}$  has been defined and is realized or  $k = 0$ , then appoint a new large follower, and initialize requirements of lower priority.

At every stage  $t \geq s$  until a follower  $x$  is cancelled or enters  $G_j$ ,  $D_{i,j,e}$  imposes restraint  $r(D, i, j, e, t)$  on  $P$  where

$$(3.5) \quad r(D, i, j, e, t) = \max\{\delta_{k,j}(W_k; x)[t] \mid M_k < D_{i,j,e} \ \& \ x \text{ follower of } D_{i,j,e}\}.$$

If  $x_k$  is defined and  $D \upharpoonright k$  changes, then put  $x_k$  into  $G_j$ . Initialize all lower priority requirements.

3.  $M_i : [\Theta_i(R) = W_i \ \& \ \Phi_i(W_i \oplus P) = Q] \Rightarrow \exists j \leq n(i) \ G_j \leq_T W_i \oplus E$ ,

$M_i$  builds and protects a potentially infinite list  $y_0 < y_1 < \dots$  of numbers ready for enumeration into  $Q$ . Suppose  $p > 0$  and  $y_{p-1}$  has been defined, or  $p = 0$ . We now proceed as in [11], possibly defining  $y = y_p$ : if  $s$  is not  $i$ -expansionary or  $M_i$  has put a number into  $Q$  since  $r$ , we go on to the next substage of stage  $s$ . Otherwise, as before go through the procedure of defining computations  $\Delta_{i,j}(x)$  summarized above and described in detail in [11, p. 263]. If we reach a point where there are no appropriate chits to pass on to the next level of functionals, we terminate this substage of stage  $s$ . Otherwise, if there is a number  $y$  such that  $y > y_{p-1}$ , and

there is a failure of  $\Delta_{i,n(i)}$  at some  $x$  and the uncanceled chit for  $y$  is assigned to  $\Delta_{i,n}(W_i; x)$  then define  $y_p = y$ . Initialize requirements of lower priority.

At any  $s$ , if  $y_p$  is defined and  $D \upharpoonright p$  changes, then put  $y_p$  into  $Q$ ,  $\gamma_j(y_p)$  into  $P$  for each  $j \leq n(i)$ . [[ As before, the point to verify will be that  $\gamma_j(y) \geq \varphi_i(y)$  for each  $j \leq n(i)$ . Thus, if  $M_i$  is not initialized, we have now permanently diagonalized, because we initialized lower priority requirements when we appointed  $y_p$ .] Also, for each  $R$ -positive strategy  $U$  of lower priority than  $M_i$  with a  $\Delta$ -correction number  $w$ , we put  $w$  into  $B$ . [[We will verify that  $w > \theta_i(\varphi_i(y_p))$  so that these enumerations do not destroy the successful diagonalization of  $M_i$ .]]

4.  $K_{e,x}$ . The strategies are as before: if  $\Xi_e(R \oplus Q; x)$  has just converged for another time, then initialize the lower priority requirements.

**3.1. Verifications.** We refer to the verifications in [11]. Clearly,  $E \leq_T D$ ,  $P \leq_T Q \leq_T D$  and  $R \leq_T D$  by direct permitting. For the verification that the strategies succeed, the same comments as in [11, 4.3] apply, but we interpret “ $i$ -expansionary stage” in the way defined here and delete the remarks on the restraint functions  $r(M, i, s)$  (since  $M_i$  no longer imposes restraint on  $P$ , except by initialization after it diagonalizes).

The only changes are in the verification that the  $M_i$ -requirements succeed. However, *Lemma 4.2* goes through unchanged, and so does the first half of the proof of *Lemma 4.3*. Let  $x, y, s \leq t$  be as in that Lemma, thus,  $\Delta_{i,j}(W_i; x)$  was defined at stage  $s$  and  $x$  enters  $G_j$  at stage  $t \geq s$  through the action of an  $R$ -positive requirement  $U$ . The change is in the argument that no number  $z < \varphi_i(y)[s]$  can enter  $P$  at a stage  $v, t < v \leq u$ , where  $u$  is the next  $i$ -expansionary (in our sense) stage after  $t$ . Assume such a  $z$  enters  $P$ . Choose  $v$  minimal, suppose  $x$  is enumerated (at stage  $t$ ) into  $G_j$  by the strategy  $R$  while  $z$  is enumerated (at stage  $v$ ) into  $P$  by an  $M_k$ -strategy. By the initializations, necessarily  $M_i < M_k < U$ . Thus, by stage  $v$ , the current  $\Delta$ -correction number  $w$  of  $U$  has entered  $E$ . Since  $w < x$ , this allows us to redefine  $\Delta_{i,j}(W_i; x) = 1$ , contrary to the assumption that  $x$  was a failure point (i.e.,  $\Delta_{i,j}(W_i; x) = 0$ ).

The rest of the proof of *Lemma 4.3* and the proof of *Lemma 4.4* are as before. The proof of *Lemma 4.5* becomes simpler, since there is no minimal pair type  $P$  restraint imposed by  $M_i$ . Recall that  $r$  is the last stage when  $M_i$  is initialized. The argument is as follows, when taking into account the need for permitting by  $D$ . Arguing by induction and using the above Lemmas, we see that  $\Delta_{i,n(i)}$  is defined for every  $x > r$ . If there are infinitely many failures of  $\Delta_{i,n(i)}$ , then  $M_i$  appoints arbitrarily many  $y_p$ . By initializing requirements of lower priority, the chit of such a number  $y_p$  remains uncanceled (hence permanently  $\Phi_i(W_i; y_p) = 0$ ). Since  $D$  is nonrecursive,  $M_i$  eventually enumerates a number  $y_p$  into  $Q$ .

In the proof of *Lemma 4.6* we add the following observation: when  $M_i$  puts a number  $y_p$  into  $Q$  at stage  $s > r$ , then the  $\Delta$ -correction number  $w$  of a lower priority  $R$ -positive strategy  $U$  (which is put into  $E$  at stage  $s$ ) is greater than  $\theta_i(\varphi_i(y_p))$ . For, the strategy  $U$  was initialized when  $y_p$  was appointed at a stage  $t \leq s$ . Thus  $w$  was chosen after  $s$ , hence  $w > \theta_i(\varphi_i(y_p))[t]$ . Since the strategies of lower priority than  $M_i$  were also initialized at  $t$ ,  $\theta_i(\varphi_i(y_p))[t] = \theta_i(\varphi_i(y_p))[s]$ . As before, we argue that, at stage  $s$ ,  $\gamma_k(y_p) > \varphi_i(y_p)$  for each  $k \leq n(i)$ , and conclude that the enumeration of  $y_p$  diagonalizes against  $\Phi_i(P \oplus W_i) = Q$ . This concludes the proof of (i).

We now consider the extensions for (ii) and (iii). For (ii), we want to ensure  $G_j \leq_T G_i \oplus L \Leftrightarrow j \preceq i$ . We proceed exactly as in [11, 4.4]: for the direction “ $\Leftarrow$ ”, we adjust the construction so that any number  $x$  put into  $G_j$  is also simultaneously put into  $L$ , unless it is put in by one of the following  $N$  requirements discussed next. These ensure the direction “ $\Rightarrow$ ”:

5.  $N_{i,j,e}$ : If  $i \not\preceq j$  then  $\Psi_e(G_i \oplus L) \neq G_j$ .

The strategy for  $N_{i,j,e}$  is similar to the strategy for  $D_{i,j,e}$ . In particular, after an initialization, first a new  $\Delta$ -correction number is appointed. A follower  $x$  is *realized* when  $\Psi_e(G_i \oplus L; x) = 0$ . The restraint in (3.5) is replaced by

$$(3.6) \quad r(N, i, j, e, t) = \max\{\delta_{l,j}(W_l; x)[t] \mid l \succeq j \ \& \ M_l < N_{i,j,e} \ \& \ x \text{ follower of } N_{i,j,e}\}.$$

If  $x_k$  is realized and  $D \upharpoonright k$  changes, then we put  $x_k$  into  $G_l$  for all  $l \succeq j$ . For  $l \neq k$ , we call this an *indirect enumeration* into  $G_l$ . We initialize all lower priority requirements.

(iii). We choose sets in the given degrees  $\mathbf{u}_0, \dots, \mathbf{u}_n, \mathbf{d}$  in a way that  $U_j \cap 2\mathbb{N} = D$  for each  $j \leq n$ . We ensure  $G_{2j} \leq_T U_j$  ( $j \leq n$ ) by direct permitting, using that there is no indirect enumeration into  $G_{2j}$ .

For  $G_{2i} \not\leq_T U_j$  when  $i \neq j$ ,  $i, j \leq n$ , we meet the requirements

6.  $Z_{i,j,e} : G_{2i} \neq \Psi_e(U_j)$  ( $i, j \leq n$ ,  $U_i \not\leq_T U_j$ )

The strategy for  $Z_{i,j,e}$  is almost exactly as the strategy for  $Z_{i,j,e}$  in [11, p. 271], exploiting the fact that each  $U_j$ ,  $j \leq n$ , is low. (We need to carry out two merely notational changes: replace  $G_i$  in [11, p. 271] by  $G_{2i}$ , and  $V_{i,j}$  by  $U_j$ .) Thus, we appoint followers  $x_p$  which are realized in the sense that  $\Psi_e(U_j; x_p) = 0$ . We maintain the suitability of  $x_p$  for  $G_{2i}$  by imposing  $P$ -restraint. When  $U_i \upharpoonright x_p$  changes we run a guessing procedure based on the lowness of  $U_j$  to see if we eventually have a chance to diagonalize successfully. If so, we enumerate  $x_p$  into  $G_{2i}$ . See [11, p. 272] for details on the guessing procedure. (In the Proof of Theorem 2.4 below, we will use a simplified guessing procedure which could be applied here as well.)

Note that, as for all  $R$ -positive strategies, after an initialization, the  $Z_{i,j,e}$ -strategy first appoints a large  $\Delta$ -correction number.

The verification for the  $Z$ -type requirements is as before. In particular, if  $U_i \not\leq_T U_j$  and infinitely many followers are realized, then from some stage on the guessing procedure always answers yes, whence we diagonalize. The enumeration of  $\Delta$ -correction numbers into  $E$  is permitted by  $D$ . Thus  $\mathbf{g}_{2i} \leq \mathbf{u}_i$  as required.

Note that  $R, L \leq_T B$ , since  $D \leq_T U_0, \dots, U_n \leq_T B$  and an enumeration of  $y$  into  $R$  or  $L$  depends on the change of one of those sets below  $y$ .

Finally, to make  $E$  nonrecursive, we add requirements

7.  $S_k : E \neq \{k\}$ .

The strategy, while unsatisfied, appoints followers  $x_0 < x_1 < \dots$ . Once  $\{k\}(x_p) = 0$  and  $D \upharpoonright p$  changes, we put  $x_p$  into  $E$  and declare the strategy satisfied.

Note that this new enumeration of numbers into  $E$  will go into all the sets  $G_j$ . However, this enumeration is permitted by  $D$ , so that we can argue that  $G_{2i} \leq_T U_i$ . Moreover the numbers  $x_p$  do not need an associated  $P$ -restraint to ensure that the minimality requirements  $M_i$  are satisfied, since they also go into the set  $W_i$ , whence the relevant  $\Delta$ -computations of  $M_i$  (computing some set  $G_j$  from the oracle  $W_i$ ) become undefined.  $\diamond$

Extending the underlying coding methods (and using a slight addition to (ii) of Theorem 3.3) we obtain a proof of Theorem 3.1. As in [11], we use definable comparison maps, i.e., we ensure that certain isomorphisms between initial segments of coded models  $M$  are uniformly definable. Also we have to pay special attention to ensuring that all parameters are below  $\mathbf{c}$  and all schemes are evaluated within  $[\mathbf{0}, \mathbf{c}]$ . In the following, all degrees are in  $[\mathbf{0}, \mathbf{c}]$ .

We say that  $M$  is coded below  $\mathbf{b}$  if all parameters involved are below  $\mathbf{b}$ . We provide a schemes for coding functions  $S_h$  such that each low  $\mathbf{b}$  is *good* in the following sense: *there is an  $M^*$  coded below  $\mathbf{b}$  such that, for each  $M$  coded below  $\mathbf{b}$ , there is an isomorphism  $h$  between  $M^*$  and an initial segment of  $M$ .*

Note that by (ii) of Theorem 3.3, some  $M$  coded below  $\mathbf{b}$  is standard, so that  $M^*$  is automatically standard. Moreover, the property “ $\mathbf{b}$  is good via (parameters coding)  $M^*$ ” can be expressed in first-order logic. Now we already obtain an interpretation of  $\text{Th}(\mathbb{N}, +, \times)$  in  $\text{Th}([\mathbf{0}, \mathbf{c}])$  as follows: for each sentence  $\varphi$  in the language of arithmetic,

$$\mathbb{N} \models \varphi \Leftrightarrow \exists \mathbf{b} \leq \mathbf{c} [\mathbf{b} \text{ good via some } M^* \text{ such that } \varphi \text{ holds in } M^*].$$

It remains to be shown:

CLAIM 3.4. *Low degrees are good.*

*Proof.* We actually show that a low  $\mathbf{b}$  is good via any standard  $M_0$  coded below  $\mathbf{b}$ . If  $M_1$  is coded below  $\mathbf{b}$ , we obtain a uniform definition of the embedding  $h : M_0 \mapsto M_1$  as a union of isomorphisms between finite initial segments of three interpolating coded models. First, we make an addition to (ii) of Theorem 3.3, which will be applied with  $\mathbf{d} = \mathbf{b}$  and  $\mathbf{e}_r$  being the lower bound for the numbers of  $M_r$  ( $r = 0, 1$ ), which is part of the parameter list coding  $M_r$ .

LEMMA 3.5. *Suppose that, in addition to the hypotheses in (ii) of Theorem 3.3, we are given nonzero degrees  $\mathbf{e}_0, \mathbf{e}_1 \leq \mathbf{d}$ . Then we can build parameters coding  $M$  (including a lower bound  $\mathbf{e}$ ) such that, in addition to (ii), there are nonzero  $\mathbf{d}_0, \mathbf{d}_1$  satisfying  $\mathbf{d}_r \leq \mathbf{e}_r, \mathbf{e}$  ( $r = 0, 1$ ). Moreover, the parameter  $\mathbf{r}$  in the list coding  $M$  is low.*

We postpone the proof of the Lemma till the end of this Section. Suppose  $M$  is given by the Lemma. We use an auxiliary scheme  $S_f^*$  to define, for each  $m \in \mathbb{N}, r = 0, 1$ , the isomorphisms  $f_r : [0, k]^{M_r} \mapsto [0, k]^M$ . Say  $r = 0$ . We apply (iii) of Theorem 3.3 with  $n = 2k$ ,  $\mathbf{u}_i = i^{M_0}$  ( $i < k$ ) and  $\mathbf{u}_i = (i - k)^M$  ( $k \leq i < 2k$ ), so that  $(\mathbf{u}_i)_{i < k}$  and  $(\mathbf{u}_j)_{k \leq i < 2k}$  form antichains. Since  $\mathbf{d}_0 \leq \mathbf{e}_0, \mathbf{e}$  the hypothesis of (iii) is satisfied via  $\mathbf{d}_0$ , and applying (iii) we obtain a model  $M'_0$ . The map  $f_0$  is obtained by adding within this model  $k$ . Beyond the parameters to code the three models involved, we use the parameters  $\mathbf{k}_0 = k_0^M, \mathbf{k}_1 = k_1^{M'_0}$  and  $\mathbf{k}_2 = k^M$ . Let  $f_0(\mathbf{x}) = \mathbf{y} \Leftrightarrow$

$$\mathbf{x} \leq_{M_0} \mathbf{k}_0 \ \& \ \mathbf{y} \leq_M \mathbf{k}_2 \ \& \ \exists \mathbf{z} \leq_{M'_0} \mathbf{k}_1 [\mathbf{z} \leq \mathbf{x} \ \& \ \mathbf{z} +_{M'_0} \mathbf{k}_1 \leq \mathbf{y}].$$

Since this definition can be expressed in first-order logic, we obtain the desired scheme. The scheme  $S_h$  is now obtained by taking unions of isomorphisms of the form  $f_1^{-1} \circ f_0$ : note that, for  $\mathbf{x}_r \in M_r$ ,  $h(\mathbf{x}_0) = \mathbf{x}_1 \Leftrightarrow$

$$\exists M \exists \mathbf{z} \in M \exists f_0, f_1 [f_r \text{ is an isomorphism } [0, \mathbf{x}_r]_{M_r} \mapsto [0, \mathbf{z}]_M \ (r = 0, 1)].$$

The latter property is first-order.  $\diamond$

We conclude the proof of the model theoretic result for  $[\mathbf{0}, \mathbf{c}]$  Theorem 3.1 that there is a parameterless interpretation of  $(\mathbb{N}, +, \times)$  in  $[\mathbf{0}, \mathbf{c}]$ . Recall that we work within  $[\mathbf{0}, \mathbf{c}]$ . We extend the scheme  $S_M$  to a scheme  $S_M^*$  by adding an upper bound (for all parameters)  $\mathbf{b}$  to the list of parameters, and require as a further correctness condition that  $M$  be coded below  $\mathbf{b}$  and  $\mathbf{b}$  be good via  $M$ . For the rest of the proof, all models  $M$  are coded via  $S_M^*$ . Then each such  $M$  is standard. It suffices (see [11, proof of Thm. 2.7]) to give a first-order definition of the equivalence relation

$$\bar{\mathbf{p}}, \mathbf{x} \bar{Q} \bar{\mathbf{q}}, \mathbf{y} \Leftrightarrow \mathbf{x} \in M(\bar{\mathbf{p}}) \ \& \ \mathbf{y} \in M(\bar{\mathbf{q}}) \ \& \ \exists n \in \mathbb{N}[\mathbf{x} = n^{M(\bar{\mathbf{p}})} \ \& \ \mathbf{y} = n^{M(\bar{\mathbf{q}})}].$$

To do so, we introduce a scheme  $S_q$  to uniformly define the isomorphism between any two  $M_0, M_1$  coded below  $\mathbf{c}$ . Let  $\mathbf{b}_r$  be the upper bound included in the parameter list for  $M_r$ , and let  $\widetilde{M}_r$  be a model coded below  $\mathbf{b}_r$ , with a low upper bound  $\widetilde{\mathbf{b}}_r \leq \mathbf{b}_r$ . Since  $\mathbf{b}_r$  is good, the isomorphism  $h_r : M_r \mapsto \widetilde{M}_r$  is definable via  $S_h$ . Since  $\widetilde{\mathbf{b}}_r$  is low, by the same method as in the proof of Claim 3.4 (but with  $\widetilde{M}_r$  in place of  $M_r$ ), we can uniformly define the isomorphism  $\widetilde{M}_0 \mapsto \widetilde{M}_1$ . Thus, we interpolate with a model  $M$  provided by Lemma 3.5, and further models between  $\widetilde{M}_r$  and  $M$  provided by (iii). Unlike to the proof of Claim 3.4, here we need that  $\mathbf{r}$  in the parameter list for  $M$  is low in order to apply (iii).  $\diamond$

*Proof of Lemma 3.5.* We may suppose that  $E_r = D \cap (3\mathbb{N} + r)$ . In addition to the sets from (ii), we build sets  $D_r \leq_T E_r, E$  and meet the requirements

$$S_{r,k} : D_r \neq \{k\} \ (r = 0, 1).$$

The strategy is to similar to the strategy for the  $S_k$  requirements above. We appoint even followers  $x_0 < x_1 < \dots$  targeted for  $D_r$  and  $E$ . Once  $\{k\}(x_p) = 0$  and  $E_r \upharpoonright p$  changes, we put  $x_p$  into  $D_r, E$  and declare the strategy satisfied.

As before, no extra  $P$ -restraint is needed. Moreover, by the success of the  $K$ -type requirements,  $R$  is low.  $\diamond$

#### 4. Proof of Theorem 2.4

Again we begin with a simpler result: for each interval  $[\mathbf{d}, \mathbf{b}]$ , where  $\mathbf{d} \neq \mathbf{0}$  and  $\mathbf{b}$  is low, and each u.r.e. sequence of degrees  $(\mathbf{u}_m)_{m \in \mathbb{N}}$  in  $[\mathbf{d}, \mathbf{b}]$ , there is a uniformly definable map from a coded standard  $M$  onto this sequence. Some easy adjustments will then lead to the full result: given  $\mathbf{d} \neq \mathbf{0}$ , one maps from a standard  $M$  onto an appropriate u.r.e. sequence in  $[\mathbf{d}, \mathbf{b}^0] \cup [\mathbf{d}, \mathbf{b}^1]$ , where  $\mathbf{b}^0, \mathbf{b}^1$  are low degrees such that each degree in  $[\mathbf{d}, \mathbf{1}]$  is the join of two degrees in that sequence.

**THEOREM 4.1.** *There is a scheme  $S_M$  for coding a model of  $Q$  and a scheme  $S_f$  for coding functions with the following property. Suppose  $\mathbf{0} < \mathbf{d} < \mathbf{b}$  and  $\mathbf{b}$  is low. Suppose further that  $(\mathbf{u}_m)_{m \in \mathbb{N}}$  is a u.r.e. sequence of degrees in  $[\mathbf{d}, \mathbf{b}]$ . Then there is a standard  $M$  and a map  $f$  such that  $\mathbf{u}_m = f(m^M)$  for each  $m$ .*

We wish to construct a standard  $M$  which can distinguish a degree  $\mathbf{u}_m$  from any degree  $\mathbf{v} \in [\mathbf{d}, \mathbf{b}]$  such that  $\mathbf{u}_m \not\leq \mathbf{v}$ , in the sense that there is an  $\mathbf{n} \in M$  below  $\mathbf{u}_m$  but not below  $\mathbf{v}$ . Let  $(V_k, \Upsilon_k)$  be a list of all r.e. sets and all Turing reductions to  $B$  (with an additional property specified later on). For each  $m, k$ , we let  $j = 2\langle m, k \rangle$  (where  $\langle, \rangle$  is the standard pairing function on  $\mathbb{N}$ ). We introduce a set  $G_j \leq_T U_m$  such that, if  $V_k = \Upsilon_k(B)$ , then  $G_j \not\leq_T V_k$  unless  $U_m \leq_T V_k$ . Thus

$$(4.1) \quad \mathbf{u}_m = \sup_{[\mathbf{d}, \mathbf{b}]} \{ \mathbf{d} \vee \mathbf{g}_{2\langle m, k \rangle} : k \in \mathbb{N} \}$$

(where the supremum is taken in  $[\mathbf{d}, \mathbf{b}]$ ). We build a standard model  $M$  such that  $\forall n \in \mathbb{N} \ n^M = \mathbf{g}_{2n}$ . Then the following map is as desired:

$$(4.2) \quad f(m^M) = \sup_{[\mathbf{d}, \mathbf{b}]} \{ \mathbf{d} \vee \langle m, k \rangle^M : k \in M \}.$$

We satisfy the same requirements of type  $T, D, M, N$  and  $S$  as in Theorem 3.3. For (4.1), we meet requirements  $Z_{j,e}$  ( $j = 2\langle m, k \rangle$ ) corresponding to the  $Z_{i,j,e}$  requirements in the proof of (iii), as well as auxiliary requirements  $Y_j$ . Let  $\Phi_e$  be a list of Turing functionals. Using the lowness of  $B$ , we let  $\Upsilon_e$  be a list of functionals such that  $\Phi_e^B \text{ total} \Rightarrow \Upsilon_e^B = \Phi_e^B$  and, for each  $x$ , if  $\Upsilon_e^B(x)[s]$  is defined for infinitely many  $s$ , then the computation is stable from some  $s$  on. To do so, we employ the Robinson certification procedure [13, p.225].

Let  $(V_k)$  be an effective list of all r.e. sets  $V$  such that  $D = V^{[0]}$ , and let  $(\Psi_e)$  be an effective list of all Turing functionals which (for each oracle) are defined on an initial segment of  $\mathbb{N}$  and have use monotonic in the input and greater than twice the input. Then we can assume the same properties hold for the approximations  $\Psi_e^B[s]$ . The requirements are

6.  $Y_j : G_j \leq_T U_m$  ( $j = 2\langle m, k \rangle$ )

7.  $Z_{j,e} : [\Upsilon_k(B) = V_k \ \& \ \Psi_e(V_k) = G_j] \Rightarrow U_m \leq_T V_k$

Thus  $U_m$  now plays the role of  $U_i$  before, and  $V_k$  plays the role of  $U_j$ . As before, the basic idea for the  $Z_{j,e}$ -strategy is to wait for an  $U_m$ -permission of a realized follower, namely of an  $x$  such that  $\Psi_e(V_k; x) = 0$ . However, in (iii) of Theorem 3.3 we knew in advance that  $U_i \not\leq_T U_j$ , guaranteeing the ability to diagonalize. In the present situation, we may fail to diagonalize, in which case we show  $U_m \leq_T V_k$ . In this case, the  $Z_{j,e}$ -strategy appoints a potentially infinite list of realized followers  $(x_p)$ , and thus may have an infinitary effect on the lower priority strategies. The worst scenario would be an infinite enumeration of followers, where each potential diagonalization is later destroyed by a  $V_k$ -change. This can be avoided exploiting the lowness of  $V_k$ : we use a  $\Delta_2^0$ -guessing procedure to predict whether we ever can diagonalize successfully (assuming by the Recursion Theorem that an index for the construction is given), and only diagonalize if the answer seems to be yes. Since we eventually will obtain the correct answer, only finitely many followers are enumerated.

Note that we seem to need a lowness index for  $V_k$  to implement this, while  $V_k$  is actually given only by a (possibly partial) reduction  $\Upsilon_k(B)$ . We will use a tree construction in order to guess at whether  $\Upsilon_k(B) = V_k$  (this is the actual purpose of the  $Y_j$ -strategies).

A more serious problem is that, since we allow the  $Z_{j,e}$ -strategy to appoint an infinite list of followers targeted for  $G_j$ , the overall  $P$ -restraint imposed by the strategy to guarantee their suitability goes to infinity. It is here that we make use of the fact that the lower bound  $D$  is nonrecursive. As in the proof of Theorem 3.3, we change  $Q$  and  $P$  only upon  $D$ -permission. Roughly speaking, we will arrange that the same  $D$  permission which allows us to change  $Q$  and  $P$  and destroys the suitability of a  $Z_{j,e}$ -candidate for enumeration into  $G_j$  lets us eliminate the candidate anyway, since  $D$  is part of  $V_k$ , so that we can redefine the (implicit) reduction from  $U_m$  to  $V_k$ . Clearly, it is important here that the  $M$ -type requirements do not generate a minimal pair type restraint on  $P$ . As before, we use  $\Delta$ -correction numbers instead.

We give more details on this after discussing the tree construction. We will need to guess at the outcomes of the  $M$ -type strategies. In our tree construction, we use standard notation as in [13]. Nodes are identified with strategies. A strategy  $\beta$  has *higher priority than*  $\gamma$  if  $\beta \subset \gamma$  or  $\beta <_L \gamma$ . We write  $\alpha : R$  to indicate that the strategy  $\alpha$  works on the requirement  $R$ , and  $\alpha : X$  to indicate that  $\alpha$  works on an unspecified  $X$ -type requirement.

We assign a priority ordering to the requirements so that  $Y_j$  precedes each  $Z_{j,e}$ . The strategies with an outcome other than “ $f$ ” on the tree are the ones for  $M$  and  $Y$ -type requirements. A  $Y$ -type strategy  $\eta$  has outcomes  $\infty$  and  $f$  corresponding to infinitely many  $\Upsilon_k(B) = V_k$ -expansionary stages (in which case in fact  $\Upsilon_k(B) = V_k$ ), or not. An  $M_i$ -strategy  $\eta$  has outcomes  $n(i) < n(i) - 1 < \dots < 0 < f$ . The outcome  $j$  indicates that  $\Delta_{\eta,j}(W_i) = G_j$ , while  $f$  indicates that  $\Phi_i(W_i \oplus P) = Q$  fails.

The tree  $T$  and an effective assignment of nodes to requirements other than  $T_j$  is defined inductively. We assign the empty node to the first requirement in our list. If  $\eta \in T$  has been newly assigned to requirement  $U$ , put the nodes  $\mu = \eta \hat{\circ}$  on  $T$ , where  $o$  is a possible outcome of  $U$ . For each such  $\mu$ , let  $R$  be the highest priority requirement which has not yet been assigned to any  $\nu \subset \mu$ , and such that, if  $R = Z_{j,e}$ , then there is  $\beta \hat{\circ} \in \mu$  such that  $\beta$  has been assigned to  $Y_j$ . Assign  $\mu$  to  $R$ .

We explain the solution to the problem of a potentially infinite  $P$ -restraint in some more detail. Consider the interaction of strategies for requirements  $Z_{j,e} < M_h$ , such  $\Psi_e(V_k) = G_j$  ( $j = 2\langle m, k \rangle$ ), and the strategy for  $Z_{j,e}$  works in the environment given by various higher priority  $M_i$  strategies such that there infinitely many  $i$ -expansionary stages. Suppose  $\beta : Z_{j,e}$  is on the true path, and  $\beta \subset \mu$  where  $\mu : M_h$ . Then  $\beta$  appoints a list  $(x_p^\beta)$  targeted for  $G_j$ , and  $\mu$  appoints a list  $(y_p^\mu)$  targeted for  $Q$  (note that  $x_p^\beta$  may be cancelled and later redefined with new value, due to violation of its associated  $P$ -restraint).

The strategy  $\beta$  may disregard those  $\alpha : M_i$  such that  $\alpha \hat{\circ} q \subseteq \beta$  for  $q \neq j$ , since they build reductions of different sets  $G_q$  to  $W_i$ . However, we only allow  $\beta$  to define  $x = x_p^\beta$  when all the computations  $\Delta_{\alpha,j}(x)$ ,  $\alpha \hat{\circ} j \subseteq \beta$ ,  $\alpha : M$ , have been defined (this will happen since  $\beta \subseteq TP$ ). Thus  $r(\beta, x) = \max\{\delta_{\alpha,j}(x) : \alpha \hat{\circ} j \subseteq \beta \ \& \ \alpha : M\}$  cannot increase due to the definition of new  $\Delta_{\alpha,j}(x)$ -computations.

Now  $\mu$  is allowed to appoint  $y_p^\mu$  only when  $x_p^\beta$  is defined, and chooses  $y_p^\mu > r(\beta, x_p^\beta)$  (where  $y_p^\mu$  is a number which already went through the process of clearing  $\Phi_h(W_h \oplus P; y)$  from  $\Gamma$  uses). At this point,  $\mu$  initializes lower priority strategies, so that  $r(\beta, x_p^\beta)$  also cannot increase because of the cancellation of chits via which  $\Delta_{\alpha,j}(x_p^\beta)$ -computations are defined (which would make us later redefine those computations with larger use). Thus at all later stages  $y_p^\mu > r(\beta, x_p^\beta)$ , unless  $\mu$  itself is initialized. This will imply that  $x_p^\beta$  is not cancelled (due to violation of its  $P$ -restraint by some  $\gamma(y_a^\mu)$ ) unless  $D \upharpoonright p$  changes. Since  $D$  is part of  $V_k$ , this implies  $U_m \leq_T V_k$  (see the proof of Lemma 4.13 for technical details).

**4.1. The formal construction.** A strategy  $\eta$  is *initialized* by making all its parameters and restraints undefined, declaring it unsatisfied, cancelling any possible test procedure involving  $\eta$  (see below), and making the functionals the strategy is building totally undefined.

At Phase 1 of stage  $s$ , we define an approximation  $\delta_s$  to the truth path TP. A  $\beta$ -stage is a stage  $s$  such that  $\beta \subseteq \delta_s$ . At Phase 2, nodes which have been accessible at Phase 1 of this or an earlier stage may act without delay, for instance when they receive a  $D$ -permitting. When strategies *act*, they initialize the lower priority strategies. Acting is always a finitary process for strategies on or to the left of the true path. A strategy may do other things besides acting, like appointing followers or continuing to build a functional.

We partition the odd numbers into recursive sets  $\omega^{[\alpha]}$ ,  $\alpha$  a strategy on the tree. The strategy  $\alpha$  only enumerates numbers from  $\omega^{[\alpha]}$  into the sets  $Q$  and  $G_i$ . The even numbers are reserved for  $E$ .

**Stage  $s$ , Phase 1.**

First, we work on  $T_j$  for each  $j < s$ . Define  $\Gamma_j(G_j \oplus P; y)$  with large use, where  $y$  is least such that the functional is undefined.

Carry out Substage  $t$  for ascending  $t < s$ . Let  $\eta = \delta_s \upharpoonright t$ . Do one of the following, according to the requirement  $\eta$  works on, and define  $\delta_s \upharpoonright t + 1 = \eta \hat{o}$ , where  $o$  is the outcome given by  $\eta$ .

**Cases  $D_{i,j,e}$ :**  $\Psi_e(G_i) \neq G_j$ ,

$N_{i,j,e}$ :  $i \not\leq j \Rightarrow \Psi_e(G_i \oplus L) \neq G_j$ .

As before, the strategy builds a list of followers  $x_0^\eta < x_1^\eta < \dots$  targeted for  $G_j$ . A follower  $x$  is *realized* when  $\Psi_e(G_i; x) = 0$ . Suppose the strategy is not yet satisfied. If  $p > 0$  and  $x_p^\eta$  has been defined and is realized or  $p = 0$ , then appoint a large odd number as a new follower and initialize requirements of lower priority.

At every stage  $t \geq s$ ,  $\eta$  imposes a restraint  $r(\eta, x, t)$  on  $P$  to preserve the suitability of  $x$  for enumeration into  $G_j$ , which is defined as follows. If  $\eta : D$ ,

$$r(\eta, t) = \max\{\delta_{\alpha,j}(W_k; x)[t] : x \text{ follower of } \eta \ \& \ \alpha \hat{j} \subseteq \eta \ \& \ \alpha : M\}$$

while, if  $\eta : N$ ,

$$r(\eta, t) = \max\{\delta_{\alpha,l}(x)[t] \mid x \text{ follower of } \eta \ \& \ j \preceq l \ \& \ \alpha \hat{l} \subseteq \eta \ \& \ \alpha : M\}$$

**Case  $S_k$ :**  $E \neq \{k\}$ .

If the strategy is still unsatisfied, appoint a new follower  $x_p^\eta$ .

**Case  $K_{e,x}$ .** If the strategy is not satisfied and not currently activated, declare it *activated*.

**Case  $Z_{j,e}$ :**  $\Upsilon_k(B) = V_k \ \& \ \Psi_e(V_k) = G_j \Rightarrow U_m \leq_T V_k$  ( $j = 2\langle m, k \rangle$ ). If  $\eta$  has no  $\Delta$ -correction number, appoint a large such number  $w \in 2\mathbb{N}$  (targeted for  $E$ ). Initialize all strategies of lower priority.

Let  $p$  be such that either  $p = 0$ , or  $p > 0$  and  $x_{p-1}^\eta$  is defined. If there is  $z \in \omega^{[\eta]}$  such that  $z$  is greater than all previously appointed followers of  $\eta$  and

$$(4.3) \quad \forall \alpha [(\alpha \hat{j} \subseteq \eta \ \& \ \alpha : M) \Rightarrow \Delta_{\alpha,j}(z)[t] \downarrow]$$

then appoint  $z$  as a new follower  $x_p^\eta$ . [[Note that, for technical reasons, we don't require that the previous followers be realized, and we continue to appoint followers even if it seems the strategy has diagonalized.]] The  $P$ -restraint imposed by  $\eta$  at stage  $t$  to preserve the suitability of a follower  $x = x_q^\eta$  for enumeration into  $G_j$  is given by

$$(4.4) \quad r(\eta, x, t) = \max\{\delta_{\alpha \hat{j}}(x)[t] \mid \alpha \hat{j} \subseteq \eta \ \& \ \alpha : M\}$$

Whenever this restraint is violated, then  $x_{q'}$  is cancelled for all  $q' \geq q$  [[but, unlike the cases of  $D$  and  $N$ -type strategies, we do not initialize  $\eta$ ]].

**Case  $Y_j : G_j \leq_T U_m$  ( $j = 2\langle m, k \rangle$ ).** *Outcomes:*  $\infty, f$ . If the length of agreement  $\max\{x : \forall y < x \Upsilon_k(B, y) = V_k(y)[s]\}$  is not greater than at any previous  $\eta$ -stage, then give outcome  $f$  and proceed to the next substage. Otherwise give outcome  $\infty$ , and proceed as follows.

By the Recursion Theorem and since  $B$  is low, we can choose a recursive function  $g(\beta, r, s)$  such that  $\lim_{s \rightarrow \infty} g(\beta, r, s)$  is 0 or 1 for every  $\beta, r$ , and, when  $\beta : Z_{j,e}$  and  $\eta \subseteq \beta$  is the corresponding  $Y_j$ -strategy,  $\lim_{s \rightarrow \infty} g(\beta, r, s) = 1$  iff there is an  $\widehat{\eta^\infty}$ -stage  $t > r$  of our construction and a realized follower  $x = x_p^\eta$  such that

(4.5)

- some number  $< p$  enters  $U_m$  between  $t'$  and  $t$ , where  $r \leq t'$  and  $t'$  is the greatest  $\widehat{\eta^\infty}$ -stage  $< t$
- $\Psi_e(V_k; x) = 0 [t]$  and  $G_j \upharpoonright x \subseteq \Psi_e(V_k) \upharpoonright x [t]$
- there is an  $\widehat{\eta^\infty}$ -stage  $u \geq t$  of our construction such that

$$V_k \upharpoonright \psi_e(x)[t] = V_k \upharpoonright \psi_e(x)[u] = \Upsilon_k(B) \upharpoonright \psi_e(x)[u]$$

via  $B$ -correct computations.

For each  $Z_{j,e}$ -strategy  $\beta \supseteq \widehat{\eta^\infty}$ , let  $r \leq s$  be the last stage where  $\beta$  was initialized, and see if the conditions above appear to hold, namely there is  $p$  such that there is follower  $x = x_p^\beta$  such that some number  $< p$  entered  $U_m$  between  $s'$  and  $s$ , where  $r \leq s'$  and  $s'$  is the greatest  $\widehat{\eta^\infty}$ -stage  $< s$ ,  $\Psi_e(V_k; x) \downarrow = 0 [s]$  and  $G_j \upharpoonright x \subseteq \Psi_e(V_k) \upharpoonright x [s]$ . In that case, start a *test procedure*  $T(\beta, x, s)$  which will run for finitely many  $\widehat{\eta^\infty}$ -stages, trying to collect evidence that  $\Psi_e(V_k; x) \downarrow = 0 [s]$  is correct.

At each  $\widehat{\eta^\infty}$ -stage  $u \geq s$  until stopped, the procedure checks whether

$$g(\beta, r, u) = 1 \text{ or } V_k \upharpoonright \psi_e(x) \text{ changed since } s.$$

If so, it terminates, otherwise it keeps running. If it terminates and  $V_k \upharpoonright \psi_e(x)$  did not change, we enable  $\beta$  to diagonalize (which means it may enumerate  $x$  into  $G_j$  and  $L$  in Phase 2).

[[For each  $\beta$ , several test procedures with different parameters  $x$  can be running simultaneously. We will verify that, if  $\widehat{\eta^\infty}$  is on the true path, then each test procedure stops.]]

**Case  $M_i$ .** *Outcomes:*  $n(i), n(i) - 1, \dots, 0, f$ . Let

$$l(\eta, s) = \max\{y \in \omega^{[\eta]} : y < l(i, s)\}$$

( $l(i, s)$  was defined in (3.4)). Each number  $y \in \omega^{[\eta]}$  obtains a chit once  $y < l(i, s)$ , which is cancelled forever in case later  $W_i \oplus P \upharpoonright \varphi_i(y)$  changes.

If  $l(\eta, s)$  is less than or equal to its value at the last  $\widehat{\eta^\infty}$ -stage, then give outcome  $f$  and proceed to the next substage. Otherwise let  $j \leq n(i)$  be the greatest number such that there is a number  $y \in \omega^{[\eta]}$  whose uncanceled chit has not yet been associated with a  $\Delta_{\eta,j}$ -computation, and, if  $j > 0$ , there is a computation  $\Delta_{\eta,j-1}(W_i, z) = 0$  associated with the chit for  $y$  such that  $z \in G_{j-1}$  (called a  $\Delta_{\eta,j-1}$  failure).

If  $j \leq n(i)$ , let  $x$  be least such that  $\Delta_{\eta,j}(W_i, x)$  is undefined, and define it with large use, and associate the chit for  $y$  with this new computation. *Give outcome  $j$ .*

If  $j = n(i) + 1$ , give outcome  $f$ , declare  $y$   $\eta$ -cleared and attempt to use  $y$  as a new follower to diagonalize: let  $p \geq 0$  be least such that  $y_p^n$  is undefined. If

$$(4.6) \quad \forall \beta \subseteq \eta [\beta : Z \Rightarrow (x_p^\beta \downarrow \ \& \ y > r(\beta, x_p^\beta, s))]$$

then  $\eta$  acts by defining  $y_p^n = y$ , and initializing lower priority strategies. [[Such followers  $y_p^n$  may be enumerated into  $Q$  in Phase 2.]]

**Phase 2.** Let  $\eta$  be the leftmost strategy which requires attention, let it carry out its desired action and initialize strategies  $> \eta$ . Here, we define “ $\eta$  requires attention” and the desired action according to the requirement  $\eta$  is a strategy for.

**Case**  $D_{i,j,e}, N_{i,j,e}, S_k$  : The strategy is unsatisfied and there is a realized follower  $x = x_p^n$  and  $D_{s-1} \upharpoonright p \neq D_s \upharpoonright p$ .

*Desired action:* (D) Enumerate  $x$  into  $G_j$  and  $L$ . Declare the strategy satisfied.

(N) Enumerate  $x$  into  $G_l$  for every  $l \succeq j$ . Declare the strategy satisfied.

(S) Put  $x_p^n$  into  $E$  and declare the strategy satisfied.

**Case**  $Z_{j,e}$  :  $\eta$  has been enabled to diagonalize via  $x$  in the preceding Phase 1.

*Desired action:* Enumerate the minimal such  $x$  into  $G_j$  and  $L$ .

**Case**  $K_{e,x}$  :  $\Xi_e(R \oplus Q; x)[s] \downarrow$  and this is the first stage we see convergence after it has been activated the last time.

*Desired action:* Initialize lower priority requirements, and declare the strategy satisfied.

**Case**  $M_i$  :  $y_p^n$  is defined and  $D \upharpoonright p$  changes.

*Desired action:* Enumerate  $y_p^n$  into  $Q$ , and  $\gamma_j(y_p^n)$  into  $P$ , for each  $j \leq n(i)$ .

### Verifications

Note that  $E \leq_T D$  and  $P \leq_T Q \leq_T D$  as in Theorem 3.3 (but  $R, L \leq_T B$  may fail).

LEMMA 4.2. *If  $n \preceq r$ , then  $G_n \leq_T G_r \oplus L$ .*

*Proof.* The required Turing reductions are obtained by direct coding. ◇  
The true path is defined by

$$TP = \bigcup \{ \delta : \exists n [\delta \text{ is } <_L\text{-least s.t. } |\delta| = n \ \& \ \exists^\infty s [\delta \subseteq \delta_s]] \}.$$

LEMMA 4.3. *Suppose  $\alpha$  is a  $Y_j$ -strategy such that  $\alpha \hat{\infty} \subseteq TP$ . Then each test procedure  $T(\beta, x, t)$  started by  $\alpha$  at a stage  $t$  terminates.*

*Proof.* By the the definition of the functionals  $\Upsilon_k$  and since  $\alpha \hat{\infty} \subseteq TP$ ,  $\Upsilon_k^E = V_k$ . If  $\beta$  is initialized after  $t$ , the test procedure terminates at that stage. Otherwise, we can suppose that  $V_k \upharpoonright \psi_e(x)$  is stable from  $t$  on, else again the procedure terminates. Then, at a stage  $u \geq t$  of our construction,  $V_k \upharpoonright \psi_e(x)[t] = V_k \upharpoonright \psi_e(x)[u] = \Upsilon_k(B) \upharpoonright \psi_e(x)[u]$  via  $B$ -correct computations. Thus, the conditions (4.5) hold, whence  $\lim_s g(\beta, r, s) = 1$ , where  $r$  is the last stage before  $s$  when  $\beta$  was initialized. Therefore, again, the procedure terminates. ◇

LEMMA 4.4. *Suppose the strategy  $\eta$  is initialized for the last time at stage  $r$ . Then  $\eta$  acts at most finitely often after stage  $r$ .*

*Proof.* It is clear that if  $\eta : K$ , then  $\eta$  acts at most once after stage  $r$  (in Phase 2). If  $\eta : D, \eta : N$ , or  $\eta : Z$ ,  $\eta$  acts at most once at a stage  $> r_\eta$  to appoint a new  $\Delta$ -correction number.

Suppose  $\eta : M, \eta : D$  or  $\eta : N$ . If  $\eta$  acts infinitely often, then  $\eta$  appoints infinitely many followers  $x_p^n$ , which remain uncanceled since  $\eta$  is not initialized any more. Since  $D$  is nonrecursive, eventually, for some  $p$ ,  $D \upharpoonright p$  changes after  $z_p^n$  has been appointed. If  $\eta : D$  or  $\eta : N$ , it is clear that  $\eta$  is satisfied from now on. If  $\eta : M_i$ , we need to verify that the enumeration of the follower into  $Q$  actually diagonalizes against  $\Phi_i(W_i \oplus P) = Q$ .

CLAIM 4.5. *If  $\eta : M_i$  puts a number  $y$  into  $Q$  at a stage  $s > r$ , then  $\Phi_i(W_i \oplus P; y) = 0$  and hence  $\lim_s l(i, s) < \infty$ .*

We extend the proof of [11, Lemma 4.6]. We first claim that no number used in the computations associated with  $y$  at  $s$  can ever be put into  $R$  or  $P$ . As no higher priority requirement ever puts a number into  $R \oplus P$  at a stage  $> r$  and no lower priority one puts one in which is used in the computations at  $s$  by initialization, it suffices to prove that

- (a) if  $w$  is the  $\Delta$ -correction number of a  $\bigoplus_n G_n$ -positive strategy  $\beta > \eta$  (so that  $w$  is enumerated into  $E$  at stage  $s$ ), then  $w > \theta_i(\varphi_i(y))[s]$ , and
- (b) each  $\gamma_j(y)[s]$  put into  $P$  at  $s$  is larger than  $\varphi_i(W_i \oplus P)(y)[s]$ .

For (a), note that the strategy  $\beta$  was initialized when  $y$  was appointed at stage  $t$ ,  $r < t \leq s$ . Thus  $w$  was chosen after  $t$ , hence  $w > \theta_i(\varphi_i(y_p))[t]$ . Since the  $M$ -type strategies of lower priority than  $\eta$  were also initialized,  $\theta_i(\varphi_i(y))[t] = \theta_i(\varphi_i(y))[s]$ .

For (b), we argue exactly as in the proof of [11, Lemma 4.6]. In order for us to put  $y$  into  $Q$  at  $s$ , we must have a chit for  $y$  which is uncanceled and associated with failed computations  $\Delta_{\eta_j}(W_i; x_j) = 0 \neq G_j(x)$  for each  $j \leq n(i)$ . Consider the stage  $s_j < s$  at which  $x_j$  was put into  $G_j$  by some  $G_j$ -positive strategy  $\beta > \eta$  (as otherwise  $s_j < r$  and so the chit for  $y$  would have been cancelled at  $r$ ). Suppose  $\Delta_{\eta, j}(W_i; x_j)$  was last defined before  $s_j$  at the  $\eta$ -expansionary stage  $t_j$  with a chit for  $y$  and use  $\varphi_i(y)[t_j]$ . At the beginning of stage  $s_j + 1$  we redefine  $\Gamma_j(G_j \oplus P; y)$  with a large use and so one larger than  $\varphi_i(y)[t_j], t_j$ . If there were any later  $\eta$ -expansionary stage (including  $s$ ) at which  $\varphi_i(y)$  increased above this  $\Gamma$  use, we would cancel the chit for  $y$  by construction and so  $y$  could not later go into  $Q$  as assumed. Since  $\gamma_j(y)[t]$  is nondecreasing in  $t$ , when we put  $y$  into  $Q$ ,  $\gamma_j(y) > \varphi_i(y)[s]$  as required. This proves the Claim and concludes the case  $\eta : M$ .

If  $\eta : Z_{j,e}$ , suppose for the sake of a contradiction that infinitely often we put a follower of  $\eta$  into  $G_j$ . By construction, this can happen at a stage  $s$  only if there is a  $u \geq s$  such that  $g(\eta, r, u) = 1$ . Thus it can happen infinitely often only if  $\lim_u g(\eta, r, u) = 1$ . In this case, the conditions (4.5) hold for some  $t > r$  via some  $x = x_p^n$ . By Lemma 4.3, the procedure  $T(\eta, x, t)$  terminates at a stage  $u$  and we would put  $x$  into  $G_j$  by construction (since  $V_k$  did not change), hence  $\Psi_e(V_k; x) = 0$ . From stage  $u$  on, no new test procedures with followers  $> x$  will be started by the second condition in (4.5), so no follower  $> x$  of  $\eta$  is enumerated into  $G_j$ , contradiction. Note that this argument also shows that  $Z_{j,e}$  is met in case  $\lim_u g(\eta, r, u) = 1$ , since  $\Psi_e(V_k) = G_j$  fails.  $\diamond$

LEMMA 4.6. *If  $\eta \subseteq TP$ , then there is a stage  $r_\eta$  so that  $\eta$  is initialized for the last time at  $r_\eta$ .*

*Proof.* Note that  $\eta$  is only initialized if some  $\mu < \eta$  acts or some  $\mu <_L \eta$  is accessible in Phase 1. Let  $s_0$  be a stage such that  $\delta_s \not\prec_L \eta$  for  $s \geq s_0$ . If there is a strategy  $<_L \eta$  which acts at a stage  $s \geq s_0$ , let  $\mu$  be the  $<$ -least such (a strategy must have

been accessible in Phase 1 before it can act). When  $\mu$  acts at  $s$ , all the remaining strategies  $<_L \eta$  are initialized, so by Lemma 4.4, after finitely many stages there is no more action to the left of  $\eta$ . By induction and Lemma 4.4 again, also strategies  $\subset \eta$  cease to act.  $\diamond$

LEMMA 4.7. *All the  $K$ ,  $D$ ,  $N$  and  $S$ -type requirements are met.*

*Proof.* Suppose that  $\eta \subset TP$  is a strategy for the requirement in question.

$K_{e,x}$ : The strategy  $\eta$  will be activated after stage  $r_\eta$ . Then, if the relevant computation  $\Xi_e(R \oplus Q; x)$  converges at a stage  $\geq r_\eta$  for the first time, it will be preserved from then on.

$D_{i,j,e}, N_{i,j,e}, S_k$ : If the requirement is not met, then  $\eta$  appoints an infinite list of followers  $x_p^\eta$  after stage  $r_\eta$ , which remain uncanceled since otherwise  $\eta$  would be initialized. As in the proof of Lemma 4.4, eventually  $D$  changes below  $p$  after  $x_p^\eta$  has been appointed, and  $x_p^\eta$  is enumerated into  $G_j$ . For  $D$  and  $N$ -type requirements, note that, since the relevant computations on input  $x_p^\eta$  have been protected by the initialization of lower priority requirements when  $x_p^\eta$  was appointed, the requirement is met.  $\diamond$

LEMMA 4.8. *Each requirement  $T_j$  is met.*

*Proof.* For each  $y$ ,  $\Gamma_j(G_j \oplus P; y)$  is defined infinitely often at the beginning of stage  $s$ . As this corresponds to some  $\Xi_e(R \oplus Q; x)[s] \downarrow$  (using that  $P \leq_T Q$  via direct permitting), the success of the  $K$ -type requirements guarantees that  $\Gamma_j(G_j \oplus P; y) \downarrow$ . Let  $\eta = TP \upharpoonright j$ . Since the numbers put into  $Q$  by strategies  $\not\subseteq \eta$  form a recursive set, it suffices to show that  $\Gamma_j(G_j \oplus P; y) = Q(y)$  for each  $y \in \omega^{[\alpha]}$ , where  $\alpha : M_i$  for some  $i$  and  $\eta \subseteq \alpha$ . But when such an  $\alpha$  enumerates  $y$  into  $Q$ , then  $\Gamma_j(G_j \oplus P; y)$  is made undefined since  $j \leq n(i)$ , so  $\gamma_j(y)$  is put into  $P$ .  $\diamond$

It remains to be shown that the  $M$ -,  $Y$ - and  $Z$ -type requirements are met. To do so, we prove two Claims by a joint induction along the true path.

CLAIM 4.9. *Suppose  $\beta \subseteq TP$  where  $\beta : Z$ . Then, for each  $p$ ,  $\lim_s x_p^\beta[s]$  and  $\lim_s r(\beta, x_p^\beta, s)$  exist.*

*Proof.* Inductively, suppose that, for all  $q$ ,  $0 \leq q < p$ ,  $x_q^\beta[s]$  has come to its limit by stage  $t$ . Since  $\beta \subset TP$ , by the inductive hypothesis for Claim 4.10 below,  $\Delta_{\alpha,j}$  is total for each  $\alpha \hat{=} j \subseteq \beta$ . Thus we will arbitrarily often at a stage  $\geq t$  find a new value  $x$  such that (4.3) holds, so that we can set  $x_p^\beta = x$ . Once defined, this value can only be cancelled if the associated  $P$ -restraint  $r(\beta, x_p^\beta, t)$  is violated, because all the relevant  $\Delta_{\alpha,j}(x)$  computations in (4.4) are defined, so that the restraint cannot go up due to the definition of more  $\Delta$ -computations. Consider the functional  $\Xi_e(R \oplus Q)$  which becomes defined on input  $p$  at stage  $s$  when a new value for  $x_p^\beta$  is appointed, so that its use is greater than  $r(\beta, x_p^\beta, s)$ . By Lemma 4.7,  $\Xi_e(R \oplus Q; p)$  is permanently defined from some  $u$  on, so that  $R \oplus P \upharpoonright u$  is stable. So the  $P$ -restraint associated with  $x_p$  is stable from  $u$  on and hence respected.

CLAIM 4.10. *Suppose  $\alpha \subseteq TP$  and  $\alpha : M$ . If there are infinitely many  $\alpha$ -expansionary stages, then there is a  $j \leq n(i)$  such that  $\alpha \hat{=} j \subseteq TP$ . Moreover,  $\Delta_{\alpha,j}(W_i)$  is total and  $\Delta_{\alpha,j}(W_i) =^* G_j$ .*

*Proof.* By hypothesis, there is  $j \leq n(i) + 1$ , which we choose maximal, such that, in Phase 1, Case  $M_i$ , infinitely often we choose  $j$  (note that  $j = 0$  is an option by the definition of " $\alpha$ -expansionary").

**Case 1.**  $j \leq n(i)$ . Then, by definition,  $\alpha \hat{j} \subseteq TP$ . Moreover,  $\Delta_{\alpha,j}(W_i)$  is total: otherwise, let  $x$  be least such that  $\Delta_{\alpha,j}(x)$  is undefined, and let  $s$  be a stage by which  $\Delta_{\alpha,j}(z)$  has settled down for every  $z < x$ . It is clear from the choice of  $j$  that  $\Delta_{\alpha,j}(W_i; x)[t] \downarrow$  for infinitely many  $t > s$ . As this corresponds to some  $\Xi_e(R \oplus Q; x)[t] \downarrow$ , the success of the K-type requirements guarantees that  $\Delta_{\alpha,j}(W_i; x)$  is defined.

Suppose there are infinitely many  $x$  such that  $\Delta_{\alpha,j}(W_i; x) \neq G_j(x)$ . We will derive a contradiction to the assumption that  $j$  is maximal. Let  $r$  be a stage after which no strategy  $\mu <_L \alpha \hat{\infty}$  is accessible in Phase 1 or acts, and we no longer elect a number  $> j$  in Phase 1, Case  $M_i$ . Choose  $x > r$  such that  $\Delta_{\alpha \hat{j}}(W_i; x) \neq G_j(x)$ . Then  $x$  enters  $G_j$  at some stage  $t > r$ , and we do not newly define  $\Delta_{\alpha,j}(W_i; x)$  at the first  $\alpha$ -expansionary stage  $u \geq t$ . Thus  $\Delta_{\alpha,j}(W_i; x)$  was already defined (with value 0) at the beginning of  $t$ , whence we have a  $\Delta_{\alpha,j}$ -failure at  $t$ . Suppose  $s < u$  is the stage where  $\Delta_{\alpha,j}(x)$  was defined last, via a chit for  $y$  (so that  $\varphi_i(y) \leq \delta_{\alpha,j}(x)[s]$ ). Then  $x < s < t$  and  $s$  is an  $\alpha \hat{j}$ -stage. We claim that the chit for  $y$  is uncanceled by stage  $u$ . Then, since  $y$  has not been assigned to a  $\Delta_{\alpha,j}$ -computation so far, at stage  $u$ , we would elect a number  $> j$  for another time, contrary to the choice of  $r$ . To show that the chit for  $y$  is uncanceled by stage  $u$ , we consider a possible cancellation at a stage (a) before  $t$  and (b) at or after  $t$ . By the choice of  $u$ , the cancellation is due to a change in  $P$ , not  $W_i$ , below  $\varphi_i(y)$ .

(a) Since  $x < s$ , at stage  $s$ ,  $x$  is already a follower of some  $G_j$ -positive strategy  $\beta > \alpha$ . Then in fact  $\alpha \hat{j} \subseteq \beta$  by our choice of  $r$ . Thus by, (4.4),  $r(\beta, x, s) \geq \delta_{\alpha,j}(x)[s] = \varphi_i(y)[s]$ . If  $P$  changes below this restraint before  $x$  enters  $G_j$ ,  $x$  would be cancelled: if  $\eta : D$  or  $\eta : N$ , then the restraint can only be violated when  $\eta$  is initialized. If  $\beta : Z$ , then the individual follower  $x$  is cancelled. Thus when  $x$  enters  $G_j$  at  $t \geq s$ ,  $P \upharpoonright \varphi_i(y)[s]$  has not changed since  $s$ .

(b) Now suppose a number  $z < \varphi_i(y)[s]$  enters  $P$  at a stage  $v, t \leq v \leq v$ . Choose  $v$  minimal such that such a number is enumerated (at stage  $v$ ) into  $P$ , by an  $M_i$ -strategy  $\mu$ . By the initializations, necessarily  $\alpha < \mu < \beta$ . Thus, by the end of stage  $v$ , the  $\Delta$ -correction number  $w$  of  $\beta$  has entered  $E$ . Since  $w < x$ , this makes  $\Delta_{\alpha,j}(W_i; x)[u-1]$  undefined, contrary to our assumption on  $u$ .

**Case 2.**  $j = n(i) + 1$ . We claim that this case cannot occur. If  $j = n(i) + 1$ , then in Phase 1, Case  $M_i$ , infinitely many numbers  $y$  are declared  $\alpha$ -cleared. Then  $\alpha$  diagonalizes successfully against  $\Phi_i(W_i \oplus P) = Q$ , contrary to the hypothesis that there are infinitely many  $\alpha$ -expansionary stages: Since  $D$  is nonrecursive, by Claim 4.5 it suffices to show that for each  $p$ ,  $\alpha$  eventually appoints a follower  $y_p^\eta$  at a stage  $> r_\eta$  (whose chit remains uncanceled from then on). Inductively, suppose  $p > 0$  and  $y_{p-1}^\eta$  has been appointed, or  $p = 0$ . By Claim 4.9, applied to each  $Z$ -type strategy  $\beta \subset \alpha$ , there is a stage where  $x_p^\beta[s]$  and  $r(\beta, x_p^\beta, s)$  have come to a limit. Thus eventually some  $y$  is declared  $\alpha$ -cleared such that (4.6) holds, and we define  $y_p^\eta = y$ .

LEMMA 4.11. *The  $M$ -type requirements are met.*

*Proof.* Immediate by Claim 4.10.

LEMMA 4.12. *Suppose  $j = 2\langle m, k \rangle$  for some  $k$ . Then  $G_j \leq_T U_m$ .*

*Proof.* Let  $\alpha \subseteq TP$  and  $\alpha : Y_j$ . If  $\alpha \hat{f} \subseteq TP$ , then only finitely many  $Z_{j,e}$ -strategies act, so that  $G_j \leq_T D$  (and hence  $G_j \leq_T U_m$ ) by direct permitting.

Now suppose  $\hat{\alpha}^\infty$  is on the true path. To show  $G_j \leq_T U_m$ , given an input  $x$ , suppose that, by stage  $x$ ,  $x = x_p^\beta$  for some  $G_j$ -positive strategy  $\beta$  (otherwise,  $x \notin G_j$ ). We can assume that  $\hat{\alpha}^\infty \subseteq \beta$ . Using the oracle, compute an  $\hat{\alpha}^\infty$  stage  $s$  such that  $U_m \upharpoonright p$  (and hence  $D \upharpoonright p$ ) is stable. Then, if  $\beta$  is a  $D$ - or  $N$ -type strategy,  $\beta$  cannot enumerate  $x$  into  $G_j$  after stage  $s$ . Otherwise,  $\beta$  is a  $Z$ -type strategy. If there is a test procedure  $T(\beta, x, s')$  which was started at stage  $s' \leq s$  and has not terminated yet, by Lemma 4.3, we can compute the stage  $t \geq s$  by which the procedure has terminated. Then  $x \in G_j$  iff  $x \in G_j[t]$ .

LEMMA 4.13. *The  $Z$ -type requirements are met.*

Suppose  $\beta \subseteq TP$  is a  $Z_{j,e}$ -strategy. Then  $V_k = \Upsilon(B)$ , since  $\hat{\alpha}^\infty$  is on the true path where  $\hat{\alpha} \subseteq TP$  is the corresponding  $Y_j$ -strategy. If  $\lim_s g(\beta, r_\beta, s) = 1$ , then  $Z_{j,e}$  is met, as noted at the end of the proof of Lemma 4.4. Now suppose  $\lim_s g(\beta, r_\beta, s) = 0$  and  $G_j = \Psi_e(V_k)$ . We show that  $U_m \leq_T V_k$ . Let

$$l_Z(m, e, t) = \mu x \{ \neg [\Psi_e(V_k; x) \downarrow = G_j(x)[t]] \}.$$

We omit the superscript  $\beta$  in what follows. By Claim 4.9, for each  $p$ ,  $\lim_s x_p[s]$  exists. Let  $s_0$  be a stage by which  $g(\beta, r_\beta, s)$  has reached its limit and such that  $\Psi_e(V_k; x) = 0$  for every  $x = \lim_s x_p > s_0$ . Recursively in  $V_k$ , we can compute stages  $t = t_p \geq s_0$  where  $x_p[t]$  is defined,  $l_Z(m, e, t) > x_p[t]$  and  $\Psi_e(V_k; x_p) = 0[t]$  via  $V_k$ -correct computations.

CLAIM 4.14.  *$x_p$  is not cancelled after stage  $t_p$ .*

Otherwise, suppose that  $p$  is least such that  $x_p > s_0$  is cancelled at a stage  $s \geq t_p$ . This cancellation of  $x_p$  is caused by the enumeration of a number  $z$  into  $P$  at stage  $s$  which violates the restraint  $r(\beta, x_p, s-1)$ . Then at a stage  $u$ ,  $r_\beta \leq u < s$ , a strategy  $\mu$  appointed a candidate  $y = y_a^\mu$  targeted for  $Q$ , and, for some  $n$ ,  $z = \gamma_n(y_a^\mu) \geq y$ . By (4.6),

$$(4.7) \quad w \geq y_a^\mu > r(\beta, x_a, u).$$

At stage  $u$ ,  $\mu$  initializes the lower priority strategies, so the  $P$ -restraint associated with  $x_a$  cannot increase unless  $\mu$  itself is initialized (in which case  $y_a^\mu$  is cancelled). Therefore (4.7) still holds at stage  $s$ , whence  $p > a$ , because the enumeration of  $z$  cancels  $x_p$ . Since  $y_a$  is enumerated into  $Q$ , a number  $< a$  enters  $D$  (and hence  $V_k$ ) at stage  $s$ , contrary to the assumption on  $t_p$ . This proves the Claim.

If  $U_m \upharpoonright p$  ever changed at some stage  $t > t_p$  we would contradict the assumption that  $\lim_s g(\beta, r_\beta, s) = 0$  by providing a stage  $t$  which is a witness to (4.5). The second condition in (4.5) holds by our choice of  $s_0$ , and because  $l_Z(e, m, t) > x$  and  $G_j = \Psi_e(V_k)$ . The third condition is true because  $V_k = \Upsilon(B)$ , so that at some stage  $u \geq t$ ,  $V_k \upharpoonright \psi_e(x_p)$  is computed via  $B$ -correct computations.  $\diamond$

We now give the changes to obtain a *Proof of Theorem 2.4*. We may suppose that  $\mathbf{d}$  is low. By the proof of the Robinson Splitting Theorem in [13, Thm XI.3.2], there are low  $\mathbf{b}^0, \mathbf{b}^1 > \mathbf{d}$  and a u.r.e. sequence  $(\mathbf{u}_i)$  such that  $\mathbf{d} \leq \mathbf{u}_{2i+q} \leq \mathbf{b}^q$  ( $q = 0, 1$ ) and  $\{\mathbf{u}_{2i} \vee \mathbf{u}_{2i+1} : i \in \mathbb{N}\} = [\mathbf{d}, \mathbf{1}]$ . (Let  $B = \bigoplus_e W_e$ , and obtain a set splitting  $B = A_0 \cup A_1$  as in that proof, so that  $\mathbf{b}^q = \deg_T(D \oplus A_q)$  is low. The sequence  $\mathbf{u}_{2i+q} = \deg_T(D \oplus (A_q \cap \omega^{[i]}))$ ,  $i \in \mathbb{N}$ ,  $q = 0, 1$ , is as desired.)

The following Claim is an extension of Theorem 4.1, where the u.r.e. sequence is now contained in the union of two low intervals  $[\mathbf{d}, \mathbf{b}^0]$  and  $[\mathbf{d}, \mathbf{b}^1]$ .

CLAIM 4.15. *In a setting as above, there is a standard  $M$  such that*

$$(4.8) \quad \mathbf{u}_m = \sup_{[\mathbf{d}, \mathbf{b}^q]} \{ \mathbf{d} \vee \langle m, k \rangle^M : k \in \mathbb{N} \} \quad (q = m \pmod{2}).$$

Then, by the choice of the sequence  $(\mathbf{u}_i)$ , the following map is as desired:

$$h(n^M) = \sup_{[\mathbf{d}, \mathbf{b}^0]} \{ \mathbf{d} \vee (\langle 2n, k \rangle)^M : k \in \mathbb{N} \} \vee \sup_{[\mathbf{d}, \mathbf{b}^1]} \{ \mathbf{d} \vee (\langle 2n+1, k \rangle)^M : k \in \mathbb{N} \}.$$

Claim 4.15 is proved by making a few adjustments to the proof of Theorem 4.1. Only the  $Z$ -type requirements are affected. The modified requirements are

7'.  $Z_{j,e} : [\Upsilon_k^q(B^q) = V_k \ \& \ \Psi_e(V_k) = G_j] \Rightarrow U_m \leq_T V_k$   
 ( $j = 2\langle m, k \rangle$ ,  $q = m \pmod{2}$ ), where, for  $q = 0, 1$ ,  $(\Upsilon_e^q)_{e \in \mathbb{N}}$  is a list of functionals such that  $\Phi_e^q(B^q)$  total  $\Rightarrow \Upsilon_e^q(B^q) = \Phi_e(B^q)$ , and  $\Upsilon_e^q(B^q)$  has the same stability property as before.

The strategies and verifications are as before with some obvious modifications. When  $j$  is given,  $j = 2\langle m, k \rangle$ , let  $q = m \pmod{2}$ . All symbols  $\Upsilon, B, g$  obtain a superscript  $q \in \{0, 1\}$ . Thus in the discussion of the  $Y_j$ -strategy  $\eta$ , we now claim that there are functions  $g^q(\eta, r, s)$  approximating the properties in (4.5). To see that these functions exist, as before, we apply the Recursion Theorem. Given an index for a (possible partial) construction  $\mathcal{C}$ , using the lowness of each  $B^q$ ,  $q = 0, 1$ , we effectively obtain functions  $g^q(\eta, r, s)$  approximating the property (4.5), with the appropriate superscripts  $q$ , of  $\mathcal{C}$ . Based on these functions we carry out a total construction  $\mathcal{C}'$ . By the Recursion Theorem, there is a fixed point, i.e. a construction  $\mathcal{C} = \mathcal{C}'$ , which builds the required sets since the information approximated by the functions  $g^q$  is correct.

*Remark.* One can give an alternative proof that an interpretation of  $\text{Th}(\mathbb{N}, +, \times)$  in  $\text{Th}([\mathbf{0}, \mathbf{b}])$  exists for each  $b \neq \mathbf{0}$ , by removing the restriction "b" low in (iii) of Theorem 3.3, and applying Theorem 3.2. This is interesting since it might be extended to all nontrivial intervals of  $\mathcal{R}_T$ . We meet the same requirements as in the proof of (iii) except for the  $K$ -type requirements. The basic  $Z$ -type strategy is as in the proof of (iii). Since  $U_j$  is no longer low, a  $Z$ -type strategy may now infinitely often attempt to diagonalize, and each time a  $U_j$  change destroys the diagonalization. However, since  $U_i \not\leq_T U_j$ , then it shows there is a least  $q$  such that  $\neg G_{2i}(q) = \Psi_e(U_j; q)$ . We implement the strategies in a tree construction similar to the one above, and guess at this outcome of a  $Z$ -type strategy  $\eta$ . Whenever  $q$  is the approximation to the outcome of  $\eta$ , we cancel all followers  $x_p^\eta$ ,  $p \geq q$ . An  $M$ -type strategy  $\alpha \supseteq \widehat{\eta}q$  only appoints new followers  $y_r^\alpha$  when  $x_q^\eta$  is undefined, so it only has to deal with the finite  $P$ -restraint imposed by the followers  $x_p^\eta$ ,  $p < q$ . To be sure this restraint is stable, the outcomes of higher priority  $M$ -type strategies have to be put on the tree as in the proof of Theorem 2.4. To make sure that the relevant  $\Delta$ -functionals are total (in the absence of the lowness requirements of type  $K$ ), when appointing  $x_q^\eta$ ,  $\eta$  has to wait till  $\Delta_{\alpha, k}(q + |\eta| + l)$  is defined for each  $\alpha \widehat{k} \subseteq \eta, \alpha : M$ , where  $l$  is the number of times  $x_q^\eta$  has been defined so far, and similarly for the  $\Gamma$ -functionals.

## 5. Ideals and ideal lattices

1. *A new definable ideal.* For a set  $D \subseteq \mathcal{R}_T$ , let  $[D]_{id}$  be the ideal of the upper semilattice  $\mathcal{R}_T$  generated by  $D$ .

THEOREM 5.1.  $[D]_{id}$  is definable without parameters in the structure  $(\mathcal{R}_T, D)$ .

*Proof.* We may suppose that  $D \not\subseteq \{0\}$ . Then  $[D]_{id} = \{\mathbf{x} : \exists \mathbf{d} \in D - \{\mathbf{0}\} \exists F \subseteq [\mathbf{d}, \mathbf{1}]$  finite  $[\mathbf{x} \leq \sup(F) \ \& \ \forall \mathbf{y} \in F \exists \mathbf{z} \in D \ \mathbf{z} \vee \mathbf{d} = \mathbf{y}]\}$ . By Theorem 2.4, this can be expressed in first-order logic.  $\diamond$

Let CAP, be the ideal of capable degrees,  $\text{NB} \subseteq \text{CAP}$  be the (definable) class of nonbounding degrees, and let  $\text{NCup}$  be the ideal of noncuppable degrees. In a forthcoming article [7] we prove:

THEOREM 5.2. (i)  $[\text{NB}]_{id} \neq \text{CAP}$   
(ii)  $[\text{NB}]_{id} \neq \text{NCup}$ .

Then, since  $[\text{NB}]_{id}$  is distinct from CAP and  $\text{NCup}$ , it is a new definable ideal of  $\mathcal{R}_T$ .

*Ideal lattices.* Let  $\mathcal{H}$  be the lattice of ideals of  $\mathcal{R}_T$ , and for  $k \geq 4$ , let  $\mathcal{L}_k$  be the lattice of  $\Sigma_k^0$ -ideals of  $\mathcal{R}_T$ . Note that  $\mathcal{R}_T$  is embedded into all those lattices via  $\mathbf{a} \rightarrow [\mathbf{0}, \mathbf{a}]$ . Some more facts:  $\mathcal{R}_T$  is automorphism invariant in  $\mathcal{H}$  as the set of compact elements, but it is not known whether  $\mathcal{R}_T$  is automorphism invariant in the lattices  $\mathcal{L}_k$ . Moreover, in  $\mathcal{L}_k$ ,  $k \geq 6$  and  $\mathcal{H}$ ,  $\text{NCup}$  is intersection of all maximal ideals (and hence  $\text{NCup}$  is a nontrivial definable element).

THEOREM 5.3. For  $k \geq 7$ , the class of  $\Sigma_k^0$ -ideals of  $\mathcal{R}_T$  is uniformly definable.

*Proof.* We will use Corollary 2.6. First we establish that it holds in fact for each  $k \geq 7$ . By the success of the  $K$ -type requirements in the proof of Theorem 2.4,  $R \oplus Q$  is low. Since  $\text{Low}_2$  is definable by [11], we can assume as an additional correctness condition in  $S_M$  that  $\mathbf{r} \vee \mathbf{q}$  is  $\text{Low}_2$ . Since the construction in the proof of Theorem 2.4 makes  $L \leq R$ , the formulas in the scheme  $S_M$  are evaluated in an interval with  $\text{Low}_2$  top.

First we show that for  $r = 4$ ,  $\alpha \leq_T \emptyset^{(r)}$  as in Lemma 2.5. The formulas  $\varphi_{num}(x, \bar{p})$ ,  $\varphi_+(x, y, z; \bar{p})$ ,  $\varphi_\times(x, y, z; \bar{p})$  used in  $S_M$  are  $\Sigma_2$ . Let  $\alpha(0)$  be an index for  $0^N$ , and let  $\alpha(i+1)$  be some index  $e$  such that  $\mathcal{R}_T \models \varphi_\oplus(\deg(W_{\alpha(i)}), 1^K, \deg(W_e); \bar{p})$ . The recursive description of  $\alpha(i+1)$  is  $\Sigma_3^0$  since, with an appropriate indexing,  $T$ -reducibility is  $\Sigma_3^0$  in any interval with  $\text{Low}_2$  top. Inspecting (4.1) it can be checked that, for each function  $h$  (coded via  $S_h$ ),  $\text{Ind}(h)$  is  $\Sigma_7^0$ . This establishes Corollary 2.6 for each  $k \geq 7$ .

We describe a scheme  $S_I$  for uniformly defining the  $\Sigma_k^0$ -ideals. It is sufficient to consider the  $\Sigma_k^0$ -ideals  $I$  containing a nonzero degree  $\mathbf{d}$ . Recall that  $S^k$  is the scheme from Corollary 2.6 which uniformly defines the class  $D_k^1$ . It involves a list of parameters  $\bar{\mathbf{q}}$  which contains a lower bound  $\mathbf{d} \neq \mathbf{0}$  for the relation to be described. The scheme  $S_I$  involves the same parameters. Since

$$\mathbf{x} \in I \Leftrightarrow \mathbf{x} \vee \mathbf{d} \in I,$$

we may use the scheme  $S^k$  to define  $I \cap [\mathbf{d}, \mathbf{1}]$ . Then we add as an extra correctness condition in  $S_I$  that the set defined is an ideal.

Notice that there is a first-order condition on parameters in  $S_I$  which holds if and only if the parameters code a principal ideal. Thus

PROPOSITION 5.4. For  $k \geq 7$ , the structure  $L_k$  consisting of the lattice of  $\Sigma_k^0$ -ideals of  $\mathcal{R}_T$  with an additional predicate for being principal, can be interpreted in  $\mathcal{R}_T$  without parameters.

Using the proposition one can show that each  $L_k$ ,  $k \geq 7$ , is biinterpretable with  $\mathcal{R}_T$  without parameters in the sense of Hodges [5]. Then, if the BI-conjecture holds for  $L_k$ , it also holds for  $\mathcal{R}_T$ .

### References

- [1] William C. Calhoun. Incomparable prime ideals of the recursively enumerable degrees. *Ann. Pure Appl. Logic*, 63:36–56, 1993.
- [2] P. Cholak, M. Groszek, and T. Slaman. An almost deep degree. Submitted, Draft Available.
- [3] L. A. Harrington and A. Nies. Coding in the lattice of enumerable sets. *Adv. in Math.*, 133:133–162, 1998.
- [4] Leo A. Harrington and S. Shelah. The undecidability of the recursively enumerable degrees. *Bull. Amer. Math. Soc.*, 6:79–80, 1982.
- [5] Wilfrid Hodges. *Model Theory*. Enzyklopedia of Mathematics. Cambridge University Press, Cambridge, 1993.
- [6] Richard Kaye. *Models of Peano arithmetic*, volume 15 of *Oxford Logic Guides*. The Clarendon Press Oxford University Press, New York, 1991. Oxford Science Publications.
- [7] A. Nies. Definable ideals of the recursively enumerable degrees. To appear.
- [8] A. Nies. The last question on recursively enumerable many-one degrees. *Algebra i Logika*, 33(5):550–563, 1995. English Translation July 1995.
- [9] A. Nies. Definability in the computably enumerable degrees: Questions and result. In Peter Cholak, Steffen Lempp, Manny Lerman, and Richard Shore, editors, *Computability Theory and Its Applications: Current Trends and Open Problems*. American Mathematical Society, 2000.
- [10] A. Nies. Interpreting  $\mathbb{N}$  in the c.e. weak truth table degrees. *Ann. Pure. Appl. Logic*, 107:35–48, 2001.
- [11] A. Nies, R. Shore, and T. Slaman. Interpretability and definability in the recursively enumerable turing-degrees. *Proc. Lond. Math. Soc.*, 3(77):241–291, 1998.
- [12] Richard Shore. Natural definability in degree structures. In Peter Cholak, Steffen Lempp, Manny Lerman, and Richard Shore, editors, *Computability Theory and Its Applications: Current Trends and Open Problems*. American Mathematical Society, 2000.
- [13] Robert I. Soare. *Recursively Enumerable Sets and Degrees*. Perspectives in Mathematical Logic, Omega Series. Springer-Verlag, Heidelberg, 1987.

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