The Extended Turing Model As Contextual Tool

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Abstract. Computability concerns information with a causal – typically algorithmic – structure. As such, it provides a schematic analysis of many naturally occurring situations. We look at ways in which computability-theoretic structure emerges in natural contexts. We will look at how algorithmic structure does not just emerge mathematically from information, but how that emergent structure can model the emergence of very basic aspects of the real world.

The adequacy of the classical Turing model of computation — as first presented in [18] — is in question in many contexts. There is widespread doubt concerning the reducibility to this model of a broad spectrum of real-world processes and natural phenomena, from basic quantum mechanics to aspects of evolutionary development, or human mental activity.

In 1939 Turing [19] described an extended model providing mathematical form to the algorithmic content of structures which are presented in terms of real numbers. Most scientific laws with a computational content can be framed in terms of appropriate Turing reductions. This can be seen in implicit form in Newton's *Principia* [14], published some 272 years before Turing's paper. Newton's work was formative in established a more intimate relationship between mathematics and science, and one which held the attention of Turing, in various guises, throughout his short life (see Hodges [10]). Just as the history of arithmetically-based algorithms, underlying many human activities, eventually gave rise to models of computation such as the Turing machine, so the oracle Turing machine schematically addresses the scientific focus on the extraction of predictions governing the form of computable relations over the reals. Whereas the inputting of data presents only time problems for the first model, the second model is designed to deal with possibly incomputable inputs, or at least inputs for which we do not have available an algorithmic presentation. One might reasonably assume that data originating from observation of the real world carries with it some level of computability, but we are yet to agree a mathematical model of physical computation which dispenses with the relativism of the oracle Turing machine. In fact, even as the derivation of recognisable incomputability in mathematics arises from quantification over algorithmic objects, so definability may play an essential role in fragmenting and structuring the computational

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content of the real world. The Turing model of computability over the natural numbers appears to many people to be a poor indicator of what to expect in science.

1 The Turing Landscape: from Local to Global

The oracle Turing machine, which made its first appearance in Turing [19], should be familiar enough. The details are not important, but can be found in most reasonable introductions to computability (see for instance [3]).

The basic form of the questioning permitted is modelled on that of everyday scientific practice. This is seen most clearly in today's digital data gathering, whereby one is limited to receiving data which can be expressed, and transmitted to others, as information essentially finite in form. But with the model comes the capacity to collate data in such a way as enable us to deal with arbitrarily close approximations to infinitary inputs and hence outputs, giving us an exact counterpart to the computing scientist working with real-world observations. If the different number inputs to the oracle machine result in 0-1 outputs from the corresponding Turing computations, one can collate the outputs to get a binary real computed from the oracle real, the latter now viewed as an input. This gives a partial computable functional Φ, say, from reals to reals.

As usual, one cannot computably know when the machine for Φ computes on a given natural number input, so Φ may not always give a fully defined real output. So Φ may be partial. One can computably list all oracle machines, and so index the infinite list of all such Φ, but one cannot computably sift out the partial Φ's from the list.

Anyway, put $\mathbb{R}$ together with this list, and we get the Turing Universe. Depending on one's viewpoint, this is either a rather reduced scientific universe, or a much expanded one. The familiar mathematical presentation of it is due to Emil Post, in his search for the informational underpinnings of computational structure.

Post's first step was to gather together binary reals which are computationally indistinguishable from each other, in the sense that they are mutually Turing computable from each other. Mathematically, this delivered a more standard mathematical structure to investigate — the familiar upper semi-lattice of the *degrees of unsolvability*, or *Turing degrees*. There is no simple scientific counterpart of the mathematical model, or any straightforward justification for what Post did with the Turing universe for perfectly good mathematical reasons — if one wants to get a material avatar of the Turing landscape one needs both a closer and a more comprehensive view of the physical context.

In approaching with this, we are presented with very real and inescapable causal structure, accompanied by the information content of its particular instantiations, and the problem is to explain and characterise this connection. The difficulty is that recognition of these causal structures entails us taking a global view of an environment of which we ourselves are a component. When we look at the mysterious emergence of structure in nature, either subatomic laws, or the richness of life forms, or large-scale galactic or super-galactic structures, we are not just looking at information, but at expressions of patterns of a universal nature. And patterns the origins of which science is as yet unable to to explain.

When we inspect the intricacies of the Cat's Eye Nebula, say, as revealed by the Hubble Space Telescope, we feel we should be able to explain the remarkable complexity observed on the basis of our understanding of the local physics. The intuition is that it should be possible to describe global relations in terms of local structure, so capturing the emergence of large-scale structure. The mathematics pertaining to any particular example will be framed in terms of the specific interactive structure on which it is based. But if one wants to reveal general characteristics, and approach deep problems around the emergence of physical laws and constants, which current theory fails to do, one needs something more fundamental.

Schematically, as we have argued, any causal context framed in terms everyday computable mathematics can be modelled in terms of Turing reductions. Then emergence can be formalised as definability over the appropriate substructure of the Turing universe; or more generally, as invariance under automorphisms of the Turing universe. Simple and fundamental as the notions of definability and invariance are, and basic as they are to everyday thought and discourse, as concepts they are not well understood outside of mathematics. This is seen most strikingly in the physicists' apparent lack of awareness of the concept in interpreting the collapse of the wave function. Quantum decoherence and the many-worlds hypothesis comprise a far more outlandish interpretive option than does speculating that measurements, in enriching an environment, merely lead to an assertion of invariance. It appears a sign of desperation to protect consistent histories by inventing new universes, when the mathematics of our observable universes already contains a straightforward explanation. We argue that many scientific puzzles can be explained in terms of failures of invariance in different contexts, and that the key task is to identify useful theoretical models within which to investigate the nature of invariance more fully. One of the most relevant of these models has to be that of Turing, based as it is on a careful analysis of the characteristics of algorithmic computation.

This brings us to a well-known research programme, initiated by Hartley Rogers in his 1967 paper [16], in which he drew attention to the fundamental problem of characterising the Turing invariant relations. Again, the intuition is that these are key to pinning down how basic laws and entities emerge as mathematical constraints on causal structure. It is important to notice how the richness of Turing structure discovered so far becomes the raw material for a multitude of non-trivially definable relations, matching in its complexity what we attempt to model.

Unfortunately, the current state of Rogers' programme is not good. For a number of years research in this area was dominated by a proposal originating with the Berkeley mathematician Leo Harrington, which can be (very) roughly stated:

Bi-interpretability Conjecture: *The Turing definable relations are exactly those with information content describable in second-order arithmetic.*

Most importantly, bi-interpretability is not consistent with the existence of non-trivial Turing automorphisms. Despite decades of work by a number of leaders in the field, the exact status of the conjecture is still a matter of controversy.

For those of us who have grown up with Thomas Kuhn's 1962 book [13] on the structure of scientific revolutions, such difficulties and disagreements are not seen as primarily professional failures, or triggers to collective shame (although they may be that too), but rather signs that something scientifically important is at stake. A far more public controversy currently shapes developments around important issues affecting theoretical physics — see, for example the recent books of Lee Smolin [17] and Peter Woit [21].

As Peter Woit [21, p.1] describes, according to purely pragmatic criteria particle physics has produced a standard model which is remarkably successful, and has great predictive power:

> By 1973, physicists had in place what was to become a fantastically successful theory of fundamental particles and their interactions, a theory that was soon to acquire the name of the standard model. Since that time, the overwhelming triumph of the standard model has been matched by a similarly overwhelming failure to find any way to make further progress on fundamental questions.

The reasons why people are dissatisfied echo misgivings going back to Einstein himself [8, p.63]:

> ... I would like to state a theorem which at present can not be based upon anything more than upon a faith in the simplicity, i.e. intelligibility, of nature ... nature is so constituted that it is possible logically to lay down such strongly determined laws that within these laws only rationally completely determined constants occur (not constants, therefore, whose numerical value could be changed without destroying the theory) ...

If one really does have a satisfying description of how the universe is, it should not contain arbitrary elements with no plausible explanation. In particular, a theory containing arbitrary constants, which one adjusts to fit the intended interpretation of the theory, is not complete. And as Woit observes:

> One way of thinking about what is unsatisfactory about the standard model is that it leaves seventeen non-trivial numbers still to be explained, ...

At one time, it had been hoped that string theory would supply a sufficiently fundamental framework to provide a much more coherent and comprehensive description, in which such arbitrary ingredients were properly pinned down. But despite its mathematical attractions, there are growing misgivings about its claimed status as "the only game in town" as a unifying explanatory theory. Here is how one time string theorist Daniel Friedan [9] combatively puts it:

> The longstanding crisis of string theory is its complete failure to explain or predict any large distance physics. ... String theory is incapable of determining the dimension, geometry, particle spectrum and coupling constants of macroscopic spacetime. ... The reliability of string theory cannot be evaluated, much less established. String theory has no credibility as a candidate theory of physics.

Smolin starts his book [17]:

> From the beginning of physics, there have been those who imagined they would be the last generation to face the unknown. Physics has always seemed to its practitioners to be almost complete. This complacency is shattered only during revolutions, when honest people are forced to admit that they don't know the basics.

He goes on to list what he calls the "five great [unsolved] problems in theoretical physics". Gathering these together, and slightly editing, they are [17, pp.5-16]:

1. Combine general relativity and quantum theory into a single theory that can claim to be the complete theory of nature.
2. Resolve the problems in the foundations of quantum mechanics.
3. The unification of particles and forces problem: Determine whether or not the various particles and forces can be unified in a theory that explains them all as manifestations of a single, fundamental entity.
4. Explain how the values of the free constants in the standard model of physics are chosen in nature.
5. Explain dark matter and dark energy. Or, if they dont exist, determine how and why gravity is modified on large scales.

That each of these questions can be framed in terms of definability is not so surprising, since that is exactly how, essentially, they are approached by researchers. The question is the extent to which progress is impeded by a lack of consciousness of this fact, and an imperfect grip of what is fundamental. Quoting Einstein again (from a letter to Robert Thornton, dated 7 December 1944, Einstein Archive 61-754), this time on the relevance of a philosophical approach to physics:

> So many people today – and even professional scientists – seem to me like someone has seen thousands of trees but has never seen a forest. A knowledge of the historical and philosophical background gives that kind of independence from prejudices of his generation from which most scientists are suffering. This independence created by philosophical insight is – in my opinion – the mark of distinction between a mere artisan or specialist and a real seeker after truth.

Smolin's comment [17, p.263] is in the same direction, though more specifically directed at the string theorists:

> The style of the string theory community ... is a continuation of the culture of elementary-particle theory. This has always been a more brash, aggressive, and competitive atmosphere, in which theorists vie to respond quickly to new developments ... and are distrustful of philosophical issues. This style supplanted the more reflective, philosophical style that characterized Einstein and the inventors of quantum theory, and it triumphed as the center of science moved to America and the intellectual focus moved from the exploration of fundamental new theories to their application.

So what is it that is fundamental that is being missed? For Smolin [17, p.241], it is *causality*:

> It is not only the case that the spacetime geometry determines what the causal relations are. This can be turned around: Causal relations can determine determine the spacetime geometry ... Its easy to talk about space or spacetime emerging from something more fundamental, but those who have tried to develop the idea have found it difficult to realize in practice. ... We now believe they failed because they ignored the role that causality plays in spacetime. These days, many of us working on quantum gravity believe that *causality itself is fundamental* – and is thus meaningful even at a level where the notion of space has disappeared.

Citing Penrose as an early champion of the role of causality, he also mentions Rafael Sorkin, Fay Dowker, and Fotini Markopoulou, known in this context for their interesting work on causal sets (see [1]), which abstract from causality relevant aspects of its underlying ordering relation. Essentially, causal sets are partial orderings which are locally finite, providing a model of spacetime with built-in discreteness. Despite the apparent simplicity of the mathematical model, it has had striking success in approximating the known characteristics of spacetime. An early prediction, in tune with observation, concerned the value of Einstein's cosmological constant.

Of course, this preoccupation with causality might suggest to a logician a need to also look at its computational content. Smolin's comment that "Causal relations can determine determine the spacetime geometry" touches on one of the biggest disappointments with string theory, which turns out to be a 'background dependant' theory with a vengeance — one has literally thousands of candidate Calabi-Yau spaces for shaping the extra dimensions of superstring theory. In current superstring models, Calabi-Yau manifolds are those qualifying as possible space formations for the six hidden spatial dimensions, their undetected status explained by the assumption of their being smaller than currently observable lengths.

Ideally, a truly fundamental mathematical model should be background independent, bringing with it a spacetime geometry arising from within.

2 Turing Invariance and the Laws of Physics

There are obvious parallels between the Turing universe and the material world. Each of which in isolation, to those working with specific complexities, may seem superficial and unduly schematic. But the lessons of the history of mathematics and its applications is that the simplest of abstractions can yield unexpectedly far-reaching and deep insights into the nature of the real world.

Most basic, science describes the world in terms of real numbers. This is not always immediately apparent, any more that the computer on ones desk is obviously an avatar of a universal Turing machine. Nevertheless, scientific theories consist, in their essentials, of postulated relations upon reals. These reals are abstractions, and do not come necessarily with any recognisable metric. They are used because they are the most advanced presentational device we can practically work with. There is no faith that reality itself consists of information presented in terms of reals. In fact, those of us who believe that mathematics is indivisible, no less in its relevance to the material world, have a due humility about the capacity for our science to capture more than a surface description of reality.

Some scientists would take us in the other direction, and claim that the universe is actually finite, or at least countably discrete. We have argued elsewhere (see for example [7]) that to most of us a universe without algorithmic content is inconceivable. And that once one has swallowed that bitter pill, infinitary objects are not just a mathematical convenience (or inconvenience, depending on ones viewpoint), but become part of the mathematical mold on which the world depends for its shape. As it is, we well know how essential algorithmic content is to our understanding of the world. The universe comes with recipes for doing things. It is these recipes which generate the rich information content we observe, and it is reals which are the most capacious receptacles we can humanly carry our information in, and practically unpack.

Globally, there are still many questions concerning the extent to which one can extend the scientific perspective to a comprehensive presentation of the universe in terms of reals — the latter being just what we need to do in order to model the immanent emergence of constants and natural laws from an entire universe. Of course, there are many examples of presentations entailed by scientific models of particular aspects of the real world. But given the fragmentation of science, is fairly clear that less natural presentations may well have an explanatory role, despite their lack of a role in practical computation.

The natural laws we observe are largely based on algorithmic relations between reals. Newtonian laws of motion will computably predict, under reasonable assumptions, the state of two particles moving under gravity over different moments in time. And, as previously noted, the character of the computation involved can be represented as a Turing functional over the reals representing different time-related two-particle states. One can point to physical transitions which are not obviously algorithmic, but these will usually be composite processes, in which the underlying physical principles are understood, but the mathematics of their workings outstrip available analytical techniques. Over forty years ago, Georg Kreisel [11] distinguished between classical systems and *cooperative phenomena* not known to have Turing computable behaviour, and proposed [12, p.143, Note 2] a collision problem related to the 3-body problem, which might result in "an analog computation of a non-recursive function (by repeating collision experiments sufficiently often)". However, there is a qualitatively different apparent breakdown in computability of natural laws at the quantum level — the *measurement problem* challenges us to explain how certain quantum mechanical probabilities are converted into a well-defined outcome following a measurement. In the absence of a plausible explanation, one is denied a computable prediction. The physical significance of the Turing model depends upon its capacity for explaining what is happening here. If the phenomenon is not composite, it does need to be related in a clear way to a Turing universe designed to model computable causal structure. We will need to talk more about definability and invariance.

For the moment, let us think in terms of what an analysis of the automorphisms of *any* sufficiently comprehensive, sufficiently fundamental, mathematical model of the material universe might deliver.

Let us first look at the relationship between automorphisms and many-worlds. When one says "I tossed a coin and it came down heads, maybe that means there is a parallel universe where I tossed the coin and it came down tails", one is actually predicating a large degree of correspondence between the two parallel universes. The assumption that *you* exist in the two universes puts a huge degree of constraint on the possible differences — but nevertheless, some relatively minor aspect of our universe has been rearranged in the parallel one. There are then different ways of relating this to the mathematical concept of an automorphism. One could say that the two parallel worlds are actually isomorphic, but that the structure was not able to *define* the outcome of the coin toss. So it and its consequences appear differently in the two worlds. Or one could say that what has happened is that the worlds are *not* isomorphic, that actually we were able to change quite a lot, without the parallel universe looking very different, and that it was these fundamental but hidden differences which forces the worlds to be separate and not superimposed, quantum fashion. The second view is more consistent with the view of quantum ambiguity displaying a failure of definability. The suggestion here being that the observed existence of a particle (or cat!) in two different states at the same time merely exhibits an automorphism of our universe under which the classical level is rigid (just as the Turing universe displays rigidity above $\mathbf{0}''$) but under which the sparseness of defining structure at the more basic quantum level enables the automorphism to re-represent our universe, with everything at our level intact, but with the particle in simultaneously different states down at the quantum level. And since our classical world has no need to decohere these different possibilities into parallel universes, we live in a world with the automorphic versions superimposed. But when we make an observation, we establish a link between the undefined state of the particle and the classical level of reality, which destroys the relevance of the automorphism. To believe that we now get parallel universes in which the alternative states are preserved, one now needs to decide how much else one is going to change about our universe to enable the state of the particle destroyed as a possiblity to survive in the parallel universe — and what weird and wonderful things one must accommodate in order to make that feasible. It is hard at this point to discard the benefits brought by a little mathematical sophistication. Quantum ambiguity as a failure of definability is a far more palatable alternative than the invention of new worlds of which we have no evidence or scientific understanding.

Another key conceptual element in the drawing together of a global picture of our universe with a basic mathematical model is the correspondence between emergent phenomena and definable relations. This gives us a framework within which to explain the particular forms of the physical constants and natural laws familiar to us from the standard model science currently provides. It goes some way towards substantiating Penrose's [15, pp.106-107] 'strong determinism', according to which "all the complication, variety and apparent randomness that we see all about us, as well as the precise physical laws, are all exact and unambiguous consequences of one single coherent mathematical structure" — and repairs the serious failure of the standard model pointed to by researchers such as Smolin and Woit. It also provides a hierarchical model of the fragmentation of the scientific enterprise. This means that despite the causal connections between say particle physics and the study of living organisms, the corresponding disciplines are based on quite different basic entities and natural laws, and there is no feasible and informative reduction of one to another. The entities in one field may emerge through phase transitions characterised in terms of definable relations in the other, along with their distinct causal structures. In this context, it may be that the answer to Smolin's first 'great unsolved problem in theoretical physics' consists of an explanation of why there is no single theory (of the kind that makes useful predictions) combining general relativity and quantum theory.

The following table provides a summary of some of the main features of the Turing interpretation, drawing out parallels between scientific activity and what the Turing model provides. For further discussion of such issues, see [2], [4], [5], [6] and [7].

Science	**Turing landscape**
Physical entities treated as information	Structures information
Theories describing relations over the reals, enabling calculations	Functionals over the reals modelled on real computational capabilities
An extensive basic causal structure which is algorithmic	Models computable causal relations over the reals
Descriptions of globally emerging laws and constants elusive	Problems pinning down the nature of Turing invariance and definability
Features quantum ambiguity and non-locality	Explanation in terms of putative breakdown in Turing definability
Theoretical fragmentation involving phase transitions	Incomputability, and algorithmic relations over emergent objects

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