

Off to Infinity in Finite Time

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Without collisions could the Newtonian n -body problem of point masses eject a particle to infinity in finite time? This intriguing century-old concern, which has motivated several interesting and deep mathematical conclusions, was recently resolved by Xia ([X1, 2]) in his Ph.D. dissertation; he proved that three-dimensional examples exist for all $n \geq 5$. Later, Gerver [G] asserted that a similar behavior occurs with the planar $3n$ body problem but with an unknown and very large n value.

Even the suggestion that our familiar Newtonian inverse square force law might allow such a counter-intuitive behavior is so surprising that it is reasonable to wonder how such an esoteric-sounding question was first raised. As we show in this brief survey, Xia's result resolves a natural, fundamental problem raised by Poincaré and Painlevé about a century ago. The issue is to characterize the nature of "singularities" of n -body systems. Here, a singularity is a "time" value $t = t^*$ where analytic continuation of the solution fails.

So, what constitutes a singularity? Let m_j, \mathbf{r}_j be, respectively, the mass and position vector of the j th particle, and let $r_{ij} = \|\mathbf{r}_i - \mathbf{r}_j\|$. From the equations of motion

$$(1) \quad \begin{aligned} m_j \mathbf{r}_j'' &= \sum_{i \neq j} \frac{m_i m_j (\mathbf{r}_i - \mathbf{r}_j)}{r_{ij}^3} \\ &= \frac{\partial U}{\partial \mathbf{r}_j}, j = 1, \dots, n, \end{aligned}$$

where the self-potential (the negative of the potential energy) is

$$(2) \quad U = \sum_{i < j} \frac{m_i m_j}{r_{ij}},$$

it is clear that a singularity requires some $r_{ij}(t)$ distance to become arbitrarily small as $t \rightarrow t^*$. Trivially, a collision is a singularity. But, are all singularities collisions? A possible scenario, considered near the end of the nineteenth century, was whether a singularity orbit could exhibit some sort of oscillatory behavior where the limit infimum of $r_{min}(t) = \min_{i \neq j} (r_{ij}(t))$ approaches zero while the limit superior of this minimum spacing between particles remains positive. Namely, could the particles flirt with colliding without ever doing so?

Reexpressing this possibility in terms of configuration space, if

$$\Delta_{ij} = \{\mathbf{r} = (\mathbf{r}_1, \dots, \mathbf{r}_n) \in (R^3)^n \mid \mathbf{r}_i = \mathbf{r}_j\},$$

then $\Delta = \cup_{i < j} \Delta_{ij}$ identifies all $(R^3)^n$ points where Eq. 1 is not defined. The scenario, then, is equivalent to an orbit admitting a subsequence $\{t_i\}$, $t_i \rightarrow t^*$ whereby $\mathbf{r}(t_i) = (\mathbf{r}_1(t_i), \dots, \mathbf{r}_n(t_i))$ approaches Δ but $\mathbf{r}(t)$ does not. During his 1895 Swedish lectures [Pa], Painlevé proved the impossibility of this oscillatory behavior.

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Painlevé's proof is a nice application of the standard existence theorem which ensures that a solution for $\mathbf{x}' = f(\mathbf{x})$ exists in a time interval of length determined by an upper bound on $\|f(\mathbf{x})\|$. To see where the bounds for Eq. 1 come from, observe that a solution heading for a singularity and allowing $\limsup_{t \rightarrow t^*} (r_{\min}(t)) > d > 0$ admits a sequence $\{t_k\}$, $t_k \rightarrow t^*$, where all distances satisfy $r_{ij}(t_k) \geq d$. By bounding these distances away from zero, both the right-hand side of Eq. 1 and U are bounded above. Bounds on the \mathbf{v}_j velocity terms come from the U bound and the energy integral

$$(3) \quad T = \frac{1}{2} \sum_{j=1}^n m_j \mathbf{v}_j^2 = U + h,$$

(where h is a constant of integration). Thus, for each t_k value, the existence theorem ensures that the solution exists beyond t_k for an extended time that depends only on d and h . By choosing t_k so that $t^* - t_k$ is less than half this guaranteed value, we contradict the assumption that t^* is a singularity.

Theorem (Painlevé). *The n -body problem has a singularity at $t = t^*$ iff*

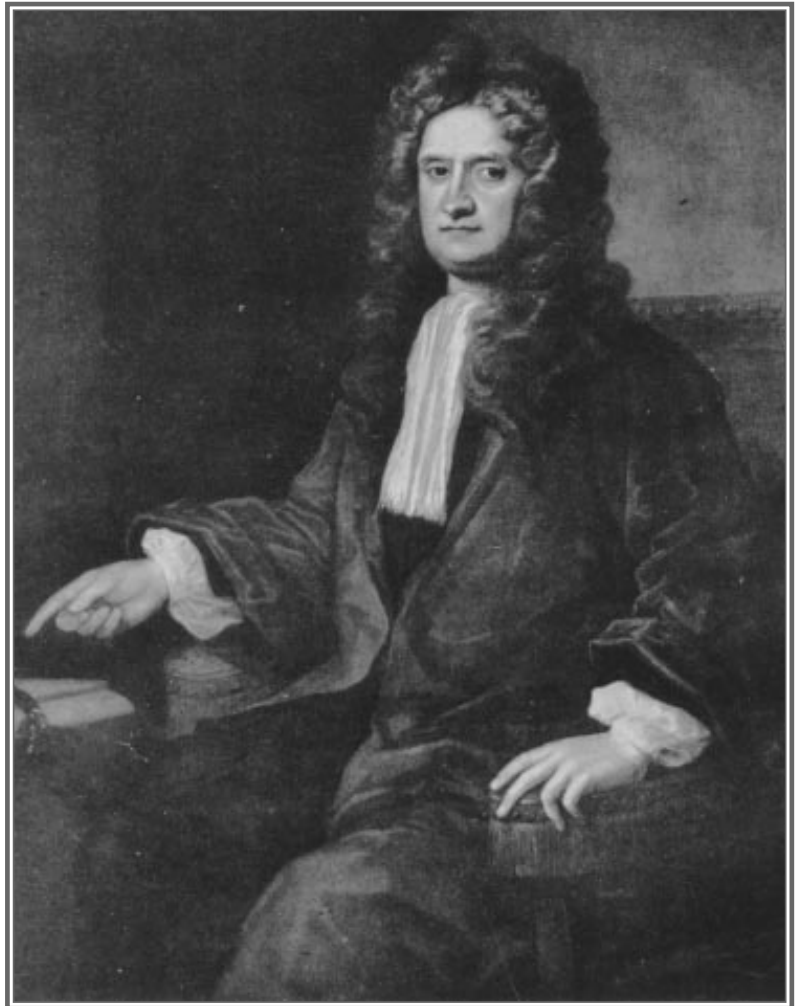
$$(4) \quad \mathbf{r}(t) \rightarrow \Delta \quad \text{as } t \rightarrow t^*.$$

Even though Painlevé tells us that a singularity requires $\mathbf{r} \rightarrow \Delta$, it remains unclear whether the particles must collide. After all, as indicated by Figure 1 (p. 541), the $r_{\min}(t) \rightarrow 0$ as $t \rightarrow t^*$ condition might be satisfied without any distance approaching zero. Instead, it still might be possible for a singularity to be generated by particles flirting with collisions without committing to do so. By a collision, we mean

Definition. *A singularity at time t^* is a collision if there is $\mathbf{q} \in \Delta$ so that $\mathbf{r}(t) \rightarrow \mathbf{q}$ as $t \rightarrow t^*$. Otherwise, the singularity is called a noncollision singularity.*

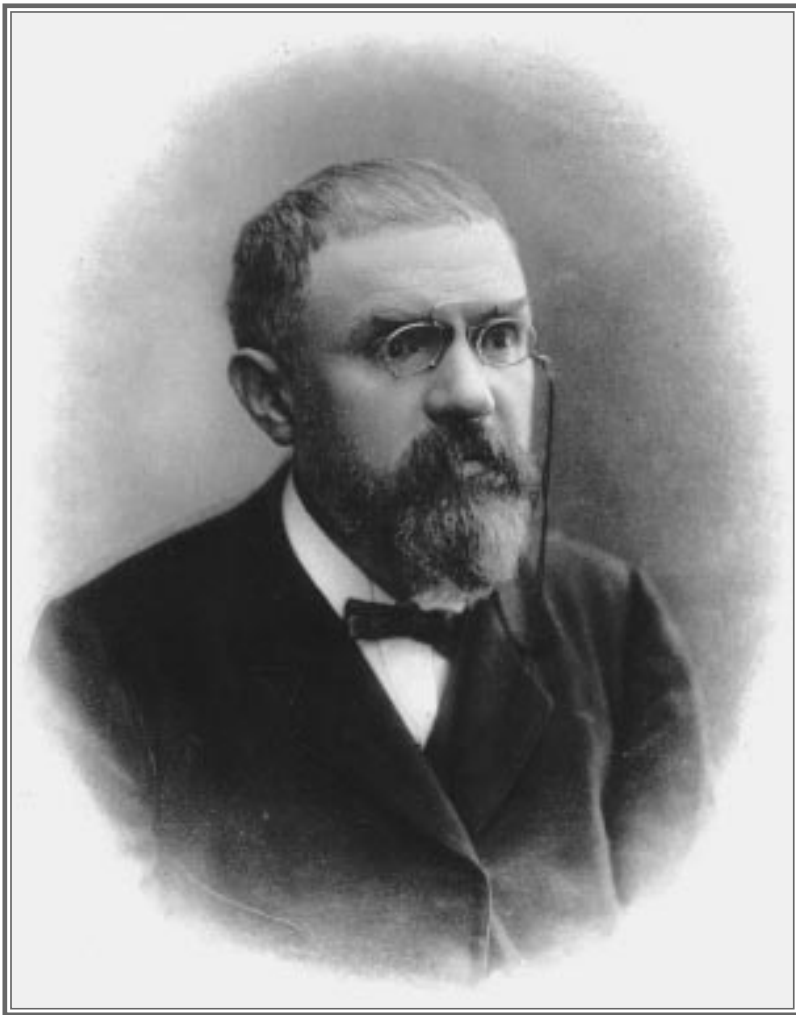
Using the triangle inequality, Painlevé proved that the three-body problem is free from the pathology depicted in Figure 1; i.e., for $n = 3$, all singularities are collisions. To describe why, we need to relate the maximum and minimum spacing between particles. Clearly, U^{-1} is a measure for $r_{\min}(t)$. With the center of mass at the origin, the maximum spacing between particles is measured by $I^{1/2}$ where $I = \frac{1}{2} \sum_{j=1}^n m_j \mathbf{r}_j^2$. It turns out (by differentiating $I(t)$ twice and using Eq. 3) that these measures are connected through the Lagrange-Jacobi equation [P1]

$$(5) \quad I'' = U + 2h.$$



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This relationship specifies, for instance, that whenever particles come close to each other (so U has a large value), this excites the acceleration of our measure for the radius of the universe (i.e., I'' becomes positive). An extreme example is a singularity where $r_{\min}(t) \rightarrow 0$, or $U \rightarrow \infty$. All we need from $I'' \rightarrow \infty$ (Eq. 5) is that $I''(t)$ eventually is positive, because this requires $I \rightarrow A$, $A \in [0, \infty]$, as $t \rightarrow t^*$. The $A = 0$ possibility, where $I \rightarrow 0$, clearly represents a collision as all particles collide at the center of mass. Otherwise $I \rightarrow A > 0$, which means that two legs of the triangle defined by the three particles are bounded away from zero. The accompanying $r_{\min}(t) \rightarrow 0$ condition requires the last triangle leg to shrink to zero. But, once r_{\min} becomes and remains sufficiently small, the triangle inequality prohibits different pairs of particles from trading the role of defining r_{\min} . As $r_{\min}(t)$ eventually is defined by a single pair of particles, the Figure 1 scenario cannot occur. It now is easy to show that all particles approach a limiting position.



Poincaré raised the problem a century ago.

After proving his results, Painlevé wondered whether noncollision singularities could exist for $n \geq 4$; namely, could $\mathbf{r}(t)$ approach Δ without approaching any point on this set? This is the question Xia resolved by showing that such solutions exist for $n \geq 5$.

Behavior and Likelihood of Noncollision Singularities

After Painlevé, the next major contribution occurred in 1908 when von Zeipel [VZ] discovered a stunning consequence of a noncollision singularity. His argument is based on the observation that the inverse square law imposes a negligible acceleration on particles when they are far apart. Consequently, over short time spans distant particles essentially move along a straight line with only miniscule velocity changes. Thus von Zeipel separated the analysis into how nearby particles interact and how clusters of neighboring particles separate from one another. By showing that this cluster argument contradicts the $I \rightarrow A < \infty$ condition, he¹ proved the surprising conclusion that

Theorem (von Zeipel). A noncollision singularity occurs at time t^* iff $I \rightarrow \infty$ as $t \rightarrow t^*$.

Namely, von Zeipel escalated the stakes by showing that if noncollision singularities exist, then Newton's law of motion would allow particles to separate infinitely far apart in finite time! How could this be? This bizarre requirement probably caused Painlevé's concern to become somewhat dormant for a half-century.

The singularity problem was resurrected in the late 1960s when Saari [S1] characterized the behavior of all collisions as part of his study with Pollard about the asymptotic behavior of the Newtonian n -body problem. These results were sharpened [PS1, PS2] to assert that all colliding particles tend toward each other like $(t^* - t)^{2/3}$. (This was previously known only for binary collisions (Sundman [Su]) and complete collapse orbits where $I \rightarrow 0$ (Wintner [W]).)

Why the $\frac{2}{3}$ exponent? Actually, the value reflects the choice of a force law because the exponent is $2/(p+1)$ for the inverse p force law, $p > 1$. (Newton's law is $p = 2$.) This is easily seen from the collinear equations $x'' = -(p-1)x^{-p}$. By multiplying both sides by x' and integrating, we obtain the energy integral $\frac{1}{2}(x')^2 = x^{1-p} + h$, or $\frac{1}{2}(x')^2 x^{p-1} = 1 + hx^{p-1}$. Thus the $x \rightarrow 0$ collision condition converts the energy integral to $x' x^{(p-1)/2} \sim -\sqrt{2}$ as $t \rightarrow t^*$. The conclusion (for the simple collinear problem) follows from integration.

Substituting this necessary and sufficient condition [PS1] for a collision,

$$(6) \quad U \sim A(t - t^*)^{-\frac{2}{3}} \quad \text{as } t \rightarrow t^*,$$

into Eq. 5 shows, after integration, that not only is I bounded, but so is I' . Clearly, to create a noncollision singularity, I' needs to be more actively excited by having $r_{\min}(t)$ approach zero much more rapidly [PS2]. But how fast could such a universe explode? By experimenting with Eq. 5 and $U(t)$ rates that allow $I \rightarrow \infty$, is it reasonable to wonder whether, say, $I \sim \ln((t^* - t)^{-1})$ as $t \rightarrow t^*$? The growth is faster; as shown in [S3], I goes to infinity more rapidly than a large class of similar functions.

Of importance to our tale is the highly oscillatory nature of a noncollision motion that was established for the argument of [S3]. It turns out that particles must approach other distant particles infinitely often and arbitrarily closely. The intuition is that a particle flying off to infinity by itself has nearly zero acceleration, so the ve-

¹ Chazy [Ch], Sperling [Sp], and Saari [S3] have proofs that clean up and extend portions of von Zeipel's presentation. Also see McGehee's expository paper [MG1].

locity remains essentially constant. As a constant velocity precludes any possibility of reaching infinity in finite time, the acceleration needs to be boosted, and this requires a close visit by another particle. For instance, consider the four-body problem as depicted in Figure 2. The $I(t) \rightarrow \infty$ condition forces some particle, say, m_1 , to satisfy $\limsup_{t \rightarrow t^*} (r_j(t)) = \infty$. If for a time period before t^* no particle comes within distance, say, 10^{-20} of m_1 , then r_1' is bounded. But, by integration, this contradicts the $\limsup(r_1(t)) = \infty$ assertion. Thus, for m_j to enjoy the $\limsup(r_j(t)) = \infty$ property, it must be that in any time interval (t, t^*) , m_j is approached arbitrarily closely by another particle. With a little extra work and applying this argument to the equations of motion for the center of mass of a binary, it turns out that if no particle visits this binary, then the binary and its center of mass remain bounded. Thus we obtain that if m_j has the $\limsup_{t \rightarrow t^*} (r_j(t)) = \infty$ property, then in any (t, t^*) interval m_j is visited by another particle. Similarly, a binary repeatedly satisfying the $r_{min} \rightarrow 0$ condition must also be visited arbitrarily often and closely by another particle.

Of course, because the center of mass is fixed, whenever a particle is far from the origin, so is another particle in an opposing direction. Consequently, at least two distant particles must be involved with other bodies. To keep $I \rightarrow \infty$, there always are two distant particles so other particles have to commute to them. With $n = 4$ where at least two particles are needed to define $r_{min}(t)$, the only way to realize these “visiting” conditions is with some combination of the scenarios where either two particles separate and a binary shuttles between them (top diagram of Figure 2) or a binary and a particle separate while the last particle shuttles between them (bottom diagram of Figure 2). Then, as each particle needs to be visited in any time interval before t^* , all of this has to happen infinitely often.

Recall that during the traversing process, the commuting particle(s) move(s), essentially, on a straight line which is carefully aimed to meet the target. As one might suspect, this action quickly forces the system to approach a fixed line in physical space. Similarly, the direction of most velocity terms also is dictated by this line. So, because a $n = 4$ noncollision singularity squeezes the motion down to approach a fixed line in phase space, we might expect the measure-preserving properties of the system to render $n = 4$ noncollision singularities as unlikely. This is the case; using this intuition and the method he developed [S2] earlier to prove Littlewood’s conjecture [L] that collisions of any kind and for all n are unlikely, Saari showed [S4] that four-body

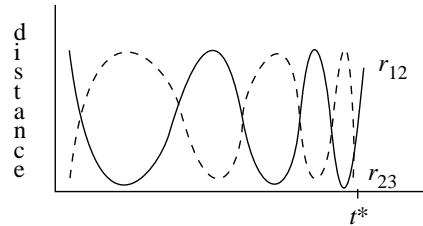


Figure 1: Oscillatory motion where minimum spacing goes to zero.

noncollision singularities constitute a set of Lebesgue measure zero.

Combining the [S2] and [S4] results, we have then that singularities are unlikely for $n \leq 4$; most orbits exist for all time. It is reasonable to expect the same conclusion to hold for all $n \geq 5$. To prove such an assertion, because collisions are unlikely [S2], it remains to show that noncollision singularities are in a set of Lebesgue measure zero. Modifications of the proof of [S4] show that this is true for those noncollision orbits where the particles eventually line up along a line (as in Xia’s construction). In fact, it appears (but has yet to be shown) that the [S4] approach and conclusion extend to all noncollision singularities. This is because the required “visiting” behavior of such an orbit forces the particles to rapidly approach a lower-dimensional hyperplane in phase space.

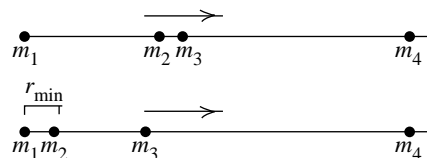


Figure 2: Two choices for shuttling particles.

The Mather-McGehee Construction

Any sense of skepticism concerning the existence of noncollision singularities vanished with a surprising 1975 paper by Mather and McGehee [MM]. They showed for the collinear four-body problem that binary collisions could accumulate in a way to eject particles to infinity in finite time. This did not resolve the Painlevé problem (because a noncollision singularity must be the first singularity of the system), but it strongly hinted that such motion exists. Indeed, Anosov [An] suggested that a four-body example of a noncollision singularity might exist in a neighborhood of the Mather-McGehee example; this ap-

proach has yet to be made successful. The Mather-McGehee construction was based on McGehee's earlier work concerning the behavior of near-triple collision orbits for the collinear three-body problem. These notions are outlined next.

From Sundman [Su] we know that a binary collision is an algebraic branch point where the dynamics mimic an elastic collision. Siegel [Se] showed, however, that triple collisions generally define a logarithmic singularity which prohibits the solution from being continued. An alternative goal, then, is to analyze what happens near a triple collision. To do so, McGehee [McG2] developed a form of "spherical coordinates" where the radius is defined by $r_{max}(t) = I^{1/2}$; with this scaling, the "angular coordinates" represent the $\frac{1}{r_{max}}(\mathbf{r}_1, \dots, \mathbf{r}_n)$ configuration formed by the particles. Important for this construction is that the force law is homogeneous. This allows the "radius" term to factor out of key equations and to be incorporated into the independent variable to rescale "time". The resulting system of "angular coordinates" describes changes in the configuration.

Mathematically, the new rescaled system is defined even for $r_{max}(t) = 0$; this is the zero point in Δ . This "blow-up" of the complete collapse singularity creates an invariant boundary manifold C called the "collision manifold". Because the augmented dynamical system smoothly extends to the boundary, the behavior of near triple collisions can be analyzed by using the simpler "gradient-like" flow that results on C . In

this manner, deep conclusions about the behavior of near triple-collision motion are forthcoming.

To describe these consequences, recall the high school physics experiment where a ball is dropped from a building. The more elastic the collision, the higher the ball rebounds. Near the ground, of course, the rebounding ball is moving rapidly upwards. To harness this speed, quickly drop a second, much smaller ball so it

hits the first one immediately after it starts its upward journey. Rather than hitting the static ground, the small ball is rebounding off of the rapidly moving first ball. Thus, the elastic collision converts the bigger, first ball into behaving like a baseball bat. With the second ball's extra momentum, picked up from the enhanced collision, the second ball bounces higher than it would have without the benefit of the collision.

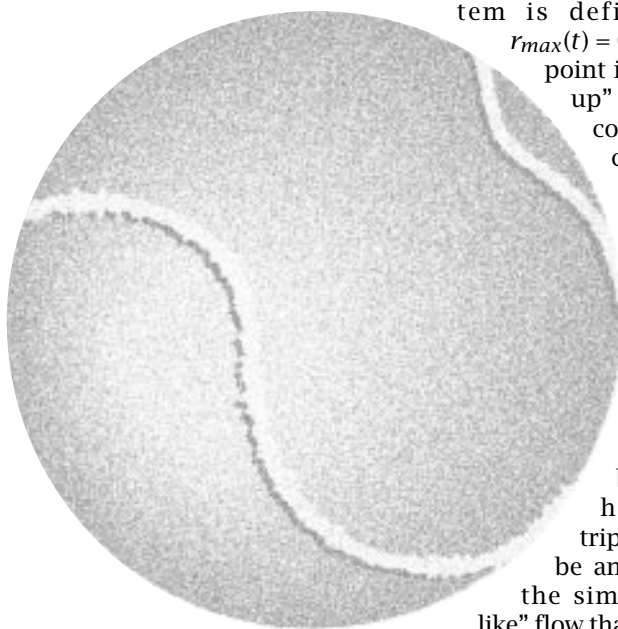
A similar effect describes a near triple collision for the Newtonian collinear three-body problem. If an initial condition leading to a complete collapse is slightly altered, one particle, m_3 , arrives a little late for the triple collision. The first colliding pair, m_1, m_2 , forms an elastic collision where, from Eqs. 3,

6, the rebounding velocity is arbitrarily large when measured sufficiently close to the collision. Thus, a rebounding particle approaches a collision with the tardy m_3 with arbitrarily large momentum. Just as with the physics experiment, the new elastic collision should cause the late arriving m_3 to leave its collision with an arbitrarily high velocity—much larger than its entering speed. While the actual situation is more complicated (e.g., just as with the balls, we need to worry about the mass values; depending on these values, there could be a series of binary collisions before one particle is expelled; the choice of the expelled particle depends upon the required number of binary collisions and the timing relative to a triple collision, etc.), this description captures the spirit of near triple collisions.

This description of motion near a triple collision (for the collinear problem) suggests that m_3 (in the bottom part of Figure 2) is ejected from the m_1, m_2 binary with an arbitrarily high velocity. To keep m_3 from being expelled to infinity, we need an obstacle—a fourth body. So, if m_3 's velocity is sufficiently large, it catches up and has an elastic collision with m_4 . Should the mass of m_3 be sufficiently small, this collision forces m_3 to rebound back to m_1, m_2 , where, if it arrives in time to nearly form another triple collision, m_3 gets batted back again. With a correct timing argument (that is, with an ap-

To try it yourself . . .

Place a tennis ball on top of a basketball and drop the pair from chest height. The tennis ball rebound is pretty dramatic — enough so that it should be done outside.



appropriate symbolic dynamic proof), this scenario repeats itself infinitely often within a finite period of time. In this manner, Mather and McGehee showed there exists a Cantor set of initial conditions defining this behavior.

McGehee's coordinates have become standard to analyze dynamical behavior for orbits near total collapse for the three-body collinear problem (e.g., see [McG2]), the isosceles three-body problem where the three particles form an isosceles triangle for all time (e.g., see Devaney [D1, D2], Moekel [M1, M2], Simó [Si]), and the anisotropic Kepler problem [D2]. In this manner, a wide selection of surprising "chaotic" behavior for three-body problems has emerged. In a related but slightly different direction, we [SX] used these coordinates to establish the existence of new kinds of orbits where the more surprising one is the "super-hyperbolic motion" discussed earlier by Pollard [P2] and then by Marchal and Saari [MS] as part of their description of how all n -body systems evolve. The concern was whether there is an upper bound for the expansion of n -body universes. Namely, is there a $f(t)$ which eventually bounds all solutions as $t \rightarrow \infty$? With special relativity, for instance, all velocities are bounded by the speed of light, so $f(t) = ct$. But Newton's universe fails to respect Einstein's formulation; once $n \geq 4$, no such $f(t)$ exists for Newtonian n -body systems! Instead, we showed that for any $f(t)$, there exist initial conditions for the four-body problem whereby $r_{max}(t)/f(t) \rightarrow \infty$ as $t \rightarrow \infty$. By choosing, for instance,

$$f(t) = \exp(\exp(\exp(\dots(\exp(t)\dots))),$$

it becomes clear that n -body systems can expand in ways that are distinctly counterintuitive.

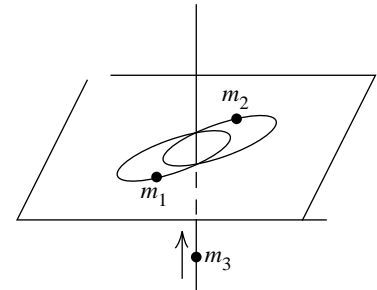
Our proof required "slowing down" the Mather-McGehee motion so that, instead of being quickly over, it lasts forever. Intuition on how this is done comes from the two-ball experiment; if that second ball is not dropped quickly enough, it hits the first one only after the rebounding momentum has decreased. Similarly, we needed to introduce a technique to capture the dynamical consequences where, rather than the particles interacting arbitrarily close to a triple collision, it happens "sufficiently late" so that the battered m_3 is not kicked out too harshly. Our technical argument creating this delay exploits the complicated manifold structure of the set of initial conditions leading to a triple collision.

Xia's Construction

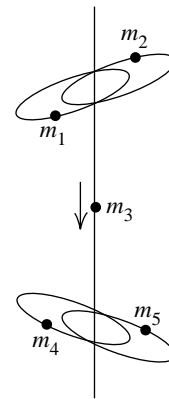
This brief history of Painlevé's problem introduces what is needed to design a noncollision singularity. First, the particles must rapidly shut-

tle among each other infinitely often causing arbitrarily close approaches. The velocity needed to allow these infinitely frequent visits comes from near multiple collisions. Herein lies part of the mathematical difficulty; to be a *noncollision* singularity, this "near multiple collision" analysis must be done without the benefit of actual collisions. But by precluding collisions, we leave the comfortable setting of the collinear problem, because it always requires bodies to bang into others. Once the directional constraints built into the collinear setting are dropped, we need to "aim" the commuting particle to direct it almost exactly where the target particles will arrive. (This is because the velocity of the visiting particle is essentially fixed until it gets arbitrarily close to the new host.) The complexity of the problem, then, involves extending all of the earlier theories to a higher-dimensional setting and then to connect them so that the required behavior arises. This is what Xia did.

To understand Xia's construction, start with a symmetry solution of the three-body problem where the motion of two equal masses, m_1, m_2 , always is parallel to the x - y plane and m_3 is restricted to the z -axis. (See Figure 3a. It is easy to show that such motion exists.) Now, should m_1, m_2 have circular orbits, the force of attraction they impose on m_3 (determined by the $r_{13} = r_{23}$ distance) is based on how far m_3 is from their plane. However, should the m_1, m_2 motion be highly elliptical, then their force on m_3 depends not only on how far m_3 is from the plane of motion but also on how close m_1, m_2 are to each other. Consider, for instance, an extreme case where the binary is so highly elliptical that it approximates a straight-line motion where the binaries approach arbitrarily close to one another, but then they separate to a comfortable distance apart.



3a Three-body encounter.



3b Xia's construction.

Figure 3: The five-body construction.



Painlevé, center with hat, wondered whether noncollision singularities could exist for $n \geq 4$.

Suppose, coming from below, m_3 passes through and is just slightly above the plane when the binary has its closest approach. If the particles are close enough to one another (and with the right choice of masses), the binary imposes an extremely powerful downward pull on m_3 . In fact, this attracting force can be made as strong as desired by adjusting the separating distances among the particles. Consequently, m_3 can be propelled downwards with an arbitrarily high velocity just when m_1, m_2 start separating. With this propitious timing, the separating binary loses any braking effect on m_3 , allowing m_3 to be launched rapidly downwards along the z -axis.

To prevent m_3 from being expelled to infinity, we need an obstacle, but it cannot be a fourth particle along the z -axis, as this would cause a collision. So replicate the above scenario by placing a second highly eccentric binary orbit of m_4, m_5 further down and orthogonal to the z -axis. (See Figure 3b.) With almost perfect timing—where the m_4, m_5 binary reaches a sufficiently close approach just after the commuting m_3 passes through their plane—the resulting

high force they impose on the commuting m_3 breaks m_3 's downward motion and thrusts it back upwards with an arbitrarily high velocity. Notice, by exploiting symmetry, the m_3 "aiming" problem is solved.

Xia's proof shows that this scenario can be repeated infinitely often in a finite time. Much like in a standard Cantor set construction, where at each stage a new "middle third" is removed, he develops a winnowing process. In other words, a set of initial conditions, which roughly assumes the shape of a wedge, is determined where the solutions perform as desired for at least one pass of the three particles. Some solutions from initial conditions in this wedge allow m_3 to interact in the indicated manner with the other binary, and some do not. Those that do not behave in the desired manner are dropped. (In particular, all orbits where m_3 fails to satisfy the careful timing requirement with the next binary are eliminated.) This process is continued. What remains in the limit then is a Cantor set of the initial conditions allowing this behavior to occur infinitely often.

To develop a flavor for how the "wedges" of initial conditions are found, notice that, in the limit, m_3 has to move infinitely fast from m_1, m_2 to m_4, m_5 ; this happens only when m_3 starts arbitrarily close to m_1 and m_2 while m_4, m_5 already are close together. Consequently, the limiting configuration is a m_1, m_2, m_3 triple collision with a simultaneous binary collision of m_4, m_5 . The idea is to exploit the stable and unstable manifold structure of this multiple collision in a way to choose sets of initial conditions with the correct behavior, at least for a while, while avoiding collisions. One way to prevent collisions is to endow each binary with a nonzero angular momentum c where the sign of c indicates whether the binary rotates in a clockwise or a counterclockwise manner. As the magnitude of c determines how close the particles can approach, to allow the necessary arbitrarily close approaches, each c must tend to zero as $t \rightarrow t^*$. To analyze these rotating interactions, the earlier collision manifold C (which is two-dimensional and does not involve rotation) needs to be extended a dimension to incorporate the c value. A true three-body problem has c as a constant of motion, so the analysis requires introducing a related variable, u , to capture the direction and speed of rotation for the binary.

As a way to introduce the next step, start with the simple system $x' = -x, y' = y$ where a solution on the x -axis—the stable manifold—gets sucked into the origin, while one on the y -axis—the unstable manifold—rapidly moves off on either the positive or negative y -axis. (See

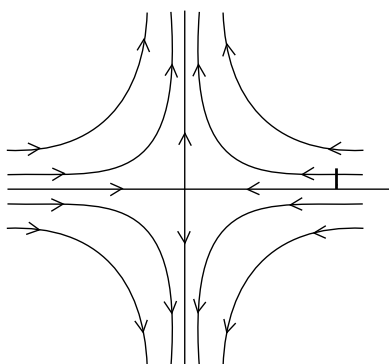


Figure 4: Behavior of the simple system.

Figure 4.) Other solutions combine this behavior; e.g., a solution starting near the x -axis stays near this axis as it moves toward the origin until it is sufficiently close to 0. Here the repelling effect in the y direction begins to dominate, so the solution begins to mimic and approach the motion on the y -axis. Notice that we can control which behavior occurs; to ensure, for instance, that the solutions eventually move near the positive (rather than negative) y -axis, just select an appropriate set of initial conditions where $y > 0$. (This is the block in Figure 4.) The near collision analysis is a higher-dimensional, more complicated version of this phenomenon, where the “wedges” correspond to the $y > 0$ selection of initial conditions.

Start with the fact that a triple collision defines an equilibrium point $x^* \in C$ with hyperbolic structure. (So, x^* replaces the origin of the simple system.) If we let Σ denote the set of initial conditions terminating in a triple collision, then Σ defines a stable manifold for $x^* \in C$. (Σ represents the x -axis in the model problem.) Using the fact x^* is hyperbolic and the inclination lemma (e.g., see Robinson [R], p. 200), we have that an orbit starting close to Σ will remain close until the orbit approaches C ; then it starts following the unstable manifold of x^* . (So the unstable manifold is a higher-dimensional version of the y -axis.) Namely, after barely missing a triple collision, the motion starts mimicking a C orbit. The subsequent behavior then is governed by the orbit structure on C near x^* .

The interesting part of this structure comes from the unstable manifold of x^* . One unstable dimension in C determines whether m_3 is propelled upwards or downwards after the three-body interaction, while a new one comes from the u variable; it represents the direction of rotation of the binary after the close interaction. By choosing the desired behavior in these unstable directions in C and using the resulting

wedge as a target, near Σ a wedge of initial conditions can be determined where the solutions will be governed by the desired C behavior for this pass of the three particles. (In the model system, this wedge choice is similar to choosing the block in Figure 4 so that these solutions follow the positive rather than the negative y -axis.) Those solutions which pass through this near triple collision and allow m_3 to reach the next binary just when all of the necessary ingredients to repeat this story are available define sub-wedges. What arises is the indicated winnowing effect.

The resulting Cantor set of initial conditions allows $r_{max}(t)$ to approach infinity in finite time without prior collisions. In this manner, the question raised by Painlevé a century ago finally is solved. Moreover, the construction makes full use of the several different contributions made by the many researchers in this fascinating area of mathematics.

While we now know that noncollision singularities exist, several mysteries remain. Any partial listing has to include whether $n = 5$ is the cutoff for this surprising behavior, or whether the four-body problem can propel particles to infinity in a finite time. Can, for instance, Anosov’s suggestion be carried out? Are there planar examples with small n values? As indicated, the mass values play an important role in the proof (for reasons similar to why the size of the balls in the physics experiment are important). Are there mass choices where noncollision singularities cannot occur? Initial conditions leading to a Xia type example are in a set of Lebesgue measure zero; are all noncollision singularities unlikely? As described, constructing examples of unbounded motion involves carefully cultivating near-collision behavior. This suggests that if $C\mathcal{O}$ represents the set of initial conditions leading to a collision of any kind, then the closure of $C\mathcal{O}$ agrees with the set of initial conditions causing any kind of singularity (including the motion described in [SX]). Is this true? (One direction in the obvious set containment argument is trivial.) More specifically, mimicking Painlevé’s concern, what is the nature of orbits generated by initial conditions in the closure of $C\mathcal{O}$? In other words, as always, the Newtonian n -body problem serves as a source of intriguing mathematical problems.

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