Constructing π Via Origami

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May 5, 2007

Abstract

We present an argument for the constructibility of the transcendental number π by paper folding, provided that curved creases are allowed.

1 Introduction

Determining which numbers are constructible by paper folding has been the heart of research in origami geometry. H. Abe's angle trisection method [1, 4] demonstrates that origami is a more powerful construction method than straightedge and compass, and as far back as the 1930s it was known, as shown by Beloch [2], that paper folding can solve general cubic equations. In 2003 R. Lang proved that the standard set of "origami axioms" people had been using for such constructions could not be made more powerful [5], which proved that quintic equations were not solvable by traditional origami. However, by "traditional" we mean only using straight creases making one fold at a time. And so, in due course R. Lang came up with an angle quintisection method [6] that employed the simultaneous creation of two creases, thus demonstrating that by using *multifolds* (aligning parts of the paper so as to produce more than one crease line simultaneously) the solutions of equations of higher degree were constructible. It now seems very plausible that, in theory, if an arbitrary number of simultaneous creases are allowed in origami constructions, then polynomials of arbitrary degree are solvable. The "in theory" part of that last sentence must be emphasized, however. In practice it seems that making more than two simultaneous creases is extremely difficult, if not impossible, even under the best of conditions.

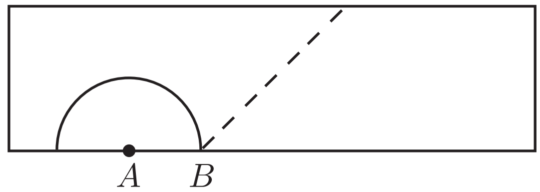

Figure 1: Crease pattern for constructing π from a strip of paper.

Despite all this activity, two techniques in paper folding which have yet to be employed in origami geometric construction explorations are non-flat folding and curved creases. (However, curved creases are non-flat by their very nature, so perhaps these two techniques should be viewed as in the same family.) In this note, we will demonstrate how the use of curved creases can allow the construction of non-algebraic numbers. In particular, we will construct π.

Please be advised, however, that the creation of curved creases is not nearly well-defined in the origami community, let alone mathematically, and thus are quite controversial to use in origami constructions.

2 Constructing π

Figure 1 shows a crease pattern for one way to construct π from a strip of paper. First valley-fold a semicircle centered at a point A on the strip's long side. (In the next section we'll discuss how to fold such a circle.) Then make a valley crease from one end of the semicircle (point B) making an angle of 45° or less from the side of the paper.

Fold the 45° crease flat and then fold the semicircle, which will make the paper form part of a cone. See Figure 2. The trick is then to slide the raw edge of the paper, brought into position by the 45° crease, into the semicircle fold. Mark with a crease where this raw edge meets the other end of the semicircle; this is the point C in Figure 2. Unfolding the paper, we see that the length of the line segment BC equals the perimeter of the semicircle. If the radius AB of the semicircle equals 1, this means that $BC = \pi$.

Performing this origami maneuver is not easy. Even if you photocopy Figure 1 so as to make the semicircle crease more easily, it is still quite tricky to maneuver the raw edge of the paper into the semicircle crease. It should literally form part of a cylinder which will intersect the cone made by the

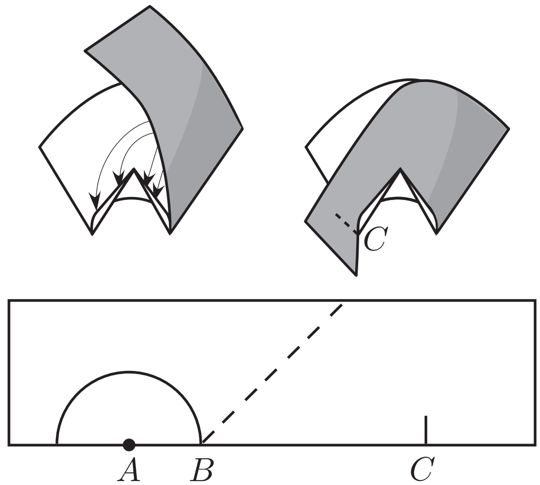

Figure 2: Making the fold and marking the point C. $BC/AB = \pi$.

rest of the paper along the semicircle. With practice, however, it can be done quite accurately. I managed to fold one starting with a semicircle of radius 25 mm and creating a segment BC of length 78.5 mm. If we assume that AB is our unit length, this means the relative length of BC is $78.5/25 = 3.14$, which is π to 2 decimals. For origami, that's pretty good accuracy.

3 Can we actually fold circles?

The above construction of π calls into question whether or not we should allow curved creases in origami constructions. The so-called Huzita-Hatori "axioms" for straight crease constructions are very exact and easily reproduced. They are determined by aligning points and lines to each other and then flattening the paper to produce a crease. Curved creases are not so easily produced. The major problem is that a curved crease cannot be folded flat, so they can't be produced by lining up points and lines.

In practice, most people seem to use tools of some sort to produce curved creases, especially when they want them to be mathematically precise. For example, a compass can be employed to carve a circular arc into the paper, making a crease. Or, if we begin with a piece of paper in the shape of a circle then we can use the boundary as a template for making circular arc creases. Former imagirian Philip Chapman-Bell's blog `http://origami.`

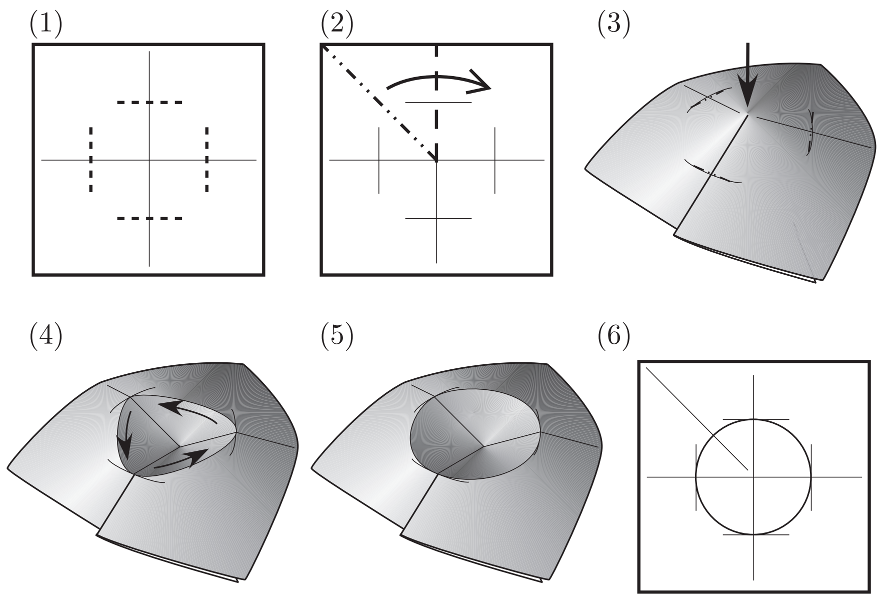

Figure 3: One way to make a circle crease without tools. Step (2) is meant to fold the paper into a cone shape. Step (3) is a non-flat, closed sink, whose crease we smooth into a circle in step (4).

`oschene.com/` has many examples of such folds (see [3]).[1]

Origami artists like Eric Joisel employ curved creases in their works, so there is no doubt that curved creases can be employed to artistic effect. But if we revert to the classic rules using a square piece of paper and only relying on our fingers to make folds, could we ever hope to make curved creases with any kind of reproducible accuracy? That is what we would need if we were to add curved creases to the origami geometric construction toolbox.

Creating even a vocabulary that begins to describe curved creases precisely sounds like a daunting task. But it does seem almost feasible that we could make circular creases from scratch. Figure 3 shows one way in which a crease in the shape of a circle could be made. The basic idea is to fold a

[1]Incidentally, numerous origamists in the past, such as Eric Kenneway and John Smith, have dismissed folding from circular paper as being silly because as soon as you make a fold you crease a straight edge to the boundary of the folded shape and thus start to loose the circle-ness. This argument fails to recognize how the presence of a circle can make some geometric objects, like a pentagon, easier to construct, not to mention using the circular edge to crease circular arc creases.

square sheet of paper into a cone, closed sink the cone, and then smooth the closed sink crease into a perfect circle. In practice, doing this feels a lot less crazy than it might at first appear. The smoothing process of the closed sink crease (step (4) in Figure 3) is done using the extrinsic, circular curvature of the folded cone. With practice I've been able to do this quite accurately. If relatively stiff paper is used, this process becomes easier because the cone's curvature is more stable, allowing you to literally carve the circle crease into the 3D cone.

There are likely other ways in which a circular crease could be created. I present the one in Figure 3 because it seems easiest way.

4 Conclusion

π is a transcendental number, which means that it is not the root of any polynomial with integer coefficients. Such numbers are not normally considered when exploring geometric constructions because the classic tools of such constructions, like a straightedge, compass, or an angle trisector, allow us to create only algebraic numbers (which are the root of some polynomial). Thus, the "game" in geometric construction research is to pick some construction tools to use and then try to figure out what the biggest field of algebraic numbers is that you can construct with them. Straightedge and compass can only produce numbers that are the roots of polynomials of degree 2 (quadratic polynomials, like $x^2 + 3x - 1$). Origami can construct roots of polynomials of degree 2, 3, and 4, and that's it if you allow only straight creases folded one-at-a-time. Alperin and Lang (in a paper that will likely be in the 4OSME proceedings) proved that with multifolds equations of degree 5 can be solved via origami, and it seems theoretically plausible, although practically infeasible, that higher-degree equations can be solved as well using multifolds.

But transcendental numbers aren't even on the radar in these construction techniques, which makes it somewhat surprising that curved creases would suddenly allow π to be constructed. But the method shown in Section 2 is not by any means ingenious; it merely uses the fact that π is, by definition, the ratio of a circle's circumference to its diameter. Thus if you can, by means of origami, create a circular crease and measure part of its circumference with a straight crease, you'll have constructed a multiple of π. In fact, the "origami move" we use in our construction of π could be formalized as this:

Given a line L and a curve C, we can align L onto C or vice-versa.

This, in a sense, is similar to the move, "Given two lines we can fold one line onto the other," in the Huzita-Hatori axioms, except that we're saying nothing about making a crease to place L onto C, just that we can align them onto each other. In practice, this is very difficult to do generally. The rest of the paper will usually get in the way and may even make the process impossible. But the operation of placing L onto C is no more taxing than some of the multifold operations that Alperin and Lang propose in their work.

I personally think that all this indicates how extremely flexible and diverse paper folding is as a method of geometric construction. Since curved creases are so difficult to do, most people would probably reject them outright as a valid construction tool. Is it not interesting, then, to see that the power of origami geometric constructions may be determined by the skill of the folder?

References

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