SOLUTION OF A LARGE-SCALE TRAVELING-SALESMAN PROBLEM*

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It is shown that a certain tour of 49 cities, one in each of the 48 states and Washington, D C , has the shortest road distance

THE TRAVELING-SALESMAN PROBLEM might be described as follows: Find the shortest route (tour) for a salesman starting from a given city, visiting each of a specified group of cities, and then returning to the original point of departure. More generally, given an n by n symmetric matrix $D=(d_{IJ})$, where d_{IJ} represents the 'distance' from I to J, arrange the points in a cyclic order in such a way that the sum of the d_{IJ} between consecutive points is minimal. Since there are only a finite number of possibilities (at most $\frac{1}{2}(n-1)!$) to consider, the problem is to devise a method of picking out the optimal arrangement which is reasonably efficient for fairly large values of n. Although algorithms have been devised for problems of similar nature, e.g., the optimal assignment problem,[3,7,8] little is known about the traveling-salesman problem. We do not claim that this note alters the situation very much; what we shall do is outline a way of approaching the problem that sometimes, at least, enables one to find an optimal path and prove it so. In particular, it will be shown that a certain arrangement of 49 cities, one in each of the 48 states and Washington, D. C., is best, the d_{IJ} used representing road distances as taken from an atlas.

* HISTORICAL NOTE· The origin of this problem is somewhat obscure. It appears to have been discussed informally among mathematicians at mathematics meetings for many years. Surprisingly little in the way of results has appeared in the mathematical literature.[10] It may be that the minimal-distance tour problem was stimulated by the so-called Hamiltonian game[1] which is concerned with finding the number of different tours possible over a specified network The latter problem is cited by some as the origin of group theory and has some connections with the famous Four-Color Conjecture [9] Merrill Flood (Columbia University) should be credited with stimulating interest in the traveling-salesman problem in many quarters. As early as 1937, he tried to obtain near optimal solutions in reference to routing of school buses. Both Flood and A W. Tucker (Princeton University) recall that they heard about the problem first in a seminar talk by Hassler Whitney at Princeton in 1934 (although Whitney, recently queried, does not seem to recall the problem) The relations between the traveling-salesman problem and the transportation problem of linear programming appear to have been first explored by M. Flood, J. Robinson, T. C. Koopmans, M. Beckmann, and later by I. Heller and H. Kuhn.[4,5,6]

In order to try the method on a large problem, the following set of 49 cities, one in each state and the District of Columbia, was selected:

1. Manchester, N H.
2. Montpelier, Vt.
3. Detroit, Mich.
4. Cleveland, Ohio
5 Charleston, W Va
6. Louisville, Ky.
7. Indianapolis, Ind.
8. Chicago, Ill
9. Milwaukee, Wis
10 Minneapolis, Minn.
11. Pierre, S. D.
12. Bismarck, N. D.
13 Helena, Mont.
14 Seattle, Wash.
15. Portland, Ore.
16. Boise, Idaho
17. Salt Lake City, Utah
18 Carson City, Nev.
19 Los Angeles, Calif.
20 Phoenix, Ariz
21. Santa Fe, N M.
22 Denver, Colo.
23. Cheyenne, Wyo.
24 Omaha, Neb
25. Des Moines, Iowa
26. Kansas City, Mo.
27. Topeka, Kans.
28 Oklahoma City, Okla
29. Dallas, Tex.
30. Little Rock, Ark.
31. Memphis, Tenn.
32 Jackson, Miss.
33 New Orleans, La.
34 Birmingham, Ala.
35. Atlanta, Ga
36. Jacksonville, Fla.
37. Columbia, S C.
38 Raleigh, N C.
39. Richmond, Va
40. Washington, D. C.
41. Boston, Mass
42. Portland, Me.
A. Baltimore, Md.
B Wilmington, Del.
C Philadelphia, Penn.
D. Newark, N. J
E New York, N. Y
F Hartford, Conn.
G. Providence, R. I.

The reason for picking this particular set was that most of the road distances between them were easy to get from an atlas. The triangular table of distances between these cities (Table I) is part of the original one prepared by Bernice Brown of The Rand Corporation. It gives $d_{IJ} = \frac{1}{17}(d'_{IJ} - 11)$,* $(I,J = 1, 2, \cdots, 42)$, where d'_{IJ} is the road distance in miles between I and J. The d_{IJ} have been rounded to the nearest integer. Certainly such a linear transformation does not alter the ordering of the tour lengths, although, of course, rounding could cause a tour that was not optimal in terms of the original mileage to become optimal in terms of the adjusted units used in this paper.

We will show that the tour (see Fig. 16) through the cities 1, 2, $\cdots$, 42 in this order is minimal for this subset of 42 cities. Moreover, since in driving from city 40 (Washington, D. C.) to city 41 (Boston, Massachusetts) by the shortest road distance one goes through A, B, $\cdots$, G, successively, it follows that the tour through 49 cities 1, 2, $\cdots$, 40, A, B, $\cdots$, G, 41, 42 in that order is also optimal.

PRELIMINARY NOTIONS

Whenever the road from I to J (in that order) is traveled, the value $x'_{IJ} = 1$ is entered into the I,J element of a matrix; otherwise $x'_{IJ} = 0$ is entered. A (directed) tour through n cities can now be thought of as a permutation matrix of order n which represents an n-cycle (we assume

* This particular transformation was chosen to make the d_{IJ} of the original table less than 256 which would permit compact storage of the distance table in binary representation; however, no use was made of this.

TABLE I

Road Distances between Cities in Adjusted Units

The figures in the table are mileages between the two specified numbered cities, less 11, divided by 17, and rounded to the nearest integer.

	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16	17	18	19	20	21	22	23	24	25	26	27	28	29	30	31	32	33	34	35	36	37	38	39	40	41
2		8																																							
3		39	45																																						
4		37	47	9																																					
5		50	49	21	15																																				
6		61	62	21	20	17																																			
7		58	60	16	17	18	6																																		
8		59	60	15	20	26	17	10																																	
9		62	66	20	25	31	22	15	5																																
0		81	81	40	44	50	41	35	24	20																															
1		103	107	62	67	72	63	57	46	41	23																														
2		108	117	66	71	77	68	61	51	46	26	11																													
3		145	149	104	108	114	106	99	88	84	63	49	40																												
4		181	185	140	144	150	142	135	124	120	99	85	76	35																											
5		187	191	146	150	156	142	137	130	125	105	90	81	41	10																										
6		161	170	120	124	130	115	110	104	105	90	72	64	34	31	27																									
7		142	146	101	104	111	97	91	85	86	75	51	59	29	53	48	21																								
8		174	178	133	138	143	129	123	117	118	107	83	84	54	46	35	26	31																							
9		185	186	142	143	140	130	126	124	128	118	93	101	72	69	58	58	43	26																						
0		164	165	120	123	124	106	106	105	110	104	86	97	71	93	82	62	42	45	22																					
1		137	139	94	96	94	80	78	77	84	77	56	64	65	90	87	58	36	68	50	30																				
2		117	122	77	80	83	68	62	60	61	50	34	42	49	82	77	60	30	62	70	49	21																			
3		114	118	73	78	84	69	63	57	59	48	28	36	43	77	72	45	27	59	69	55	27	5																		
4		85	89	44	48	53	41	34	28	29	22	23	35	69	105	102	74	56	88	99	81	54	32	29																	
5		77	80	36	40	46	34	27	19	21	14	29	40	77	114	111	84	64	96	107	87	60	40	37	8																
6		87	89	44	46	46	30	28	29	32	27	36	47	78	116	112	84	66	98	95	75	47	36	39	12	11															
7		91	93	48	50	48	34	32	33	36	30	34	45	77	115	110	83	63	97	91	72	44	32	36	9	15	3														
8		105	106	62	63	64	47	46	49	54	48	46	59	85	119	115	88	66	98	79	59	31	36	42	28	33	21	20													
9		111	113	69	71	66	51	53	56	61	57	59	71	96	130	126	98	75	98	85	62	38	47	53	39	42	29	30	12												
0		91	92	50	51	46	30	34	38	43	49	60	71	103	141	136	109	90	115	99	81	53	61	62	36	34	24	28	20	20											
1		83	85	42	43	38	22	26	32	36	51	63	75	106	142	140	112	93	126	108	88	60	64	66	39	36	27	31	28	28	8										
2		89	91	55	55	50	34	39	44	49	63	76	87	120	155	150	123	100	123	109	86	62	71	78	52	49	39	44	35	24	15	12									
3		95	97	64	63	56	42	49	56	60	75	86	97	126	160	155	128	104	128	113	90	67	76	82	62	59	49	53	40	29	25	23	11								
4		74	81	44	43	35	23	30	39	44	62	78	89	121	159	155	127	108	136	124	101	75	79	81	54	50	42	46	43	39	23	14	14	21							
5		67	69	42	41	31	25	32	41	46	64	83	90	130	164	160	133	114	146	134	111	85	84	86	59	52	47	51	53	49	32	24	24	30	9						
6		74	76	61	60	42	44	51	60	66	83	102	110	147	185	179	155	133	159	146	122	98	105	107	79	71	66	70	70	60	48	40	36	33	25	18					
7		57	59	46	41	25	30	36	47	52	71	93	98	136	172	172	148	126	158	147	124	121	97	99	71	65	59	63	67	62	46	38	37	43	23	13	17				
8		45	46	41	34	20	34	38	48	53	73	96	99	137	176	178	151	131	163	159	135	108	102	103	73	67	64	69	75	72	54	46	49	54	34	24	29	12			
9		35	37	35	26	18	34	36	46	51	70	93	97	134	171	176	151	129	161	163	139	118	102	101	71	65	65	70	84	78	58	50	56	62	41	32	38	21	9		
0		29	33	30	21	18	35	33	40	45	65	87	91	117	166	171	144	125	157	156	139	113	95	97	67	60	62	67	79	82	62	53	59	66	45	38	45	27	15	6	
1		3	11	41	37	47	57	55	58	63	83	105	109	147	186	188	164	144	176	182	161	134	119	116	86	78	84	88	101	108	88	80	86	92	71	64	71	54	41	32	25	
2		5	12	55	41	53	64	61	61	66	84	111	113	150	186	192	166	147	180	188	167	140	124	119	90	87	90	94	107	114	77	86	92	98	80	74	77	60	48	38	32	6

$n>2$ throughout). For example, for $n=5$, the first matrix displayed below

$$\|x'_{IJ}\| = \begin{Vmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \end{Vmatrix}, \qquad \|x'_{IJ}\| = \begin{Vmatrix} 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{Vmatrix}.$$

is a tour since it represents visiting the cities in the 5-cycle (1 2 4 3 5), while the other matrix is not a tour since it represents visiting the cities by means of two sub-cycles (1 2) and (3 5 4).

It is clear that all representations for directed tours satisfy the relations

$$\sum_I x'_{IJ} = \sum_J x'_{IJ} = 1, \qquad x'_{II} = 0, \qquad x'_{IJ} \geq 0.$$

The matrix may be made into a triangular array by reflecting the numbers above the diagonal in the diagonal. The sum of corresponding elements is denoted by $x_{IJ} = x'_{IJ} + x'_{JI}$. Then the matrices above become

$$\|x_{IJ}\| = \begin{Vmatrix} \cdot & & & & \\ 1 & \cdot & & & \\ 0 & 0 & \cdot & & \\ 0 & 1 & 1 & & \\ 1 & 0 & 1 & 0 & \cdot \end{Vmatrix}, \qquad \|x_{IJ}\| = \begin{Vmatrix} \cdot & & & & \\ 2 & \cdot & & & \\ 0 & 0 & \cdot & & \\ 0 & 0 & 1 & \cdot & \\ 0 & 0 & 1 & 1 & \cdot \end{Vmatrix}.$$

Consequently, the sum along the Kth row plus the sum along the Kth column must now be 2. This may be written

$$\sum_{J<I=K} x_{IJ} + \sum_{I>J=K} x_{IJ} = 2, \qquad (K=1, \cdots, n;\ x_{IJ} \geq 0) \quad (1)$$

This device yields a representation for undirected tours and is the one used throughout this paper. It will be noted that the second array above does not represent a tour but nevertheless satisfies the relation (1).

For undirected tours, the symbol x_{IJ} will be treated identically with x_{JI} so that we may rewrite (1) as

$$\sum_{J=1}^{n} x_{IJ} = 2. \qquad (x_{IJ} \geq 0;\ I=1, 2, \cdots, n;\ I \neq J;\ x_{IJ} \equiv x_{JI}) \quad (2)$$

The problem is to find the minimum of the linear form

$$D(x) = \sum_{I>J} d_{IJ}\, x_{IJ}, \quad (3)$$

where the $x_{IJ}=0$ or 1 and the $x_{IJ}=1$ form a tour, and where the summation in (3) extends over all indices (I,J) such that $I>J$.

To make a linear programming problem out of this (see ref. 2) one

needs, as we have observed, a way to describe tours by more linear restraints than that given by (2). This is extremely difficult to do as illustrated by work of I. Heller[4] and H. Kuhn.[6] They point out that such relations always exist. However, there seems to be no simple way to characterize them and for moderate size n the number of such restraints appears to be astronomical. In spite of these difficulties, this paper will describe the techniques we have developed which have been successful in solving all the problems we have tried by this approach. A surprising empirical observation is the use of only a trivial number of the many possible restraints to solve any particular problem. To demonstrate the procedure, we shall attempt to use direct elementary proofs even though they were originally motivated in many places by linear programming procedures.

There are possibly four devices we have used which have greatly reduced the effort in obtaining solutions of the problems we have attempted.

First of all, we use undirected tours. This seems to simplify the characterization of the tours when n is small and certainly cuts down the amount of computation, even for large n. Secondly, and this is decisive, we do not try to characterize the tours by the complete set of linear restraints, but rather impose, in addition to (2), just enough linear conditions on the x_{IJ} to assure that the minimum of the linear form (3) is assumed by some tour. For the 49-city problem and also for all the smaller problems we have considered, such a procedure has been relatively easy to carry through by hand computation. This may be due in part to the fact that we use a simple symbolism which permits direct representation of the algebraic relationships and manipulations on a map of the cities. This third device speeds up the entire iterative process, makes it easy to follow, and sometimes suggests new linear restraints that are not likely to be obtained by less visual methods. Finally, once a tour has been obtained which is nearly optimal, a combinatorial approach, using the map and listing possible tours which have not yet been eliminated by the conditions imposed on the problem, may be advantageous. This list can be very much shorter than one would expect, due to the complex interlocking of the restraints. However, except for short discussion in the section below, "An Estimation Procedure," this method will not be described in detail although it has worked out well for all examples we have studied.

An important class of conditions that tours satisfy, which excludes many non-tour cases satisfying (2), are the 'loop conditions' These are linear inequality restraints that exclude subcycles or loops. Consider a non-tour solution to (2) which has a subtour of $n_1 < n$ cities; we note that the sum of the x_{IJ} for those links (I,J) in the subtour is n_1. Hence we can

eliminate this type of solution by imposing the condition that the sum of x_{IJ} over all links (I,J) connecting cities in the subset S of n_1 cities be less than n_1, i.e.,

$$\sum_{S} x_{IJ} \leq n_1 - 1 \tag{4}$$

where the summation extends over all (I,J) with I and J in the n_1 cities S. From (2) we note that two other conditions, each equivalent to (4), are

$$\sum_{\bar{S}} x_{IJ} \leq n - n_1 - 1, \tag{5}$$

where $\bar{S}$ means the summation extends over all (I,J) such that neither I nor J is in S, and

$$\sum_{S\bar{S}} x_{IJ} \geq 2, \tag{6}$$

where $S\bar{S}$ means that the summation extends over all (I,J) such that I is in S and J not in S.

There are, however, other more complicated types of restraints which sometimes must be added to (2) in addition to an assortment of loop conditions in order to exclude solutions involving fractional weights x_{IJ}. In the 49-city case we needed two such conditions. However, later when we tried the combinatorial approach, after imposing a few of the loop conditions, we found we could handle the 49-city problem without the use of the special restraints and this would have led to a shorter proof of optimality. In fact, we have yet to find an example which could not be handled by using only loop conditions and combinatorial arguments.

THE METHOD

The technique will be illustrated by a series of simple examples.

Example 1

First consider a five-city map forming a regular pentagon of unit length per side and with length $\frac{1}{2}(\sqrt{5}+1) \doteq 1.7$ on a diagonal (Fig. 1). Sup-

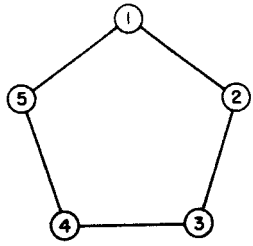


FIGURE 1

pose that the problem is to minimize (3) subject only to (2) Start with a tour which is conjectured to be optimal, obviously (1 2 3 4 5). In this case the values of x_{IJ}, denoted by $\bar{x}_{IJ}$, are $\bar{x}_{12}=\bar{x}_{23}=\bar{x}_{34}=\bar{x}_{45}=\bar{x}_{51}=1$ and all other $\bar{x}_{IJ}=0$. The variables x_{IJ} corresponding to links on the tour are called 'basic variables.' The length of the tour given by the linear form (3) for $x=\bar{x}$ is $D(\bar{x})=5$. There are five equations in (2). Multiply each by a parameter π_I to be determined, and then subtract the sum from (3). Thus, we are led to

$$D(x)=\sum_{I>J} d_{IJ}x_{IJ}-\sum_{I=1}^{n}\pi_I\left(\sum_{J=1}^{n}x_{IJ}-2\right) \qquad (x_{IJ}\equiv x_{JI};\ I\neq J)$$

$$=-\sum_{I>J}(\pi_I+\pi_J-d_{IJ})x_{IJ}+2\sum_{1}^{n}\pi_I.$$

Denote the coefficients of x_{IJ} by δ_{IJ} so that

$$D(x)=-\sum_{I>J}\delta_{IJ}x_{IJ}+2\sum_{1}^{n}\pi_I. \qquad (\delta_{IJ}=\pi_I+\pi_J-d_{IJ}) \quad (7)$$

Now determine the five π_I values so that δ_{IJ} corresponding to basic variables vanish:

$$\delta_{IJ}=0, \qquad (\text{for } \bar{x}_{IJ}=1) \quad (8)$$

i.e., if the link (I,J) is on the tour in question. Note that to solve for the π_I we have five linear equations in five unknowns.

If now we set $x_{IJ}=\bar{x}_{IJ}$ in (7), then $\bar{x}_{IJ}\delta_{IJ}=0$ for *all* (I,J) and

$$D(\bar{x})=2\sum_{1}^{n}\pi_I=5. \qquad (9)$$

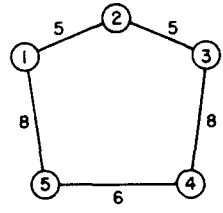
Subtracting (9) from (7) we have finally

$$D(x)-D(\bar{x})=-\sum_{I>J}\delta_{IJ}x_{IJ}. \qquad (10)$$

For the regular pentagon $\pi_I=\frac{1}{2}$ for $I=1, 2, 3, 4, 5$ solves (8), and so $\delta_{IJ}=\frac{1}{2}(1-\sqrt{5})<0$ on a diagonal, i.e., $\delta_{IJ}\leq 0$ for every (I,J). Thus, the right side of (10) is always nonnegative or $D(x)\geq D(\bar{x})$ for all x satisfying (2), and in particular all other tours are longer than the tour represented by $\bar{x}$.

Example 2

Next, take another five-city problem whose map is not a regular pentagon (Fig. 2). We start with the tour (1 2 3 4 5) of length $D(\bar{x})=32$ where the basic variables take on the values $\bar{x}_{12}=\bar{x}_{23}=\bar{x}_{34}=\bar{x}_{45}=\bar{x}_{51}=1$ and all other $\bar{x}_{IJ}=0$. Repeat the steps in the previous problem leading to (10)



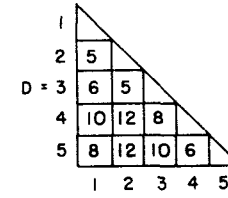


FIGURE 2

where, as before, calculate the π_I by setting $\delta_{IJ}=0$ for δ_{IJ} corresponding to basic variables x_{IJ}. The five equations that the π_I must satisfy are

$$\pi_1+\pi_2=5, \qquad \pi_2+\pi_3=5, \qquad \pi_3+\pi_4=8, \qquad \pi_4+\pi_5=6, \qquad \pi_5+\pi_1=8$$

By alternately subtracting and adding these equations one obtains.

$$2\pi_1=d_{12}-d_{23}+d_{34}-d_{45}+d_{51}=5-5+8-6+8=10,$$

or
$$\pi_1=5, \qquad \pi_2=0, \qquad \pi_3=5, \qquad \pi_4=3, \qquad \pi_5=3.$$

The factors π_I which multiply equations (2) to form (10) are called 'potentials.'* There is one such potential associated with each city I, and these are readily computed by working directly on the map of the cities (see Fig. 3).

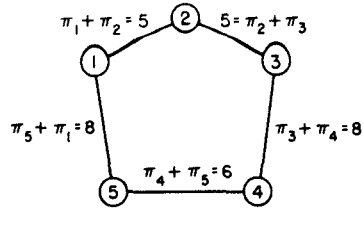


FIGURE 3

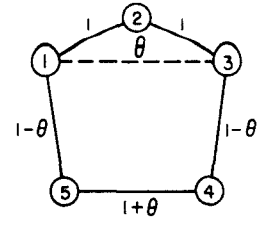


FIGURE 4

To form other δ_{IJ}, add the π_I and π_J of city I and city J and subtract off the distance d_{IJ} between them. In this case we note that except for $\delta_{31}=5+5-6=+4$, all the other δ_{IJ} are ≤ 0.

We see from (10) that if x_{31} were to take on a positive value, $x_{31}=\theta$, the other nonbasic variables remaining at zero, this may lead to a better solution. We let θ be the largest value consistent with (2). Thus, the weights x_{IJ} must add up to 2 on links from each city and no weight is negative. However, in setting $x_{31}=\theta$ we adjust only the basic set of variables, leaving all other nonbasic variables at zero value. This is

* The term potential is used by T. C. Koopmans in an analogous connection for the transportation problem.[5]

worked out on the map shown in Fig. 4. Here the maximum value of θ is 1, and this leads to a 3-cycle (1 2 3) and a 2-cycle (4 5) (Fig. 5).

This is not a tour, so we add a loop condition which excludes this solution but which is satisfied by all tours. In this case $x_{45} \leq 1$ or

$$x_{45} + y_6 - 1 = 0, \qquad (y_6 \geq 0) \quad (11)$$

is such a condition. Accordingly, we start over again using the five equations (2) and the sixth equation (11). This time we will need six basic variables and it will be convenient to have x_{13} (the one we set equal to θ previously) included with those associated with the tour. Thus, the

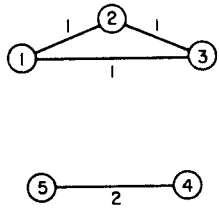


FIGURE 5

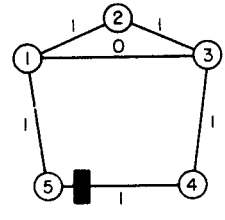


FIGURE 6

starting solution is as follows: The basic variables have values $\bar{x}_{12} = \bar{x}_{23} = \bar{x}_{34} = \bar{x}_{45} = \bar{x}_{51} = 1$, $\bar{x}_{13} = 0$. All other $\bar{x}_{IJ} = 0$. This solution is shown in Fig. 6. The presence of an upper bound on x_{45} or relation (11) is depicted in Fig. 6 by a block symbol on (4, 5). Now we multiply equation (11) by π_6, add it to $\sum_{I=1}^{5} \pi_I \left(\sum_{J=1}^{5} x_{IJ} - 2 \right)$, subtract the sum from $\sum d_{IJ} x_{IJ}$ and collect terms in x_{IJ} as before. The result is

$$\sum_{I>J} d_{IJ} x_{IJ} = - \sum_{I>J} \delta_{IJ} x_{IJ} + 2 \sum_{I=1}^{5} \pi_I + \pi_6 (1 - y_6) \qquad (12)$$

where $\delta_{IJ} = \pi_I + \pi_J - d_{IJ}$ except $\delta_{45} = \pi_4 + \pi_5 - (d_{45} - \pi_6)$.

Now determine the six values of π_I by setting $\delta_{IJ} = 0$ corresponding to basic variables x_{IJ}:

$$\delta_{12} = \delta_{23} = \delta_{34} = \delta_{45} = \delta_{51} = \delta_{13} = 0, \qquad (13)$$

from which it follows that

$$D(x) - D(\bar{x}) = - \sum \delta_{IJ} x_{IJ} - \pi_6 y_6. \qquad (14)$$

To evaluate π_I we note that there are six equations in six unknowns. These are shown on the map below (Fig. 7). The three conditions about the triangular loop (1, 2, 3) permit us to solve for π_1, π_2, π_3. Branching out from the triangle we get next π_4 and π_5 and finally π_6. Thus, we determine first that $2\pi_1 = d_{12} - d_{23} + d_{31} = 5 - 5 + 6$ so that $\pi_1 = 3$, $\pi_2 = 2$, $\pi_3 = 3$.

Working down, $\pi_4=5$, $\pi_5=5$. Thus, $\pi_4+\pi_5=-\pi_6+6$, so $\pi_6=-4$. These values are shown adjacent to each city in Fig. 7.

With these values of π_I all remaining $\delta_{IJ}=(\pi_I+\pi_J-d_{IJ})\leq 0$; hence, with $\pi_6<0$ we have the right side of (14) always positive, so the tour (1 2 3 4 5) is minimal. This illustrates the use of the simplest of the loop conditions, namely, an upper bound on the variable x_{45}.

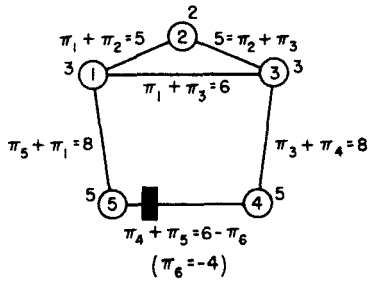


FIGURE 7

Example 3

Here we consider a six-city case (Fig. 8) where the optimal tour is not our initial choice. Let the starting tour be (1 2 3 4 5 6) of length $D(\bar{x})=23$. If we proceed as before, relation (8) implies that the π_I satisfy the relations shown in Fig. 8. In this case (and this is generally true for

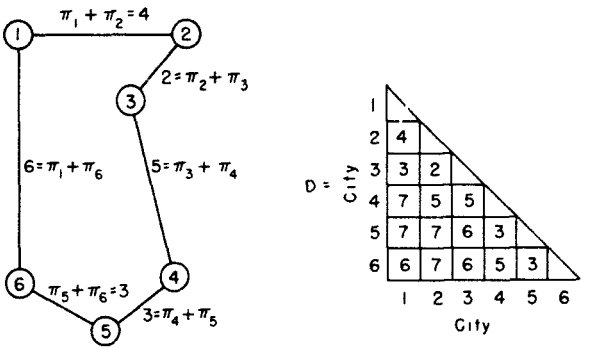


FIGURE 8

loops with an even number of links) the sum of equations on links (1, 2), (3, 4), (5, 6) is identical with the sum for (2, 3), (4, 5), (6, 1) except for different constant terms, so that the system of equations in π_I is inconsistent.

This difficulty can be avoided if the following general rule is followed: The set of basic variables must be so selected that when the remaining

x_{IJ} are fixed, the values of the basic variables are uniquely determined. This means the matrix of coefficients of the basic variables is nonsingular (i.e., their determinant is nonvanishing). Since the π_I satisfy a system of equations whose coefficient matrix is the transpose of this matrix, the π_I will be uniquely determined also. In the six-city case, one may augment system (2) with the additional upper-bound condition

$$x_{45}+y_7=1 \qquad (y_7\geq 0) \quad (15)$$

and select x_{13} as a basic variable in addition to the basic variables x_{IJ} corresponding to (I, J) on the tour. Then, letting π_7 be the weight associated with restriction (15), the π_I satisfy relations in Fig. 9.

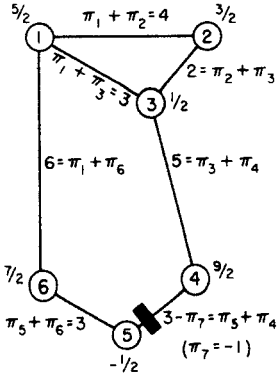


FIGURE 9

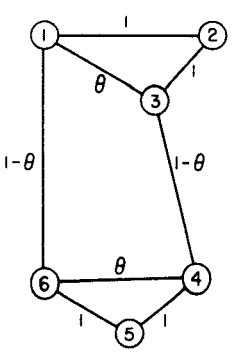


FIGURE 10

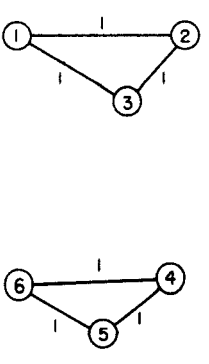


FIGURE 11

The value of $\pi_1=5\frac{1}{2}$ can be determined from the odd loop (1 2 3) by alternately adding and subtracting the equations around the loop. The others can then be evaluated immediately. In this case, we have, analogous to (14),

$$D(x)-D(\bar{x})=-\sum \delta_{IJ}\, x_{IJ}-\pi_7\, y_7, \qquad (16)$$

where $\delta_{IJ}=0$ if x_{IJ} is a basic variable and $\delta_{IJ}=\pi_I+\pi_J-d_{IJ}$ otherwise. Since $\delta_{46}=3$, increasing the value of x_{46} to θ (while all other nonbasic variables remain zero), with corresponding adjustments in the basic variables, will yield $D(x)-D(\bar{x})=-3\theta<0$. In Fig. 10 it is seen that the largest value of $\theta=1$ and the resulting solution is Fig. 11, which is not a new tour, but two loops. However, we can exclude this solution by imposing the additional restriction satisfied by all tour solutions

$$x_{12}+x_{23}+x_{31}\leq 2, \quad \text{or} \quad x_{12}+x_{23}+x_{31}+y_8=2, \qquad (y_8\geq 0) \quad (17)$$

since in Fig. 11 the inadmissible solution has $x_{12}+x_{23}+x_{31}=3$. We now start all over again augmenting relations (2) by (15) and (17). Let

the basic variables be the same as before but include x_{46} (i.e., the one we set equal to θ in Fig. 10). Let π_I for 1, 2, $\cdots$, 8 be the weights assigned to these relations respectively in forming $D(x)-D(\bar{x})$; then the π_I satisfy the relations shown in Fig. 12, where the loop condition (17) is symbolized by the dotted loop in the figure.

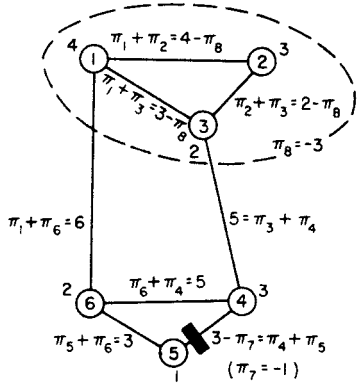


FIGURE 12

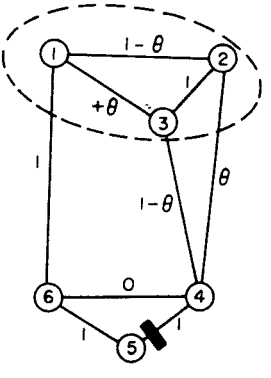


FIGURE 13

The value of $\pi_6=2$ may be evaluated from the odd loop (6 4 3 2 1) by alternately adding and subtracting the equations in π_I shown on this loop. The other π_I can then be immediately determined. This time

$$D(x)-D(\bar{x})=-\sum \delta_{IJ}x_{IJ}-\pi_7 y_7-\pi_8 y_8 \tag{18}$$

where $\delta_{IJ}=0$ for x_{IJ} a basic variable and $\delta_{IJ}=\pi_I+\pi_J-d_{IJ}$ otherwise. Since $\delta_{24}=1$ while all other $\delta_{IJ}\leq 0$, we set $x_{24}=\theta$; then the adjustments in the values of the basic variables necessary to satisfy (2), (15), (17) are

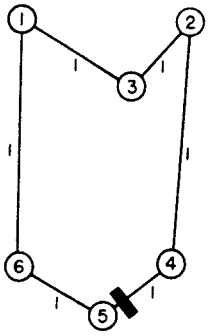


FIGURE 14

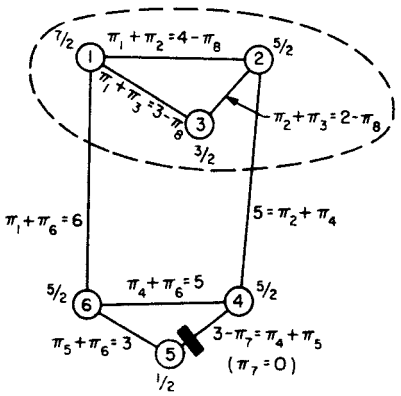


FIGURE 15

shown in Fig. 13 and the new solution for $\theta=1$ is a new tour $\bar{x}$ with length $D(\bar{x})=D(\bar{x})-1=22$, Fig. 14. We may now drop $x_{34}=0$ from the basic set of variables (or alternatively x_{12}) and replace it by x_{24} as a new basic variable. This yields the relations for π_I of Fig. 15. The expression for $D(x)-D(\bar{x})$ is similar to (18). It can now be tested that all $\delta_{IJ}\leq 0$ corresponding to non-basic x_{IJ}, and the coefficients of y_7 and y_8 are $\pi_7\leq 0$, $\pi_8\leq 0$, so that the new tour is established as optimal.

AN ESTIMATION PROCEDURE

In any linear programming problem with bounded variables, an estimate is available of how much a basic solution differs from an optimal solution. Let $D(x)$ represent a linear form to be minimized and $D(\bar{x})$ be the value for some basic solution $\bar{x}$ where variables $(x_1, x_2, \cdots, x_{n'})$, represented by the symbol x, satisfy a system of equations as well as bounds $0\leq x_J\leq r_J$. If the equations are multiplied by weights π_I and substracted from $D(x)$, then (as we have noted earlier)

$$D(x)-D(\bar{x})=-\sum_{J=1}^{n'}\delta_J x_J \qquad (x_J\geq 0) \quad (19)$$

where π_I are chosen such that $\delta_J=0$ if the corresponding x_J is a basic variable. We may now split the right side of (19) into positive and negative parts and obtain a lower bound for the difference by dropping the positive part, i.e.,

$$D(x)-D(\bar{x})=-\sum_{\delta_J>0}\delta_J x_J-\sum_{\delta_J\leq 0}\delta_J x_J, \qquad (x_J\geq 0) \quad (20)$$

$$D(x)-D(\bar{x})\geq -\sum_{\delta_J>0}\delta_J x_J\geq -E, \qquad (E\geq 0) \quad (21)$$

where $-E$ is some estimate for the negative part. By setting $x_J=r_J$, we obtain in particular

$$D(x)-D(\bar{x})\geq -\sum_{\delta_J>0}\delta_J r_J. \qquad (22)$$

For the traveling-salesman problem the variables x_{IJ} must be either 0 or 1 if x represents a tour. From (20), no link (I, J) can occur in an optimal tour if

$$\delta_{IJ}<-E, \qquad (23)$$

hence all corresponding variables x_{IJ} can be dropped from further consideration.

During the early stages of the computation, E may be quite large and very few links can be dropped by this rule; however, in the latter stages often so many links are eliminated that one can list all possible tours that

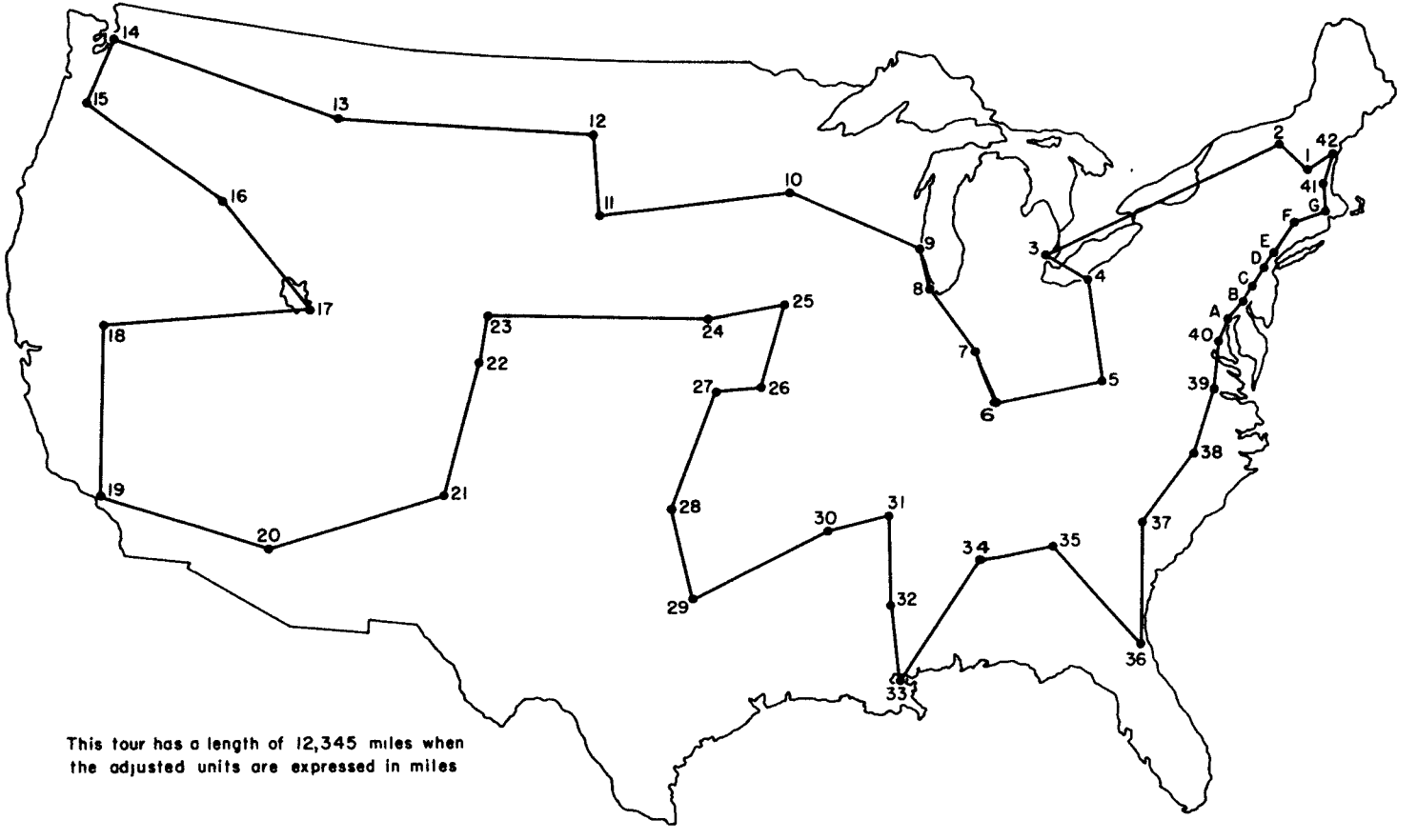


FIG. 16. The optimal tour of 49 cities

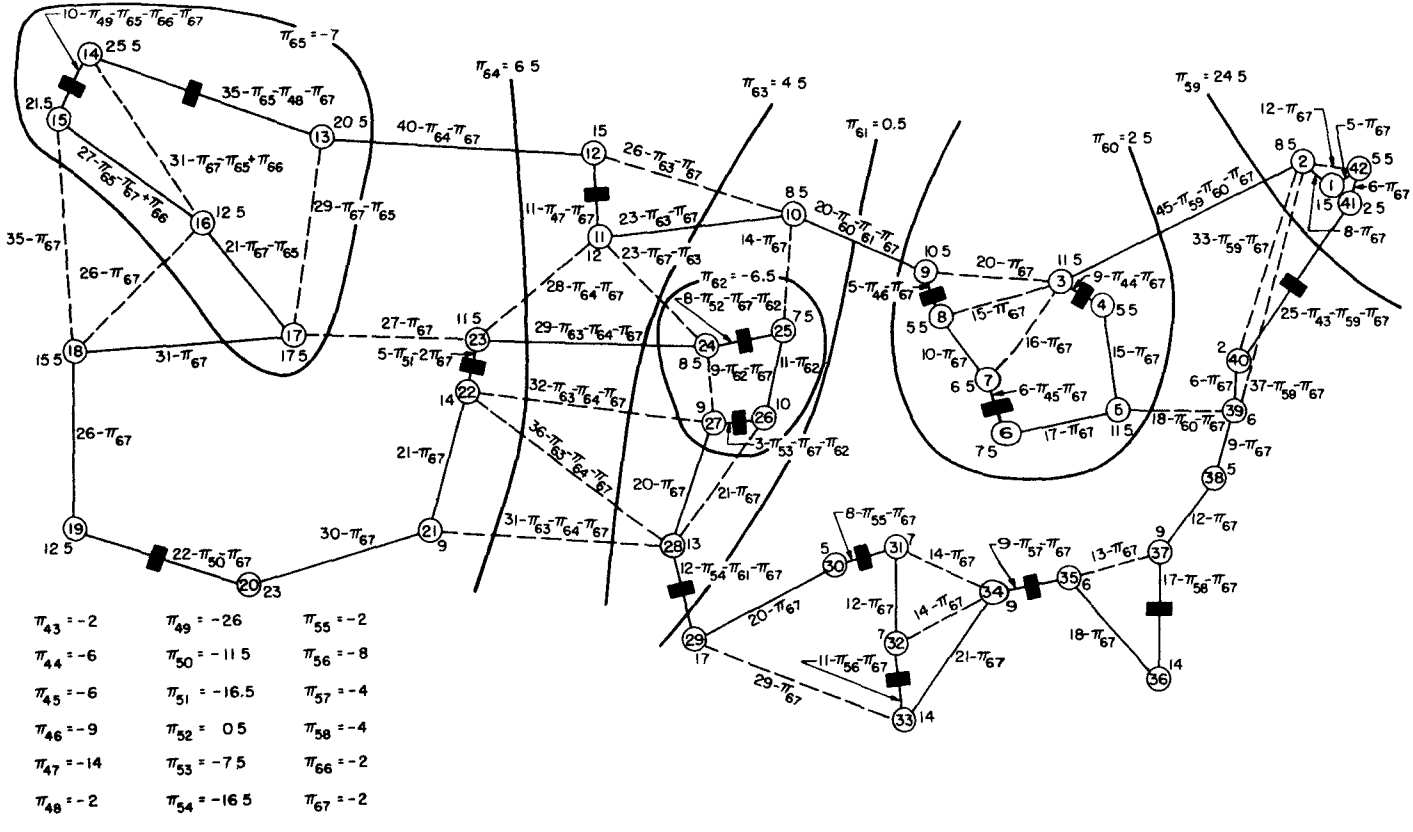


FIG. 17. Only the right-hand side of the equations satisfied by π_I are shown on the map; the left-hand side on line (I,J) is $\pi_I + \pi_J$. Dotted links (I,J) correspond to additional basic variables x_{IJ}.

use the remaining admissible links. By extending this type of combinatorial argument to the range of values of the 'slack' variables y_K, it is often possible at an earlier stage of the iterative algorithm to rule out so many of the tours that direct examination of the remaining tours for minimum length is a feasible approach.

THE 49-CITY PROBLEM*

The optimal tour $\bar{x}$ is shown in Fig. 16. The proof that it is optimal is given in Fig. 17. To make the correspondence between the latter and its programming problem clear, we will write down in addition to 42 relations in non-negative variables (2), a set of 25 relations which suffice to prove that $D(x)$ is a minimum for $\bar{x}$. We distinguish the following subsets of the 42 cities:

$S_1=\{1, 2, 41, 42\}$
$S_2=\{3, 4, \cdots, 9\}$
$S_3=\{1, 2, \cdot, 9, 29, 30, \cdot, 42\}$
$S_4=\{11, 12, \cdots, 23\}$
$S_5=\{13, 14, \cdot\cdot, 23\}$
$S_6=\{13, 14, 15, 16, 17\}$
$S_7=\{24, 25, 26, 27\}$.

Except for two inequalities which we will discuss in a moment, the programming problem may now be written as the following 65 relations:†

$$\sum_J x_{IJ}=2 \quad (I=1, \cdots, 42),$$

$$x_{41,1}\leq 1, \quad x_{4,3}\leq 1, \quad x_{7,6}\leq 1, \quad x_{9,8}\leq 1, \quad x_{12,11}\leq 1, \quad x_{14,13}\leq 1, \quad x_{15,14}\leq 1, \quad x_{20,19}\leq 1,$$

$$x_{23,22}\leq 1, \quad x_{25,24}\leq 1, \quad x_{27,26}\leq 1, \quad x_{29,28}\leq 1, \quad x_{31,30}\leq 1,$$

$$x_{33,32}\leq 1, \quad x_{35,34}\leq 1, \quad x_{37,36}\leq 1, \quad \sum_{S_1,S_1} x_{IJ}\geq 2, \quad \sum_{S_2,S_2} x_{IJ}\geq 2,$$

$$\sum_{S_3,S_3} x_{IJ}\geq 2, \quad \sum_{S_4,S_4} x_{IJ}\geq 2, \quad \sum_{S_5,S_5} x_{IJ}\geq 2, \quad \sum_{S_6} x_{IJ}\leq 4, \quad \sum_{S_7} x_{IJ}\leq 3.$$

The remaining two relations (*66* and *67*) are perhaps most easily described verbally.

66: $x_{14,15}$ minus the sum of all other x_{IJ} on links out of 15, 16, 19, except for $x_{18,15}$, $x_{18,16}$, $x_{17,16}$, $x_{19,18}$, and $x_{20,19}$, is not positive

67: $\Sigma a_{IJ}x_{IJ}\leq 42$, where $a_{23,22}=2$, $a_{26,25}=0$, all other $a_{IJ}=1$ except $a_{IJ}=0$ if x_{IJ} is a non-basic variable and either (a) I is in S_3, J not in S_3, or (b) I or J is 10, 21, 25, 26, 27, or 28.‡

These two inequalities are satisfied by all tours. For example, if a tour were to violate the first one, it must have successively $x_{15,14}=1$,

* As indicated earlier, it was possible to treat this as a 42-city problem.

† $\Sigma_{S,S} x_{IJ}$ means the sum of all variables where only one of the subscripts I or J is in S. $\Sigma_S x_{IJ}$ means the sum of all variables such that I and J are in S—see relations (4), (5), (6).

‡ We are indebted to I Glicksberg of Rand for pointing out relations of this kind to us.

$x_{18,15}=1$, $x_{18,10}=1$, but also $x_{19,18}=1$, a contradiction. The argument that each tour satisfies the second inequality is similar. If a tour x exists with $\Sigma\, a_{IJ}x_{IJ}>42$, then clearly $x_{23,22}=1$, and also $x_{10,9}=x_{29,28}=1$, since by (a) these are the only links connecting S_3 and $\bar{S}_3$ having non-zero a_{IJ}. (See Fig. 17 to distinguish between basic and non-basic variables.) Moreover, since $a_{26,25}=0$, it follows from (b) that $x_{25,10}=x_{25,24}=x_{27,26}=x_{28,27}=1$. Again, (b) and the fact that $x_{28\ 21}=0$ imply $x_{21\ 20}=x_{22,21}=1$ Now look at city 27. There are three possibilities: $x_{27\ 24}=1$, $x_{27\ 22}=1$, or $x_{28\ 27}=1$ But each of these contradicts the assumption that x is a tour.

These relations were imposed to cut out fractional solutions which satisfy all the conditions (2) and (4). A picture of such a fractional solution, which gives a smaller value for the minimizing form than does any tour, is shown in Fig. 18. Notice that it does not satisfy relation 67.

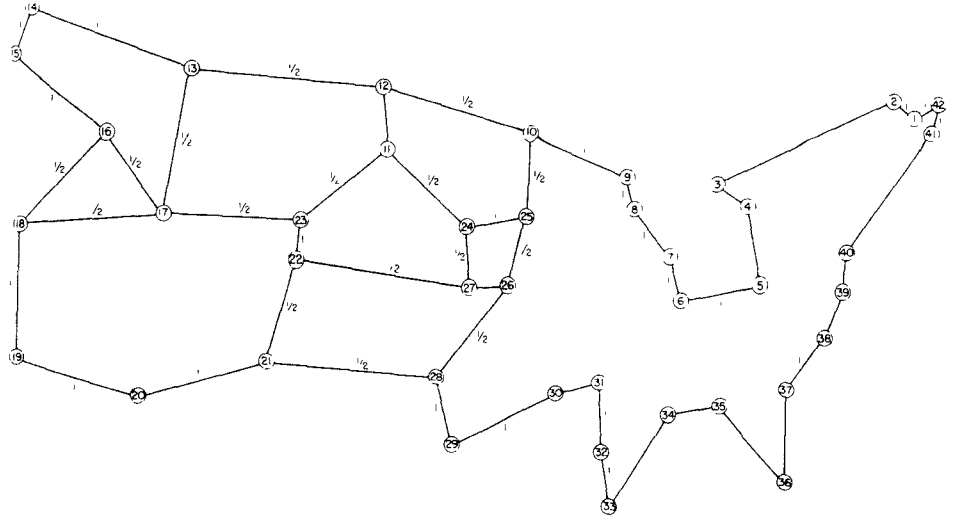


FIG 18. A fractional solution x satisfying all loop conditions with $\Sigma\, d_{IJ}\, x_{IJ}=698$.

We assert that if the weights π_I are assigned to these restraints in the order presented above, then the values as given in Fig. 17 satisfy $\delta_{IJ}=0$ for all variables x_{IJ} in the basis. With these values of π_I in the expression for $D(x)-D(\bar{x})$, all $\delta_{IJ}\leq 0$ corresponding to variables x_{IJ} and π_{43}, π_{44}, $\cdots$, π_{67} corresponding to variables y_{43}, $\cdots$, y_{67} are appropriately positive or negative (positive if its y occurs with a minus sign in the relation, negative otherwise) with the exception of $\pi_{52}=1\frac{1}{2}$ where $x_{25,24}+y_{52}=1$. This proves, since $E=\frac{1}{2}$ and all the d_{IJ} are integers, that $\bar{x}$ is minimal. The length $D(\bar{x})$ is 699 units, or 12,345 miles except for rounding errors.

It can be shown by introducing all links for which $\delta_{IJ} \geq -\frac{1}{2}$ that $\bar{x}$ is the unique minimum. There are only 7 such links in addition to those shown in Fig. 17, and consequently all possible tying tours were enumerated without too much trouble. None of them proved to be as good as $\bar{x}$.

CONCLUDING REMARK

It is clear that we have left unanswered practically any question one might pose of a theoretical nature concerning the traveling-salesman problem; however, we hope that the feasibility of attacking problems involving a moderate number of points has been successfully demonstrated, and that perhaps some of the ideas can be used in problems of similar nature.

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