A non-generic real incompatible with $0^\#$

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1. Introduction

This paper is devoted to the proof of

Theorem 1. *Let L be a minimal countable standard transitive model of ZFC + $V{=}L$. There exists a real x_{ng} having the following three properties:*

(1) $x_{\mathrm{ng}} \notin L$.
(2) $L[x_{\mathrm{ng}}]$ *satisfies ZFC.*
(3) *In the following sense, x_{ng} is not generic over any outer model of L: Assume that V is transitive and contains the same ordinals as L, and that $\mathbb{P}$ is a V-amenable partial ordering. Let $\Vdash_0$ be the forcing relation for $\mathbb{P}$ over $\langle V; \mathbb{P}\rangle$, restricted to Σ_0 sentences of the forcing language. Suppose that $\langle V; \mathbb{P}, \Vdash_0\rangle \models \mathrm{ZF}$, that G is $\langle V; \mathbb{P}, \Vdash_0\rangle$-definably generic, and that $V[G]$ is admissible. If $x_{\mathrm{ng}} \in V[G]$, then $x_{\mathrm{ng}} \in V$.*

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The standard structure L_β is said to be *minimal* when there exists an ordinal $\alpha < \beta$ such that every element in L_β is first-order definable from ordinal parameters less than α. For example, if β is the least ordinal greater than α such that $L_\beta \models \mathrm{ZFC}$, then L_β is minimal.

A class $A \subseteq V$ is *V-amenable* if $A \cap x \in V$, for all $x \in V$.

A filter G on $\mathbb{P}$ is $\langle V; \mathbb{P}, \cdots \rangle$-definably generic, when G meets every dense subclass of $\mathbb{P}$ that is definable over the standard structure $\langle V; \mathbb{P}, \cdots \rangle$.

Rather than generic, the real x_{ng} is "diagonally generic" over L: There exist L-definable class orderings $\mathbb{P}^n$, for $n \in \omega$, such that x_{ng} is partially generic over each $\mathbb{P}^n$. These orderings are designed so that were x_{ng} class generic over an outer model V, then a truth predicate for L would be definable from the Σ_0 forcing relation for that class forcing. Because L is minimal, this would contradict that V satisfies ZF.

Perhaps more perspicuous than Theorem 1 itself is its corollary,

Theorem 2. *Let L be a minimal countable standard transitive model of ZFC + $V{=}L$. There exists a real x_{ng} such that*

(1) $x_{\mathrm{ng}} \notin L$;
(2) $L[x_{\mathrm{ng}}]$ *satisfies ZFC; and*
(3) x_{ng} *is not weakly class generic over L. That is, there does not exist an outer model V, a V-amenable partial ordering $\mathbb{P}$, and a filter G on $\mathbb{P}$ meeting every $\langle V; \mathbb{P}\rangle$-definable dense subclass of $\mathbb{P}$ such that*
   (a) $x_{\mathrm{ng}} \notin V$;
   (b) $x_{\mathrm{ng}} \in V[G]$; *and*
   (c) $\langle V[G]; V, \mathbb{P}, G\rangle$ *satisfies ZFC.*

Transparently, Theorem 2 follows from Theorem 1 and the following proposition, which lies outside the scope of this paper:

Proposition 3. (c.f. [**S3**]) *Assume that V is a standard transitive model of ZFC, that $\mathbb{P}$ is an V-amenable partial ordering, and that G is a filter on $\mathbb{P}$ meeting every dense subclass of $\mathbb{P}$ that is definable over $\langle V; \mathbb{P}\rangle$. Suppose also that $\langle V[G]; V, \mathbb{P}, G\rangle$ satisfies ZFC. Then the Σ_0 forcing relation is definable on a cone of conditions rooted in G; that is, there exists a condition $\hat{p} \in G$ such that*

$$\Big\{(p, \varphi) : p \leqslant \hat{p} \ \text{ and } \ \varphi \text{ is } \Sigma_0 \ \text{ and } \ p \Vdash \varphi \Big\}$$

is first-order definable over $\langle V; \mathbb{P}\rangle$.

This section has four subsections. The first of these makes a number of definitions to clarify what is claimed in Theorems 1 and 2. The next discusses some related results and open questions. The third makes a number of definitions and cites several propositions that will be needed for the proof. The last explains the organization of the paper.

1.1. Class forcing

Let us begin with some definitions towards making precise what is claimed in Theorems 1 and 2. Much of what follows cannot be formalized in ZFC for class models; the reader who would work in first-order set theory should cast it in terms of countable models of ZFC.

Indeed, in any broad sense, class genericity is not a first-order property. Consequently, unlike propositions regarding set generic extensions, our theorems cannot be euphemized in terms of consistency.

As is customary, we confuse partial orderings with their underlying fields of conditions, and standard models of set theory with their underlying universes.

If $\mathbb{P}$ is a partial ordering, say that the condition p *extends* the condition $\bar{p}$ when $p \leqslant \bar{p}$.

Suppose that V is transitive. If $R_1, \ldots, R_k$ are relations on V, say that the structure $\langle V; R_1, \ldots, R_k\rangle$ *satisfies ZFC* when this standard structure satisfies ZFC as formulated in a language with predicate symbols for the relations $R_1, \ldots, R_k$. In particular, ZFC in such an expanded language includes all instances of collection and separation that can be formulated in the augmented language.

Suppose that V is a transitive standard model of ZFC and that $\mathbb{P}$ is a V-amenable partial ordering, that is, that both the underlying field of conditions in $\mathbb{P}$ and their ordering are V-amenable. Assume as well that $\langle V; \mathbb{P}\rangle$ satisfies ZFC. Working in this structure, the class $V^{\mathbb{P}}$ of *Shoenfield terms* can be defined:

$$\mathring{a} \in V^{\mathbb{P}} \qquad \text{iff} \qquad \mathring{a} \in V \text{ and } \mathring{a} \subseteq V^{\mathbb{P}} \times \mathbb{P}.$$

For Shoenfield terms $\mathring{a}$ and $\mathring{b}$, set

$$\mathring{a}(\mathring{b}) = \Big\{\, p \in \mathbb{P} : (\mathring{b}, p) \in \mathring{a} \,\Big\}.$$

Say that $G \subseteq \mathbb{P}$ is a *filter* on $\mathbb{P}$ iff any pair of conditions in G has a common extension in G, and any condition extended by a condition in G itself lies in G. The requirement that the compatibility of conditions in G is witnessed *in* G improves the numerology in §1.3.

If G is a filter on $\mathbb{P}$, set

$$\begin{aligned} \mathring{a}^G &= \Big\{\, \mathring{b}^G : \mathring{a}(\mathring{b}) \cap G \neq \emptyset \,\Big\}, \text{ and} \\ V[G] &= \Big\{\, \mathring{a}^G : \mathring{a} \in V^{\mathbb{P}} \,\Big\}. \end{aligned}$$

Then $V[G]$ is transitive. If G is non-empty, then the same ordinals lie in $V[G]$ as in V; also $V \subseteq V[G]$, and V is $V[G]$-amenable. Furthermore, the standard structure $V[G]$ satisfies the axioms of extensionality, pairing, infinity, and foundation. If it satisfies separation, then it satisfies the axioms of unions and choice, as well. So, to preserve ZFC in $V[G]$ it suffices to preserve the power set axiom and instances of collection and separation.

Say that a set x is *V-amenably generic* when

(1) there exists a V-amenable partial ordering $\mathbb{P}$ such that $\langle V;\mathbb{P}\rangle$ satisfies ZFC;
(2) there exists a filter G on $\mathbb{P}$ meeting every dense subclass of $\mathbb{P}$ that is first-order definable over $\langle V;\mathbb{P}\rangle$;
(3) $x \in V[G] \setminus V$; and
(4) $\langle V[G]; V, \mathbb{P}, G\rangle$ satisfies ZFC.

For current purposes, three sorts of class genericity are relevant, namely, L-amenable class genericity, weak genericity, and invisible genericity.

Say that x is *weakly generic* if there exists an outer model V such that x is V-amenably generic. And say that x is *invisibly generic over V*, if x satisfies the definition of V-amenable genericity with (4) replaced by simply

(4′) $V[G]$ satisfies ZFC.

1.2. Open problems

Theorem 1 is motivated by the following conjecture of Beller-Jensen-Welch [**BJW**]:

Conjecture. *If x is a real and $0^{\#} \notin L[x]$, then x is class generic over L.*

The real x_{ng} of Theorem 1 is non-generic in a strong sense. All that is not ruled out is that it is invisibly generic for a forcing with an undefinable forcing relation. However, x_{ng} is incompatible with the existence of $0^{\#}$. Indeed, if $0^{\#}$ exists, then L is not minimal.

Sy Friedman [**F1**] has found a real that, though non-generic in a weaker sense, is compatible with $0^{\#}$. His real is not L-amenably generic and is constructible from $0^{\#}$.

Several questions remain regarding non-generic reals.

Question 1. *Can the hypothesis that L is minimal be eliminated in Theorem 1?*

If this is the case, then perhaps the virtues of x_{ng} and Friedman's real can be combined:

Question 2. *Does $0^{\#}$ construct a real that is not weakly generic?*

If α is countable and x is a real such that $L_\alpha[x]$ satisfies ZFC + "x is $0^{\#}$," then let us call x a "countable $0^{\#}$" (even though x might be the "real" $0^{\#}$). If L_α supports a countable $0^{\#}$, then it supports 2^ω many countable $0^{\#}$'s.

A candidate for answering Question 2 is at hand:

Question 3. *Is $0^{\#}$ weakly generic?*

It is shown in [**S2**] that there exist countable $0^{\#}$'s that are invisibly generic. However, it open whether they all are.

Question 4. *Is every countable $0^{\#}$ invisibly generic?*

In fact, no set is known to not be invisibly generic.

Question 5. *Do there exist reals that are not invisibly generic?*

1.3. Ramified genericity

This subsection makes some definitions and states some results about class forcing needed for the proof of Theorem 1.

Let T^- be the result of removing the power set axiom from the set theory T. ZF_{n+1} is ZF with collection restricted to Σ_{n+1} formulas and with separation restricted to Σ_n formulas. Adjoining 'C' to the name of a set theory indicates adding the Axiom of Choice.

A standard model of ZF^-_{n+1} is said to be *Σ_{n+1}-admissible* (or simply *admissible*, if $n = 0$). A quibble is that in ZF the $\in$-well-foundedness of definable classes follows from the $\in$-well-foundedness of sets, so usually the axiom of foundation in ZF is a single sentence asserting the $\in$-well-foundedness of sets. In admissible set theory, Foundation is a scheme asserting the $\in$-well-foundedness of all definable classes. Since we shall be working with these fragments of ZF, let us adopt this foundation scheme as part of ZF.

Partial orderings are required only to be transitive and reflexive, but not necessarily anti-symmetric. Conditions in a partial ordering are *compatible* when they have a common extension. Say that $X \subseteq \mathbb{P}$ is *predense with respect to p* if every extension of p is compatible with some element of X. Similarly, X is dense with respect to p when every extension of p has an extension into X. A set or class of conditions X is *empty* with respect to a condition p if p is incompatible with every member of X.

Suppose $R_1, \ldots, R_k$ are relations on the universe V, and that $Y \subseteq V$. Say that the class $X \subseteq V$ is *$\Sigma_n(Y; R_1, \ldots, R_k)$ definable* if X is definable over $\langle V; R_1, \ldots, R_k \rangle$ by a Σ_n formula with (set) parameters from Y and predicate symbols for $R_1, \ldots, R_k$. We speak similarly of $\Pi_n(Y; R_1, \ldots, R_k)$ and $\Delta_n(Y; R_1, \ldots, R_k)$ definability. Say that X is $\Sigma_\omega(Y; R_1, \ldots, R_k)$ definable, if it is $\Sigma_n(Y; R_1, \ldots, R_k)$ definable, for some n.

Say that $\mathbb{P}$ satisfies *$\Sigma_n(Y; \mathbb{P}, R_1, \ldots, R_k)$ predensity reduction* when, given any condition $\bar{p} \in \mathbb{P}$, and any uniformly $\Sigma_n(Y; \mathbb{P}, R_1, \ldots, R_k)$ definable sequence $\langle\, D_i : i \in I \,\rangle$ of classes predense with respect to $\bar{p}$ (where $I \in V$ is a set), there exists a condition p extending $\bar{p}$, and sets $d_i \subseteq D_i$ such that $\langle\, d_i : i \in I \,\rangle \in V$, and d_i is predense with respect to p, for all $i \in I$.

If the axiom of choice holds in V (so that every set has a cardinality), say that $\mathbb{P}$ satisfies *strong $\Sigma_n(Y; \mathbb{P}, R_1, \ldots, R_k)$ predensity reduction* when, given any condition $\bar{p} \in \mathbb{P}$, and any uniformly $\Sigma_n(Y; R_1, \ldots, R_k)$ definable sequence $\langle\, D_i : i \in I \,\rangle$ of classes predense with respect to $\bar{p}$ (where $I \in V$ is a set), there exists a condition p extending $\bar{p}$, and sets d_i such that $\langle\, d_i : i \in I \,\rangle \in V$, and, for each $i \in I$, d_i is predense with respect to p, $|d_i| \leqslant |I|$, and any condition meeting d_i meets D_i.

(To see that the strong form of $\Sigma_n(Y; \mathbb{P}, R_1, \ldots, R_k)$ predensity reduction implies the weak form, apply the strong form to $D'_i = \left\{\, p \in \mathbb{P} : p \text{ meets } D_i \,\right\}$ to get d'_i reducing D'_i. Then let $d_i = f"d'_i$, where $f(p) \in D_i$ is chosen so that $f(p) \geqslant p$, for $p \in d'_i$.)

Let $\mathbb{P} \restriction \hat{p} = \{\, p \in \mathbb{P} : p \leqslant \hat{p} \,\}$.

If forcing with $\mathbb{P}$ preserves ZFC in a language with predicate symbols for V, $\mathbb{P}$, and G, then $\mathbb{P}$ satisfies $\Sigma_\omega(V; \mathbb{P})$ predensity reduction:

Proposition 1.2. (c.f. [**S3**]) *Fix a natural number $n > 0$. Assume that G is a filter on $\mathbb{P}$ meeting every $\Sigma_\omega(V; \mathbb{P})$ definable dense class. Assume also that $\langle V[G]; V, \mathbb{P}, G \rangle$*

satisfies ZF_n. *Then there exists a condition* $\hat{p} \in G$ *such that* $\mathbb{P}\restriction\hat{p}$ *satisfies* $\Sigma_n(V;\mathbb{P})$ *predensity reduction (in* $\langle V;\mathbb{P}\rangle$*).*

In turn, sufficient predensity reduction is enough to prove that $\Vdash_0$, the forcing relation restricted to Σ_0 sentences of the forcing language, is first order definable over $\langle V;\mathbb{P}\rangle$. The version of this claim we shall use is

Proposition 1.3. (c.f. [**S3**]) *Assume that*

- $\langle V;\mathbb{P}\rangle$ *is a model of ZFC; that*
- $\{ (p,q) : p$ *is compatible with* $q \}$ *is* $\Delta_1(V;\mathbb{P})$ *definable; and that*
- $\mathbb{P}$ *satisfies* $\Sigma_\omega(V;\mathbb{P})$ *predensity reduction.*

Then $\Vdash_0$ *is* $\Pi_2(\emptyset;\mathbb{P})$ *definable.*

(In fact, at the cost of a slightly more complicated definition for $\Vdash_0$, the hypothesis that compatibility is $\Delta_1(V;\mathbb{P})$ can be eliminated. Also, only a bounded amount of predensity reduction is required. Proposition 3 cited earlier follows from Proposition 1.2 and such a modified Proposition 1.3.)

Using Proposition 1.3, the following converse of Proposition 1.2 can be proved:

Proposition 1.4. (c.f. [**S3**]) *If* $\langle V;\mathbb{P}\rangle \vDash \mathrm{ZF}$ *(or ZFC), and* $\mathbb{P}$ *satisfies* $\Sigma_\omega(V;\mathbb{P})$ *predensity reduction, and* G *is* $\Sigma_\omega(V;\mathbb{P})$ *generic, then* $\langle V[G];V,\mathbb{P},G\rangle \vDash \mathrm{ZF}^-$ *(respectively,* ZFC^-*). If, in fact,* $\mathbb{P}$ *satisfies strong* $\Sigma_\omega(V;\mathbb{P})$ *predensity reduction, then* $V[G]$ *is a cardinal-preserving extension of* V*.*

In order to argue that our diagonally generic real x_{ng} preserves ZF^-, we shall need a ramified version of this. To state it, we need a ramified notion of genericity.

If X is a class of conditions, say that p *decides* X, when either p meets X, or p is incompatible with every member of X. Say that a filter G on $\mathbb{P}$ is *$\Pi_n(Y;R_1,\dots,R_k)$ generic* provided that any $\Pi_n(Y;R_1,\dots,R_k)$ definable class X is decided by some condition $p \in G$. This definition, a bit stronger than simply requiring G to meet all $\Pi_n(Y;R_1,\dots,R_k)$ definable predense classes, improves the estimates in the following two propositions.

Proposition 1.5. (c.f. [**S3**]) *Assume that*

- $\langle V;\mathbb{P}\rangle$ *is a model of ZFC; that*
- $\{ (p,q) : p$ *is compatible with* $q \}$ *is* $\Delta_1(V;\mathbb{P})$ *definable; that*
- $\mathbb{P}$ *satisfies* $\Sigma_\omega(V;\mathbb{P})$ *predensity reduction; and that*
- G *is a* $\Pi_{n+2}(V;\mathbb{P})$ *generic filter on* $\mathbb{P}$*.*

Then $V[G] \vDash \mathrm{ZFC}^-_{n+1}$*.*

Proposition 1.6. (c.f. [**S3**]) *Assume that*

- $\langle V;\mathbb{P}\rangle$ *is a model of ZFC; that*
- $\{ (p,q) : p$ *is compatible with* $q \}$ *is* $\Delta_1(V;\mathbb{P})$ *definable; that*
- $\mathbb{P}$ *satisfies* $\Sigma_\omega(V;\mathbb{P})$ *predensity reduction; and that*
- G *is a* $\Pi_{n+2}(V;\mathbb{P})$ *generic filter on* $\mathbb{P}$*.*

Then $V[G] \vDash \varphi$ *iff* $\exists p \in G\ p \Vdash \varphi$*, whenever* φ *is either a* Σ_n *or* Π_n *sentence of the forcing language.*

1.4. Organization of the proof

The heart of the proof is in §2, where $x_{\rm ng}$ is constructed and Theorem 1 is proved assuming certain facts, labeled (T1)–(T5) and (P1)–(P4). These are verified in the remaining sections.

As mentioned above, $x_{\rm ng}$ is "diagonally generic" for a certain sequence of class orderings $\mathbb{P}^n$, where $n \in \omega$. Forcing with $\mathbb{P}^n$ adds a real x, as well as branches through certain definable trees $T^n_{k,i}$ (where $k \in \omega$ and i is 0 or 1). In fact, a $\mathbb{P}^n$ generic real x codes a branch through $T^n_{k,i}$ iff $x(k) = i$. Conceptually, then, $\mathbb{P}^n$ has two components, namely the trees $T^n_{k,i}$, and coding apparatus that serves to code branches through these trees into a real.

The trees $T^n_{k,i}$ are constructed in §3, where facts (T1)–(T5) are verified.

The coding apparatus, which is based on a simplified version of Jensen coding [**S1**], is developed from scratch in §4.

The ordering $\mathbb{P}^n$ is defined in §5, and all of (P1)–(P4) except one part of (P3) are verified. On account of the simplified coding, this is almost trivial.

The one remaining fact, namely that $\mathbb{P}^n$ satisfies strong predensity reduction, is verified in §6.

The first draft of this paper was written in the fall of 1986. Though the idea for the proof remains same, this version is substantially different from the original in two ways. First of all, I am indebted to Sy Friedman for the essential idea used in §3 to construct the trees $T^n_{k,i}$. This construction is much simpler than the original.

Secondly, the coding developed in §4 allows for several simplifications because extending conditions is almost trivial. In Jensen's version [**BJW**], extending conditions is entwined with distributivity. Jensen's methods are more general and less destructive than those used here, but these features are not needed to prove Theorem 1.

2. Proof of Theorem 1

In this section, assuming certain facts that are proved in the remaining sections, the real $x_{\rm ng}$ is constructed and shown to confirm Theorem 1.

Let $2^{<\omega}$ consist of all functions $\hat{s}\colon n \to \{0,1\}$, for some natural number n.

If $\hat{s} \in 2^{<\omega}$, let

$$d(\hat{s}) = \Big(\omega \times \{0,1\}\Big) \Big\backslash \Big\{ (k, 1{-}i) : \hat{s}(k) = i \Big\}.$$

That is, $d(\hat{s})$ is the set of all pairs (k,i) such that $\hat{s}'(k) = i$, for some $\hat{s}'$ extending $\hat{s}$ in $2^{<\omega}$. Typically, this notation will be used for $\hat{s} \in 2^{<\omega}$ approximating the non-generic real $x_{\rm ng}$.

Begin by fixing a sequence $\langle\, E_{k,i} : k \in \omega \text{ and } i = 0,1 \,\rangle$ of sets of natural numbers having the following three properties:

- $E_{k,0} \cup E_{k,1} = \omega$, for all k;
- $E_{k,0} \cap E_{k,1} = k+1$, for all k; and
- $\bigcap_{k<|\hat{s}|} E_{k,\hat{s}(k)}$ is infinite, for all $\hat{s} \in 2^{<\omega}$.

For example, we could set

$$E_{k,0} = \Big\{\, n : n \leqslant k \ \text{ or } \ \text{the } k^{\text{th}} \text{ prime number does not divide } n \,\Big\} \text{ and}$$
$$E_{k,1} = \Big\{\, n : n \leqslant k \ \text{ or } \ \text{the } k^{\text{th}} \text{ prime number does divide } n \,\Big\}.$$

With such a sequence in mind, let us observe for future reference a key feature that is used in proving the distributivity facts we need to see that $L[x_{\text{ng}}]$ satisfies ZFC and preserves L-cardinals:

Lemma 2.1. *Given any natural number n and any $\hat{s} \in 2^{<\omega}$, there exists an $n' \geqslant n$ and an $\hat{s}' \supseteq \hat{s}$ such that*

$$n' \in \bigcap_{(k,i) \in d(\hat{s}')} E_{k,i}.$$

Proof: Choose $n' \in \bigcap_{k<|\hat{s}|} E_{k,\hat{s}(k)}$ such that $n' \geqslant n$. Then let $\hat{s}' \supseteq \hat{s}$ be such that $|\hat{s}'| \geqslant n'$ and $\hat{s}'(k) = i$, where i is such that $n' \in E_{k,i}$, for $k \in \text{dom}(\hat{s}') \setminus \text{dom}(\hat{s})$. □

Because we are working in a minimal model, the class of η such that $L_\eta \prec L$ is bounded in the ordinals. Let η_* be its supremum. For $n < \omega$, define Ω^n to be the class of all α such that $\alpha > \eta_*$ and $L_\alpha \prec_{\Sigma_n} L$. Then Ω^n is a $\Delta_{n+1}\big(\{\eta_*\}\big)$ definable club class of ordinals, and $\bigcap_{n<\omega} \Omega^n = \emptyset$.

If α is an ordinal, let $n(\alpha) = 0$, if α is not admissible; otherwise, let $n(\alpha)$ be the greatest $n \leqslant \omega$ such that L_α is a model of ZF_n^-.

In §3 we shall define (uniformly in k and i, but not in n) trees $T^n_{k,i}$ of closed sets of ordinals, for $n, k \in \omega$ and $i = 0, 1$. Our non-generic real $x_{\text{ng}} \colon \omega \to 2$ will code a branch through $T^n_{k,x_{\text{ng}}(k)}$, for each n and k. These trees will have the following five properties:

(T1) If $t \in T^n_{k,i}$, then t is a closed subset of $\{i\} \cup \{\, \alpha : n(\alpha) \in E_{k,i} \,\}$; $i \in t$; $(1-i) \notin t$; and $\sup(t) \in t$.

(T2) Ordered by end-extension, $T^n_{k,i}$ is a normal tree. Specifically, let $t \subseteq_{\text{e}} t'$ indicate that t' end-extends t, that is, $t \subseteq t'$ and $t' \cap \big(\sup(t) + 1\big) = t$. Then
- $T^n_{k,i} \neq \emptyset$;
- if $t \in T^n_{k,i}$ and δ is an ordinal, then there exists $t' \in T^n_{k,i}$ such that $t \subseteq_{\text{e}} t'$ and $\sup(t') \geqslant \delta$; and
- if $t \in T^n_{k,i}$, then $\big\{\, t' \in T^n_{k,i} : t \subseteq_{\text{e}} t' \,\big\}$ is not linearly ordered by $\subseteq_{\text{e}}$.

(T3) $T^{n+1}_{k,i} \subseteq T^n_{k,i}$

As mentioned above, the $T^n_{k,i}$'s are not definable uniformly in n. As n gets larger, the definition of $T^n_{k,i}$ gets logically more complex. The complexity of these definitions is controlled by the function $\sigma \colon \omega \to \omega$:

$$\sigma(0) = 1$$
$$\sigma(n+1) = \sigma(n) + n + 4$$

Why such a function works will be the subject of a certain amount of numerology; any function growing sufficiently quickly would do.

If $t, t' \in T^n_{k,i}$ are incomparable under $\subseteq_{\text{e}}$, set $\text{split}(t, t') = \sup\big\{\, \delta : t \cap \delta = t' \cap \delta \,\big\}$.

The final two properties we need are (T4) and (T5):

(T4) Suppose that $k < n$ and $t, t' \in T^n_{k,i}$ are incomparable under end-extension. Set $\eta = \min\{\, \beta \in \Omega^{\sigma(n)} : n(\beta) \in E_{k,i} \,\}$. If $\delta \in t$ and $\eta, \mathrm{split}(t,t') \leqslant \delta \leqslant \sup(t')$, then $\min(t' \setminus \delta) \in \Omega^{\sigma(n)}$.

(T5) $T^n_{k,i}$ is $\Delta_{\sigma(n)+1}(\{\eta_*\})$ definable.

The point of (T4) is that initial segments of $\Omega^{\sigma(n)}$ can be recovered from sufficiently high pairs of incomparable nodes in $T^n_{k,i}$. A feature of (T1) is that it is incompatible with ZF^- that there exist definable cofinal branches through both of $T^n_{k,i}$ and $T^n_{k,(1-i)}$, because the intersection of two such branches would be a club class consisting of ordinals that are not Σ_{k+1}-admissible. These two features of the trees $T^n_{k,i}$ are used in Lemma 2.3 to show that x_{ng} is not generic.

In §4 a decoding process that recovers a class of ordinals $\mathrm{Decode}(z,k)$ from any real z and any natural number k will be described. The class $\mathrm{Decode}(z,k)$ is $\Delta_2(\emptyset)$ definable over $L[z]$, uniformly in z and k. (Consequently, the decoding is absolute for admissible outer models that include z.)

Our non-generic real x_{ng} will have the following two properties:

(R1) $L[x_{\mathrm{ng}}] \vDash \mathrm{ZFC}$, and is a cardinal preserving extension of L.

(R2) $\mathrm{Decode}(x_{\mathrm{ng}},k)$ is a $\Pi_{n+2}(L;T^n_{k,i})$ generic branch through $T^n_{k,i}$, where $i = x_{\mathrm{ng}}(k)$.

We say that the class B is a $\Pi_{n+2}(L;T^n_{k,i})$ generic branch through $T^n_{k,i}$ when B is a class of ordinals such that if $T^n_{k,i}$ is ordered by reverse end-extension, then

$$\Big\{\, t \in T^n_{k,i} : t = B \cap \big(\sup(t)+1\big) \,\Big\}$$

is a $\Pi_{n+2}(L;T^n_{k,i})$ generic filter on $T^n_{k,i}$. (Recall this means that every $\Pi_{n+2}(L;T^n_{k,i})$ definable class is decided.) By (T1) and (T2), such a class B is closed and unbounded.

2.1. x_{ng} is not generic

Using (T1)–(T5), (R1), and (R2), we can prove that x_{ng} is not generic.

Lemma 2.2. *x_{ng} is non-constructible.*

Proof: Suppose not. Say $x_{\mathrm{ng}}(0) = 0$. Then

$$\Big\{\, t \in T^0_{0,0} : t \neq \mathrm{Decode}(x_{\mathrm{ng}},0) \cap \big(\sup(t)+1\big) \,\Big\}$$

is $\Delta_2(L)$ definable, and is dense, since $T^0_{0,0}$ is a normal tree and $\mathrm{Decode}(x_{\mathrm{ng}},0)$ is a cofinal branch through $T^0_{0,0}$. So, by (R2), it is met by $\mathrm{Decode}(x_{\mathrm{ng}},0)$, which is absurd. □

Lemma 2.3. *Suppose that V is an outer model of L, that $\langle V;\mathbb{P},\Vdash_0\rangle \vDash \mathrm{ZF}$, that G is $\Sigma_\omega(V;\mathbb{P},\Vdash_0)$ generic, that $V[G]$ is admissible, and that $x_{\mathrm{ng}} \in V[G]$. Then $x_{\mathrm{ng}} \in V$.*

Proof: The proof breaks into two claims. The first is that the branch through $T^0_{k,x_{\mathrm{ng}}(k)}$ coded by x_{ng} is definable over V (uniformly in k); the second claim is that $T^0_{k,1-x_{\mathrm{ng}}(k)}$ does not have such a V-definable branch. It follows then that $x_{\mathrm{ng}} \in V$.

Let $\mathring{x}$ be a Shoenfield term such that $\mathring{x}^G = x_{\mathrm{ng}}$.

Claim. *Fix a natural number k. There exists a condition p such that*

$$\alpha \in \mathrm{Decode}(x_{\mathrm{ng}}, k) \quad \text{iff} \quad p \Vdash \check{\alpha} \in \mathrm{Decode}(\mathring{x}, \check{k}).$$

Proof: For $q \in \mathbb{P}$, let $b(q)$ be the initial segment of $\mathrm{Decode}(\mathring{x}, \check{k})$ determined by q; that is

$$\alpha \in b(q) \quad \text{iff} \quad \forall \beta \leqslant \alpha \Big(q \Vdash \check{\beta} \in \mathrm{Decode}(\mathring{x}, \check{k}) \text{ or } q \Vdash \check{\beta} \notin \mathrm{Decode}(\mathring{x}, \check{k}) \Big) \text{ and}$$
$$q \Vdash \check{\alpha} \in \mathrm{Decode}(\mathring{x}, \check{k}).$$

If $q, q' \in \mathbb{P}$, and if $b(q)$ and $b(q')$ are incomparable under end-extension, declare that $\alpha \in c(q, q')$ iff

$$\exists \delta \in b(q) \Big(\mathrm{split}\big(b(q), b(q')\big) \leqslant \delta \leqslant \sup\big(b(q')\big) \text{ and } \alpha = \min\big(b(q') \setminus \delta\big) \Big)$$
$$\text{or}$$
$$\exists \delta \in b(q') \Big(\mathrm{split}\big(b(q), b(q')\big) \leqslant \delta \leqslant \sup\big(b(q)\big) \text{ and } \alpha = \min\big(b(q) \setminus \delta\big) \Big).$$

If one of $b(q)$ and $b(q')$ end-extends the other, let $c(q, q')$ be empty.

Fix $n > k$ and assume for a contradiction that the Claim fails. We maintain that

$$\begin{aligned}
\alpha \in \Omega^{\sigma(n)} \quad \text{iff} \quad & \alpha > \eta_*, \text{ and there exists an ordinal } \eta \\
& \text{and a condition } \hat{p} \in \mathbb{P} \text{ such that} \\
& \text{(a) } \forall \delta\ \exists q, q' \leqslant \hat{p}\ \sup\big(c(q, q')\big) \geqslant \delta; \\
& \text{(b) } \forall q, q', r, r' \leqslant \hat{p}\ \forall \beta \in c(q, q')\ \forall \gamma \in c(r, r') \\
& \qquad \big(\eta \leqslant \beta < \gamma \rightarrow L_\beta \prec_{\Sigma_{\sigma(n)}} L_\gamma\big); \text{ and} \\
& \text{(c) } \exists q, q' \leqslant \hat{p}\ \exists \beta \in c(q, q') \setminus \eta\ L_\alpha \prec_{\Sigma_{\sigma(n)}} L_\beta.
\end{aligned}$$

Note that establishing this will suffice for a contradiction. Indeed, the above provides a definition of $\Omega^{\sigma(n)}$ over $\langle V; \mathbb{P}, \Vdash_0 \rangle$ uniformly in n. This contradicts that $\langle V; \mathbb{P}, \Vdash_0 \rangle$ satisfies ZF, since the function $n \mapsto \min\big(\Omega^{\sigma(n)}\big)$ is cofinal in the ordinals of V.

To see the ($\Rightarrow$) implication, set $\eta = \min\big\{ \alpha \in \Omega^{\sigma(n)} : n(\alpha) \in E_{k,i} \big\}$ and choose $\hat{p} \in G$ such that

$$\hat{p} \Vdash \text{“}\mathrm{Decode}(\mathring{x}, \check{k}) \text{ is a cofinal branch through } T^{\check{n}}_{\check{k},\check{\imath}}\text{,”}$$

where $i = x_{\mathrm{ng}}(k)$. Then, for each $p \leqslant \hat{p}$, there exists $t \in T^n_{k,i}$ such that $b(p) \subseteq_{\mathrm{e}} t$. Using (T4), if $q, q' \leqslant \hat{p}$, then $c(q, q') \setminus \eta \subseteq \Omega^{\sigma(n)}$, and (b) is established.

Note next that on account of our assumption that the Claim fails, there exists an ordinal γ and conditions $p, p' \leqslant \hat{p}$ such that $\gamma \in b(p)$, but $\gamma \in \sup\big(b(p')\big) \setminus b(p')$. Then $b(q)$ and $b(q')$ are $\subseteq_{\mathrm{e}}$-incomparable, and $\mathrm{split}\big(b(q), b(q')\big) \leqslant \gamma$, whenever $q \leqslant p$ and $q' \leqslant p'$.

To see (a), given δ, choose $q \leqslant p$ and $q' \leqslant p'$ such that $\max(\eta, \delta, \gamma) \leqslant \sup\big(b(q)\big) \leqslant \sup\big(b(q')\big)$. This is possible by (T2) and our choice of $\hat{p}$. Then $c(q, q') \setminus \max(\eta, \delta) \neq \emptyset$, which suffices for (a).

Item (c) is similar, since by hypothesis $\alpha \in \Omega^{\sigma(n)}$.

To see the ($\Leftarrow$) implication, fix α, $\hat{p}$, and η as in (a)–(c). Set

$$C = \bigcup_{q,q' \leqslant \hat{p}} c(q,q') \setminus \eta.$$

Then C is unbounded in the ordinals by (a). If $\beta < \gamma$ in C, then $L_\beta \prec_{\Sigma_{\sigma(n)}} L_\gamma$. It follows that $C \setminus (\eta_* + 1)$ is an unbounded subclass of $\Omega^{\sigma(n)}$. Hence $\alpha \in \Omega^{\sigma(n)}$ by (c) and that $\alpha > \eta_*$. □

Set $B(p,k) = \left\{ \alpha : p \Vdash \check{\alpha} \in \mathrm{Decode}(\mathring{x}, \check{k}) \right\}$. By the previous claim, we have that for each k, there exists a $p \in \mathbb{P}$ such that $B(p,k)$ is a cofinal branch in $T^0_{k,x_{\mathrm{ng}}(k)}$.

Claim. *Fix $k \in \omega$ and $p \in \mathbb{P}$. Then $B(p,k)$ is not a cofinal branch through $T^0_{k,1-x_{\mathrm{ng}}(k)}$.*

Proof: Suppose not. Say $B(p_j,k)$ is a cofinal branch through $T^0_{k,j}$, for $j = 0,1$. Then $B(p_j,k)$ is a closed unbounded class contained in $\{ \alpha : n(\alpha) \in E_{k,j} \}$. Hence $B(p_0,k) \cap B(p_1,k)$ is a club class contained in $\{ \alpha : n(\alpha) \leqslant k \}$, contradicting that $\langle V; \mathbb{P}, \Vdash_0 \rangle \models \mathrm{ZF}$. □

Using these two Claims, the proof of Lemma 2.3 can be completed by noting that

$$x_{\mathrm{ng}}(k) = i \quad \text{iff} \quad \exists p \ \Big(B(p,k) \text{ is a cofinal branch through } T^0_{k,i} \Big),$$

and so $x_{\mathrm{ng}} \in V$. □

2.2. Construction of x_{ng}

This subsection reduces (R1) and (R2) to facts (P1)–(P4) regarding the orderings $\mathbb{P}^n$ mentioned earlier, and constructs the real x_{ng}, given a counting of L.

Working in L we shall define class forcing properties $\mathbb{P}^n$, for each natural number n. As n increases, $\mathbb{P}^n$ decreases under containment and has a definition of increasing logical complexity. The real x_{ng} is denoted by a Shoenfield term $\mathring{x}$ that is common to the forcing languages of all of the forcing properties $\mathbb{P}^n$. The ordering $\mathbb{P}^n$ is constructed so that if G is sufficiently $\mathbb{P}^n$ generic, then $\mathring{x}^G$ codes a suitably generic branch through $T^n_{k,i}$, where $\mathring{x}^G(k) = i$. In the next lemma, we construct a $G \subseteq \mathbb{P}^0$ such that $G \cap \mathbb{P}^n$ is suitably $\mathbb{P}^n$ generic, simultaneously for all n. This is how (R2) is arranged.

In order to arrange (R1), namely, that $L[x_{\mathrm{ng}}]$ is a cardinal and ZFC preserving extension of L, it suffices to arrange that $L[G \cap \mathbb{P}^n]$ is an L-cardinal preserving model of ZF^-_n, for each $n > 0$. This is because $L[x_{\mathrm{ng}}]$—an inner model of every $L[G \cap \mathbb{P}^n]$—is then an L-cardinal preserving model of ZF^-, and, consequently, a model of ZFC, since x_{ng} is a real. For this, it suffices to arrange that $\mathbb{P}^n$ satisfies strong predensity reduction, as well as that $G \cap \mathbb{P}^n$ is sufficiently generic.

Perhaps a distressing point in this outline is that $G \cap \mathbb{P}^n$ is required to be suitably generic, simultaneously for various n. This is impossible in the case of full genericity. However, if we ask only that $G \cap \mathbb{P}^n$ be partially generic, only a restricted version of the usual Truth Lemma holds, as was made precise in Proposition 1.6. If the complexity of $\mathbb{P}^{n+1}$'s definition is greater than the degree of genericity required of $G \cap \mathbb{P}^n$, suitable

$\mathbb{P}^n$ genericity does not preclude suitable $\mathbb{P}^{n+1}$ genericity. (This suggests why σ needs to be at least quadratic: $\mathbb{P}^{n+1}$ should be "invisible" to $\Pi_{n+2}(L;\mathbb{P}^n)$ genericity.)

Of course, these genericity requirements are not automatically compatible, either. We shall need to take measures while constructing the $T^n_{k,i}$'s to insure that it is possible to decide simply definable subclasses of $\mathbb{P}^n$ with conditions that lie in $\mathbb{P}^{n+1}$.

The orderings $\mathbb{P}^n$ will have the following properties. It is these, as well as properties (T1)–(T5) of the trees $T^n_{k,i}$ that the remaining sections of this paper will address.

- (P1) $\mathbb{P}^n$ is $\Delta_{\sigma(n)+1}\big(\{\eta_*\}\big)$ definable.
- (P2) The orderings $\mathbb{P}^n$ and $\mathbb{P}^{n+1}$ are related as follows:
  - (P2a) $\mathbb{P}^{n+1} \subseteq \mathbb{P}^n$;
  - (P2b) if $\alpha = \min\big(\Omega^{\sigma(n+1)}\big)$, then $\mathbb{P}^{n+1} \cap L_\alpha = \mathbb{P}^n \cap L_\alpha$; and
  - (P2c) if $m \leqslant n$ and $\bar{p} \in \mathbb{P}^n$ and D is a $\Pi_{\sigma(m)+m+2}(L)$ definable subclass of $\mathbb{P}^m$, then there exists $p \leqslant \bar{p}$ in $\mathbb{P}^n$ deciding D (in $\mathbb{P}^m$).
- (P3) $\mathbb{P}^n$ has these combinatorial properties:
  - (P3a) Compatibility in $\mathbb{P}^n$ is $\Delta_1(\emptyset;\mathbb{P}^n)$ definable; and
  - (P3b) $\mathbb{P}^n$ satisfies strong $\Sigma_\omega(L)$ predensity reduction.
- (P4) There exists a term $\mathring{x} \in L^{\mathbb{P}^n}$, for all n, such that if G is $\Pi_{n+2}(L;\mathbb{P}^n)$ generic, then
  - (P4a) $\mathring{x}^G\colon \omega \to 2$; and
  - (P4b) $\mathrm{Decode}(\mathring{x}^G, k)$ is $\Pi_{n+2}(L;T^n_{k,\mathring{x}^G(k)})$ generic, for all k.

Given that $x_{\text{ng}} = \mathring{x}^G$ when $G \cap \mathbb{P}^n$ is $\Pi_{n+2}(L;\mathbb{P}^n)$ generic, for all n, property (R2) is immediate from (P4b). (R1) is established in the second lemma below.

Lemma 2.4. (Construction of x_{ng}) *Suppose that L is a countable minimal model of* ZFC + $V{=}L$. *Then there exists a G such that $G \cap \mathbb{P}^n$ is $\Pi_{n+2}(L;\mathbb{P}^n)$ generic, for all n.*

PROOF: Let α_n be the least element of $\Omega^{\sigma(n)}$. Let $\langle\, \varphi_n : n \in \omega \,\rangle$ enumerate all formulas φ such that φ is a $\Pi_{\sigma(m)+m+2}(L)$ formula defining a subclass of $\mathbb{P}^m$, for some m. Assume, as well, that $\varphi_n \in L_{\alpha_{n+1}}$, and that it is for some $m \leqslant n$ that φ_n is a $\Pi_{\sigma(m)+m+2}(L)$ formula defining a subclass of $\mathbb{P}^m$.

Define a descending sequence of conditions $\langle\, p_n : n \in \omega \,\rangle$ such that $p_n \in \big(\mathbb{P}^n\big)^{L_{\alpha_{n+1}}}$ as follows: Let $p_0 \in \big(\mathbb{P}^0\big)^{L_{\alpha_1}}$ be arbitrary. Since $\eta_* < \alpha_1$ and $L_{\alpha_1} \prec_{\Sigma_5} L$, and since $\mathbb{P}^0$ is $\Delta_2\big(\{\eta_*\}\big)$ definable, we know that $\big(\mathbb{P}^0\big)^{L_{\alpha_1}}$ is non-empty. If $n > 0$, we have that $p_{n-1} \in \big(\mathbb{P}^{n-1}\big)^{L_{\alpha_n}}$ and $\big(\mathbb{P}^{n-1}\big)^{L_{\alpha_n}} = \mathbb{P}^{n-1} \cap L_{\alpha_n} \subseteq \mathbb{P}^n$ by (P1) and (P2b). Let $m \leqslant n$ be such that φ_n is a $\Pi_{\sigma(m)+m+2}(L)$ formula defining a subclass of $\mathbb{P}^m$. Using (P2c), there exists a condition p extending p_{n-1} in $\mathbb{P}^n$ and deciding (in $\mathbb{P}^m$) the class defined by φ_n. In fact, there exists such a condition in $\big(\mathbb{P}^n\big)^{L_{\alpha_{n+1}}}$. This is because

$$\exists p \leqslant p_{n-1} \Big(p \in \mathbb{P}^n \wedge \big(\exists q \;\; (q \in \mathbb{P}^m \wedge \varphi_n[q] \wedge p \leqslant q) \vee \\ \vee \forall q \;\, \forall q\,' \, (q, q' \in \mathbb{P}^m \wedge \varphi_n[q] \wedge q' \leqslant q \to q' \nleqslant p) \big) \Big)$$

is a $\Sigma_{\sigma(n)+n+4}(L)$ sentence, and $p_{n-1}, \varphi_n \in L_{\alpha_{n+1}} \prec_{\Sigma_{\sigma(n+1)}} L$. Let p_n be such a condition in $\left(\mathbb{P}^n\right)^{L_{\alpha_{n+1}}}$.

Set $G = \left\{ p \in \mathbb{P}^0 : p \geqslant p_n, \text{ for some } n \right\}$. Then $G \cap \mathbb{P}^m$ is a filter on $\mathbb{P}^m$, for each m. Furthermore, if X is a $\Pi_{m+2}(L;\mathbb{P}^m)$ definable subclass of $\mathbb{P}^m$, then X is $\Pi_{\sigma(m)+m+2}(L)$ definable, since $\mathbb{P}^m$ is $\Delta_{\sigma(m)+1}(L)$ definable. By construction, $G \cap \mathbb{P}^m$ includes a condition deciding X. □

Lemma 2.5. (Preservation of ZFC) *Suppose that $G \cap \mathbb{P}^n$ is a $\Pi_{n+2}(L;\mathbb{P}^n)$ generic filter on $\mathbb{P}^n$, for all n. Set $x_{\mathrm{ng}} = \mathring{x}^G$. Then $L[x_{\mathrm{ng}}] \models \mathrm{ZFC}$ and $L[x_{\mathrm{ng}}]$ is a cardinal preserving extension of L.*

Proof: We have that $G \cap \mathbb{P}^n$ is $\Pi_{n+2}(L;\mathbb{P}^n)$ generic, by hypothesis, and that $\mathbb{P}^n$ satisfies strong $\Sigma_\omega(L;\mathbb{P}^n)$ predensity reduction. Using that compatibility in $\mathbb{P}^n$ is $\Delta_1(\emptyset;\mathbb{P}^n)$ definable, it follows from Proposition 1.5 that $L[G \cap \mathbb{P}^n]$ is Σ_{n+1}-admissible. And $L[x_{\mathrm{ng}}]$ is a $\Sigma_1\left(\{x_{\mathrm{ng}}\}\right)$ definable inner model, so $L[x_{\mathrm{ng}}] \models \mathrm{ZFC}^-_{n+1}$.

Because this holds for all n, we have that $L[x_{\mathrm{ng}}] \models \mathrm{ZFC}^-$. To see that $L[x_{\mathrm{ng}}] \models \mathrm{ZFC}$, it suffices to see that $L[G \cap \mathbb{P}^1]$ is a cardinal preserving extension of L.

Claim. *Suppose that κ is an L-regular cardinal, and that $\mathring{f}$ is a term in $L^{\mathbb{P}^1}$ such that $\mathring{f}^G \colon \kappa \to \mathrm{OR}$. Then there exists a set $B \in L$ such that $|B|^L \leqslant \kappa$ and $\mathrm{rng}(\mathring{f}^G) \subseteq B$.*

Proof: Set $G^1 = \mathbb{P}^1 \cap G$. By hypothesis G^1 is a $\Pi_3(L;\mathbb{P}^1)$ generic filter on $\mathbb{P}^1$. By Proposition 1.6 we have that $L[G^1] \models \varphi$ iff $\exists p \in G^1 \;\; p \Vdash \varphi$, for all Σ_1 sentences of the forcing language. Choose $\bar{p} \in G^1$ such that $\bar{p} \Vdash \mathring{f} \colon \check{\kappa} \to \mathrm{OR}$. Using that $\mathbb{P}^1$ satisfies strong $\Sigma_\omega(L)$ predensity reduction, it follows that the class

$$\left\{ p : \exists B \;\; \left(|B| \leqslant \kappa \text{ and } p \Vdash \mathrm{rng}(\mathring{f}) \subseteq \check{B}\right) \right\}$$

is predense with respect to $\bar{p}$. And this class is $\Sigma_3(L;\mathbb{P}^1)$ definable by Proposition 1.3. It follows that there exists $p \in G^1$ meeting it. □

In fact, $L[G]$ is a cardinal preserving extension of L. This follows by the same proof, and that a $\Pi_2(L;\mathbb{P}^0)$ generic filter meets every $\Sigma_3(L;\mathbb{P}^0)$ definable dense class. Seeing this requires some work which we need not undertake.

3. Construction of the trees $T^n_{k,i}$

In this section the trees $T^n_{k,i}$ described in the previous section are constructed. Once we have constructed $\langle T^n_{k,i} : k \in \omega \text{ and } i = 0,1 \rangle$, an associated partial ordering $\mathbb{Q}^n$ can be defined. For each k, the forcing $\mathbb{Q}^n$ picks one of $T^n_{k,0}$ and $T^n_{k,1}$ and adds a cofinal branch through it. Precisely, elements of $\mathbb{Q}^n$ are functions q such that

- $\mathrm{dom}(q) = \omega$; and
- there exists a natural number $\mathrm{sp}(q)$ such that
  if $k < \mathrm{sp}(q)$, then $q(k) \in T^n_{k,0} \cup T^n_{k,1}$, and
  if $k \geqslant \mathrm{sp}(q)$, then $q(k) \in T^n_{k,0} \times T^n_{k,1}$.

We shall have that if $t \in T^n_{k,i}$, then $i \in t$ and $(1-i) \notin t$. Given $q \in \mathbb{Q}^n$, define $\hat{s}(q)\colon \mathrm{sp}(q) \to \{0,1\}$ by $\hat{s}(q)(k) = i$ iff $i \in q(k)$. Then $q(k) \in T^n_{k,\hat{s}(q)(k)}$, for all $k < \mathrm{sp}(q)$.

If $k < \mathrm{sp}(q)$ and $\hat{s}(q)(k) = i$, set $t_{k,i}(q) = q(k)$. If $k \geqslant \mathrm{sp}(q)$ and i is 0 or 1, let $t_{k,i}(q) \in T^n_{k,i}$ be such that $q(k) = \big(t_{k,0}(q), t_{k,1}(q)\big)$. Otherwise—that is, if (k,i) is not in $d\big(\hat{s}(q)\big)$—let $t_{k,i}(q)$ be undefined.

Let $\mathbb{Q}^n$ be ordered by declaring that $\bar{q} \geqslant q$ iff $\mathrm{sp}(\bar{q}) \leqslant \mathrm{sp}(q)$ and $t_{k,i}(q)$ end-extends $t_{k,i}(\bar{q})$ in $T^n_{k,i}$, for all k and i such that $t_{k,i}(q)$ is defined.

Conceptually, $\mathbb{Q}^n$ is one of two components of the partial ordering $\mathbb{P}^n$ mentioned in the previous section. The other is the coding apparatus defined in the next section. If G is a (suitably) generic filter on $\mathbb{Q}^n$, then $L[G]$ adds a Cohen real $x = \bigcup\big\{\, \hat{s}(q) : q \in G \,\big\}$, as well as a (suitably) generic branch through $T^n_{k,x(k)}$, for each k. The ordering $\mathbb{P}^n$ will constrain x so that, rather than being Cohen generic, it codes the branches added.

It is while constructing the $T^n_{k,i}$'s that the only real work to insure property (P2c) of $\mathbb{P}^n$ will be done. (Recall that (P2c) states that if $m \leqslant n$, then any condition in $\mathbb{P}^n$ can be extended in $\mathbb{P}^n$ to decide any simply definable subclass of $\mathbb{P}^m$.) Now $\mathbb{P}^n$, unlike $\mathbb{Q}^n$, imposes commitments on the growth of $\hat{s}(q)$ aimed at making the generic real x code branches through the $T^n_{k,x(k)}$'s. Because of this, we need a variant of $\mathbb{Q}^n$ in order to isolate the property of the $T^n_{k,i}$'s needed later to prove (P2c): If $z \subseteq \omega$, set

$$\mathbb{Q}^n_z = \Big\{\, q \in \mathbb{Q}^n : \hat{s}(q)(k) = 0, \text{ for all } k \in z \cap \mathrm{sp}(q) \,\Big\},$$

and let $\mathbb{Q}^n_z$ inherit the ordering of $\mathbb{Q}^n$.

The trees $T^n_{k,i}$ will be constructed to have properties (TC1)–(TC7) listed below. Listing these properties has three purposes. First of all, some serve as recursion hypotheses in the construction. Secondly, as we shall see presently, the facts (T1)–(T5) used in §2 follow easily from them. Finally, (TC1)–(TC7) will be used in §5 and §6 to prove facts (P1)–(P5).

It is convenient to set $\Omega^n_{k,i} = \big\{\, \alpha \in \Omega^n : n(\alpha) \in E_{k,i} \,\big\}$.

(TC1) $\{i\} \in T^n_{k,i}$; and if $t \in T^n_{k,i}$, then
- $t \subseteq \{i\} \cup \big\{\, \alpha : n(\alpha) \in E_{k,i} \,\big\}$ is closed;
- $i \in t$ and $(1-i) \notin t$; and
- $\sup(t) \in t$.

(TC2) If $t \in T^n_{k,i}$, then
- if $\sup(t) \geqslant \min\big(\Omega^{\sigma(n)}_{k,i}\big)$, then $\sup(t) \in \Omega^{\sigma(n)}_{k,i}$; and
- if $\alpha \in \Omega^{\sigma(n)}_{k,i} \setminus \sup(t)$, then $t \cup \{\alpha\} \in T^n_{k,i}$.

(TC3) If $m \leqslant n$, then
- $T^m_{k,i} \supseteq T^n_{k,i}$;
- $T^m_{k,i} = T^n_{k,i}$, if $k \geqslant n$; and
- $T^n_{k,i} \cap L_\alpha = T^{n+1}_{k,i} \cap L_\alpha$, if $\alpha = \min\Big(\Omega^{\sigma(n+1)}_{k,i}\Big)$.

(TC4) Suppose that $k < n$, and that $t, t' \in T^n_{k,i}$ are incomparable under end-extension. If $\delta \in t$ and

$$\min\bigl(\Omega^{\sigma(n)}_{k,i}\bigr), \operatorname{split}(t,t') \leqslant \delta \leqslant \sup(t'),$$

then $\min(t' \setminus \delta) \in \Omega^{\sigma(n)}$.

(TC5) Suppose that $\alpha \in \Omega^{\sigma(n)}_{k,i}$ and $\alpha = \sup\bigl(\Omega^{\sigma(n)}_{k,i} \cap \alpha\bigr)$. If $\langle\, t_\delta : \delta < \lambda \,\rangle$ is such that $t_\delta \in T^n_{k,i}$; $\gamma < \delta \Rightarrow t_\gamma \subseteq_{\mathrm{e}} t_\delta$; and $\sup_{\delta<\lambda} \sup(t_\delta) = \alpha$; then $\bigcup_{\delta<\lambda} t_\delta \cup \{\alpha\} \in T^n_{k,i}$.

(TC6) If $z \subseteq \omega$ and $m \leqslant n$ and D is a $\Sigma_{\sigma(m)+m+4}(L)$ definable dense subclass of $\mathbb{Q}^m_z$, then $D \cap \mathbb{Q}^n_z$ is dense in $\mathbb{Q}^n_z$.

(TC7) $T^n_{k,i}$ is $\Delta_1\bigl(\{\eta_*\}; \Omega^{\sigma(n)}\bigr)$, hence $\Delta_{\sigma(n)+1}\bigl(\{\eta_*\}\bigr)$ definable, uniformly in k and i. Thus $\mathbb{Q}^n$ is $\Delta_{\sigma(n)+1}\bigl(\{\eta_*\}\bigr)$ definable.

Note that (T1), (T3), (T4), and (T5) in §2 are immediate from (TC1), (TC3), (TC4), and (TC7), respectively. To see (T2), note first that $T^n_{k,i} \neq \emptyset$ by (TC1). If $t \in T^n_{k,i}$ and δ is an ordinal, then given any $\alpha, \beta \in \Omega^{\sigma(n)}_{k,i} \setminus \max\bigl(\delta, \sup(t)\bigr)$, both $t \cup \{\alpha\}$ and $t \cup \{\beta\}$ lie in $T^n_{k,i}$ by (TC2).

Let us now construct the trees $T^n_{k,i}$. Proceed by recursion on n. Begin by declaring that

$$
\begin{aligned}
t \in T^0_{k,i} \quad \text{iff} \quad & t \subseteq \{i\} \cup \bigl\{\, \alpha : n(\alpha) \in E_{k,i} \,\bigr\} \text{ is closed;} \\
& i \in t \text{ and } (1-i) \notin t; \\
& \sup(t) \in t; \text{ and} \\
& \sup(t) \in \Omega^{\sigma(0)}_{k,i}, \text{ if } |t| > 1.
\end{aligned}
$$

Before defining $T^{n+1}_{k,i}$, fix a $\Delta_1\bigl(\{\eta_*\}; \Omega^1\bigr)$ definable list $\langle\, (\varphi_\xi, \bar{q}_\xi, z_\xi) : \xi \in \mathrm{OR} \,\rangle$ of all triples $(\varphi, \bar{q}, z)$ such that $z \subseteq \omega$ and $\bar{q} \in \mathbb{Q}^0_z$ and φ is a $\Sigma_\omega(L)$ formula in one free variable. Let us also insist that $\varphi_\xi, \bar{q}_\xi \in L_\kappa$, where κ is the least uncountable cardinal greater than ξ. This sequence will be used for property (TC6).

If $k \geqslant n+1$, simply set $T^{n+1}_{k,i} = T^n_{k,i}$, in accord with (TC3). If $k < n+1$, proceed by induction on $\alpha \in \Omega^{\sigma(n+1)}_{k,i}$ to define $T^{n+1}_{k,i} \cap L_{\alpha^+}$. Note that such α are limit cardinals because $\sigma(n+1) \geqslant 2$.

Begin by letting $T^{n+1}_{k,i} \cap L_\alpha = T^n_{k,i} \cap L_\alpha$, if $\alpha = \min\bigl(\Omega^{\sigma(n+1)}_{k,i}\bigr)$, as required by (TC3).

Suppose now that $\alpha \in \Omega^{\sigma(n+1)}_{k,i}$ and that we have defined $T^{n+1}_{k,i} \cap L_\alpha$. Declare that $t \in S_\alpha$ iff $t \in T^{n+1}_{k,i} \cap L_\alpha$ or $t = t' \cup \{\alpha\}$, for some $t' \in T^{n+1}_{k,i} \cap L_\alpha$. In each case we shall insure that $S_\alpha \subseteq T^{n+1}_{k,i} \cap L_{\alpha^+}$.

Case 1. $\boldsymbol{\sup\bigl(\Omega^{\sigma(n+1)}_{k,i} \cap \alpha\bigr) \notin \Omega^{\sigma(n+1)}_{k,i}}$. Set $T^{n+1}_{k,i} \cap L_{\alpha^+} = S_\alpha$.

Case 2. $\boldsymbol{\alpha = \sup\bigl(\Omega^{\sigma(n+1)}_{k,i} \cap \alpha\bigr)}$. Declare that $t \in T^{n+1}_{k,i} \cap L_{\alpha^+}$ iff $t \in S_\alpha$ or there exists $\langle\, t_\delta : \delta < \lambda \,\rangle$ such that

- $t_\delta \in T^{n+1}_{k,i} \cap L_\alpha$;
- $\gamma < \delta \Rightarrow t_\gamma \subseteq_{\mathrm{e}} t_\delta$;
- $\sup_{\delta<\lambda} \sup(t_\delta) = \alpha$; and
- $t = \bigcup_{\delta<\lambda} t_\delta \cup \{\alpha\}$.

Case 3. $\boldsymbol{\sup(\Omega^{\sigma(n+1)}_{k,i} \cap \alpha) \in \Omega^{\sigma(n+1)}_{k,i} \cap \alpha}$. Set $\beta = \sup\big(\Omega^{\sigma(n+1)}_{k,i} \cap \alpha\big)$, and set $X = \big\{\, \gamma \in \Omega^{\sigma(n+1)} \cap \beta : n(\gamma) = n(\beta) \,\big\}$. Then $X \subseteq \Omega^{\sigma(n+1)}_{k,i}$. Let ξ be the order type of X.

Set $T^{n+1}_{k,i} \cap L_{\alpha^+} = S_\alpha$, *unless*

- φ_ξ is a $\Sigma_{\sigma(n)+n+4}(L_\alpha)$ formula;
- $D = \Big\{\, q \in \mathbb{Q}^n_{z_\xi} \cap L_\alpha : L_\alpha \models \varphi_\xi[q] \,\Big\}$ is dense in $\mathbb{Q}^n_{z_\xi} \cap L_\alpha$;
- $\bar{q}_\xi \in \mathbb{Q}^{n+1}_{z_\xi} \cap L_\beta$, $k < \mathrm{sp}(\bar{q}_\xi)$, and $\hat{s}(\bar{q}_\xi)(k) = i$; and
- $n(\beta) \in E_{m,\hat{s}(\bar{q}_\xi)(m)}$, for all $m < \mathrm{sp}(\bar{q}_\xi)$.

In this case, let q' be identical with $\bar{q}_\xi$, except that $q'(m) = \bar{q}_\xi(m) \cup \{\beta\}$, for each $m < \mathrm{sp}(\bar{q}_\xi)$. Then $q' \in \mathbb{Q}^n_{z_\xi} \cap L_{\beta^+}$ (by inductive appeal to (TC2)). Let q extending q' in $\mathbb{Q}^n_{z_\xi} \cap L_\alpha$ be L-least meeting D, and set $T^{n+1}_{k,i} \cap L_{\alpha^+} = S_\alpha \cup \big\{\, t_{k,i}(q) \cup \{\alpha\} \,\big\}$.

Let us now check properties (TC1)–(TC7).

Properties (TC1) and (TC2) are clear from the construction. Property (TC5) is evident for $T^0_{k,i}$ and was explicitly ensured for $T^{n+1}_{k,i}$ by Case 2 in the construction. Property (TC7) is evident, as well, since the definition of $T^{n+1}_{k,i} \cap L_{\alpha^+}$ is by recursion on $\alpha \in \Omega^{\sigma(n+1)}_{k,i}$.

The second two clauses in (TC3) were explicitly insured in the construction. To see that $T^m_{k,i} \supseteq T^{n+1}_{k,i}$ whenever $m < n+1$, note first that this is trivial if $k \geqslant m$, since then $T^m_{k,i} = T^0_{k,i}$ and certainly $T^{n+1}_{k,i} \subseteq T^0_{k,i}$. If $k < m$, let us proceed by induction on $\alpha \in \Omega^{\sigma(n+1)}_{k,i}$ to see that every $t \in T^{n+1}_{k,i} \cap L_{\alpha^+}$ lies in $T^m_{k,i}$. If $\alpha = \sup\big(\Omega^{\sigma(n+1)}_{k,i} \cap \alpha\big)$, then $\alpha = \sup\big(\Omega^{\sigma(m)}_{k,i} \cap \alpha\big)$ and $\alpha \in \Omega^{\sigma(m)}_{k,i}$. Thus α fell under Case 2 in the construction of both $T^m_{k,i}$ and $T^{n+1}_{k,i}$. Using this, as well as induction, it follows that $T^{n+1}_{k,i} \cap L_{\alpha^+} \subseteq T^m_{k,i} \cap L_{\alpha^+}$.

If α fell under Case 1 in the construction of $T^{n+1}_{k,i}$ and $t \in T^{n+1}_{k,i} \cap \big(L_{\alpha^+} \setminus L_\alpha\big)$, then $t = t' \cup \{\alpha\}$, for some $t' \in T^{n+1}_{k,i} \cap L_\alpha$. By (TC2) and induction, it follows that $t \in T^m_{k,i} \cap L_{\alpha^+}$.

Finally, if α fell under Case 3, then again (TC2) and induction suffice, unless $t = t_{k,i}(q) \cup \{\alpha\}$, in the notation of that case. But $q \in \mathbb{Q}^n$, so $t_{k,i}(q) \in T^m_{k,i}$ by induction. Hence $t \in T^m_{k,i}$.

For (TC4), assume that $k < n+1$ and again proceed by induction on $\alpha \in \Omega^{\sigma(n+1)}_{k,i}$. Note that in each of Cases 1, 2, and 3, if $t \in T^{n+1}_{k,i} \cap \big(L_{\alpha^+} \setminus L_\alpha\big)$, then $\sup(t) = \alpha$. (In fact, this must be the case by (TC2).) Suppose that $t, t' \in T^{n+1}_{k,i} \cap L_{\alpha^+}$ are incomparable under end-extension, that $\delta \in t$, and that $\min\big(\Omega^{\sigma(n+1)}_{k,i}\big), \mathrm{split}(t,t') \leqslant \delta \leqslant \sup(t')$. Set $\mu = \min(t' \setminus \delta)$. Then $\mu \leqslant \alpha$.

If α is least in $\Omega^{\sigma(n+1)}_{k,i}$, then $\alpha = \min\big(\Omega^{\sigma(n+1)}_{k,i}\big) \leqslant \delta \leqslant \mu \leqslant \alpha$, so $\mu = \alpha \in \Omega^{\sigma(n+1)}$.

Assume now that $\alpha > \min\big(\Omega^{\sigma(n+1)}_{k,i}\big)$. If $\mu = \alpha$, then $\mu \in \Omega^{\sigma(n+1)}$; so assume also that $\mu < \alpha$. If α falls under either of Cases 1 and 2, then it follows by induction that $\mu \in \Omega^{\sigma(n+1)}$. If α falls under Case 3, then, in the notation of that case, we may assume by induction that $\mu \geqslant \beta$. We maintain that $\mu = \beta$, and hence $\mu \in \Omega^{\sigma(n+1)}$, as required. Indeed, if $\mu > \beta$, then it must be that $t' = t_{k,i}(q) \cup \{\alpha\}$, since $\mu < \alpha$. It follows that $t \cap (\beta, \alpha) = \emptyset$, since $t \neq t'$. So $\delta \leqslant \beta$, since $\delta \in t$ and $\delta \leqslant \mu < \alpha$. Now $\beta \in t'$ because $\beta \in t_{k,i}(q)$ by construction. Thus $\mu = \min(t' \setminus \delta) \leqslant \beta$, contradicting our hypothesis that $\mu > \beta$.

For (TC6), note first that it suffices to see that if D is a $\Sigma_{\sigma(n)+n+4}(L)$ definable dense subclass of $\mathbb{Q}^n_z$, then $D \cap \mathbb{Q}^{n+1}_z$ is dense in $\mathbb{Q}^{n+1}_z$. This can be seen by induction on n, using (TC7) and that $\sigma(m+1) = \sigma(m) + m + 4$.

Fix a $z \subseteq \omega$, fix a $\Sigma_{\sigma(n)+n+4}(L)$ formula φ defining a dense subclass D of $\mathbb{Q}^n_z$, and fix a condition $\bar{q} \in \mathbb{Q}^{n+1}_z$. We seek a condition $\hat{q}$ extending $\bar{q}$ in $\mathbb{Q}^{n+1}_z$ and meeting D. Without loss of generality, $\mathrm{sp}(\bar{q}) \geqslant n+1$. Set $\hat{s} = \hat{s}(\bar{q})$ and choose $n' \in \bigcap_{k<\mathrm{sp}(\bar{q})} E_{k,\hat{s}(k)}$ such that $n' \geqslant \sigma(n+1)$. Let ξ be such that $(\varphi_\xi, \bar{q}_\xi, z_\xi) = (\varphi, \bar{q}, z)$, let β be the ξ^{th} element of the class $\{\, \gamma \in \Omega^{\sigma(n+1)} : n(\gamma) = n' \,\}$, and let α_k be least in $\Omega^{\sigma(n+1)}_{k,\hat{s}(k)}$ greater than β, for $k < n+1$.

Now $\varphi = \varphi_\xi$ and $\bar{q} = \bar{q}_\xi$ lie in L_{α_k}, and α_k is a limit cardinal, so certainly z_ξ lies in L_{α_k}. Also $L_{\alpha_k} \prec_{\Sigma_{\sigma(n+1)}} L$ and $\sigma(n+1) = \sigma(n) + n + 4$, so L_{α_k} satisfies that $\{\, q \in \mathbb{Q}^n_z : \varphi[q] \,\}$ is dense in $\mathbb{Q}^n_z \cap L_{\alpha_k}$.

Stage α_k in the construction of $T^{n+1}_{k,\hat{s}(k)}$ then fell under Case 3, and we chose q extending $\bar{q}$ in $\mathbb{Q}^n_z \cap L_\alpha$ such that q meets D, and put $t_{k,\hat{s}(k)}(q) \cup \{\alpha_k\}$ into $T^{n+1}_{k,\hat{s}(k)}$. Furthermore, we chose q canonically, so our choice of q at stage α_k in the construction of $T^{n+1}_{k,\hat{s}(k)}$ was the same for all $k < n+1$. Let $\hat{q}$ be identical with q, except that $t_{k,\hat{s}(k)}(\hat{q}) = t_{k,\hat{s}(k)}(q) \cup \{\alpha_k\}$, for $k < n+1$. Then $\hat{q} \in \mathbb{Q}^n_z$ and meets D. Furthermore, $t_{k,\hat{s}(k)}(\hat{q}) \in T^{n+1}_{k,\hat{s}(k)}$, for $k < n+1$, and $T^{n+1}_{k,i} = T^n_{k,i}$, for $k \geqslant n+1$. So $\hat{q} \in \mathbb{Q}^{n+1}_z$, as required.

4. Coding

This section develops the building blocks of a simplified version of Jensen coding based on that in [**S1**]. Because we shall be coding a generic extension of L, and because there is no need to preserve large cardinal properties in the extension, substantial simplifications are possible. In fact, the coding developed here is a simplified version of that in [**S1**], where one concern is preserving large cardinals.

Unlike Jensen's, our coding conditions will have Easton support. Furthermore, the coding of intervals $[\alpha, \alpha^+)$ will be essentially the same, whether α is regular or singular. Our coding does render all inaccessibles non-Mahlo. This is merely a convenience. It permits us to isolate efforts towards coding the extension on intervals $[\alpha, \alpha^+)$, for different cardinals α.

Nothing more elaborate than ordinary condensation and the statement of $\square_\infty$ will be needed. No use of our assumption that L is minimal will be made.

4.1. Threads

Threads are the price to pay for a unified treatment of regular and singular cardinals. This uniform treatment makes extending coding conditions trivial. In Jensen coding, extending conditions is entwined with distributivity.

When α is infinite and regular, the ordinary almost disjoint coding of a subset $X \subseteq [\alpha, \alpha^+)$ proceeds by fixing a ground model sequence $\langle\, b_\xi : \xi \in [\alpha, \alpha^+) \,\rangle$ of pairwise almost disjoint unbounded subsets of α, then producing generically a coding set $x \subseteq \alpha$ such that $\xi \in X$ iff $|x \cap b_\xi| < \alpha$. In order to do the same sort of thing when α is singular, instead of a sequence of unbounded sets of ordinals below α, we shall use a sequence of "indomitable" sets of "threads" below α at regular and singular α alike. These sets of threads will be pairwise almost disjoint in an appropriate sense.

Let CARD be the class of all infinite L-cardinals, together with 0. Let α^+ denote the least element of CARD greater than α. Say that a set of ordinals is *Easton* when it is bounded below every infinite regular cardinal.

A *thread* is a non-empty Easton set of ordinals u such that $|u \cap [\alpha, \alpha^+)| \leqslant 1$, for all $\alpha \in \mathrm{CARD}$. If u is a thread, define its *support* by

$$\mathrm{sp}(u) = \Big\{\, \alpha \in \mathrm{CARD} : u \cap [\alpha, \alpha^+) \neq \emptyset \,\Big\}.$$

Say that a thread u is *below* α when $u \subseteq \alpha$. If $\alpha \in \mathrm{sp}(u)$, let u_α be the ordinal β such that $u \cap [\alpha, \alpha^+) = \{\beta\}$.

If a and b are sets of ordinals, say that a is *cofinal* in b if $a \cap b \setminus \beta \neq \emptyset$, for all $\beta \in b$.

A set b of threads below α is *indomitable* iff whenever $e \subseteq \alpha$ is Easton, there exists a thread $u \in b$ such that $\sup(u) \geqslant \sup(e)$ and e is not cofinal in u.

If α is an infinite cardinal, let α^* denote the least regular cardinal greater than or equal to α.

The non-indomitable sets of threads below α form a $<\!\alpha^*$-complete ideal on the threads below α. In this sense, an indomitable set of threads below α has "positive measure."

Lemma 4.1. *If b is an indomitable set of threads below α and X is a set of fewer than α^* many threads below α, then there exist indomitably many threads in b in which no thread from X is cofinal.*

It follows that if b is an indomitable set of threads below α and $X \subseteq b$ is a set of fewer than α^* many threads, then $b \setminus X$ is indomitable.

PROOF: If α is regular, then $\cup X \setminus |X|$ is Easton.

If α is singular, set $\lambda = \mathrm{cf}(\alpha)$ and let $\langle\, \beta_i : i < \lambda \,\rangle$ be a monotonically increasing sequence of cardinals that is cofinal in α. Let $e \subseteq \alpha$ be Easton. Suppose $X = \bigcup_{i<\lambda} X_i$, where $|X_i| \leqslant \beta_i$, and set

$$e' = \Big\{\, \beta_i : i < \lambda \,\Big\} \cup \bigcup_{i<\lambda} \Big((\cup X_i) \setminus \beta_i \Big) \cup e.$$

Then $e' \setminus \lambda$ is Easton and $\sup(e') = \alpha$. If $u \in b$ is such that $\sup(u) = \alpha$ and e' is not cofinal in u, and if $u' \in X_i$, then either $u' \subseteq \beta_i$, or a tail of u' is contained in e'. In either case, u' is not cofinal in u. □

Suppose that $\hat{\alpha}$ is admissible and that $x \in L_{\hat{\alpha}}$ is a parameter. Fix a natural number $n \geqslant 1$, and define an ordinal θ and a sequence $\langle\, \alpha_i : i < \theta \,\rangle$ by setting

$$\alpha_i = \text{the least } \alpha \text{ such that } L_\alpha \prec_{\Sigma_n} L_{\hat{\alpha}} \text{ and } \{\, \alpha_j : j < i \,\} \cup \{x\} \subseteq L_\alpha,$$

for i such that this α is less that $\hat{\alpha}$; let θ be the least i such that α_i is not defined. Call the sequence $\langle\, \alpha_i : i < \theta \,\rangle$ the *canonical Σ_n-tower approximating $L_{\hat{\alpha}}$ above x* and call θ its *height*. For infinite $\gamma \in \big[|\theta|, \hat{\alpha}\big) \cap \mathrm{CARD}$, set

$$M_x^\gamma = \Sigma_n\text{-Skolem Hull}_{L_{\hat{\alpha}}}\Big(\gamma \cup \{\gamma, x\} \cup \{\, \alpha_j : j < \theta \,\}\Big).$$

Define $f_x \colon \mathrm{CARD} \cap \big[|\theta|, \hat{\alpha}\big) \setminus \omega \to \hat{\alpha}$ by setting $f_x(\gamma) = M_x^\gamma \cap \gamma^+$.

Lemma 4.2. (Thread Lemma) *Suppose that α is an infinite cardinal. Then there exists a (uniformly $\Delta_1\big(\{\alpha\}\big)$ definable over L_{α^+}) sequence $\langle\, b_\xi : \xi \in [\alpha, \alpha^+) \,\rangle$ having the following four properties:*

(1) *b_ξ is an indomitable collection of threads below α.*

(2) *Suppose that $\alpha \leqslant \xi, \zeta < \alpha^+$ and that $\xi \neq \zeta$. Then*

$$\Big|\{\, u \in b_\zeta : u \text{ is cofinal in some } u' \in b_\xi \,\}\Big| < \alpha.$$

(3) *Suppose that $\hat{\alpha} > \alpha$ is a limit cardinal, that $n \geqslant 1$ is a natural number, that $x \in L_{\hat{\alpha}}$ is a parameter, and that $|\theta| \leqslant \alpha$, where θ is the height of the canonical Σ_n-tower approximating $L_{\hat{\alpha}}$ above x. Suppose that $\xi \in [\alpha, \alpha^+)$, that $u \in b_\xi$, and that $u_\delta < f_x(\delta)$, for δ cofinal in $\mathrm{sp}(u)$. Then $u \in M_x^\delta$, for some $\delta \in \mathrm{sp}(u)$.*

(4) *If $u \in b_\xi$, where $\xi \in [\omega, \omega_1)$, then $|u| = 1$.*

Conclusions (1) and (2) hold that the codes b_ξ are analogous to ordinary almost disjoint codes with "indomitable" in place of "unbounded in α," and with conclusion (2) in place of "pairwise almost disjoint." Conclusion (3) is only used in verifying predensity reduction in §6. Conclusion (4) facilitates dealing with $\mathbb{Q}^n$ without coding constraints in $\mathbb{P}^n$ interfering, *cf.*, (TC6) in §3.

PROOF: ***Case 1. α is regular.*** Let F be the L-least one-to-one function from $2^{<\alpha}$ into the collection of threads u below α such that $|u| = 1$. For $\xi \in [\alpha, \alpha^+)$, let $\hat{b}_\xi$ be the L-least path through $2^{<\alpha}$ such that $\hat{b}_\xi \neq \hat{b}_\zeta$, for all $\zeta < \xi$. Finally, set $b_\xi = F``\hat{b}_\xi$.

Properties (1), (2), and (4) are insured by our choice of b_ξ. Property (3) is trivial, because if $u_\delta < f_x(\delta)$, for some $\delta \in \mathrm{sp}(u)$, then $u \in M_x^\delta$, since $|u| = 1$.

Case 2. α is singular. For $i < \alpha^+$, set

$$\mu_i = \begin{cases} \text{the least } \mu > \alpha, \sup_{j<i} \mu_j \text{ such that } L_\mu \text{ is a model} \\ \text{of } \mathrm{ZF}^- + \forall x\ |x| \leqslant \alpha + \text{“}\alpha \text{ is singular.”} \end{cases}$$

Then every element of L_{μ_i} is definable over L_{μ_i} from parameters in $\alpha \cup \{\alpha, \sup_{j<i} \mu_j\}$. (In fact, the parameter α is unnecessary.) By induction on i, note also that $\mu_i > i$, for all $i < \alpha^+$.

Set $N_i = L_{\mu_i}$ and $N_i^\gamma = \Sigma_\omega\text{-Skolem Hull}_{N_i}\big(\gamma \cup \{\gamma, \alpha, \sup_{j<i} \mu_j\}\big)$, for $\gamma \in \mathrm{CARD} \cap \alpha$. Then $N_i = \bigcup_{\gamma \in \mathrm{CARD} \cap \alpha} N_i^\gamma$. Set $u^i = \{\, N_i^\gamma \cap \gamma^+ : \gamma \in \mathrm{CARD} \cap \alpha \,\}$.

Note that if $g \in N_i$ is a function with $\mathrm{dom}(g) \subseteq \mathrm{CARD}$ satisfying $g(\gamma) < \gamma^+$, for all $\gamma \in \mathrm{dom}(g)$, then there exists a bound $\beta < \alpha$ such that $g(\gamma) < u^i_\gamma$, for all $\gamma \in \mathrm{dom}(g) \setminus \beta$. (Choose β such that $g \in N^\beta_i$.) Note also that if $j < i < \alpha^+$, then $u^j \in N_i$; consequently, $u^i \cap u^j$ is bounded below α.

Let $\ll\xi,\zeta\gg$ indicate the Gödel pairing of the ordinals ξ and ζ. Fix an Easton set of cardinals $E \in L_{\mu_0}$ that is cofinal in α. (It is for this that we insisted that α be singular in L_{μ_0}.) For $\xi \in [\alpha, \alpha^+)$, declare that

$$u \in b_\xi \quad \text{iff} \quad \mathrm{sp}(u) = E \text{ and, for some } i < \alpha^+,\ u_\delta = u^{\ll\xi,i\gg}_\delta, \text{ for all } \delta \in E.$$

Then b_ξ is indomitable, since $\{\, \ll\xi,i\gg : i < \alpha^+ \,\}$ is cofinal in α^+, and if $e \in N_j$ is Easton, then e is not cofinal in u^j. (Consider $g(\gamma) = \sup(e \cap \gamma^+)$, for $\gamma \in \mathrm{CARD} \cap \alpha$.)

If $\alpha \leqslant \zeta, \xi < \alpha^+$ and $\xi \neq \zeta$, and if $u \in b_\zeta$ and $u' \in b_\xi$, then $u \cap u'$ is bounded below α. Hence $\{\, u \in b_\zeta : u \text{ is cofinal in some } u' \in b_\xi \,\}$ is empty.

Finally, to verify (3) in the statement of the lemma, suppose that $u \in b_\xi$. Say that $u_\delta = u^i_\delta$, for all $\delta \in E$. Assume that

$$u \notin M' = \bigcup_{\delta\in E} M^\delta_x = \Sigma_n\text{-Skolem Hull}_{L_{\hat\alpha}}\Big(\alpha \cup \{x\} \cup \{\, \alpha_i : i < \theta \,\}\Big),$$

where $\langle\, \alpha_i : i < \theta \,\rangle$ is the canonical Σ_n-tower approximating $L_{\hat\alpha}$ above x. We must see that $u^i_\delta \geqslant f_x(\delta)$, for all sufficiently large $\delta \in E$.

Set $\beta = M' \cap \alpha^+$ and choose η so that $L_\eta \cong M'$. Then $\alpha \leqslant \beta \leqslant \eta < \alpha^+$. Furthermore, $u^j \in L_\eta$, for all $j \in [\alpha, \beta)$. It follows that $\mu_i > i \geqslant \beta$. (If $i < \beta$, then $\beta > \alpha$. But then $\alpha \in M'$ and $M' \vDash$ "α is singular;" consequently E, hence u, lies in M'.)

But then $\mu_i > \eta$. Indeed, we may assume that $\eta > \alpha$. On the one hand, if $\beta > \alpha$, then $L_\eta \nvDash |\beta| = \alpha$, and, on the other, if $\beta = \alpha$, then $L_\eta \vDash$ "α is regular."

Let $\pi\colon L_\eta \to L_{\hat\alpha}$ be the inverse of the transitive collapse of M' to L_η. Then π is a Σ_n-elementary embedding. Set $\bar x = \pi^{-1}(x)$ and let $\langle\, \bar\alpha_i : i < \bar\theta \,\rangle$ be the canonical Σ_n-tower approximating L_η above $\bar x$. Then $\{\, \bar\alpha_i : i < \bar\theta \,\} \in L_{\mu_i}$, since $\mu_i > \eta$. Using that $\alpha_i \in \mathrm{rng}(\pi)$, for all $i < \theta$, and that $L_\delta \prec_{\Sigma_n} L_\eta$ iff $L_{\pi(\delta)} \prec_{\Sigma_n} L_{\hat\alpha}$, it follows that $\bar\theta = \theta$ and that $\pi^{-1}(\alpha_i) = \bar\alpha_i$, for all $i < \theta$. Consequently, $f_x \restriction \alpha$ is definable over L_{μ_i}; hence $f_x(\delta) < u^i_\delta$, for all sufficiently large $\delta \in E$. □

4.2. Colors

Different sorts of threads will be used for coding different intervals $[\alpha, \alpha^+)$ so that efforts to code one interval do not interfere with those to code another. Two mechanisms will be useful, namely, *colors* and *supports.*

Recall that a thread u has support $\mathrm{sp}(u) = \{\, \alpha \in \mathrm{CARD} : u \cap [\alpha, \alpha^+) \neq \emptyset \,\}$.

For each ordinal δ, let Z_δ be the class of all ordinals of the form $\ll\delta,\xi\gg$, and say that ordinals in Z_δ have *color* δ. Say that a thread u has color δ when $u \subseteq Z_\delta$. Clearly, if u and u' are threads of different colors, then $u \cap u' = \emptyset$.

Our assignment of colors and possible supports to infinite cardinals will be one-to-one. The assignment of possible supports takes place in the ground model. The assignment of colors is generic, though for successor and singular cardinals it effectively takes place in the ground model. The following table summarizes these assignments.

Colorization Table

cardinal	possible supports	possible colors
α^+	$\{\alpha\}$	$\{\ll 2, \alpha \gg\}$
singular α	$\{\, C_\alpha \setminus \mathrm{ot}(C_\alpha)\, \}$	$\{\ll 1, \mathrm{ot}(C_\alpha) \gg\}$
inaccessible α	$\mathrm{CARD} \cap \alpha$	$\{\ll 0, \gamma \gg : \gamma < \alpha\}$

Here $\langle C_\lambda : \lambda \text{ is a singular cardinal} \rangle$ is a fixed $\Delta_1(\emptyset; \mathrm{CARD})$ definable $\square_\infty$-sequence with the property that $C_\lambda \in L_\mu$, where μ is the least ordinal ZF^--ordinal such that λ is singular in L_μ. (Standard $\square_\infty$-sequences have this property.)

Let SP_α be the set of possible supports and COL_α the set of possible colors assigned to α in the above table.

Lemma 4.3. (Colorized Thread Lemma) *Let $c\colon \mathrm{CARD} \to \mathrm{OR}$ be a partial function assigning colors as in the above table. (That is, $c(\alpha) \in \mathrm{COL}_\alpha$, for $\alpha \in \mathrm{dom}(c)$.) There exists a sequence (uniformly $\Delta_1(\{\alpha\})$ definable over L_{α^+})*

$$\left\langle b_\xi^{c(\alpha)} : \alpha \in \mathrm{dom}(c) \text{ and } \xi \in [\alpha, \alpha^+) \right\rangle$$

such that

(1) *$b_\xi^{c(\alpha)}$ is a collection of threads below α such that if $u \in b_\xi^{c(\alpha)}$, then u has color $c(\alpha)$ and $\mathrm{sp}(u) \in \mathrm{SP}_\alpha$.*

(2) *Conclusions (1)–(4) of the Thread Lemma hold with $b^{c(\alpha)}$ in place of b.*

The proof of the Colorized Thread Lemma is a minor variation on the proof of the Thread Lemma. (In the case of successor cardinals α^+, require that the range of F includes only threads above α. In the case of singular α, let $E = C_\alpha \setminus \mathrm{ot}(C_\alpha)$. In each case, use only threads of the correct color.)

The following consequence of our colorization scheme will be used later.

Lemma 4.4. *Assume that $c\colon \mathrm{CARD} \to \mathrm{OR}$ is an Easton support partial function assigning colors as in the colorization table, and that c is one-to-one on inaccessible cardinals. Suppose that $\alpha, \alpha' \in \mathrm{dom}(c)$ and $\alpha \neq \alpha'$; that $\xi \in [\alpha, \alpha^+)$ and $\xi' \in [\alpha', \alpha'^+)$; and that $u \in b_\xi^{c(\alpha)}$ and $u' \in b_{\xi'}^{c(\alpha')}$. Then u is not cofinal in u'.*

Proof: The lemma is clear unless α and α' are singular cardinals such that $\mathrm{ot}(C_\alpha) = \mathrm{ot}(C_{\alpha'})$. But then $\mathrm{sp}(u) = C_\alpha \setminus \mathrm{ot}(C_\alpha)$ and $\mathrm{sp}(u') = C_{\alpha'} \setminus \mathrm{ot}(C_{\alpha'})$ are eventually disjoint. $\square$

4.3. Decoding

If $x\colon\omega \to \{0,1\}$ is a sufficiently generic real added by the forcing $\mathbb{P}^n$, then a branch through $T^n_{k,x(k)}$ is definable from x, for each k. In this section we make explicit how these branches are defined; that is, we define the class $\mathrm{Decode}(x,k)$ mentioned in §2.

A branch through $T^n_{k,x(k)}$ is recovered in three steps. First, a sequence $\langle\, P^k_\omega : k \in \omega \,\rangle$ of subsets of $[\omega,\omega_1)$ is recovered from x. Next, sets $B^k_\gamma \subseteq [\gamma,\gamma^+)$ are recovered from P^k_ω by recursion on CARD. Finally, a branch through $T^n_{k,x(k)}$ is extracted from the B^k_γ's.

To be as described in §2, the class $\mathrm{Decode}(x,k)$ must be uniformly $\Delta_2(\emptyset)$ definable, for all reals x preserving admissibility in the universe, not just those which happen to code branches through $T^n_{k,x(k)}$.

Assume that the ambient universe is at least admissible, and fix any real x. Declare that

$$\begin{aligned} \xi \in P^k_\omega \quad \text{iff} \quad & \xi \in [\omega,\omega_1^L) \text{ and} \\ & \big\{\, u \in b^{\ll 2,0\gg}_{\ll \xi, \ll k, x(k)\gg\gg} : x^{-1}(1) \text{ is cofinal in } u \,\big\} \text{ is finite.} \end{aligned}$$

(Recall that the Colorization Table gives $\omega = 0^+$ the color $\ll 2,0 \gg$.)

Next, we define sets of ordinals $B^k_\gamma, P^k_\gamma \subseteq [\gamma,\gamma^+)$ by recursion on CARD. Once we have succeeded in this, we shall declare that

$$\zeta \in \mathrm{Decode}(x,k) \qquad \text{iff} \qquad \omega_1 + \zeta \in \textstyle\bigcup_{\gamma \in \mathrm{CARD}\cap[\omega,\infty)} B^k_\gamma.$$

Because colors are assigned generically by $\mathbb{P}^n$, we shall need to recover simultaneously a coloring of cardinals. Consequently, we must simultaneously define B^k_γ, P^k_γ, and a function c^k such that $c^k(\gamma) < \gamma$. (In fact, if x is $\mathbb{P}^n$ generic, then $c^k = c^{k'}$, for all k and k'.)

We have already defined P^k_ω. Set $B^k_\omega = \emptyset$, and set $c^k(\omega) = \ll 2, 0 \gg$.

Suppose now that $\gamma \in \mathrm{CARD}$ is greater than ω. Define $c^k(\gamma)$ as follows: If $\gamma = \beta^+$, set $c^k(\gamma) = \ll 2, \beta \gg$. If γ is singular in L, set $c^k(\gamma) = \ll 1, \mathrm{ot}(C_\gamma) \gg$, where C_γ is provided by our fixed $\Delta_1(\emptyset;\mathrm{CARD})$ definable $\square_\infty$-sequence.

The interesting case is that of inaccessible γ. Set $P^k \cap \gamma = \bigcup_{\delta \in \mathrm{CARD}\cap[\omega,\gamma)} P^k_\delta$. If δ is least such that $Z_{\ll 0,\delta\gg} \cap P^k \cap \gamma$ is unbounded in γ, then set $c(\gamma) = \ll 0, \delta \gg$. If there is no such $\delta < \gamma$, set $c(\gamma) = \ll 0, 0 \gg$. (If x is sufficiently $\mathbb{P}^n$ generic, then there will exists such a δ.)

Now let us define B^k_γ and P^k_γ for $\gamma > \omega$. Declare that

$$\begin{aligned} \xi \in P^k_\gamma \quad \text{iff} \quad & \xi \in [\gamma,\gamma^+), \text{ and there exists a constructible Easton} \\ & \text{set of ordinals } e \text{ such that if } u \in b^{c^k(\gamma)}_{\ll \xi, 1 \gg} \text{ and } P^k \cap \gamma \\ & \text{is cofinal in } u, \text{ then } e \text{ is cofinal in } u. \end{aligned}$$

The definition of B^k_γ is the same as that of P^k_γ, except the subscript "$\ll \xi, 1 \gg$" is replaced by "$\ll \xi, 0 \gg$."

Using that $\big\langle\, (c^k(\gamma), B^\kappa_\gamma, P^k_\gamma) : \gamma \in \mathrm{CARD}\cap[\omega,\infty) \,\big\rangle$ is defined by recursion on CARD, it can be seen that $\mathrm{Decode}(x,k)$ is $\Delta_1\big(\emptyset;\mathrm{CARD}\big)$ definable, uniformly in x and k. It follows that $\mathrm{Decode}(x,k)$ is $\Delta_2(\emptyset)$ definable, uniformly in x and k.

5. The conditions $\mathbb{P}^n$

Before giving $\mathbb{P}^n$'s precise definition, let us describe it informally. Conditions in $\mathbb{P}^n$ are quintuples $\boldsymbol{p} = (q, c, \hat{p}, \dot{p}, \dot{s})$, where $q \in \mathbb{Q}^n$ and the remaining components help $\hat{s}(q)$ grow into a real x_G that codes a branch through $T^n_{k,x_G(k)}$, for each $k \in \omega$.

Below $\boldsymbol{p}$ there are effectively $d\big(\hat{s}(q)\big)$ many copies of the coding apparatus. (Recall that $(k,i) \in d(\hat{s})$ when $\hat{s}'(k) = i$, for some $\hat{s}' \supseteq \hat{s}$.) The copy indexed by (k,i) is devoted to coding into a subset of $[\omega, \omega_1)$ the branch through $T^n_{k,i}$ that is approximated by $t_{k,i}(q)$. For $(k,i) \in d\big(\hat{s}(q)\big)$, $\hat{p}_{k,i}$ is an Easton support characteristic function approximating the $(k,i)^{\text{th}}$ coding class. Let us temporarily call this class $P^{k,i}$. The component $\dot{p}_{k,i}$ is a function with an Easton set of cardinals as its domain; $\dot{p}_{k,i}(\alpha)$ imposes commitments that control the growth of $\hat{p}_{k,i} \restriction \alpha$ so that $P^{k,i} \cap \alpha$ codes both $P^{k,i} \cap [\alpha, \alpha^+)$ and $t_{k,i}(q) \cap [\alpha, \alpha^+)$.

The component c is a function assigning colors to cardinals in accord with the Colorization Table.

Finally, $\dot{s}$ is a finite set of coding commitments that controls the growth of $\hat{s}(q)$, so that the generic real x_G codes $P^{k,i} \cap [\omega, \omega_1)$ when $i = x_G(k)$.

To be precise, a quintuple $\boldsymbol{p} = (q, c, \hat{p}, \dot{p}, \dot{s})$ is in $\mathbb{P}^n$ when it meets the following requirements:

(1) $q \in \mathbb{Q}^n$.

(2) c is a function with an Easton set of uncountable cardinals as its domain.

(a) If $\alpha \in \operatorname{dom}(c)$, then

$$c(\alpha) = \begin{cases} \ll 2, \beta \gg, & \text{if } \alpha = \beta^+; \\ \ll 1, \operatorname{ot}(C_\alpha) \gg, & \text{if } \alpha \text{ is singular; and} \\ \ll 0, \gamma \gg, \text{ for some } \gamma < \alpha, & \text{if } \alpha \text{ is inaccessible.} \end{cases}$$

(b) If $\beta < \alpha$ lie in $\operatorname{dom}(c)$, and both are inaccessible, then $c(\alpha)$ does not lie in the interval $\big[c(\beta), \beta\big]$.

(3) $\hat{p}$ and $\dot{p}$ are sequences of functions indexed by $d\big(\hat{s}(q)\big)$. For $(k,i) \in d\big(\hat{s}(q)\big)$,

(a) $\hat{p}_{k,i}$ is a partial function from $[\omega, \infty)$ into $\{0,1\}$ having an Easton domain; and

(b) $\dot{p}_{k,i}$ is a function with the same domain as c. For $\alpha \in \operatorname{dom}(\dot{p}_{k,i})$,

(*i*) $\big|\dot{p}_{k,i}(\alpha)\big| < \alpha^*$ (where α^* is the least regular cardinal greater than or equal to α);

(*ii*) $\dot{p}_{k,i}(\alpha)$ is a subset of

$$\alpha \cup \Big\{ \ll \xi, 0 \gg : \xi \in [\alpha, \alpha^+) \text{ and } \xi = \omega_1 + \zeta, \text{ for some } \zeta \in t_{k,i}(q) \Big\} \cup \Big\{ \ll \xi, 1 \gg : \xi \in [\alpha, \alpha^+) \text{ and } \hat{p}_{k,i}(\xi) = 1 \Big\};$$

and

(*iii*) $\dot{p}_{k,i}(\alpha) \cap \alpha = \emptyset$, unless α is inaccessible.

(c) If $\hat{p}_{k,i}(\xi) = 1$ and ξ has color $\ll 0, \gamma \gg$, then there exists an $\alpha \in \operatorname{dom}(c) \setminus (\xi + 1)$ such that $c(\alpha) = \ll 0, \gamma \gg$.

(4) $\dot{s}$ is a finite subset of

$$\Big\{ \ll\xi, \ll k, i\gg\gg : \xi \in [\omega, \omega_1), \text{ and either } (k,i) \notin d\big(\hat{s}(q)\big) \text{ or } \hat{p}_{k,i}(\xi) = 1 \Big\}.$$

Most clauses in the definition of $\mathbb{P}^n$ are transparent, but three have technical motivations that should be mentioned. Clause (2b) implies that c is one-to-one on inaccessible cardinals, and also insures that no condition uses up too many colors. This is used in Lemma 5.5 to see that conditions can be extended to absorb arbitrary uncountable cardinals into the domain of c.

If α is inaccessible, then that possibly $\dot{p}_{k,i}(\alpha) \cap \alpha \neq \emptyset$ and clause (3c) work together (with clause (6) in the definition of $\mathbb{P}^n$'s ordering) to insure that $c(\alpha)$ is the least $\ll 0, \gamma\gg$ such that $Z_{\ll 0,\gamma\gg} \cap P^{k,i} \cap \alpha$ is unbounded in α. As described in §4.3, this is used when recovering the color assigned to α from a generic real x_G. These clauses figure in the proof of Lemma 5.10.

The elements of $\mathbb{P}^n$ are ordered by declaring that $\bar{\boldsymbol{p}} = (\bar{q}, \bar{c}, \hat{\bar{p}}, \dot{\bar{p}}, \dot{\bar{s}}) \geqslant (q, c, \hat{p}, \dot{p}, \dot{s}) = \boldsymbol{p}$ when the following seven conditions are met:

(1) $\bar{q} \geqslant q$ in $\mathbb{Q}^n$; and if $\xi \in \dot{\bar{s}}$ and $u \in b_\xi^{\ll 2,0\gg}$ and $k \in u \cap \big[\mathrm{sp}(\bar{q}), \mathrm{sp}(q)\big)$, then $\hat{s}(q)(k) = 0$.
(2) The function c extends the function $\bar{c}$, literally $\bar{c} \subseteq c$.
(3) If $(k,i) \in d\big(\hat{s}(q)\big)$, then the function $\hat{p}_{k,i}$ extends the function $\hat{\bar{p}}_{k,i}$, literally $\hat{\bar{p}}_{k,i} \subseteq \hat{p}_{k,i}$.
(4) If $(k,i) \in d\big(\hat{s}(q)\big)$, then $\mathrm{dom}(\dot{\bar{p}}_{k,i}) \subseteq \mathrm{dom}(\dot{p}_{k,i})$; and if $\alpha \in \mathrm{dom}(\dot{\bar{p}}_{k,i})$, then $\dot{\bar{p}}_{k,i}(\alpha) \subseteq \dot{p}_{k,i}(\alpha)$.
(5) $\dot{\bar{s}} \subseteq \dot{s}$.
(6) If $\xi < \sup\big(\dot{\bar{p}}_{k,i}(\alpha) \cap \alpha\big)$, and if $\xi \in \mathrm{dom}(\hat{p}_{k,i}) \setminus \mathrm{dom}(\hat{\bar{p}}_{k,i})$ and has color $\bar{c}(\alpha)$, then $\hat{p}_{k,i}(\xi) = 0$.
(7) If $\xi \in \dot{\bar{p}}_{k,i}(\alpha) \setminus \alpha$ and $u \in b_\xi^{\bar{c}(\alpha)}$ and $\mathrm{dom}(\hat{p}_{k,i}) \setminus \mathrm{dom}(\hat{\bar{p}}_{k,i})$ is cofinal in u, then $\hat{p}_{k,i}^{-1}(1)$ is not cofinal in u.

By analogy to (7), the *prima facie* weaker clause (1′) might be expected in place of (1):

(1′) $\bar{q} \geqslant q$ in $\mathbb{Q}^n$; and if $\xi \in \dot{\bar{s}}$ and $u \in b_\xi^{\ll 2,0\gg}$ and $\big[\mathrm{sp}(\bar{q}), \mathrm{sp}(q)\big)$ is cofinal in u, then $\hat{s}(q)^{-1}(1)$ is not cofinal in u.

In fact, (1′) is equivalent to (1), since $|u| = 1$ when $u \in b_\xi^{\ll 2,0\gg}$. The simpler characterization of the coding commitments $\dot{\bar{s}}$ offered by (1) is used in the proof of Lemma 5.2, which verifies property (P2).

If $\boldsymbol{p} = (q, c, \hat{p}, \dot{p}, \dot{s})$ is a condition, then set

$$\begin{aligned} q(\boldsymbol{p}) &= q \\ \hat{s}(\boldsymbol{p}) &= \hat{s}\big(q\big) \\ c(\boldsymbol{p}) &= c \\ \hat{p}_{k,i}(\boldsymbol{p}) &= \hat{p}_{k,i} \\ \dot{p}_{k,i}(\boldsymbol{p}) &= \dot{p}_{k,i} \\ \dot{s}(\boldsymbol{p}) &= \dot{s} \end{aligned}$$

We can immediately verify several of properties (P1)–(P4) used in §2.

Lemma 5.1. (Property (P1)) *$\mathbb{P}^n$ is $\Delta_{\sigma(n)+1}(\{\eta_*\})$ definable.*

PROOF: This is evident, given that $\mathbb{Q}^n$ is $\Delta_{\sigma(n)+1}(\{\eta_*\})$ definable, that CARD is $\Pi_1(\emptyset)$ definable, and that $\sigma(n)+1 \geqslant 2$. □

Lemma 5.2. (Property (P2)) *The orderings $\mathbb{P}^n$ and $\mathbb{P}^{n+1}$ are related as follows:*

(a) $\mathbb{P}^{n+1} \subseteq \mathbb{P}^n$;
(b) *if* $\alpha = \min(\Omega^{\sigma(n+1)})$, *then* $\mathbb{P}^{n+1} \cap L_\alpha = \mathbb{P}^n \cap L_\alpha$; *and*
(c) *if* $m \leqslant n$ *and* $\bar{\boldsymbol{p}} \in \mathbb{P}^n$ *and* D *is a* $\Pi_{\sigma(m)+m+2}(L)$ *definable subclass of* $\mathbb{P}^m$, *then there exists* $\boldsymbol{p} \leqslant \bar{\boldsymbol{p}}$ *in* $\mathbb{P}^n$ *deciding* D *(in* $\mathbb{P}^m$*).*

PROOF: Conclusions (a) and (b) are consequences of (TC3), which holds the analogous facts regarding $T^{n+1}_{k,i}$ and $T^n_{k,i}$.

For conclusion (c), begin by noting that if D is a $\Pi_{\sigma(m)+m+2}(L)$ definable subclass of $\mathbb{P}^m$, then the class of $\boldsymbol{p} \in \mathbb{P}^m$ deciding D, namely,

$$\Big\{ \boldsymbol{p} \in \mathbb{P}^m : \exists \boldsymbol{p}'' \geqslant \boldsymbol{p}\, (\boldsymbol{p}'' \in D) \text{ or } \forall \boldsymbol{p}' \leqslant \boldsymbol{p}\, \forall \boldsymbol{p}'' \geqslant \boldsymbol{p}'\, (\boldsymbol{p}'' \notin D) \Big\},$$

is dense in $\mathbb{P}^m$, and is $\Sigma_{\sigma(m)+m+4}(L)$ definable (in fact, $\Delta_{\sigma(m)+m+4}(L)$ definable).

Thus it suffices to see that if $D' \subseteq \mathbb{P}^m$ is dense in $\mathbb{P}^m$ and is $\Sigma_{\sigma(m)+m+4}(L)$ definable, then $D' \cap \mathbb{P}^n$ is dense in $\mathbb{P}^n$. Given $\bar{\boldsymbol{p}} \in \mathbb{P}^n$, let $z \subseteq \omega$ capture the coding commitments imposed by $\dot{s}(\bar{\boldsymbol{p}})$ by declaring that

$$k \in z \quad \text{iff} \quad k \geqslant \operatorname{sp}(q(\bar{\boldsymbol{p}})), \text{ and } k \in u, \text{ for some } u \in b_\xi^{\ll 2,0 \gg} \text{ and some } \xi \in \dot{s}(\bar{\boldsymbol{p}}).$$

(Clause (1) in the definition of $\mathbb{P}^n$'s ordering was stated so that the restriction imposed by $\dot{s}(\bar{\boldsymbol{p}})$ on $q(\bar{\boldsymbol{p}})$'s growth can be characterized in this simple manner.) Then

$$\Big\{ q \in \mathbb{Q}^m_z : \exists \boldsymbol{p} \in \mathbb{P}^m\, \big(\boldsymbol{p} \leqslant \bar{\boldsymbol{p}} \text{ and } \boldsymbol{p} \text{ meets } D' \text{ and } q = q(\boldsymbol{p})\Big\}$$

is $\Sigma_{\sigma(m)+m+4}(L)$ definable, and is dense with respect to $q(\bar{\boldsymbol{p}})$ in $\mathbb{Q}^m_z$. (The key point is that if $q \leqslant q(\bar{\boldsymbol{p}})$ in $\mathbb{Q}^m_z$, then $\big(q, c(\bar{\boldsymbol{p}}), \hat{p}, \dot{p}, \dot{s}(\bar{\boldsymbol{p}})\big) \leqslant \bar{\boldsymbol{p}}$, where $\hat{p}$ and $\dot{p}$ are the sequences $\hat{p}(\bar{\boldsymbol{p}})$ and $\dot{p}(\bar{\boldsymbol{p}})$ restricted to $d(\hat{s}(q))$, respectively.)

So there exists a condition $q \in \mathbb{Q}^n_z$ meeting this class by (TC6). Consequently, there exists $\boldsymbol{p} \in \mathbb{P}^m$ below $\bar{\boldsymbol{p}}$ such that $\boldsymbol{p}$ meets D' and $q(\boldsymbol{p}) \in \mathbb{Q}^n$. Since $q(\boldsymbol{p}) \in \mathbb{Q}^n$, in fact we have $\boldsymbol{p} \in \mathbb{P}^n$. □

Lemma 5.3. (Property (P3a)) *Compatibility in $\mathbb{P}^n$ is $\Delta_1(\emptyset; \mathbb{P}^n)$ definable.*

PROOF: Two conditions $\boldsymbol{p}'$ and $\boldsymbol{p}''$ in $\mathbb{P}^n$ are compatible iff $\hat{s} = \hat{s}(\boldsymbol{p}') \cup \hat{s}(\boldsymbol{p}'') \in 2^{<\omega}$, and $\boldsymbol{p} \in \mathbb{P}^n$ and extends both $\boldsymbol{p}'$ and $\boldsymbol{p}''$, where $\boldsymbol{p} = (q, c, \hat{p}, \dot{p}, \dot{s})$ is defined as follows:

- $c = c(\boldsymbol{p}') \cup c(\boldsymbol{p}'')$ and $\dot{s} = \dot{s}(\boldsymbol{p}') \cup \dot{s}(\boldsymbol{p}'')$; and

- for all $(k,i) \in d(\hat{s})$,

$$t_{k,i}(q) = t_{k,i}\big(q(\boldsymbol{p}')\big) \cup t_{k,i}\big(q(\boldsymbol{p}'')\big),$$
$$\hat{p}_{k,i} = \hat{p}_{k,i}(\boldsymbol{p}') \cup \hat{p}_{k,i}(\boldsymbol{p}''), \text{ and}$$
$$\dot{p}_{k,i}(\alpha) = \dot{p}_{k,i}(\boldsymbol{p}')(\alpha) \cup \dot{p}_{k,i}(\boldsymbol{p}'')(\alpha), \text{ for } \alpha \in \operatorname{dom}(c).$$

The operation $(\boldsymbol{p}', \boldsymbol{p}'') \mapsto \boldsymbol{p}$ is $\Sigma_0(\emptyset)$ definable, so the relation "$\boldsymbol{p}'$ is compatible with $\boldsymbol{p}''$" is $\Delta_1(\emptyset; \mathbb{P}^n)$ definable. (In fact, it can be seen to be $\Sigma_0(\emptyset; \mathbb{P}^n)$ definable, but we do not need this sharper estimate.) □

The properties that remain to be verified are (P3b) and (P4). Property (P4) will be checked in §5.4, and property (P3b), in §6.

5.1. Easy extension lemmas

There are no hard extension lemmas, but with the modest exception of adding inaccessibles to $\operatorname{dom}\big(c(\bar{\boldsymbol{p}})\big)$, the sorts of extensions described in the following four lemmas are trivial.

Lemma 5.4. *Suppose that $\bar{\boldsymbol{p}}$ is a condition in $\mathbb{P}^n$. If e is an Easton set of infinite ordinals, then there exists a condition $\boldsymbol{p}$ extending $\bar{\boldsymbol{p}}$ that is identical with $\bar{\boldsymbol{p}}$ except that $\operatorname{dom}\big(\hat{p}_{k,i}(\boldsymbol{p})\big) = \operatorname{dom}\big(\hat{p}_{k,i}(\bar{\boldsymbol{p}})\big) \cup e$, for all $(k,i) \in d\big(\hat{s}(\bar{\boldsymbol{p}})\big)$.*

Proof: Set $\hat{p}_{k,i}(\boldsymbol{p})(\xi) = 0$, for $\xi \in e \setminus \operatorname{dom}\big(\hat{p}_{k,i}(\bar{\boldsymbol{p}})\big)$. □

Lemma 5.5. *Suppose that $\bar{\boldsymbol{p}}$ is a condition in $\mathbb{P}^n$ and that α is an uncountable cardinal. Then $\bar{\boldsymbol{p}}$ has an extension $\boldsymbol{p}$ such that $\alpha \in \operatorname{dom}\big(c(\boldsymbol{p})\big)$ and $\alpha \in \operatorname{dom}\big(\dot{p}_{k,i}(\boldsymbol{p})\big)$, for all $(k,i) \in d\big(\hat{s}(\boldsymbol{p})\big)$.*

Proof: We may assume that $\alpha \notin \operatorname{dom}\big(c(\bar{\boldsymbol{p}})\big)$, and that α is inaccessible, since otherwise we can simply set $c(\boldsymbol{p})(\alpha)$ to the value prescribed by the Colorization Table, and set $\dot{p}_{k,i}(\boldsymbol{p})(\alpha) = \emptyset$, for $(k,i) \in d\big(\hat{s}(\bar{\boldsymbol{p}})\big)$.

Assume, then, that α is inaccessible. Set $\bar{c} = c(\bar{\boldsymbol{p}})$. Note first that

$$\Big\{\, \bar{c}(\beta) : \beta \text{ is inaccessible and } \bar{c}(\beta) < \alpha < \beta \,\Big\}$$

is finite. Indeed, if $\bar{c}(\beta)$ and $\bar{c}(\beta')$ lie in this set and $\beta < \beta'$, then $\bar{c}(\beta') < \bar{c}(\beta)$. (This is the main purpose of clause (2b) in the definition of $\mathbb{P}^n$.)

Choose $\delta < \alpha$ such that

$$\ll 0, \delta \gg \; > \sup\Big(\big\{\, \bar{c}(\beta) : \beta \text{ is inaccessible and } \bar{c}(\beta) < \alpha < \beta \,\big\} \cup \big(\operatorname{dom}(\bar{c}) \cap \alpha\big)\Big).$$

Let $\boldsymbol{p}$ be identical with $\bar{\boldsymbol{p}}$ except that $\operatorname{dom}\big(c(\boldsymbol{p})\big) = \operatorname{dom}\big(\dot{p}_{k,i}(\boldsymbol{p})\big) = \operatorname{dom}(\bar{c}) \cup \{\alpha\}$, and $c(\boldsymbol{p})(\alpha) = \ll 0, \delta \gg$ and $\dot{p}_{k,i}(\boldsymbol{p})(\alpha) = \emptyset$, for $(k,i) \in d\big(\hat{s}(\bar{\boldsymbol{p}})\big)$. □

Lemma 5.6. *Suppose that $\bar{\boldsymbol{p}} \in \mathbb{P}^n$, that $(k,i) \in d\big(\hat{s}(\bar{\boldsymbol{p}})\big)$, and that $\alpha \in \operatorname{dom}\big(c(\bar{\boldsymbol{p}})\big)$ is an uncountable cardinal.*

(a) *If ξ lies in the interval $[\alpha, \alpha^+)$ and $\hat{p}_{k,i}(\bar{\boldsymbol{p}})(\xi) = 1$, then $\bar{\boldsymbol{p}}$ has an extension $\boldsymbol{p}$ such that $\ll\xi, 1\gg \in \dot{p}_{k,i}(\boldsymbol{p})(\alpha)$.*

(b) *If $\zeta \in t_{k,i}\big(q(\bar{\boldsymbol{p}})\big)$ and $\omega_1 + \zeta \in [\alpha, \alpha^+)$, then $\bar{\boldsymbol{p}}$ has an extension $\boldsymbol{p}$ such that $\ll\omega_1{+}\zeta, 0\gg \in \dot{p}_{k,i}(\boldsymbol{p})(\alpha)$.*

(c) *If α is inaccessible and $\xi < \alpha$, then $\bar{\boldsymbol{p}}$ has an extension $\boldsymbol{p}$ with $\xi \in \dot{p}_{k,i}(\boldsymbol{p})(\alpha)$.*

PROOF: Let $\boldsymbol{p}$ be identical with $\bar{\boldsymbol{p}}$ except that $\dot{p}_{k,i}(\boldsymbol{p})(\alpha) = \dot{p}_{k,i}(\bar{\boldsymbol{p}})(\alpha) \cup \{\eta\}$, where $\eta = \ll\xi, 1\gg$, $\eta = \ll\omega_1{+}\zeta, 0\gg$, or $\eta = \xi$, in (a), (b), and (c), respectively. □

Lemma 5.7. *Suppose that $\bar{\boldsymbol{p}}$ is a condition in $\mathbb{P}^n$. If ξ lies in the interval $[\omega, \omega_1)$ and $\hat{p}_{k,i}(\bar{\boldsymbol{p}})(\xi) = 1$, then $\bar{\boldsymbol{p}}$ has an extension $\boldsymbol{p}$ such $\ll\xi, \ll k, i\gg\gg \in \dot{s}(\boldsymbol{p})$.*

PROOF: Set $\dot{s}(\boldsymbol{p}) = \dot{s}(\bar{\boldsymbol{p}}) \cup \big\{\ll\xi, \ll k, i\gg\gg\big\}$. □

5.2. Coding lemmas

The following three lemmas are used in the proof of Lemma 5.11 to see that a $\mathbb{P}^n$ generic real x codes a branch through each $T^n_{k,x(k)}$.

Lemma 5.8. *Assume that $\bar{\boldsymbol{p}} = (\bar{q}, \bar{c}, \hat{\bar{p}}, \dot{\bar{p}}, \dot{\bar{s}})$ is a condition in $\mathbb{P}^n$, that α is an uncountable cardinal, and that ξ is an ordinal in the interval $[\alpha, \alpha^+)$.*

(a) *If $\xi \in \dot{\bar{p}}_{k,i}(\alpha)$, then there exists an Easton $e \subseteq \alpha$ such that if $\boldsymbol{p}$ extends $\bar{\boldsymbol{p}}$ and $u \in b^{\bar{c}(\alpha)}_{\xi}$ and $\hat{p}_{k,i}(\boldsymbol{p})^{-1}(1)$ is cofinal in u, then e is cofinal in u.*

(b) *If $\xi \notin \dot{\bar{p}}_{k,i}(\alpha)$ and $e \subseteq \alpha$ is Easton, then there exists a condition $\boldsymbol{p}$ extending $\bar{\boldsymbol{p}}$ such that $\hat{p}_{k,i}(\boldsymbol{p})^{-1}(1)$ is cofinal in some $u \in b^{\bar{c}(\alpha)}_{\xi}$ in which e is not cofinal.*

PROOF OF (a): Set $e = \operatorname{dom}(\hat{\bar{p}}_{k,i})$ and apply clause (7) in the definition of the ordering of $\mathbb{P}^n$.

PROOF OF (b): Choose $u \in b^{\bar{c}(\alpha)}_{\xi}$ such that

(a) u is not cofinal in u', for all $u' \in \bigcup_{\eta \in \dot{\bar{p}}_{k,i}(\alpha)} b^{\bar{c}(\alpha)}_{\eta}$;

(b) $\sup\big(\dot{\bar{p}}_{k,i}(\alpha) \cap \alpha\big) < \sup(u)$; and

(c) $e \cup \operatorname{dom}(\hat{\bar{p}}_{k,i})$ is not cofinal in u.

To see that these requirements can be met, note that there exist indomitably many threads $u \in b^{\bar{c}(\alpha)}_{\xi}$ as in (a), because (a) fails for fewer than α^* many $u \in b^{\bar{c}(\alpha)}_{\xi}$. Also $\sup\big(\dot{\bar{p}}_{k,i}(\alpha) \cap \alpha\big) < \alpha$. And the set of ordinals in (c) is an Easton subset of α.

Let $\boldsymbol{p}$ be identical with $\bar{\boldsymbol{p}}$, except that $\operatorname{dom}\big(\hat{p}_{k,i}(\boldsymbol{p})\big) = \operatorname{dom}\big(\hat{\bar{p}}_{k,i}\big) \cup u$, and

$$\hat{p}_{k,i}(\boldsymbol{p})(\zeta) = \begin{cases} \hat{\bar{p}}_{k,i}(\zeta), & \text{if } \zeta \in \operatorname{dom}(\hat{\bar{p}}_{k,i}); \\ 1, & \text{if } \zeta \in u \setminus \operatorname{dom}(\hat{\bar{p}}_{k,i}) \text{ and } \zeta \geqslant \sup\big(\dot{\bar{p}}_{k,i}(\alpha) \cap \alpha\big); \text{ and} \\ 0, & \text{otherwise.} \end{cases}$$

Note that $\boldsymbol{p}$ is a condition in $\mathbb{P}^n$, specifically, note clause (3c) in the definition.

In seeing that $\boldsymbol{p}$ extends $\bar{\boldsymbol{p}}$, the only matters requiring attention are clauses (6) and (7) in the definition. For clause (6), fix $\zeta \in u \setminus \operatorname{dom}(\hat{\bar{p}}_{k,i})$, and suppose that ζ

has color $\bar{c}(\beta)$ and $\zeta < \sup\big(\dot{\bar{p}}_{k,i}(\beta) \cap \beta\big)$. Then $\bar{c}(\beta) = \bar{c}(\alpha)$, since $\zeta \in u$, which has color $\bar{c}(\alpha)$. Hence $\beta = \alpha$, since $\bar{c}$ is one-to-one on inaccessibles. But then $\hat{p}_{k,i}(\boldsymbol{p})(\zeta) = 0$ by definition.

For clause (7), suppose that $\xi' \in \dot{\bar{p}}_{k,i}(\alpha') \setminus \alpha'$, that $u' \in b_{\xi'}^{\bar{c}(\alpha')}$. It suffices to see that $\mathrm{dom}\big(\hat{p}_{k,i}(\boldsymbol{p})\big) \setminus \mathrm{dom}\big(\hat{\bar{p}}_{k,i}\big)$ is not cofinal in u'. This follows by (a) in our choice of u, if $\alpha' = \alpha$. Otherwise, it follows from Lemma 4.4. $\square$

Lemma 5.9 is the analog of Lemma 5.8 for the coding that produces a $\mathbb{P}^n$ generic real.

Lemma 5.9. *Assume that $\bar{\boldsymbol{p}} = (\bar{q}, \bar{c}, \hat{\bar{p}}, \dot{\bar{p}}, \dot{\bar{s}})$ is a condition in $\mathbb{P}^n$ and that ξ is an ordinal in the interval $[\omega, \omega_1)$.*

- (a) *If $\xi \in \dot{\bar{s}}$, then there exists a natural number m such that if $\boldsymbol{p}$ extends $\bar{\boldsymbol{p}}$ and $u \in b_\xi^{\ll 2,0 \gg}$, then $\hat{s}(\boldsymbol{p})^{-1}(1) \cap u \subseteq m$.*
- (b) *If $\xi \notin \dot{\bar{s}}$ and $m \in \omega$, then there exists a condition $\boldsymbol{p}$ extending $\bar{\boldsymbol{p}}$ such that $\hat{s}(\boldsymbol{p})^{-1}(1)$ is cofinal in some $u \in b_\xi^{\ll 2,0 \gg}$ such that $u \nsubseteq m$.*

Proof of (a): Set $m = \mathrm{dom}\big(\hat{s}(\bar{\boldsymbol{p}})\big)$ and apply (1) in the definition of $\mathbb{P}^n$'s ordering.

Proof of (b): Choose $u \in b_\xi^{\ll 2,0 \gg}$ such that $u \nsubseteq m \cup \mathrm{dom}\big(\hat{s}(\bar{\boldsymbol{p}})\big)$ and u is not cofinal in u', for all $u' \in \bigcup_{\eta \in \dot{\bar{s}}} b_\eta^{\ll 2,0 \gg}$. Then, in fact, $u \cap u' = \emptyset$, for all $u' \in \bigcup_{\eta \in \dot{\bar{s}}} b_\eta^{\ll 2,0 \gg}$, because $|u| = 1$. (This is the only point at which this fact is used.) Say $u = \{r\}$. Let $\hat{s} \in 2^{<\omega}$ be such that $\mathrm{dom}(\hat{s}) = r + 1$ and

$$\hat{s}(k) = \begin{cases} \hat{s}(\bar{\boldsymbol{p}})(k), & \text{if } k \in \mathrm{dom}\big(\hat{s}(\bar{\boldsymbol{p}})\big); \\ 1, & \text{if } k = r; \text{ and} \\ 0, & \text{otherwise.} \end{cases}$$

Then set $\boldsymbol{p} = (q, c, \hat{p}, \dot{p}, \dot{s})$, where

$$\begin{aligned} q(k) &= \begin{cases} t_{k,\hat{s}(k)}(\bar{q}), & \text{if } k \in \mathrm{dom}(\hat{s}); \\ \bar{q}(k), & \text{otherwise;} \end{cases} \\ c &= \bar{c}; \\ \hat{p}_{k,i} &= \hat{\bar{p}}_{k,i}, \text{ for } (k,i) \in d(\hat{s}); \\ \dot{p}_{k,i} &= \dot{\bar{p}}_{k,i}, \text{ for } (k,i) \in d(\hat{s}); \text{ and} \\ \dot{s} &= \dot{\bar{s}}. \end{aligned}$$

Then $\boldsymbol{p}$ is as required. $\square$

Lemma 5.10 is used to recover the coloring of inaccessible cardinals in a generic extension.

Lemma 5.10. *Assume that $\bar{\boldsymbol{p}} = (\bar{q}, \bar{c}, \hat{\bar{p}}, \dot{\bar{p}}, \dot{\bar{s}})$ is a condition in $\mathbb{P}^n$, that $\alpha \in \mathrm{dom}(\bar{c})$ is an inaccessible cardinal, and that $(k,i) \in d\big(\hat{s}(\bar{\boldsymbol{p}})\big)$.*

- (a) *If $\beta < \alpha$, then there exists a condition $\boldsymbol{p}$ extending $\bar{\boldsymbol{p}}$ and an ordinal ξ of color $\bar{c}(\alpha)$ such that $\beta < \xi < \alpha$ and $\hat{p}_{k,i}(\xi) = 1$.*
- (b) *Suppose that $\ll 0, \gamma \gg < \bar{c}(\alpha)$. Then there exists a condition $\boldsymbol{p}$ extending $\bar{\boldsymbol{p}}$ and a bound $\beta < \alpha$ such that if $\boldsymbol{p}'$ is any condition extending $\boldsymbol{p}$, then $Z_{\ll 0,\gamma \gg} \cap \hat{p}_{k,i}(\boldsymbol{p}')^{-1}(1) \subseteq \beta$.*

Proof of (a): This is immediate from the proof of part (b) of Lemma 5.8.

Proof of (b): *Case 1. There is no* $\boldsymbol{p}'' \leqslant \bar{\boldsymbol{p}}$ *such that* $c(\boldsymbol{p}'')(\alpha') = \ll 0,\gamma \gg$, *for some inaccessible* $\alpha' > \alpha$. Set $\boldsymbol{p} = \bar{\boldsymbol{p}}$ and set $\beta = \bar{c}(\alpha)$. Suppose that $\boldsymbol{p}'$ extends $\boldsymbol{p}$ and that $\hat{p}_{k,i}(\boldsymbol{p}')(\xi) = 1$, where ξ has color $\ll 0,\gamma \gg$. Then there exists an $\alpha' \in \mathrm{dom}\big(c(\boldsymbol{p}')\big) \setminus (\xi + 1)$ such that $c(\boldsymbol{p}')(\alpha') = \ll 0,\gamma \gg$. (This is the reason for (3c) in the definition of $\mathbb{P}^n$.) Because $\ll 0,\gamma \gg < c(\boldsymbol{p}')(\alpha)$, it follows that either $\alpha < \alpha'$ or $\alpha' < c(\boldsymbol{p}')(\alpha)$. The former of these possibilities is ruled out by the case hypothesis; the latter implies that $\xi < \beta$.

Case 2. There does exist a condition $\boldsymbol{p}'' \leqslant \bar{\boldsymbol{p}}$ *such that* $c(\boldsymbol{p}'')(\alpha') = \ll 0,\gamma \gg$, *for some inaccessible* $\alpha' > \alpha$. Let $\boldsymbol{p} \leqslant \bar{\boldsymbol{p}}$ be such a condition $\boldsymbol{p}''$. We may assume that $\alpha \in \dot{p}_{k,i}(\boldsymbol{p})(\alpha')$. Set $\beta = \sup\big(\mathrm{dom}(\hat{p}_{k,i}(\boldsymbol{p})) \cap \alpha\big)$. Then $\boldsymbol{p}$ is as required on account of clause (6) in the definition of $\mathbb{P}^n$'s ordering. □

5.3. The term $\mathring{x}$

If G is a filter on $\mathbb{P}^n$, let x_G be defined by

$$x_G = \bigcup \Big\{ \hat{s}(\boldsymbol{p}) : \boldsymbol{p} \in G \Big\}.$$

The task at hand is defining a term $\mathring{x}$ that is common to forcing languages of all of the $\mathbb{P}^n$'s and is such that $\mathring{x}^G = x_G$, whenever G is a filter on $\mathbb{P}^n$.

For $\hat{s} \in 2^{<\omega}$, define $\boldsymbol{p}_{\hat{s}} \in \mathbb{P}^n$ by setting $\boldsymbol{p}_{\hat{s}} = (q, \emptyset, \hat{p}, \dot{p}, \emptyset)$, where

$$q(k) = \begin{cases} \big\{\hat{s}(k)\big\}, & \text{if } k \in \mathrm{dom}(\hat{s}); \\ \big(\{0\},\{1\}\big), & \text{otherwise;} \end{cases}$$

and $\hat{p}_{k,i} = \dot{p}_{k,i} = \emptyset$, for all $(k,i) \in d(\hat{s})$. Note that $q \in \mathbb{Q}^n$ by (TC1). Note also that if G is a filter on $\mathbb{P}^n$, then $\hat{s} \subseteq x_G$ iff $\boldsymbol{p}_{\hat{s}} \in G$.

Set

$$\mathring{x} = \Big\{ \big((k,i)^\vee, \boldsymbol{p}_{\hat{s}}\big) : \hat{s} \in 2^{<\omega} \text{ and } \hat{s}(k) = i \Big\}.$$

(Note that $\mathbb{P}^n$ has a weakest condition, namely $\boldsymbol{p}_\emptyset$, so $\vee$-terms can be defined as usual.) Then $\mathring{x}^G = x_G$ whenever G is a filter on $\mathbb{P}^n$.

Finally, note that $\mathring{x}^G\colon \omega \to 2$ if G is even $\Sigma_0(L;\mathbb{P}^n)$ generic.

5.4. Decoding branches

Suppose that G is a $\Pi_{n+2}(L;\mathbb{P}^n)$ generic filter on $\mathbb{P}^n$. Fix a natural number k, and suppose that $x_G(k) = i$. Define the following classes:

$$P^k_G = \Big\{ \xi : \hat{p}_{k,i}(\boldsymbol{p})(\xi) = 1, \text{ for some } \boldsymbol{p} \in G \Big\}$$
$$D^k_G = \Big\{ \zeta : \zeta \in t_{k,i}\big(q(\boldsymbol{p})\big), \text{ for some } \boldsymbol{p} \in G \Big\}$$
$$c_G = \bigcup \Big\{ c(\boldsymbol{p}) : \boldsymbol{p} \in G \Big\} \cup \Big\{ \big(\omega, \ll 2, 0 \gg\big) \Big\}$$
$$B^k_G = \Big\{ \omega_1 + \zeta : \zeta \in D^k_G \Big\}$$

Lemma 5.11. (Property (P4)) *Suppose that G is a $\Pi_{n+2}(L;\mathbb{P}^n)$ generic filter on $\mathbb{P}^n$. Then* $\mathrm{Decode}(\mathring{x}^G, k) = D^k_G$ *is a $\Pi_{n+2}(L;T^n_{k,i})$ generic branch, where $i = \mathring{x}^G(k)$.*

PROOF: Note first that D^k_G is a $\Pi_{n+2}(L;T^n_{k,i})$ generic branch. Indeed, if $E \subseteq T^n_{k,i}$ is $\Pi_{n+2}(L;T^n_{k,i})$ definable, then

$$\Big\{\boldsymbol{p} \in \mathbb{P}^n : (k,i) \notin d\big(\hat{s}(\boldsymbol{p})\big) \text{ or } t_{k,i}\big(q(\boldsymbol{p})\big) \in E\Big\}$$

is $\Pi_{n+2}(L;\mathbb{P}^n)$ definable, hence decided by G.

In the previous subsection, we saw that $\mathring{x}^G = x_G$. So it suffices to see that $\mathrm{Decode}(x_G, k) = D^k_G$. Because $\mathrm{Decode}(x_G, k)$ is defined in three steps, we shall need to retrace those steps to see this. Recall that the decoding process begins by declaring that

$$\begin{aligned} \xi \in P^k_\omega \quad \text{iff} \quad & \xi \in [\omega, \omega^L_1) \text{ and} \\ & \big\{u \in b^{\ll 2,0\gg}_{\ll \xi, \ll k,i\gg\gg} : x_G^{-1}(1) \text{ is cofinal in } u\big\} \text{ is finite.} \end{aligned}$$

Using Lemmas 5.7 and 5.9, G is sufficiently generic that

$$\begin{aligned} \xi \in P^k_G \cap [\omega, \omega^L_1) \quad \text{iff} \quad & \xi \in [\omega, \omega^L_1) \text{ and there exists a natural number } m \\ & \text{such that if } u \in b^{\ll 2,0\gg}_{\ll \xi, \ll k,i\gg\gg} \text{ and } x_G^{-1}(1) \text{ is cofinal} \\ & \text{in } u, \text{ then } u \subseteq m. \end{aligned}$$

It follows that $P^k_\omega = P^k_G \cap [\omega, \omega^L_1)$.

The next step in defining $\mathrm{Decode}(x_G, k)$ was defining $c^k(\gamma)$, P^k_γ, and B^k_γ by recursion on infinite $\gamma \in \mathrm{CARD}$. We maintain that

$$\begin{aligned} c^k(\gamma) &= c_G(\gamma); \\ P^k_\gamma &= P^k_G \cap [\gamma, \gamma^+); \text{ and} \\ B^k_\gamma &= B^k_G \cap [\gamma, \gamma^+). \end{aligned}$$

Proceed by induction on γ. If γ is not inaccessible, it is immediate from the definitions that $c^k(\gamma) = c_G(\gamma)$. (The $c(\boldsymbol{p})$'s have only uncountable cardinals in their domains, but we set $c_G(\omega) = \ll 2, 0\gg$ above.) Suppose that γ is inaccessible. Then $c^k(\gamma) = \ll 0, \delta\gg$, if δ is least such that $Z_{\ll 0,\delta\gg} \cap \bigcup_{\eta \in \mathrm{CARD} \cap [\omega,\gamma)} P^k_\eta$ is unbounded in γ. (If there is no such δ, then $c^k(\gamma) = \ll 0, 0\gg$.) Now, using Lemma 5.10, G is sufficiently generic that $c_G(\gamma)$ is the least $\ll 0, \delta\gg$ such that $Z_{\ll 0,\delta\gg} \cap P^k_G$ is unbounded in γ. By induction, $P^k_G \cap \gamma = \bigcup_{\eta \in \mathrm{CARD} \cap [\omega,\gamma)} P^k_\eta$. And Gödel pairing is monotone: $\ll 0, \delta\gg < \ll 0, \eta\gg$ iff $\delta < \eta$. So $c_G(\gamma) = c^k(\gamma)$.

For $\gamma > \omega$, by definition

$$\begin{aligned} \xi \in P^k_\gamma \quad \text{iff} \quad & \xi \in [\gamma, \gamma^+) \text{ and there exists a constructible Easton} \\ & e \subseteq \gamma \text{ such that if } u \in b^{c^k(\gamma)}_{\ll \xi, 1\gg} \text{ and } \textstyle\bigcup_{\eta \in \mathrm{CARD} \cap [\omega,\gamma)} P^k_\eta \\ & \text{is cofinal in } u, \text{ then } e \text{ is cofinal in } u. \end{aligned}$$

Using Lemmas 5.6 and 5.8, G is sufficiently generic that

$$\xi \in P^k_G \cap [\gamma, \gamma^+) \quad \text{iff} \quad \xi \in [\gamma, \gamma^+) \text{ and there exists a constructible Easton } e \subseteq \gamma \text{ such that if } u \in b^{c_G(\gamma)}_{\ll \xi, 1 \gg} \text{ and } P^k_G \cap \gamma \text{ is cofinal in } u, \text{ then } e \text{ is cofinal in } u.$$

So $P^k_G \cap [\gamma, \gamma^+) = P^k_\gamma$ by induction and that $c^k(\gamma) = c_G(\gamma)$.

Similarly, $B^k_G \cap [\gamma, \gamma^+) = B^k_\gamma$. (Begin by noting that $B^k_G \cap \omega_1 = \emptyset = B^k_\omega$.)

Finally,

$$\begin{aligned} \zeta \in \operatorname{Decode}(x_G, k) \quad &\text{iff} \quad \omega_1 + \zeta \in \textstyle\bigcup_{\gamma \in \mathrm{CARD} \cap [\omega, \infty)} B^k_\gamma \\ &\text{iff} \quad \omega_1 + \zeta \in B^k_G \\ &\text{iff} \quad \zeta \in D^k_G. \ \square \end{aligned}$$

6. Predensity reduction

This section proves (P3b): $\mathbb{P}^n$ satisfies strong predensity reduction. This is the one remaining fact needed to complete the proof of Theorem 1.

In outline, the proof is typical: Fix an infinite regular cardinal κ. A condition in $\mathbb{P}^n$ can be broken into an "upper part" and a "lower part." (There will also be a small "middle part," but this is unimportant for now.) There exist only κ many possible lower parts. Given a uniformly definable κ-sequence of predense classes, in κ many steps, the upper part of a given condition $\bar{\boldsymbol{p}}$ will be extended to meet each of these predense classes relative to each possible alternative lower part that can be adjoined to $\bar{\boldsymbol{p}}$. In this way, we obtain a condition $\boldsymbol{p}_\kappa$ such that any extension of $\boldsymbol{p}_\kappa$ can be extended to meet any of the predense classes in the given sequence simply by adjoining a suitable lower (and middle) part.

Two facts get us through limit stages in the construction of $\boldsymbol{p}_\kappa$: One is (TC5), which implies that $\mathbb{Q}^n$ is closed at certain levels; the other is conclusion (3) of the Thread Lemma, which can be used to show that properly constructed limit conditions do not inadvertently violate prior coding commitments.

Let us begin by breaking conditions into upper and lower parts. Fix an infinite regular cardinal κ and a condition $\bar{\boldsymbol{p}} \in \mathbb{P}^n$. Suppose that $\sup\big(t_{k,i}(q(\bar{\boldsymbol{p}}))\big) \geqslant \kappa^+$, for all $(k,i) \in d\big(\hat{s}(\bar{\boldsymbol{p}})\big)$. It is such a condition $\bar{\boldsymbol{p}}$ that we shall extend to reduce a given uniformly definable κ-sequence of predense classes. Mention of $\bar{\boldsymbol{p}}$ is suppressed in the following notation, since we shall always be working below it.

Define the "lower part" $(\boldsymbol{p})^\kappa$ of a condition $\boldsymbol{p}$ extending $\bar{\boldsymbol{p}}$ by setting $\hat{s} = \hat{s}(\boldsymbol{p})$ and $(\boldsymbol{p})^\kappa = (q, c, \hat{p}, \dot{p}, \dot{s})$, where

$$\begin{aligned} t_{k,i}(q) &= t_{k,i}\big(q(\boldsymbol{p})\big) \cap \kappa^+, \text{ for } (k,i) \in d(\hat{s}); \\ c &= c(\boldsymbol{p}) \restriction \kappa^+; \\ \hat{p}_{k,i} &= \hat{p}_{k,i}(\boldsymbol{p}) \restriction \kappa; \\ \dot{p}_{k,i} &= \dot{p}_{k,i}(\boldsymbol{p}) \restriction \kappa; \\ \dot{s} &= \begin{cases} \dot{s}(\boldsymbol{p}), & \text{if } \kappa > \omega; \text{ and} \\ \emptyset, & \text{if } \kappa = \omega. \end{cases} \end{aligned}$$

Let $(\mathbb{P}^n)^\kappa$ be the set of all lower parts of conditions in $\mathbb{P}^n$ below $\bar{\boldsymbol{p}}$:

$$(\mathbb{P}^n)^\kappa = \Big\{ (\boldsymbol{p})^\kappa : \boldsymbol{p} \in \mathbb{P}^n \text{ and } \boldsymbol{p} \leqslant \bar{\boldsymbol{p}} \Big\}.$$

Note that $\big|(\mathbb{P}^n)^\kappa\big| = \kappa$.

Let $[\boldsymbol{p}]^\kappa$ have the same definition, except that $\dot{p}_{k,i} = \dot{p}_{k,i}(\boldsymbol{p}) \restriction \kappa^+$ (rather than κ) and $\dot{s} = \dot{s}(\boldsymbol{p})$, even if $\kappa = \omega$. The "middle part" of $\boldsymbol{p}$ consists of the $\dot{p}_{k,i}(\boldsymbol{p})(\kappa)$'s, if $\kappa > \omega$, and is $\dot{s}(\boldsymbol{p})$, if $\kappa = \omega$. So $[\boldsymbol{p}]^\kappa$ combines the lower and middle parts of $\boldsymbol{p}$.

Declare that

$$\boldsymbol{p} \leqslant_\kappa \bar{\boldsymbol{p}} \qquad \text{iff} \qquad \boldsymbol{p} \leqslant \bar{\boldsymbol{p}} \text{ and } [\boldsymbol{p}]^\kappa = [\bar{\boldsymbol{p}}]^\kappa.$$

Then $\boldsymbol{p} \leqslant_\kappa \bar{\boldsymbol{p}}$ asserts that $\boldsymbol{p}$ extends $\bar{\boldsymbol{p}}$, properly at most on its "upper part."

Define this upper part of $\boldsymbol{p}$ by setting $\hat{s} = \hat{s}(\boldsymbol{p})$ and $(\boldsymbol{p})_\kappa = (q, c, \hat{p}, \dot{p}, \dot{s})$, where

$$\begin{aligned} q &= q(\boldsymbol{p}); \\ c &= c(\boldsymbol{p}) \restriction [\kappa^+, \infty); \\ \hat{p}_{k,i} &= \hat{p}_{k,i}(\boldsymbol{p}) \restriction [\kappa, \infty); \\ \dot{p}_{k,i} &= \dot{p}_{k,i}(\boldsymbol{p}) \restriction [\kappa^+, \infty); \text{ and} \\ \dot{s} &= \emptyset. \end{aligned}$$

Note that $(\boldsymbol{p})_\kappa$ is a condition in $\mathbb{P}^n$, so $(\boldsymbol{p})_\kappa$'s literally are ordered by $\mathbb{P}^n$'s ordering. Note also that if $\boldsymbol{p}' \leqslant \boldsymbol{p}$, then $(\boldsymbol{p}')_\kappa \leqslant (\boldsymbol{p})_\kappa$.

If $\hat{s} \in 2^{<\omega}$ and $\hat{s} \supseteq \hat{s}(\boldsymbol{p})$, let $\boldsymbol{p} * \hat{s}$ be the result of restricting q, $\hat{p}$, and $\dot{p}$ to $d(\hat{s})$ (as in the proof of Lemma 5.9). Then $\boldsymbol{p} * \hat{s}$ is a condition in $\mathbb{P}^n$, though it may not extend $\boldsymbol{p}$ because $\hat{s}$ may violate the coding commitments imposed by $\dot{s}(\boldsymbol{p})$. On the other hand, if $\boldsymbol{p}' \leqslant \boldsymbol{p}$, then $\boldsymbol{p}' \leqslant \boldsymbol{p} * \hat{s}(\boldsymbol{p}')$.

Finally, note that if $(\boldsymbol{p}')^\kappa = (\boldsymbol{p})^\kappa$ and $(\boldsymbol{p}')_\kappa \leqslant (\boldsymbol{p})_\kappa$ and the middle part of p' extends that of p, then $\boldsymbol{p}' \leqslant \boldsymbol{p}$.

Lemma 6.1. *Suppose that $\boldsymbol{p}' \leqslant \bar{\boldsymbol{p}}$ and that $\dot{s}(\boldsymbol{p}') = \dot{s}(\bar{\boldsymbol{p}})$. Then there exists a condition $\boldsymbol{p} \leqslant \bar{\boldsymbol{p}}$ such that $\hat{s}(\boldsymbol{p}) = \hat{s}(\bar{\boldsymbol{p}})$ and $\boldsymbol{p}' = \boldsymbol{p} * \hat{s}(\boldsymbol{p}')$.*

PROOF: Let $\boldsymbol{p}$ be identical with $\boldsymbol{p}'$, except that

$$\begin{aligned} t_{k,i}\big(q(\boldsymbol{p})\big) &= t_{k,i}\big(q(\bar{\boldsymbol{p}})\big), \\ \hat{p}_{k,i}(\boldsymbol{p}) &= \hat{p}_{k,i}(\bar{\boldsymbol{p}}), \text{ and} \\ \dot{p}_{k,i}(\boldsymbol{p}) &= \dot{p}_{k,i}(\bar{\boldsymbol{p}}), \end{aligned}$$

for $(k, i) \in d\big(\hat{s}(\bar{\boldsymbol{p}})\big) \setminus d\big(\hat{s}(\boldsymbol{p}')\big)$.

Using that $\dot{s}(\boldsymbol{p}') = \dot{s}(\bar{\boldsymbol{p}})$ for clause (4) in the definition, note that $\boldsymbol{p}$ is a condition in $\mathbb{P}^n$. Note then that $\boldsymbol{p}$ extends $\bar{\boldsymbol{p}}$, using that $\boldsymbol{p}' \leqslant \bar{\boldsymbol{p}}$. $\square$

Lemma 6.2. (Property (3b)) $\mathbb{P}^n$ *satisfies strong* $\Sigma_\omega(L)$ *predensity reduction.*

PROOF: Fix an infinite regular cardinal κ, and suppose that $\langle D_\xi : \xi < \kappa \rangle$ is a uniformly $\Sigma_m\big(\{w\}\big)$ definable sequence of classes predense in $\mathbb{P}^n$. Fix a condition $\bar{\boldsymbol{p}} \in \mathbb{P}^n$ and suppose, as we have above, that $\sup\big(t_{k,i}\big(q(\bar{\boldsymbol{p}})\big)\big) \geqslant \kappa^+$, for all $(k,i) \in d\big(\hat{s}(\bar{\boldsymbol{p}})\big)$. We may also assume that $m > \sigma(n)$, and, extending $\hat{s}(\bar{\boldsymbol{p}})$ if necessary, that $m \in \bigcap_{(k,i)\in d(\hat{s}(\bar{\boldsymbol{p}}))} E_{k,i}$, by Lemma 2.1.

Let $\langle \alpha_\xi : \xi \leqslant \kappa \rangle$ enumerate the first $\kappa+1$ many elements of the canonical Σ_m-tower approximating L above $\{\bar{\boldsymbol{p}}, \kappa, w, \eta_*\}$. That is, define $\langle \alpha_\xi : \xi \leqslant \kappa \rangle$ by

$$\alpha_\xi = \text{the least } \alpha \text{ such that } L_\alpha \prec_{\Sigma_m} L \text{ and } \{\bar{\boldsymbol{p}}, \kappa, w, \eta_*\} \cup \{ \alpha_\zeta : \zeta < \xi \} \subseteq L_\alpha.$$

Then $\langle \alpha_\xi : \xi \leqslant \kappa \rangle$ is a continuous, monotonically increasing sequence of limit cardinals. Furthermore, $n(\alpha_\xi) = m$, for all $\xi \leqslant \kappa$. Thus $\alpha_\xi \in \Omega^{\sigma(n)}_{k,i}$, for all $(k,i) \in d\big(\hat{s}(\bar{\boldsymbol{p}})\big)$.

Set $M_\xi = L_{\alpha_\xi}$, for $\xi \leqslant \kappa$. Then for $\gamma \in \text{CARD} \cap [\kappa, \alpha_\xi)$, set

$$M^\gamma_\xi = \Sigma_m\text{-Skolem Hull}_{M_\xi}\Big(\gamma \cup \{\gamma, \bar{\boldsymbol{p}}, \kappa, w, \eta_*\} \cup \{ \alpha_\zeta : \zeta < \xi\}\Big).$$

If $B \subseteq \text{CARD} \cap [\kappa, \alpha_\xi)$ is an Easton set of cardinals, set

$$B^{(\xi)} = \bigcup\Big\{ e \setminus \alpha : \alpha \in B \text{ and } e \in M^\alpha_\xi \text{ is Easton} \Big\}.$$

Note that then $B^{(\xi)}$ is Easton. Indeed, if β is regular, then

$$B^{(\xi)} \cap \beta = \bigcup\Big\{ e \cap [\alpha, \beta) : \alpha \in B \cap \beta \text{ and } e \in M^\alpha_\xi \text{ is Easton} \Big\}$$

is a union of fewer than β many bounded subsets of β.

Let $\varphi(v_1, v_2)$ be a $\Sigma_m\big(\{w\}\big)$ formula such that $D_\xi = \big\{ \boldsymbol{p} \in \mathbb{P}^n : \varphi(\xi, \boldsymbol{p}) \big\}$. And let $\langle (\boldsymbol{p}'_\xi, \varphi_\xi) : \xi < \kappa \rangle$ enumerate all pairs $\big(\boldsymbol{p}', \varphi(\eta, v_2)\big)$ such that $\boldsymbol{p}' \in (\mathbb{P}^n)^\kappa$ and $\eta < \kappa$. We may assume that $\langle (\boldsymbol{p}'_\xi, \varphi_\xi) : \xi < \kappa \rangle$ lies in all of the M^γ_ξ's, or, equivalently, that $\langle (\boldsymbol{p}'_\xi, \varphi_\xi) : \xi < \kappa \rangle \in M^\kappa_0$. Let $E_\xi = \big\{ \boldsymbol{p} \in \mathbb{P}^n : \varphi_\xi(\boldsymbol{p}) \big\}$.

Next, let us define simultaneously a $\leqslant_\kappa$-descending sequence $\langle \boldsymbol{p}_\xi : \xi \leqslant \kappa \rangle$ of conditions in $\mathbb{P}^n$ and a sequence $\langle \boldsymbol{r}_{\xi+1} : \xi < \kappa \rangle$ of conditions in $\mathbb{P}^n$. Inductively, we shall maintain that $\boldsymbol{p}_\xi \in M^\kappa_{\xi+1}$. The idea is that $\boldsymbol{p}_{\xi+1}$ will reduce meeting E_ξ below a condition extending $\boldsymbol{p}_{\xi+1}$ and having lower part $\boldsymbol{p}'_\xi$ to extending $\boldsymbol{r}_{\xi+1}$, if this is possible.

Set $\hat{\hat{s}} = \hat{s}(\bar{\boldsymbol{p}})$.

Begin by setting $\boldsymbol{p}_0 = \bar{\boldsymbol{p}}$. At stage $\xi + 1$, set

$\boldsymbol{r}_{\xi+1} =$ the L-least $\boldsymbol{r} \leqslant \boldsymbol{p}_\xi$ such that $\boldsymbol{r}$ meets E_ξ and $(\boldsymbol{r})^\kappa = \boldsymbol{p}'_\xi$, if such a condition $\boldsymbol{r}$ exists; otherwise, set $\boldsymbol{r}_{\xi+1} = \bar{\boldsymbol{p}}$.

$\boldsymbol{p}_{\xi+1} =$ the L-least $\boldsymbol{p} \leqslant_\kappa \boldsymbol{p}_\xi$ such that

- $(\boldsymbol{p})_\kappa * \hat{s}(\boldsymbol{r}_{\xi+1}) \leqslant (\boldsymbol{r}_{\xi+1})_\kappa$;
- for each $(k,i) \in d(\hat{\hat{s}}\,)$,
  $\Big(\text{sp}\big(\hat{p}_{k,i}(\boldsymbol{p}_\xi)\big) \cap [\kappa, \alpha_\xi)\Big)^{(\xi)} \subseteq \text{dom}\big(\hat{p}_{k,i}(\boldsymbol{p})\big)$, where
  $\text{sp}\big(\hat{p}_{k,i}(\boldsymbol{p})\big) = \Big\{ \gamma \in \text{CARD} : \text{dom}\big(\hat{p}_{k,i}(\boldsymbol{p})\big) \cap \big[\gamma, \gamma^+\big) \neq \emptyset \Big\}$; and
- $\sup\big(t_{k,i}(q(\boldsymbol{p}))\big) \geqslant \alpha_\xi$, for all $(k,i) \in d(\hat{\hat{s}}\,)$.

To obtain such a condition $\boldsymbol{p}$, first let $\boldsymbol{p}^\dagger$ be $\boldsymbol{p}_\xi$, if $\boldsymbol{r}_{\xi+1} = \bar{\boldsymbol{p}}$; otherwise, let $\boldsymbol{p}^\dagger$ be obtained by setting $(\boldsymbol{p}^\ddagger)_\kappa = (\boldsymbol{r}_{\xi+1})_\kappa$ and $[\boldsymbol{p}^\ddagger]^\kappa = [\boldsymbol{p}_\xi * \hat{s}(\boldsymbol{r}_{\xi+1})]^\kappa$. Then $\boldsymbol{p}^\ddagger$ is a condition extending $\boldsymbol{p}_\xi$. Furthermore, $\dot{s}(\boldsymbol{p}^\ddagger) = \dot{s}(\boldsymbol{p}_\xi)$. Now use Lemma 6.1 to obtain $\boldsymbol{p}^\dagger \leqslant_\kappa \boldsymbol{p}_\xi$ such that $\boldsymbol{p}^\dagger * \hat{s}(\boldsymbol{r}_{\xi+1}) = \boldsymbol{p}^\ddagger$.

To obtain $\boldsymbol{p}$ from $\boldsymbol{p}^\dagger$, extend $q(\boldsymbol{p}^\dagger)$ to satisfy the final requirement, and extend $\hat{p}_{k,i}(\boldsymbol{p}^\dagger)$ to include the Easton set $\big(\mathrm{sp}(\hat{p}_{k,i}(\boldsymbol{p}_\xi)) \cap [\kappa, \alpha_\xi)\big)^{(\xi)}$ in its domain, for each $(k,i) \in d(\hat{\hat{s}})$.

Note that it follows from the second requirement on $\boldsymbol{p}_{\xi+1}$ that

$$M^\gamma_\xi \cap [\gamma, \gamma^+) \subseteq \mathrm{dom}\big(\hat{p}_{k,i}(\boldsymbol{p}_{\xi+1})\big), \text{ for all } \gamma \in \mathrm{sp}\big(\hat{p}_{k,i}(\boldsymbol{p}_\xi)\big) \cap [\kappa, \alpha_\xi),$$

because $[\gamma, \eta) \in M^\gamma_\xi$ is an Easton set whenever $\eta < M^\gamma_\xi \cap \gamma^+$.

If ξ is a limit ordinal, let $\boldsymbol{p}_\xi$ be the limit of the conditions $\boldsymbol{p}_\zeta$, for $\zeta < \xi$. That is, set $\boldsymbol{p}_\xi = (q, c, \hat{p}, \dot{p}, \dot{s})$, where

$$\begin{aligned}
t_{k,i}(q) &= \textstyle\bigcup_{\zeta<\xi} t_{k,i}\big(q(\boldsymbol{p}_\zeta)\big) \cup \{\alpha_\xi\}, \text{ for } (k,i) \in d(\hat{\hat{s}});\\
c &= \textstyle\bigcup_{\zeta<\xi} c_\zeta;\\
\hat{p}_{k,i} &= \textstyle\bigcup_{\zeta<\xi} \hat{p}_{k,i}(\boldsymbol{p}_\zeta), \text{ for } (k,i) \in d(\hat{\hat{s}});\\
\mathrm{dom}\big(\dot{p}_{k,i}\big) &= \textstyle\bigcup_{\zeta<\xi} \mathrm{dom}\big(\dot{p}_{k,i}(\boldsymbol{p}_\zeta)\big) = \mathrm{dom}(c), \text{ for } (k,i) \in d(\hat{\hat{s}});\\
\dot{p}_{k,i}(\alpha) &= \textstyle\bigcup_{\zeta<\xi} \dot{p}_{k,i}(\boldsymbol{p}_\zeta)(\alpha), \text{ for } \alpha \in \mathrm{dom}(\dot{p}_{k,i}) \text{ and } (k,i) \in d(\hat{\hat{s}}); \text{ and}\\
\dot{s} &= \textstyle\bigcup_{\zeta<\xi} \dot{s}(\boldsymbol{p}_\zeta) = \dot{s}(\bar{\boldsymbol{p}}).
\end{aligned}$$

Claim. *$\boldsymbol{p}_\xi \in \mathbb{P}^n$, for all $\xi \leqslant \kappa$; and $\boldsymbol{p}_\xi \in M^\kappa_{\xi+1}$, for all $\xi < \kappa$.*

Proof: Both claims are established by induction on ξ.

That $\boldsymbol{p}_\xi \in \mathbb{P}^n$ is trivial unless ξ is a limit ordinal. In this case, it suffices to see that $q(\boldsymbol{p}_\xi) \in \mathbb{Q}^n$. Now $\alpha_\zeta \in \Omega^{\sigma(n)}_{k,i}$, for all ζ and all $(k,i) \in d(\hat{\hat{s}})$. So $\alpha_\xi \in \Omega^{\sigma(n)}_{k,i}$ and $\alpha_\xi = \sup\big(\Omega^{\sigma(n)}_{k,i} \cap \alpha_\xi\big)$ when ξ is a limit ordinal. It follows from (TC5) that $q(\boldsymbol{p}_\xi) \in \mathbb{Q}^n$.

Now, let us see that $\boldsymbol{p}_\xi \in M^\kappa_{\xi+1}$. If $\xi < \kappa$ is a limit ordinal, then $\langle \boldsymbol{p}_\zeta : \zeta < \xi \rangle$ is definable from the parameters $\bar{\boldsymbol{p}}$, κ, w, and η_* over $\big\langle M^\kappa_\xi, \langle \alpha_\zeta : \zeta < \xi \rangle \big\rangle$. (This assumes that $\boldsymbol{p}_\zeta \in M^\kappa_\xi$, for $\zeta < \xi$, of course.) Because $\alpha_\xi \in M^\kappa_{\xi+1}$, both M^κ_ξ and $\langle \alpha_\zeta : \zeta < \xi \rangle$ line in $M^\kappa_{\xi+1}$. It follows that if $\xi < \kappa$ is a limit ordinal, then $\boldsymbol{p}_\xi \in M^\kappa_{\xi+1}$.

To see that $\boldsymbol{p}_{\xi+1} \in M^\kappa_{\xi+2}$, note first that $(\boldsymbol{r}_{\xi+1})^{M^\kappa_{\xi+2}} = \boldsymbol{r}_{\xi+1}$, because $\mathbb{P}^n$ is $\Delta_{\sigma(n)+1}\big(\{\eta_*\}\big)$ definable, $m > \sigma(n)$, $M^\kappa_{\xi+2} \prec_{\Sigma_m} L$, and E_ξ is $\Sigma_m\big(\{w\}\big)$ definable.

Then note that the steps taken to obtain a suitable extension $\boldsymbol{p}$ of $\boldsymbol{p}_\xi$ can be carried out inside $M^\kappa_{\xi+2}$, hence $\boldsymbol{p}_{\xi+1} \in M^\kappa_{\xi+2}$. (In particular, note that there exists a q extending $q(\boldsymbol{p}^\dagger)$ in $\mathbb{Q}^n \cap M^\kappa_{\xi+2}$ such that $\sup\big(t_{k,i}(q)\big) \geqslant \alpha_{\xi+1}$, for all $(k,i) \in d(\hat{\hat{s}})$, because $\mathbb{Q}^n$ is $\Delta_{\sigma(n)+1}\big(\{\eta_*\}\big)$ definable and $M^\kappa_{\xi+2} \prec_{\Sigma_{\sigma(n)+1}} L$.) □

It follows from $\boldsymbol{p}_\xi \in M^\kappa_{\xi+1}$ that

$$\operatorname{dom}\bigl(\hat{p}_{k,i}(\boldsymbol{p}_\xi)\bigr) \cap [\gamma, \gamma^+) \subseteq M^\gamma_{\xi+1},$$

for all $(k,i) \in d(\hat{\hat{s}}\,)$ and all $\gamma \in \operatorname{sp}\bigl(\hat{p}_{k,i}(\boldsymbol{p}_\xi)\bigr) \cap [\kappa, \alpha_{\xi+1})$.

Claim. *If $\zeta \leqslant \xi \leqslant \kappa$, then $\boldsymbol{p}_\xi \leqslant_\kappa \boldsymbol{p}_\zeta$.*

Proof: Proceed by induction on $\xi \leqslant \kappa$ to prove the claim for all $\zeta \leqslant \xi$. The claim is trivial by induction, unless ξ is a limit ordinal. In this case, it is clause (7) in the definition of $\mathbb{P}^n$'s ordering which we must verify.

Fix $(k,i) \in d(\hat{\hat{s}}\,)$. To simplify the notation a bit, set $\hat{p}_\zeta = \hat{p}_{k,i}(\boldsymbol{p}_\zeta)$, for $\zeta \leqslant \xi$. It suffices to see that if $\operatorname{dom}(\hat{p}_\xi)$ is cofinal in $u \in b^{c(\boldsymbol{p}_\xi)(\beta)}_\eta$, for some β and η, then $u \setminus \delta \subseteq \operatorname{dom}(\hat{p}_{\zeta+1})$, for some $\delta \in \operatorname{sp}(u)$ and some $\zeta < \xi$.

Set $f_\xi(\delta) = M^\delta_\xi \cap \delta^+$, for $\delta \in \mathrm{CARD} \cap [\kappa, \alpha_\xi)$. Setting $\hat{\alpha} = \alpha_\xi$ and $x = \{\bar{\boldsymbol{p}}, \kappa, w, \eta_*\}$, note that M^δ_ξ is M^δ_x, and that f_ξ is $f_x \restriction [\kappa, \alpha_\xi)$ in the notation of the Thread Lemma.

Note also that if $\delta \in \operatorname{sp}(\hat{p}_\xi) \cap [\kappa, \alpha_\xi)$, then $\operatorname{dom}(\hat{p}_\xi) \cap [\delta, \delta^+) = \bigl[\delta, f_\xi(\delta)\bigr)$. This is because $M^\delta_\xi = \bigcup_{\zeta<\xi} M^\delta_\zeta$ and $\hat{p}_\xi = \bigcup_{\zeta<\xi} \hat{p}_\zeta$ and, for $\zeta < \xi$ large enough that $\delta \in \operatorname{sp}(\hat{p}_\zeta)$,

$$M^\delta_\zeta \cap [\delta, \delta^+) \subseteq \operatorname{dom}(\hat{p}_{\zeta+1}) \cap [\delta, \delta^+) \subseteq M^\delta_{\zeta+2} \cap [\delta, \delta^+).$$

Thus $u_\delta < f_\xi(\delta)$ for δ cofinal in $\operatorname{sp}(u)$. By (3) of the Thread Lemma, it follows that $u \in M^\delta_\xi$, for some $\delta \in \operatorname{sp}(u)$. We may assume that $\delta \in \operatorname{sp}(\hat{p}_\xi) \cap [\kappa, \alpha_\xi)$, since $\operatorname{sp}(\hat{p}_\xi)$ is cofinal in $\operatorname{sp}(u)$ and $M^\delta_\xi \subseteq M^\gamma_\xi$, when $\delta \leqslant \gamma$.

Now $M^\delta_\xi = \bigcup_{\zeta<\xi} M^\delta_\zeta$ and $\operatorname{sp}(\hat{p}_\xi) = \bigcup_{\zeta<\xi} \operatorname{sp}(\hat{p}_\zeta)$ and $\alpha_\xi = \bigcup_{\zeta<\xi} \alpha_\zeta$, and each of these decompositions is $\subseteq$-monotone, so there exists an ordinal $\zeta < \xi$ such that $u \in M^\delta_\zeta$ and $\delta \in \operatorname{sp}(\hat{p}_\zeta)$ and $\delta < \alpha_\zeta$. But then $u \setminus \delta \subseteq \bigl(\operatorname{sp}(\hat{p}_\zeta) \cap [\kappa, \alpha_\zeta)\bigr)^{(\zeta)} \subseteq \operatorname{dom}(\hat{p}_{\zeta+1})$. □

Claim. *The condition $\boldsymbol{p}_\kappa$ reduces each predense class D_η to size at most κ.*

Proof: We may assume that $\bar{\boldsymbol{p}}$ does not meet D_η. Set

$$d_\eta = \Bigl\{ \boldsymbol{r}_{\xi+1} : \xi < \kappa \text{ and } E_\xi = D_\eta \text{ and } \boldsymbol{r}_{\xi+1} \neq \bar{\boldsymbol{p}} \Bigr\}.$$

Clearly, $|d_\eta| \leqslant \kappa$. Furthermore, if $\boldsymbol{p} \leqslant \boldsymbol{p}_\kappa$ and $\boldsymbol{p} \leqslant \boldsymbol{r}_{\xi+1} \in d_\eta$, then $\boldsymbol{p}$ meets D_η, by the definition of $\boldsymbol{r}_{\xi+1}$.

We must see that d_η is predense with respect to $\boldsymbol{p}_\kappa$. Suppose that $\boldsymbol{p} \leqslant \boldsymbol{p}_\kappa$. We may assume that $\boldsymbol{p}$ meets D_η and that $\kappa \in \operatorname{dom}\bigl(\dot{p}_{k,i}(\boldsymbol{p})\bigr)$, for all $(k,i) \in d\bigl(\hat{s}(\boldsymbol{p})\bigr)$. Choose $\xi < \kappa$ such that $\boldsymbol{p}'_\xi = (\boldsymbol{p})^\kappa$ and $E_\xi = D_\eta$. Note that $\boldsymbol{r}_{\xi+1} \in d_\eta$ and $(\boldsymbol{r}_{\xi+1})^\kappa = \boldsymbol{p}'_\xi = (\boldsymbol{p})^\kappa$. Also $\hat{s}(\boldsymbol{p}) = \hat{s}(\boldsymbol{r}_{\xi+1})$ and $(\boldsymbol{p})_\kappa \leqslant (\boldsymbol{p}_\kappa)_\kappa * \hat{s}(\boldsymbol{r}_{\xi+1}) \leqslant (\boldsymbol{p}_{\xi+1})_\kappa * \hat{s}(\boldsymbol{r}_{\xi+1}) \leqslant (\boldsymbol{r}_{\xi+1})_\kappa$. Let $\boldsymbol{p}^*$ be identical with $\boldsymbol{p}$, except that it absorbs $\boldsymbol{r}_{\xi+1}$'s middle part:

- If $\kappa > \omega$ and $(k,i) \in d\bigl(\hat{s}(\boldsymbol{p})\bigr)$ and $\kappa \in \operatorname{dom}\bigl(\dot{p}_{k,i}(r_{\xi+1})\bigr)$, then $\dot{p}_{k,i}(\boldsymbol{p}^*)(\kappa) = \dot{p}_{k,i}(\boldsymbol{p}_\kappa)(\kappa) \cup \dot{p}_{k,i}(\boldsymbol{r}_{\xi+1})(\kappa)$, and
- if $\kappa = \omega$, then $\dot{s}(\boldsymbol{p}^*) = \dot{s}(\boldsymbol{p}) \cup \dot{s}(\boldsymbol{r}_{\xi+1})$.

Then $\boldsymbol{p}^* \leqslant \boldsymbol{p}$. And $\boldsymbol{p}^* \leqslant \boldsymbol{r}_{\xi+1}$, since $(\boldsymbol{p}^*)^\kappa = (\boldsymbol{p})^\kappa = (\boldsymbol{r}_{\xi+1})^\kappa$ and $(\boldsymbol{p}^*)_\kappa = (\boldsymbol{p})_\kappa \leqslant (\boldsymbol{r}_{\xi+1})_\kappa$. Thus $\boldsymbol{p}^* \leqslant \boldsymbol{p}$ and meets d_η. □

7. Index

Notation

Nomenclature

8. References

[**BJW**] A. Beller, R.B. Jensen, and P. Welch, *Coding the Universe*, Lon. Math. Soc. Lec. Note Ser., vol. 47, Camb. Univ. Press (1982).
[**F1**] Sy Friedman, *The genericity conjecture*, Jour. Sym. Logic **59** (1994), *pp.* 606–614.
[**F2**] ———, *Class forcing* (to appear).
[**S1**] M.C. Stanley, *Coding a Generic Extension of L* (to appear).
[**S2**] ———, *Invisible Genericity and $0^{\#}$* (to appear).
[**S3**] ———, *Class forcing* (to appear).

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