

DAVID WILLIAMSON SHAFFER and JAMES J. KAPUT

**MATHEMATICS AND VIRTUAL CULTURE:
AN EVOLUTIONARY PERSPECTIVE
ON TECHNOLOGY AND MATHEMATICS EDUCATION¹**

ABSTRACT. This paper suggests that from a cognitive-evolutionary perspective, computational media are qualitatively different from many of the technologies that have promised educational change in the past and failed to deliver.

Recent theories of human cognitive evolution suggest that human cognition has evolved through four distinct stages: episodic, mimetic, mythic, and theoretical. This progression was driven by three cognitive advances: the ability to “represent” events, the development of symbolic reference, and the creation of external symbolic representations. In this paper, we suggest that we are developing a new cognitive culture: a “virtual” culture dependent on the externalization of symbolic *processing*.

We suggest here that the ability to externalize the manipulation of formal systems changes the very nature of cognitive activity. These changes will have important consequences for mathematics education in coming decades. In particular, we argue that mathematics education in a virtual culture should strive to give students generative fluency to learn varieties of representational systems, provide opportunities to create and modify representational forms, develop skill in making and exploring virtual environments, and emphasize mathematics as a fundamental way of making sense of the world, reserving most exact computation and formal proof for those who will need those specialized skills.

1. PRINTING PRESS OR STEREOSCOPE: AN INTRODUCTION

The history of the printing press and its transformative impact on Western culture is a well-known story. The introduction of the printing press, and with it mass-produced and widely-available books, led to the standardization of vernacular languages, the spread of literacy, the development of new literary forms (including the “novel”), and ultimately contributed to the growth of a middle class and with it many of the social and economic transformations of the last few centuries (see McLuhan, 1962).

A slightly less well-known episode in the history of technology is the stereoscope. The stereoscope was first described in 1838 as a means to capture three-dimensional images by directing separate photographs of the same scene to each of the viewer’s eyes, thus simulating binocular vision. The trade in stereoscopic images boomed after Queen Victoria viewed them at the Crystal Palace exhibition in 1851; by 1853 a New York Tribune writer proclaimed with excitement that “the day must soon come when nearly all important photographic pictures... will be produced double, that is, by couples, [for] stereoscopic reproduction, in all the exact truth of living nature” (New York Daily Tribune, 1853). But history proved this journalist misplaced in his enthusiasm, as still photography and stereoscopic images were replaced first by motion pictures and later by television and video as the dominant visual medium. Stereoscopic images today are the preserve of collectors and specialists, or relegated to the status of children’s toys.

We raise these examples because discussions of computational media in education often take on the tone of the enthralled journalist (see Oppenheimer, 1997), with advocates claiming that the introduction of computers into education will bring profound and lasting changes—and arguing that putting computers and internet-access into schools should be our highest priority in educational policy. At the same time, skeptics look to recent history and suggest that technologies come and technologies go without making any dramatic impact on educational practices. These skeptics point out that some of the same kinds of

rhetoric that we hear about computers today was used in the past about motion pictures, radio, film strips, television, and other “new media” (see Cuban, 1986; Tyack and Cuban, 1996; Oppenheimer, 1997).

This paper looks at the introduction of computational media from an even longer-term perspective. Recent work by psychologist Merlin Donald (1991) argues that human cognition has developed over evolutionary time through a series of four distinct stages: starting with episodic (ape-like) memory and progressing through mimetic, mythic, and theoretical revolutions. This paper suggests that viewed in terms of the growth of symbolic understanding, computational media are the manifestation of a fifth stage of cognitive development—and thus in their long-term impact on our culture and society more like the profoundly-transformative printing press than the relatively-insignificant stereoscope. As mathematics educators we are particularly interested in the critical role that mathematics has played in the development of this new stage of cognition, and on the implications of this new cognitive culture for mathematics learning.

2. FOUR STAGES OF MENTAL EVOLUTION: AN OVERVIEW

In his book Origins of the Modern Mind (Donald, 1991), Merlin Donald argues that anatomical evidence of human evolutionary development (both current and in the fossil record) suggests that human culture has gone through four distinct stages of development.² Donald suggests that each of these stages of *cultural* development was driven by a specific *cognitive* advance, and that these changes in cognition led to changes in brain development as well as new kinds of communication and social interaction.

The first stage Donald outlines is essentially that of primate (ape-like) cognition with origins among early primates more than three million years ago. This stage is based on “episodic” thought, which Donald describes as thinking based on literal recall of events. Apes can remember details of, for example, a social interaction, and can even recall those details in context—thus an ape might “remember” that a larger male is dominant because he

can recall a fight where the dominant male won. But, as Donald and studies of primate behavior make clear, apes do not represent events in any way. They do not attach labels to events, meanings to events, or generalize from events. They do not process events other than storing their images in episodic memory. Donald argues that apes who have learned rudimentary sign language are essentially storing and using the signs in much the same way as they would process any kind of conditioning—they remember signs as responses leading in certain circumstances to pleasure or away from pain (p. 154).

Episodic cognition provided a basis for social interaction by giving early hominids the ability to recall previous events and respond accordingly. This rudimentary socialization was extended by the development of the fundamental ability to “represent” events dating from homo erectus about 1.5 million years ago. Donald describes this as “mimesis,” or “the ability to produce conscious, self-initiated, representational acts that are intentional but not linguistic” (p. 168), comprising the second stage. For example, following the gaze or pointing gesture of another requires an understanding that their gestures are referring to something of interest. Or, more dramatically, reenacting or replaying events using objects (like a child who spansks a doll after getting a spanking him or her self) shows a basic ability to process events and to communicate about them to oneself and to others.

Donald argues that this ability to represent events was not (and is not) dependent on language. The morphological changes required for the development of speech are quite dramatic, and therefore unlikely to occur without some evolutionary pressure favoring the ability to communicate using language. Donald believes that the evolution of language was dependent on this prior cognitive development: namely, the development of symbolic reference. Donald distinguishes iconic representation, where the representation shares some property with the thing being referred to, from symbolic representation, where the symbol can be any arbitrary gesture, sign, or sound.³ The first symbols were probably, according to Donald, standardized or ritualized gestures. From simple vocalization to more complex articulation, language developed as the most efficient system for creating and

communicating symbols of the world. In other words, the social need to communicate ideas (symbolic representation of events) drove the evolutionary development of language, not the other way around (p. 255). The development of language marked the arrival of a “mythic” culture based on narrative transmission of cultural understanding, comprising the third stage beginning about 300,000 years ago (see also Bruner, 1973, 1986, 1996).

In her recent book on language and development, Katherine Nelson (1996) accepts Donald’s categorization of stages of mental development, but argues that in individual (as opposed to evolutionary) development, the relationship Donald describes between representation and language is reversed. That is, Nelson argues that language drives individual development of symbolic competence. Language provides an external structure that scaffolds a child’s ability both to represent events, and later to develop narrative and categorical understanding of the world. Although Nelson does not say so explicitly, this reversal makes sense *for a child who is raised in a culture that has already developed language*.⁴ In other words, it seems reasonable that evolutionary development of a cognitive ability and individual development of the same ability might differ—and that the evolutionary development of a new form of representation might have profound developmental consequences. We will return to this idea later in the discussion.

The fourth stage Donald identifies is that of “theoretic culture,” or culture based on written symbols and paradigmatic thought. Again, Donald argues that the principal change here was in the development of new cognitive ability rather than a new means of expression—in this case, the need to work with complex phenomena drove the development of external representations beginning 30-50 thousand years ago. The record-keeping needs of commerce and astronomy drove the creation of external symbol systems (p. 333ff), of which mathematical notations were probably the first (Donald, 1991, p. 287; see also Schmandt-Besserat, 1978, 1992, 1994)⁵ The existence of external representations, according to Donald, made it possible for humans to begin reflecting on the interrelationships among recorded ideas in an analytic fashion. Donald refers to this use of

notation to augment thinking as the “external memory field” or EXMF. The accumulation of recorded knowledge, which Donald calls “external symbolic storage” or ESS, provides a larger context in which analytic thinking takes place.

Donald suggests that modern scientific culture developed from and depends on the existence of external notations for thinking and of external records for ideas. Much of schooling is about learning to access parts of the cultural record and to manipulate them using the tools of external working memory such as writing and mathematical notation (p. 320). Our theoretic culture is dependent on large-scale storage of information as a kind of database for analytic thinking, as well as on a set of external tools that help us control the flow of this information to our biological processors—that is, to our minds, which evaluate and transform that information (p. 329).

Donald argues that all of these ways of thinking—episodic, mimetic, narrative, and theoretic—exist in our minds simultaneously, and that we move among them and use them in a fluid way. So, for example, a song-and-dance involves both mimetic and linguistic representation, often in a mythic context. The process of scientific discovery is a dialog between particular (episodic) events and general (theoretic) models. Donald refers to this combining and shifting of representational perspectives as the “hybrid mind.”

3. INDIVIDUAL AND CULTURAL DEVELOPMENT: FOUR THEMES

There are four themes that emerge from this summary of Donald’s work. The first is that, *at the evolutionary level*, changes in cognition drive changes in representation rather than the other way around. At each stage in development, a new way of thinking about (modeling) the experienced world gradually creates new means of instantiating that model. Language evolves as a consequence of symbolic thinking, not the other way around; a point of view consistent with the deep analyses offered by Deacon (1997).

The second idea that emerges from Donald's work (or, at least from this overview of it), is that our current, "theoretic" culture depends on the external storage of information. We use external media to record ideas and to act as an external memory buffer while we are processing ideas. The generation, translation, and transformation of ideas are done internally, but depend on the presence of shared external information and the external tools to augment our working memories.⁶

A third idea from Donald's work is that this theoretic culture based on external storage of information arose, in large part, from the need to deal with quantitative information (Schmandt-Besserat, 1978, 1992, 1994; Donald, 1991). Whether it was records of harvests, of business transactions, or of the movements of celestial bodies, the storage and computation of numerical information was a driving force in the development of our current culture.

A fourth and final point worth noting here comes from Nelson's study of Donald's theories in the *developmental* (rather than evolutionary) realm. This is the idea that new cognitive processes affect the way older modes of thought emerge in individual development. The "hybrid mind" does not just refer to the interaction of modes of thinking *within* an individual. The presence of various modes of representation within a culture changes the way we learn to understand our worlds as individuals.

The remainder of this paper explores these ideas in relation to our posited fifth stage of the development of human cognition. Computational media make it possible to externalize not only information, but also the *processing* of information. This has profound implications for the nature of human cognition in general and mathematics learning in particular⁷.

4. INFORMATION AND PROCESSING: A NEW MEDIUM

Donald is quite explicit in his claim that the major cognitive development involved in the creation of theoretic culture is the appearance of the “external memory field” as a external memory loop—that is, as an externalization of working memory (p. 329ff).⁸ Cognitive theorists, and particularly those whose information processing perspective matches Donald’s analysis (see, e.g., Block, 1981; Akin, 1986; Rowe, 1987; Kosslyn and Koenig, 1992; Simon 1996), refer to short term or working memory as a kind of scratch pad or data storage buffer for mental processing. Working memory holds pieces of information for processing, but is not necessarily the part of the mind that does the actual transformation of information.

Whether or not one believes that mental activity can be as neatly segmented as such an information processing perspective suggests (and we do not), it is clear that Donald is arguing that theoretic culture depends not on external *processing* of information, but on external devices to store information as a substitute for long- and short-term memory. That such storage has cognitive consequences is clear. When a writer makes an outline for a paper, it helps organize his or her thinking. But the transformation of outline into text is still at every level a function of the working of his or her biological mind.

Or is it? Clearly, when a person is working with pen and paper, the external tools are recording the products of his or her thinking. That these inert products (i.e., the outline or emerging text) feed back into thinking is obvious. But when a person writes with a word processor, before finishing a paper he or she can also run it through a spell-checker and grammar-checker. These programs alter the text (or more accurately in most cases, make suggestions for altering the text) based on rules of standard usage for formal spelling and writing. The computer running a spell-checker is not just recording a person’s thinking in a loop of production and expression of thought. It is actually performing some of the functions that a mind might take on in a similar circumstance—in this case, the functions of

a (particularly literally-minded) editor. It is true that the author might be able to do this editing him- or herself, but the point is that he or she does not have to. The computer not only records the writer's thinking, it actually does some of the information processing; similarly, a calculator—either numeric or symbolic—processes information outside the user's head.

To take a more dramatic example, when a researcher performs a statistical analysis, he or she may ask the computer to compute a multivariate regression, or perhaps a principal components analysis, on a set of data. The software analyzes the raw data and, using an iterative technique, produces output that the researcher can use to understand the structure of the data in question—perhaps even by producing a highly visual representation of the newly organized data. What makes this example so compelling is that for all practical purposes, the researcher *could not* perform the same manipulation of that information. This is true partly because of the time it would take to run the same calculations by hand. But a researcher can even use software to compute statistics that he or she knows how to use and interpret but does not know how to compute by hand.

In all of these examples (and there are many others possible), the computer is doing something quite different from Donald's model of the "external *memory* field." The computer is, essentially, acting as an external *processing* field, taking information in one form and returning it in another form without action by the writer (or researcher) in the interim. Whether the computer "understands" the information in any sense is irrelevant here. The point is that a person can use the computer as a tool to augment or replace not only memory, but substantial mental processing of information as well.

In both of these examples, the computer is externalizing mental processing by adding algorithms to an underlying store of information. That is, the computer has an external store of information, and a set of procedures for acting on that information. In the case of the spell-checker, for example, the computer has the equivalent of (parts of) a dictionary.

But it also has a set of rules for comparing words in a text to entries in the dictionary. As is the case with a traditional printed dictionary, it can “remember”—but it can also “act.”⁹

To take a concrete illustration, if a great chef dies without ever telling anyone about or recording his or her secret recipes, the meals die with him or her. No one can recreate exactly the dishes he or she cooked. If he or she writes down the recipes, then as long as the book exists, someone (with some degree of culinary skill) can recreate the dishes. But if that chef were also a master engineer, he or she could (conceivably) create a machine or machines that could reproduce the actual dishes—that is, it record not only the information for the food, but actually replicate the processes for its production. The book of recipes would be an example of Donald’s external symbolic storage. The machine for food production would be more like a modern computer.¹⁰

This suggests that we are on the verge of a new culture in Donald’s sense of the term, one dependent on the externalization of pieces of mental process as well as on externalization of pieces of representation. Computational media make it possible to externalize algorithm as well as information.¹¹ According to Donald, the development of an ability to represent events created a “mimetic” culture based on communication mediated by the exchange of physical gestures. The addition of language made possible a “mythic” culture based on the exchange of narrative stories. The creation of written symbols led to a “theoretical” culture based on external symbolic storage. Continuing the progression, we suggest that the computational media are in the process of creating a virtual culture based on the externalization of algorithmic processing.¹²

5. VIRTUAL CULTURE AND FORMAL SYSTEMS: THE POWER OF THE EMPTY SIGN

Donald’s thesis suggests that we should look for the roots of the development of a fifth stage of cognition not in the media of representation but in a change in the way we represent or model our experience of the world. That is, we should understand the cultural

development of computational media by looking at the cognitive processes that made possible their creation. The development of computational media depends on three factors: the existence of discrete notations, the creation of rules of transformation, and an external system capable of autonomously applying those rules.

Nelson Goodman (Goodman, 1968; see also Gardner, 1982) argues that a principal feature of any notation system is the extent to which the symbols it uses are both disjoint and well-defined, as opposed to syntactically and semantically dense. Put in less jargony terms, symbol systems differ in the extent to which a given symbol can be precisely identified and precisely interpreted. The canonical example is a spiked line, which might be the trace of a function on a graph or a drawing of a line of hills. In one case, the meaning of the mark would be unambiguous and its translation into a representation of numerical values clear. In the other, the line would hint at spaces present as well as spaces missing and have shades of meaning for the viewer.

<Insert Fig. 1 here>

Perhaps the first, and certainly the most well-explored, system of notation with precise symbols and unambiguous meanings is the representation of numbers, particularly of whole numbers. In the case of integers, from a cultural, if not a psychological perspective, a specific symbol (or combination of symbols) represents a specific number and no other. The symbols of the number system are also one of the first instances of a system of notation with rules of transformation. Algorithms for addition and subtraction of numerals have existed for thousands of years. The process of addition (or subtraction, multiplication, etc.) can be represented by a set of rules for manipulating strings of numerical symbols (digits). Terminology from our most current algorithms like “carry the one” or “borrow from the next column” reflect the extent to which we think of the processes of arithmetic as a set of rules for manipulating numerals. The base-ten placeholder system of numerals and

the algorithms build upon it have been a critical factor in the development of our culture (Swetz, 1987).

A prodigious advance in the development of mathematics was the creation of another, more general and therefore more powerful set of algorithms for representing and manipulating quantitative relationships: namely, the development of algebra and the rules for manipulating algebraic symbols to solve equations, transform character strings into one or another canonical form, and so on (Bochner, 1966).

In even a relatively simple “toy” problem, such as the one shown in Figure 2, we can see how an algebraic solution represents a situation as a set of equations within a notation system and then applies transformation rules to the equations to produce a solution. As the transformation rules are being applied, in many cases the intermediate states of solution do not make much sense in terms of the original problem (note the bold cells in Figure 2)¹³. The rules of transformation operate on the *equations*, not on the situation the equations represent. In Bruner’s terms, the symbols are being treated as “opaque:” that is, they act as autonomous objects with their own identity and rules of transformation. This is different from a use based on what the symbols stand for, which Bruner refers to as “transparent” (Bruner, 1973).

<Insert Fig. 2 here>

The combination of a discrete (i.e., unambiguous) notation and set of transformation rules can be thought of as a formal system. Or, perhaps more precisely, as the *core* of a formal system. Euclid created the first known example of a set of explicitly defined objects and the relations on them. But Euclid (and the rest of the mathematical community for centuries) believed that this system was an ideal representation of some aspect of the real world. It was not until the development of non-Euclidean geometries that this notion was seriously questioned. Among many other events, this helped lead to a twentieth century crisis in the foundations of “truth” in mathematics that, in turn, led to a genuinely

“formalist” view of mathematics as a domain where symbols and actions on them no longer needed concrete referents.

Davis and Hersh (1982) describe in some detail the development of a formalist philosophy of mathematics. Briefly, the formalist position is that mathematical inquiry is, at heart, a “game.” In this game, we define a set of symbols, a set of legal strings of symbols (axioms), and a set of rules for manipulating those symbols. The game is to determine what possible well-formed combinations of symbols (theorems) can be made from the starting set and the rules of combination.¹⁴ What is particularly important to note about this vision of mathematics is that *the symbols don’t necessarily refer to any specific external referent*. I can interpret “line” as having a meaning in a plane, on a sphere, or as an abstract entity with no physical or conceptual equivalent. One might define an operation on a set of symbols in such a way that you might recognize the set of symbols and operation as, say, an abstract semigroup. But whether or not it is recognized as a conceptually familiar object is independent of its status as a formally defined system.

A formal system, then, is an arbitrary but well-defined set of symbols and, most importantly, rules of transformation on those symbols.¹⁵ A critically important feature of such systems is their operative nature—the existence of internally coherent rules for transforming the allowable (“well-formed”) symbols into other symbols.

The power of computational media is in using such formal systems to model aspects of the experienced world.¹⁶ In Donald’s analysis, mimetic culture developed from the ability to represent the world using iconic gestures. Narrative culture developed from the ability to use abstract (arbitrary) symbols for representing aspects of the world. Theoretic culture developed from humans’ ability to use symbols to refer to other symbols using an external medium. In our extension of this scheme, the next stage of algorithmic or virtual culture develops from the ability (1) to use operative systems of symbols that do not have fixed

external reference, and (2) to effect autonomous operations on those symbols in a dynamic medium.¹⁷

One might object at this point that the symbols the computer uses are not arbitrary—that in fact they do refer to something particular in the world, like the array of pixel values in a computer graphics image or the entries in a balance sheet created in a spreadsheet (Newell, 1980). It is, of course, true that in many cases the symbols on which a computer is operating have come from the external world. And in many cases, the symbols that result from processing will be interpreted as having meaning in the real world—as, for example, would be the case in a simulation program (Smith, 1996). And, of course, most such systems arise from more concretely referenced systems (e.g., the matrix system mentioned below). But the key idea is that it doesn't matter whether the symbols refer to any particular thing or not. The computation is the same whether the image starts as a random assortment of colors or a digitized image of your cousin Bert. The cognitive leap is in thinking about “processes” on their own, with an underlying set of symbols as merely a representation of unspecified inputs and outputs of the process—that is, without reference to any specific domain, concrete or abstract.¹⁸

One might also complain that we have not said anything in particular about the machine that uses this formal system to externalize processing of information. Just so. There are many media with which one can externalize the storage of symbols, and while one of them (alphabetic writing) has been particularly powerful and influential historically, the key developments in theoretic culture are the cognitive causes and cognitive consequences of the externalization of memory. The contemporary hardware and software of computers are not the only way that one can externalize process as described above. For example, the formal system of linear matrix algebra over some field can be used as a non-computerized system for externalizing pieces (albeit small ones) of mental processing required to solve systems of linear equations with coefficients in the field—you can solve a system of equations by following a simpler series of transformation rules on the “equivalent” matrix.

Another particularly simple example is the abacus, which involves purely physical actions on a highly structured physical system to effect arithmetic operations--with a human partner. What the computer presents is a particularly effective—and therefore particularly transformative—instance of this underlying cognitive revolution that enables *autonomous* processing.

Obviously, the fact that a mathematical model or system might have more than one interpretation in the “real world” does not mean that relationships *within* the model are arbitrary. What is significant is that, over the past several hundred years, modern mathematics has created a way of thinking about rules of transformation *separate from* any particular set of symbolic references¹⁹. A “formal system,” describes a set of abstract relationships and procedures that can produce a specified result based on any set of objects or symbols that fit its initial conditions. The ability to specify explicitly the rules of such systems makes it possible to externalize the processing as well as the storage of information—and thus make a virtual culture possible.

6. MATHEMATICS AND VIRTUAL CULTURE: SOME IMPLICATIONS

One striking feature of this description of virtual culture is that like theoretic culture, the impetus for a new cognitive mode came from the world of mathematics. If theoretic culture had its roots in the need to record quantitative information, then virtual culture has its roots in the need to conduct more and more complex processing of information using formal mathematical systems.²⁰ Perhaps more interesting, though, is an examination of the consequences—and particularly the consequences for mathematics and mathematics education—that come about as a result of the development of externalizable algorithmic thinking.

6.1. New Representational Forms

Papert and Kaput (Papert, 1980; Kaput, 1992) among others have written about the way in which computational media make it possible to externalize algorithms and thus make processes of thinking available as explicit objects for reflection. Whether this takes place through the “construction by example” of scripts for automating actions (as in the Geometer’s Sketchpad), through the explicit programming of procedures (as in Logo), or through the recording of editable algorithms from actions on physical manipulatives (Kaput, 1996a), the existence of an external representation of an algorithm that can be built, tested, discussed, and changed by students makes it possible to talk about mental activities that were once difficult to describe, much less investigate.

Computational media also create new representational forms, such as dynamic geometry environments or manipulable Cartesian graphs separate from algebraic notations (see Kaput & Roschelle, 1998). Such new representations enable students to work on problems using different cognitive modes, allowing students, for example, to take more concrete approaches to abstract problems (see Turkle & Papert, 1990; Papert, 1993). In general, we might say that computers widen the available range of different approaches for generating, collecting, processing, and interpreting information. One can solve a problem of linear relations using algebra and a pencil. Or find a graphic solution using graphing calculator. Or use iterated estimation with a spreadsheet. Or use a geometric representation in a dynamic geometry environment. Or use a motion simulation. Or use symbolic manipulation software (see Fey, 1989).

This representational pluralism (see Turkle & Papert, 1990; Papert, 1993) suggests that one of the key changes in mathematics education will be a move away from the traditional focus on algorithmic fluency in a few extremely concise and notationally compressive

representational systems (notably arithmetic and algebra). The pedagogy of mathematics in a virtual culture should logically shift towards fluency in representing problem situations in a variety of systems, and towards students' ability to coordinate among representations as well as create and interpret novel ones. Kaput (1986) has suggested in earlier work that one of the important features of computational media in mathematics learning is their ability to help students see the relationship among different representations of the same mathematical situation. Others have similarly suggested that mathematics is (or should be) more about the ability to move among and use a variety of representations than about facility in performing arithmetic or algebraic manipulations (Confrey & Smith, 1994; Davis & Hersh, 1982; Yerushalmy, 1991).

The association of "mathematics" with "fluency in arithmetic and algebraic algorithms" makes perfect sense in the context of a theoretic culture. Theoretic culture is about using external symbolic memory to support complex mental processing. Thus doing mathematics in theoretic culture is clearly about doing the manipulations (whether simple addition of numbers or complex integration by parts) that move from one character string (external symbolic representation) to the next. Virtual culture, on the other hand, is about off loading such symbolic processing from the biological mind to some external device. From a virtual perspective, the mathematics is not in performing the formal manipulations, but in imaginatively framing the problem in a way that uses clear and compelling representations, often interactively, leading to efficient and convincing solutions through interpretation of external action by technological tools.

To take a different example, from a theoretic perspective reading is not about committing a text to memory. That is the role of the external symbolic storage system. Reading is about decoding a text, making sense of it, and being able to use the information it contains in a meaningful way. A curriculum that focused for years on students' ability to memorize prose would be seen as pedagogically backward in a theoretic culture. Similarly, from a virtual perspective, mathematics is not about calculations. That is the role of the

external symbolic processing system. Mathematics is about understanding a problem, representing it in an external processing system, and being able to use the information produced by the external calculations in a meaningful way—as Richard Hamming put it: “The purpose of computation is not numbers, but insight.” Curricula that focus on students’ ability to perform routine calculations are thus regressive in a virtual culture (see also Dorfler, 1993). This is not to say that students in a virtual culture should not learn to do some calculations by hand or in their heads—just as students in a theoretic culture still commit some particularly important ideas to memory, and still learn to do some calculations without pencil and paper. As Donald suggests, the mind is a hybrid of its many cognitive cultures. But the balance of mathematics education in a virtual culture would reasonably shift towards an emphasis on representational versus computational fluency.²¹

6.2. New Pedagogical Approaches

Another implication of virtual culture for mathematics education is that computational media make it possible to think of mathematics education in a much more inductive and naturalistic way. This is not to say that the concept of mathematics as a deductive enterprise should be abandoned—although as Davis and Hersh (1982) point out, mathematics’ claim to deductive “truth” is increasingly shaky. Rather, it suggests that students can profitably experience mathematics as an experimental enterprise leading to the need and desire for more formal justification. That is, they can encounter mathematical ideas in a way that lets them manipulate phenomena and expose possible underlying mathematical relationships. For example, early work by Yerushalmy suggests that students who used the Geometric Supposer evolved a significantly stronger expressed need for proof than did their more traditionally taught peers (Yerushalmy, 1986).

It becomes possible, in other words, to put a student in a highly interactive mathematical situation and let him or her “play” as a way of developing mathematical understanding because the feedback loops are much tighter and less dependent on students’ own ability to transform symbols. It is also possible to put a student in a manipulable

simulation and support the student in extending his or her understanding of the simulated situation and of the underlying mathematics. Just as Nelson describes the power of language to drive development once it exists in the external culture, so algorithmic processing can drive understanding once it exists as a common tool in a child's environment. The ability to think about and with "live" external representations of processes can scaffold the development of mathematical understanding, which was a fundamental point of Papert's *Mindstorms* (1980).

Such mathematical play might be scientific and experimental, as described above. Or it might be artistic and expressive. The ability of computational media to represent artistic expression within formal systems (as, for example, in image-manipulation tools such as Adobe PhotoShop) means that students can explore mathematical and aesthetic concerns simultaneously (Shaffer, 1997). Part of the power of a computer is that a sufficiently fast formal process can take on artistic dimensions—just as a sufficiently fast static image can take on dynamic qualities, as in a motion picture or a dynamic geometry environment.

6.3. New Objects of Study

New virtual tools not only change the way students can approach traditional mathematical ideas, they also make it possible to address new and different phenomena in the world around us—phenomena that were difficult or impossible to model using non-computational methods. In particular, computational tools provide visual, manipulable models of dynamical systems (Heim, 1993; Cohen & Stewart, 1994; Hall, 1994; Holland, 1995; Kauffman, 1995; Casti, 1996). These models approach complex, often non-linear phenomena with iterative, dynamic representations—representations that can only produce meaningful results in a realistic time-frame in a computational medium that supports massive and rapid external processing. These new dynamic models make it possible to examine social, physical, and biological situations such as traffic jams, oscillating springs, or the behavior of biological systems such as ant colonies that embody many strongly interacting elements. These kinds of situations can be

modeled in entirely new ways by and for learners using computational environments (see Resnick, 1994).

Mathematical experience in a virtual culture will thus be more intimately connected with students' wider world of experience. Inductive explorations with manipulable modeling systems will make it possible for learners to make more lively, more compelling, and more intimate connections between "mathematics" and what they experience as the "real world" (Kaput, 1996b).

In response to these new tools, new pedagogical approaches, and new content areas, we may need to reexamine the idea of "mathematical abstraction." This is a topic that Nemirovsky (1998), Noss and Hoyles (1996), and Wilensky (1991) have addressed in other work and that we plan to pursue in subsequent papers. Briefly, as these authors variously suggest, we may need to make room in our notion of mathematical understanding for a kind of "concrete abstraction" that builds mathematical meaning from an active web of meaningful associations rather than an relatively meaningless set of empty formal rules.

6.4. Towards a Virtual Mathematics

All of the above suggests that one of the implications of virtual culture for mathematics education will be an increased emphasis on embedded and situated mathematics—on mathematics as a way of knowing the world—rather than on mathematics as computation or mathematics as formal proof. To understand this mathematics we will need ways of thinking about mathematical knowledge, especially the new forms that themselves require the computational medium, that include a broader range of thinking styles and that are situated more firmly in the real-world experiences of mathematics learners. Again, this is not to say that computation and proof will or should disappear from the curriculum. After all, older forms of thinking did not disappear with the appearance of writing, nor should they have disappeared. Rather, the broadening of mathematical forms in a virtual culture suggests that computation and proof are better suited to a supporting role—rather than their

traditional starring role—as mainstream students learn to make sense of quantitative information through mathematical modeling and exploration.

7. COMPUTERS AND COMPUTATION: IN CONCLUSION

We began this paper by suggesting that “computational media” are a transformative technology, more like the printing press than the stereoscope in terms of their long-term impact on our culture. We chose the broad term “computational media” rather than the more usual term “computer” deliberately, because the point of this paper is not that “computers” in their current form at the end of the twentieth century are necessarily transformative. Rather, we suggest that the ability to externalize significant pieces of the processing of information—of which the modern computer is only the best example to date—has the power to transform the way we approach mathematics (as well as a host of other areas of cognition and culture). We expect that this transformation will only be fully realized after computation moves off the desktop and is distributed more widely in the space around us (Roschelle, Kaput, Stroup, & Kahn, 1998). But a cognitive-evolutionary perspective on computational media and their relation to earlier forms of cognition suggests that computational media are qualitatively different from many of the technologies that have promised educational change in the past and failed to deliver.

When and how—and perhaps whether—change will come from computational media and the virtual culture they make possible will of course depend on questions of politics and policy as much as cognitive and educational theory. But all technologies are not created equal. Some, like the stereoscope, are relatively insignificant in their epistemological implications. Others, like the printing press and computational media, have profound cognitive and social consequences—profound enough, in some cases, to lead to substantial cultural and social transformation.

In making this claim, we recognize that this account of mathematics and virtual culture is far from complete. In particular, we have focused primarily on individual cognition and

how virtual culture changes the way individual students might approach mathematical thinking. In doing so we have only touched briefly on how a virtual culture creates new relations between individual and social experience, and new cultural forms. Nor can we reliably predict the dynamics of evolution that will occur in the virtual culture, and what new representational forms may arise (Dyson, 1997)—after all, who really predicted the WorldWide Web? It seems clear, however, that in a virtual culture students will have new ways of sharing new forms of mathematical experiences, mathematical representations, and ultimately, mathematical understanding. We plan to approach this issue—and to look more generally at the social implications of virtual culture for mathematics learning and cognition—in subsequent publications. We hope that this paper has helped illuminate some current issues in a new light.

NOTES

¹ Work in the paper, by the second author, was supported by Department of Education OERI grant #R305A60007.

² As might be expected, a bold and comprehensive thesis such as Donald's will generate a body of reaction and criticism. See, for example, (Mithen, 1997) for a critique from anthropology, and a broader set of responses in the special issue of *Behavioral and Brain Sciences* (Donald, 1993). The structure and implications of Donald's thesis remain intact. See, for example, the review of Mithen's book by Howard Gardner (1997).

³ In this sense, Donald makes the same kind of distinction that Jerome Bruner does between an iconic and symbolic stage of representation (Bruner, 1973). However, in Donald's scheme, linguistic symbols at first refer to concrete events. For Bruner, the significance of the symbolic stage is that abstract symbols can refer to other symbols. See also the discussion below of Goodman's analysis of symbolic systems (Goodman, 1968).

⁴ Papert has made a similar point about the development of mathematical understanding in the context of a mathematically-rich surrounding culture (Papert, 1980)

⁵ Other authors have similarly argued for the record-keeping as the impetus for the development of written languages. See (Kaput & Roschelle, 1998) for a more detailed discussion of this issue.

⁶ Less significant in Donald's argument—but certainly of great importance in understanding “theoretic culture”—are the social and institutional roles of the external inscriptions (Latour, 1979). The impact of computational media on forms and patterns of communication will be the subject of a subsequent paper on virtual culture.

⁷ A subsequent paper will address the question of changes in the nature of mathematics (content) itself.

⁸ In a similar way, theorists of social or distributed cognition suggest that cognition is best understood not as a phenomenon “in the mind,” but as a complex interaction between an individual and the people and things around him or her (Vygotsky, 1934/1986, 1978; Bruner, 1990, 1996; Rogoff, 1990; Pea, 1993).

⁹ Ultimately, the algorithms come from human beings who have programmed the computer—or, more accurately, who designed another machine that creates and programs the machines we use. In this sense the algorithms are not that different from the dictionary itself, which was recorded over time by other people and comes to us as part of the cultural surround—Donald's external symbolic storage system. What is important is not where the algorithms come from as much as the fact that they allow independent, external processing of information.

¹⁰ Dennett (1996) suggests that a cookbook is actually recording algorithms. The point here is that while a cookbook records algorithms, it still requires a human being to carry them out—and is thus an example of external information storage rather than external information processing. It is interesting to note that much of the current food-processing industry operates by automation in the production of ready-to-eat (or heat) meals—which serves as a gustatorial reminder of the potential power of external processing.

¹¹ We should be careful here to point out that we are not advocating or assuming a traditional information-processing view of the mind or of all mental activity. Rather, we are arguing that virtual

culture results from the development of a new means of carrying out a *particular kind* of mental function: namely, the execution of well-specified algorithms.

¹² In making this argument, we recognize that the creation of this “virtual culture” is in its infancy. In particular, we recognize that the creation of a true “virtual culture” may also require the simulation of direct sensory information, as described in the fiction of authors such as in Gibson’s *Neuromancer* (Gibson 1984). The ability to directly experience sight, sound, or other simulated sensory information would surely complete a “virtual” culture. A point of view strongly arguing for the importance of more direct experience and less heavily mediated experience is offered by Reed (Reed 1996).

¹³ Given the relative referential “senselessness” of the intermediate steps in an algebraic solution, the process of algebraic manipulation usually requires the temporary suspension of referential sense-making in favor of syntactical sense-making. Of course it is also the case that the syntactically generated strings might help reveal previously hidden relationships. Such is the magic and the mystery of formalism.

¹⁴ Douglas Hofstadter provides a particularly clear example of such an ideal system with his MIU game *Godel, Escher, Bach* (Hofstadter, 1979).

¹⁵ A more precise definition can, of course, be offered relative to a choice of logic (see, for example, Roman, 1990), but this level of precision is beyond the scope of this paper.

¹⁶ It is worth pointing out in this regard that the external *representation* of algorithmic processes (with or without formal systems) is distinctly and importantly different from the external *execution* of symbolic manipulation (which in the case of computational media depends on the existence of formal systems). The cognitive advance we are describing here was not only in the external storage of algorithms in formal systems (which remove memory requirements on human cognition), but also in the external operation of those algorithms using formal systems, which make it possible to offload symbolic *processing* as well as symbolic *storage*.

¹⁷ In the mid 1970’s the notion of a genetic algorithm was invented by John Holland and others, wherein the program modifies itself across iterations by way of random mutations of its operation strings. This form of external symbol processing amounts to a new level of processing autonomy, different in kind

from self-modifying programs as had been developed by John MacCarthy and others in the context of LISP during the 1960's (see Holland, 1995).

¹⁸This is, of course, the formal idea of a computer program as described by Turing and von Neumann (Von Neumann, 1966, Turing, 1992).

¹⁹ It is worth remembering that operations on quantities were very much tied to dimensionality and other physical referential constraints till Descartes' time (Kline, 1972).

²⁰ In suggesting that mathematics has played a significant role in cultural and cognitive evolution, we certainly do not mean to imply that mathematics was the sole cause of cultural or intellectual change. As is appreciated by the readers of this journal, mathematics operates within a cultural context, and is only one—perhaps currently under-appreciated—factor in cultural development.

²¹ It is worth noting that the forms of knowledge developed in a computational medium may be subtly different from those developed in static, inert media (see Sherin, 1996), just as the forms of knowledge developed in mental calculations may be different from those developed using pencil and paper.

REFERENCES

- Akin, O.: 1986, *Psychology of Architectural Design*, Pion, London.
- Block, N. (ed.): 1981, *Imagery*, MIT Press, Cambridge, MA.
- Bochner, S.: 1966, *The Role of Mathematics in the Rise of Science*, Princeton University Press, Princeton.
- Bruner, J.: 1986, *Actual Minds, Possible Worlds*, Harvard University Press, Cambridge, MA.
- Bruner, J.: 1990, *Acts of Meaning*, Harvard University Press, Cambridge, MA.
- Bruner, J. S.: 1973, *Beyond the Information Given: Studies in the Psychology of Knowing*, Norton, New York.
- Bruner, J. S.: 1996, *The Culture of Education*, Harvard University Press, Cambridge, MA.
- Casti, J.: 1996, *Would-be Worlds: How Simulation Is Changing the Frontiers of Science*, John Wiley & Sons, New York.
- Cohen, J. & Stewart, I.: 1994, *The Collapse of Chaos: Discovering Simplicity in a Complex World*, Viking Books, New York.
- Confrey, J., & Smith, E.: 1994, 'Exponential functions, rates of change, and the multiplicative unit', *Educational Studies in Mathematics* 26, 135-168.
- Cuban, L.: 1986, *Teachers and Machines: The Classroom Use of Technology Since 1920*, Teachers College Press, New York.
- Davis, P. J. & Hersh, R.: 1982, *The Mathematical Experience*, Houghton Mifflin, Boston.
- Deacon, T.: 1997, *The Symbolic Species: The Co-evolution of Language and the Brain*, Norton, New York.
- Dennett, D. C.: 1996, *Darwin's Dangerous Idea: Evolution and the Meanings of Life*, Simon and Schuster, New York.
- Donald, M.: 1991, *Origins of the Modern Mind: Three Stages in the Evolution of Culture and Cognition*, Harvard University Press, Cambridge, MA.
- Donald, M.: 1993, 'Precis of Origins of the Modern Mind: Three Stages in the Evolution of Culture and Cognition', *Behavioral and Brain Sciences* 16, 737-791.
- Dorfler, W.: 1993, 'Computer use and use of the mind', in C. Keitel & K. Ruthven (eds.). *Learning From Computers: Mathematics Education and Technology*, Springer-Verlag, Berlin.
- Dyson, G. B.: 1997, *Darwin Among the Machines: The Evolution of Global Intelligence*. Addison-Wesley, Reading, MA.
- Fey, J.: 1989, 'Technology and mathematics education: a survey of recent developments and important problems', *Educational Studies in Mathematics* 20, 237-272.
- Gardner, H.: 1982, *Art, Mind, and Brain: A Cognitive Approach to Creativity*, Basic Books, New York.
- Gardner, H.: 1997, 'Thinking About Thinking', *New York Review of Books*, October 9, 1997, 23-27.
- Gibson, W.: 1984, *Neuromancer*, Ace Books, New York.

- Goodman, N: 1968, *Languages of Art: An Approach to a Theory of Symbols*, Bobbs-Merrill Co, Indianapolis, IN.
- Hall, N: 1994, *Exploring Chaos: A Guide to the New Science of Disorder*, Norton, New York.
- Heim, M: 1993, *The Metaphysics of Virtual Reality*, Oxford University Press, New York.
- Hofstadter, D. R: 1979, *Godel, Escher, Bach: An Eternal Golden Braid*, Basic Books, New York.
- Holland, J. H: 1995, *Hidden Order: How Adaptation Builds Complexity*, Addison-Wesley, New York.
- Kaput, J. J: 1986, 'Information technology and mathematics: Opening new representational windows', *The Journal of Mathematical Behavior*, 5(2), 187-207.
- Kaput, J. J: 1992, 'Technology and mathematics education', in D. A. Grouws (ed.), *Handbook of Research on Mathematics Teaching and Learning*, Maxwell Macmillan International, New York.
- Kaput, J. J: 1996a, 'Overcoming physicality and the eternal present: Cybernetic manipulatives', in R. S. J. Mason (ed.), *Technology and Visualization in Mathematics Education*, Springer Verlag, London.
- Kaput, J. J: 1996b, 'Technology, curriculum and representation: Rethinking the foundations and the future', in W. Doerfler, et al (eds.), *Schriftenreihe Didaktik der Mathematik, Trends und Perspektiven*, Hoelder-Pichler-Tempsky, Vienna.
- Kaput, J. J. & Roschelle, J: 1998, 'The mathematics of change and variation from a millennial perspective: New content, new context', in C. Hoyles, C. Morgan, & G. Woodhouse (Eds.), *Rethinking the mathematics curriculum*, pp. 155-170, Springer Verlag, London.
- Kauffman, S: 1995, *At Home in the Universe: The Search for the Laws of Self-Organization and Complexity*, Oxford University Press, New York.
- Kline, M: 1972, *Mathematical Thought from Ancient to Modern Times*, Oxford University Press, New York.
- Kosslyn, S. M. & Koenig, O: 1992, *Wet Mind: The New Cognitive Neuroscience*, Free Press, New York.
- Latour, B: 1979, *Laboratory Life: The Social Construction of Scientific Facts*, Sage Publications, Beverly Hills.
- McLuhan, M: 1962, *The Gutenberg Galaxy: The Making of Typographic Man*, University of Toronto Press, Toronto.
- Mithin, S: 1997, *The Prehistory of the Mind: The Cognitive Origins of Art, Religion, and Science*, Thames & Hudson, London.
- Nelson, K: 1996, *Language in Cognitive Development: Emergence of the Mediated Mind*, Cambridge University Press, Cambridge.
- Nemirovsky, R: 1998, *How One Experience Becomes Another*, Paper given to the International Conference on Symbolizing and Modeling in Mathematics Education, Utrecht, June, 1998.
- Nemirovsky, R., Kaput, J. & Roschelle, J: 1998, 'Enlarging mathematical activity from modeling phenomena to generating phenomena', in A. Olivier & K. Newstead (eds.), *Proceedings of the 22nd Conference of the International Group for the Psychology of*

Mathematics Education, Vol. 3, University of Stellenbosch, Stellenbosch, South Africa, pp. 287-294.

New York Daily Tribune: 1853, 'Photography in the United States', New York.

Newell: 1980, 'Physical symbol systems', *Cognitive Science* 4, 135-183.

Noss, R. & Hoyles, C: 1996, *Windows on Mathematical Meanings: Learning Cultures and Computers*, Kluwer Academic Publishers, Dordrecht, The Netherlands.

Oppenheimer, T: 1997, 'The computer delusion', *The Atlantic Monthly*, 280(1), 45-62.

Papert, S: 1980, *Mindstorms: Children, Computers, and Powerful ideas*, Basic Books, New York.

Papert, S: 1993, *The Children's Machine: Rethinking School in the Age of the Computer*, Basic Books, New York.

Papert, S: 1996, *The Connected Family: Bridging the Digital Generation Gap*, Longstreet Press, Atlanta, GA.

Pea, R: 1993, 'Practices of distributed intelligence and designs for education', in G. Salomon. (ed.), *Distributed Cognitions: Psychological and educational considerations*, Cambridge University Press, Cambridge.

Reed, D. S: 1996, *The Necessity of Experience*, Yale University Press, New Haven.

Resnick, M: 1994, *Turtles, Termites, and Traffic Jams: Explorations in Massively Parallel Microworlds*, MIT Press, Cambridge, MA.

Rogoff, B: 1990, *Apprenticeship in Thinking: Cognitive Development in Social Context*, Oxford University Press, New York.

Roman, S: 1990, *Logic*, Innovative Textbooks, Irvine, CA.

Roschelle, J., Kaput, J., Stroup, W: in submission, 'Scalable integration of educational software: Exploring the promise of component architectures', *Journal of Interactive Media in Education*.

Rowe, P. G: 1987, *Design Thinking*, MIT Press, Cambridge, MA.

Schmandt-Besserat, D: 1978, 'The earliest precursor of writing', *Scientific American*, 238.

Schmandt-Besserat, D: 1992, *Before Writing*, University of Texas Press, Austin, TX.

Schmandt-Besserat, D: 1994, Before numerals. *Visible Language* 18.

Shaffer, D. W: 1997, 'Learning mathematics through design: the anatomy of Escher's world', *Journal of Mathematical Behavior*, 16(2).

Shaffer, D. W: in preparation, *Society of Design: The Development of Mathematical Thinking in a Digital Design Studio*, MIT Media Lab, Cambridge, MA.

Shaffer, D. W., J. Kaput, et al: 1997, *Expressive Mathematics: Making Mathematics Meaningful*, National Council of Teachers of Mathematics Research Preconference, Minneapolis, MN.

Sherin, B: 1996, *The Symbolic Basis of Physical Intuition: A Study of Two Symbol Systems in Physics Instruction*, University of California, Berkeley, CA.

Simon, H. A: 1996, *The Sciences of the Artificial*, MIT Press, Cambridge, MA.

Smith, B. C: 1996, *On the Origin of Objects*, MIT Press, Cambridge, MA.

Swetz, F: 1987, *Capitalism and Arithmetic: The New Math of the 15th Century*, Open Court, La Salle.

- Turing, A. M: 1992, *Mechanical Intelligence*, Elsevier Science Pub, Amsterdam.
- Turkle, S. & Papert, S: 1990, 'Epistemological pluralism: Styles and voices within the computer culture', *Signs*, 16(1).
- Tyack, D. & Cuban, L: 1996, *Tinkering Towards Utopia*, Harvard University Press, Cambridge, MA.
- Von Neumann, J: 1966, *Theory of Self-Reproducing Automata*, University of Illinois Press Urbana, IL.
- Vygotsky, L: 1934/1986, *Thought and Language*, MIT Press, Cambridge, MA.
- Vygotsky, L. S: 1978, *Mind in Society*, Harvard University Press, Cambridge, MA.
- Wilensky, U: 1991, 'Abstract meditations on the concrete and concrete implications for mathematics education', in I. Harel & S. Papert (eds.), *Constructionism: Research Reports and Essays*, Ablex, Norwood, NJ.
- Wilensky, U: 1995, 'Paradox, programming, and learning probability: a case study in a connected mathematics framework', *Journal of Mathematical Behavior*, 14, 253-280.
- Yerushalmy, M: 1986, *Induction and Generalization: An Experiment in the Teaching and Learning of High School Geometry*, Harvard Graduate School of Education, Cambridge, MA.
- Yerushalmy, M: 1991, 'Student perceptions of aspects of algebraic function using multiple representation software', *Journal of Computer Assisted Learning*, 7, 42-57.

DAVID WILLIAMSON SHAFFER

*Project PACE, Harvard University
Cambridge, MA, USA*

JAMES J. KAPUT

*Mathematics Department, The University of Massachusetts-Dartmouth
No. Dartmouth, MA, USA*

CAPTIONS

Figure 1: Illustration from USA Today of a graph that uses lines in both pictorial and notational modes.

Figure 2: Solving a problem using the notation and transformation rules of algebra. Note that the intermediate steps of the solution (see the bold area) “make sense” as transformations of discrete symbols rather than as statements about the original problem situation. Also notice that in the last step, the variable “B” is canceled from both sides of the equation, giving us additional information that the number of girls at the party is not dependent on the number of boys at the party.

Why we bake for the holidays



Problem: Mary is having a birthday party, and her mother decides to rent extra tables and chairs. She rents as many chairs as there are boys and girls coming to the party, and she rents one table for every four chairs. When the children come to the party, the boys sit 8 to a table, and take up one less than half of the tables. How many girls came to Mary's party?

Equation	Transformation Rule	Interpretation
$C=B+G$		There were as many chairs as boys and girls at the party.
$4T=C$		There were 4 chairs per table.
$T/2 - 1 = B/8$		When the boys sit 8 to a table, they take up one less than half of the tables.
$T - 2 = B/4$	Multiply both sides by 2	If the boys sit 4 to a table, they use two less than the number of tables.
$T = B/4 + 2$	Add two to both sides	If the boys sit 4 to a table and use 2 more tables for holding presents, that uses up all of the tables.
$4 (B/4 + 2) = C$	Substitution of equations	You take four times the number of groups you get when you put the boys in groups of 4 and add two more groups, then that total number of groups will be equal to the total number of chairs at the party.
$B + 8 = C$	Distributive property	The number of chairs is eight more than the number of boys.
$B + 8 = B + G$	Substitution of equations	The number of boys and girls is eight more than the number of boys.
$8 = G$	Subtract B from both sides of the equation	There were 8 girls at the party.