

Computable Content of Vaughtian Models

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0.1 Preface

These notes are in preparation for four lectures by Robert Soare to MATH-LOGAPS in Leeds, August 21–25, 2006. The outlines of the lectures are presented in another file.

The first two chapters of this tutorial are background material in model theory and computability theory which will be reviewed briefly at the start of each of the lectures, and used during the rest of the lecture.

Part I: Prerequisites in Model Theory

Chapter 1. Vaughtian Models

The main background material in model theory is Vaught [1961] where the basic results on prime, saturated, and homogeneous models were introduced. The material here is not strictly on the computable content of Vaught's models, but elementary remarks are made about what happens if the theory T is decidable.

This fundamental material has also been covered in standard model theory books, such as Chang and Keisler [1990] and Marker [2002]. The main difference is that here: (1) we are only interested in countable models; (2) we immediately use *trees* $\mathcal{T}_n(T)$ of types for Henkin's canonical model and for the Stone Space; (3) we introduce homogeneous models first.

Countability Convention Like Vaught we shall assume that all theories are countable, complete, and have only infinite models. For the analysis of computable content of models we shall assume that the theory is complete and decidable (CD). The models we study will all be countable.

Homogeneous Models. We have rewritten these notes starting from the beginning in order to present them in a slightly different order and style. For example, we present homogeneous models in §1.6 before prime or saturated models in order to use homogeneous uniqueness to prove prime and saturated uniqueness, and to stress that every model considered by Vaught or by us will be homogeneous. It is useful to think in terms of the spectrum of homogeneous models in §1.9 by analogy with the spectrum of countable models of ACF_0 the theory of algebraically closed fields of characteristic zero.

Trees of Types. Most standard references do not present these Vaught results using trees, but trees are very useful in analyzing countable models, our present concern. To this end, we immediately recast Lindenbaum's Lemma and the Henkin proof on a tree \mathcal{T}_0 in §1.1.1 where we introduce extendible Π_1^0 classes and effectively closed subsets of Cantor space. We continue a detailed examination of trees in §1.2 using the tree of types $\mathcal{T}_n(T)$ to introduce the Stone space of types $S_n(T)$. Notions of atom, isolated (principal) path, generator of an isolated path, and atomic tree in §1.2 are all defined first for *trees* and only later for theories and types which are viewed entirely through the associated trees.

To further apply trees $\mathcal{T} \subseteq 2^{<\omega}$ we index formulas $\theta_\alpha(x)$ on the tree \mathcal{T}_n of types by strings $\alpha \in \widehat{\mathcal{T}}_n$ of length n to get an effectively isomorphic tree $\widehat{\mathcal{T}}_n \subseteq 2^{<\omega}$ on which we can carry out all the usual computability operations.

This prepares for the later computability theoretic analysis of finite forcing in trees and 1-generic sets constructed on trees. Viewing all the model theoretic notions in terms of trees will make it easy to combine with standard computability theoretic methods.

Part II: Prerequisites in Computability Theory

Chapter 2. Basic Computability Theory

This contains some elementary results, such as the Limit Lemma for Δ_2 sets, 1-generic sets and finite forcing, low sets, domination and Martin's theorem on high sets and dominating all computable functions, and similar results.

References The material in Part 1 is largely self-contained, but a deeper understanding can be found by consulting the reference. The references for Part 1 are Marker [2002], Soare [1987], and part of his Soare new book [cta] (not yet published, but parts attached here). The other references such as Vaught [1961] and other papers are mentioned for historical reasons.

Part I

Prerequisites in Model
Theory

Chapter 1

The Classical Theory of Vaughtian Models

Ryll-Nardzewski [1959] proved that a complete theory T with infinite models is ω -categorical (all countable models of T are isomorphic) iff for every n the Lindenbaum algebra $B_n(T)$ is finite. (As Vaught observed the latter is equivalent to having the set of n -types $S_n(T)$ be finite for every n .) In what Marker [2002, p. 172] described as “one of the most beautiful papers in model theory,” Vaught [1961] developed the theory of types and used them to define and study the classes of models he introduced: prime, saturated, and homogeneous. Morley, who had been working with Vaught at Berkeley at the time, significantly deepened the study of types and used it to prove his theorem on categoricity in power [1965] which transformed model theory into its modern form and laid the foundation for stability theory.

We begin with the Henkin proof of the Gödel completeness theorem since the method of building Henkin models is used extensively in the Vaught theorems and in most of our later ones in computable model theory.

Although we refer to other references for historical accuracy, the only references needed for mathematical content in Chapters 1 and 2 are Marker [2002], Soare [cta] and perhaps Soare [1987] for parts of [cta] not yet available.

Convention 1.0.1. [Countability Convention] *Unless otherwise stated all theories T here will be consistent, countable, and complete, as in Vaught [1961], and all models \mathcal{A} of T will be countable, as in Vaught [1961], §1–4.*

1.1 Building Henkin Models

A (first order) language \mathcal{L} consists of the usual logical symbols plus a set of nonlogical symbols: relation symbols, function symbols, and constant symbols (called *signature* of \mathcal{L}). A formula in \mathcal{L} is a *sentence* if it has no free variables. A *theory* in \mathcal{L} (i.e., \mathcal{L} -theory) T is a set of \mathcal{L} -sentences closed under first order derivability, i.e., $\sigma \in T$ and $\sigma \vdash \tau$ implies $\tau \in T$. An \mathcal{L} -theory T is *consistent* if there is no sentence such that $T \vdash \sigma$ and $T \vdash \neg\sigma$, and T is *complete* if for every \mathcal{L} -sentence σ either $T \vdash \sigma$ or $T \vdash \neg\sigma$. An \mathcal{L} -structure \mathcal{A} is a *model* of an \mathcal{L} -theory T ($\mathcal{A} \models T$) if every \mathcal{L} -sentence σ in T is true in \mathcal{A} ($\mathcal{A} \models \sigma$).

In his Ph.D. dissertation at Princeton [1949] Leon Henkin, later a professor at Berkeley for many years, gave an easy proof of Gödel's completeness theorem from Gödel's own thesis [1930]. Henkin's method for constructing models is the basis for Vaught's methods of building prime, saturated, and homogeneous models, as well as our more effective models.

Theorem 1.1.1. Gödel's Completeness Theorem [1930] *Let T be a consistent \mathcal{L} -theory. Then T has a model.*

Corollary 1.1.2. *If σ is an \mathcal{L} -sentence, then $T \vdash \sigma \iff T \models \sigma$. (That is, T proves σ iff T semantically implies σ).*

Proof. (of Gödel Completeness Theorem 1.1.1). Fix a consistent \mathcal{L} -theory T . Expand \mathcal{L} to a language $\mathcal{L}_c = \mathcal{L} \cup \{c_n\}_{n \in \omega}$ where $C = \{c_n\}_{n \in \omega}$ is a new set of constant symbols not in \mathcal{L} . For every formula $\theta(x)$ with one free variable of \mathcal{L}_c (not just of \mathcal{L}) we adjoin a *Henkin axiom* of the form

$$(\exists x)\theta(x) \longrightarrow \theta(c)$$

for some $c \in C$ which does not occur in $\theta(x)$. Let T_c denote T together with these new axioms. Any theory in \mathcal{L}_c which includes these axioms is called a *Henkin theory* and we say that it has the *witness property*.

It follows from the axioms of logic that T_c is a *conservative extension* of T , i.e., any \mathcal{L} -sentence provable in T_c is already provable in T . (Therefore, it does not hurt to add the Henkin axioms, and we normally do this first before applying Lindenbaum's Lemma.)

Lemma 1.1.3. [Lindenbaum's Lemma] *If S is a consistent \mathcal{L} -theory then it has a complete consistent extension S' in \mathcal{L} .¹*

¹If S is uncountable we must apply the Axiom of Choice, but we assume the Countability Convention 1.0.1 and hence we can give a more constructive proof.

Proof. (of Lindenbaum's Lemma). Let $\{\sigma_i\}_{i \in \omega}$ be a listing of all \mathcal{L}_c -sentences. Let S_{k+1} be $S_k \cup \{\sigma_k\}$ if $S_k \cup \{\sigma_k\}$ is consistent, and $S_{k+1} = S_k \cup \{\neg\sigma_k\}$ otherwise. Define $S = \cup_k S_k$. Clearly, S is consistent and complete. \square

Now apply Lindenbaum's Lemma obtain a complete consistent extension of T'_c of T_c . Use T'_c to define a model as follows.

Definition 1.1.4. [Canonical Henkin Model]. If T'_c is a complete Henkin theory define the *canonical Henkin model* for T'_c as follows. The universe A of \mathcal{A} consists of the equivalence classes $\{c^* : c \in \mathcal{L}_c\}$ where

$$c^* = \{d : T'_c \vdash c = d\}.$$

Define function $F^{\mathcal{A}}(c^*) = d^*$ iff $F(c) = d$ is a sentence of T'_c . The relations $R^{\mathcal{A}}$ are defined similarly. (These functions and relations are well-defined.)

It is easy to check that \mathcal{A} is a model of T'_c and therefore of T . \square

1.1.1 Recasting Lindenbaum's Lemma on a Tree $\mathcal{T}_0(T)$

Definition 1.1.5. [Elementary Diagram] If T is a theory and \mathcal{A} is a model of a theory T ($\mathcal{A} \models T$) with universe $A = |\mathcal{A}|$, then the *elementary diagram*, $D^e(\mathcal{A})$, of \mathcal{A} is the set of all \mathcal{L}_A -sentences true in \mathcal{A} , and the *atomic diagram* $D(\mathcal{A})$ is the set of all *quantifier free* sentences of \mathcal{L}_A true in \mathcal{A} .

Notice that *any* complete extension of T_c determines (the elementary diagram of) a Henkin model. The set of complete extensions is best viewed as the set of paths through a tree. We first give a quick sketch and then develop trees more rigorously in the next section.

Definition 1.1.6. (i) Let $2^{<\omega}$ be the set of finite sequences of 0's and 1's. For any $\alpha \in 2^{<\omega}$ let $|\alpha|$ be the length of α , and $\alpha(i)$ the i^{th} bit of α .

(ii) Fix a countable language \mathcal{L}_c with Henkin constants as above. Let $\{\sigma_n\}_{n \in \omega}$ be a listing of all \mathcal{L}_c -sentences. Let σ^1 denote σ and σ^0 denote $\neg\sigma$. Define $\sigma_\alpha =_{\text{dfn}} \bigwedge_{i < |\alpha|} \sigma_i^{\alpha(i)}$.

(iii) Fix T_c an \mathcal{L}_c -theory T_c . Define $\mathcal{T}_0(T_c) = \{\sigma_\alpha : T_c \vdash \sigma_\alpha\}$, the tree of sentences provable in T_c .

(iv) Let $[\mathcal{T}_0(T_c)]$ denote the set of paths of $\mathcal{T}_0(T_c)$.

In the proof of Lindenbaum's Lemma 1.1.3 given the conjunction of sentences σ_α added to S so far we let $i = |\alpha|$ and we chose σ_i if consistent and $\neg\sigma_i$ otherwise. This produces the lexicographically greatest path $g \in [\mathcal{T}_0(T_c)]$.

However, if T_c is a Henkin theory, then *any* path $f \in [\mathcal{T}_0(T_c)]$ gives a complete Henkin theory and hence the corresponding Henkin model \mathcal{A}_f .

As we produce models of theories the trivial parts of the Henkin construction will always be the same: (1) the addition of the constant symbols c_k and Henkin axioms at the beginning to obtain the consistent (but not yet complete) Henkin theory T_c , and (2) the definition of the canonical Henkin model \mathcal{A}_f from a path $f \in [\mathcal{T}_0(T_c)]$. The more subtle part will always be exactly *how* to obtain the path f and with which additional properties, such as making it determine a prime or saturated model, or have a certain Turing degree.

1.2 Trees, Types, and the Stone Space

1.2.1 Trees and Π_1^0 Classes

The Vaught theorems and our computability theoretic results make considerable use of trees. Types will be viewed as paths on trees, and all the familiar terminology and results about types will be developed first for trees. We begin the tree definitions here and continue with more properties on Definition 1.2.5 on atomic trees, and atomic theories and models where we define atoms and isolated (principal) types.

Definition 1.2.1. [Trees Part 1] (i) A *tree* $\mathcal{T} \subseteq 2^{<\omega}$ is a subset of $2^{<\omega}$ closed under initial segment, *i.e.*, $\tau \subset \sigma \in \mathcal{T}$ implies $\tau \in \mathcal{T}$. Define the set of (infinite) *paths*,

$$(1.1) \quad [\mathcal{T}] = \{ f : f \in 2^\omega \ \& \ (\forall x)[f \upharpoonright x \in \mathcal{T}] \}.$$

(ii) *Cantor space* is 2^ω with the following topology. In the context of trees of types with the same topology this is called the *Stone Space*. Let $\sigma \subset f$ denote that f extends σ . For every $\sigma \in 2^{<\omega}$ define the *basic open set*,

$$\mathcal{U}_\sigma = \{ f : f \in 2^\omega \ \& \ \sigma \subset f \}$$

(iii) A class $\mathcal{C} \subset 2^\omega$ is *closed* if $2^\omega - \mathcal{C}$ is open or equivalently if $\mathcal{C} = [\mathcal{T}]$ for some tree as in (1.1).

(iv) If $\mathcal{C} = [\mathcal{T}]$ for some *computable* tree \mathcal{T} , then \mathcal{C} is *effectively closed*, and is called a Π_1^0 -*class*, a very important type of classes.

(v) The Π_1^0 -class $\mathcal{C} = [\mathcal{T}]$ is *extendible* if every node σ can be extended to an infinite path, *i.e.*,

$$(\forall \sigma \in \mathcal{T})(\exists f \supset \sigma)[f \in \mathcal{C}],$$

and \mathcal{C} is *nonextendible* otherwise.

1.2.2 The Lindenbaum Algebra $B_n(T)$ of Formulas

From now on let \mathcal{L} be a computable language, meaning we can effectively determine the arity of the function, relation, and constant symbols.

Definition 1.2.2. [Lindenbaum Algebra] (i) Let $F_n(\mathcal{L})$ be set of the formulas $\theta(x_0, \dots, x_{n-1})$ of \mathcal{L} with free variables included in x_0, \dots, x_{n-1} . Let $F(\mathcal{L}) = \cup_{n \geq 0} F_n(\mathcal{L})$. For an \mathcal{L} -theory T let $F_n(T) = F_n(\mathcal{L})$.

(ii) The *equivalence class* of $\theta(\bar{x}) \in F_n(T)$ under T -provability \vdash_T is,

$$\theta^*(\bar{x}) = \{ \gamma(\bar{x}) : \vdash_T (\forall \bar{x}) [\theta(\bar{x}) \leftrightarrow \gamma(\bar{x})] \}$$

and the *Lindenbaum algebra* $B_n(T)$ consists of these equivalence classes under the induced operations. We often identify $\theta(\bar{x})$ and its equivalence class $\theta^*(\bar{x})$.

(iii) Let $\{ \theta_i(\bar{x}) \}_{i \in \omega}$ be an effective listing of $F_n(T)$. From i we can effectively find the unique n such that $\theta_i \in F_n(T)$. For every string $\alpha \in 2^{<\omega}$ define

$$\theta_\alpha(\bar{x}) = \bigwedge \{ \theta_i^{\alpha(i)}(\bar{x}) : i < |\alpha| \}$$

where $\theta^1 = \theta$ and $\theta^0 = \neg\theta$.

1.2.3 The Stone Space $S_n(T)$ as paths in the tree $\mathcal{T}_n(T)$

Definition 1.2.3. Let T be a complete theory.

(i) A formula $\theta(\bar{x}) \in F_n(\mathcal{L})$ is *consistent with T* if $T \cup (\exists \bar{x})\theta(\bar{x})$ is consistent, *i.e.*, if $T \vdash (\exists \bar{x})\theta(\bar{x})$, because T is complete.

(ii) Define the tree of n -ary formulas consistent with T

$$\mathcal{T}_n(T) = \{ \theta_\alpha(\bar{x}) : \alpha \in 2^{<\omega} \ \& \ (\exists \bar{x})\theta_\alpha(\bar{x}) \in F_n(T) \}.$$

If $\alpha \subset \beta$, then we say that β lies *below* α , that θ_β *extends* θ_α and contains more information. (Note that the equivalence classes $\{[\theta_\alpha] : \alpha \in \mathcal{T}_n(T)\}$ generate the Lindenbaum algebra $B_n(T)$ under the Boolean operations.)

(iii) We regard α as an *index* of θ_α . Define the tree of indices,

$$\widehat{\mathcal{T}}_n(T) = \{ \alpha : \theta_\alpha \in \mathcal{T}_n(T) \}.$$

The trees $\mathcal{T}_n(T)$ and $\widehat{\mathcal{T}}_n(T)$ are effectively isomorphic but $\widehat{\mathcal{T}}_n(T) \subseteq 2^{<\omega}$ and is notationally simpler. Hence, any definitions or results on trees $\widehat{\mathcal{T}}_n(T) \subseteq 2^{<\omega}$ automatically carry over to $\mathcal{T}_n(T)$. We mostly suppress explicit mention of $\widehat{\mathcal{T}}_n(T)$ and we identify α and $\theta_\alpha(\bar{x})$.

Definition 1.2.4. [Types and the Stone Space] (i) An n -type of T is a maximal consistent subset p of formulas of $F_n(T)$. There is a 1:1 correspondence between paths $f \in [\widehat{\mathcal{T}}_n(T)] \subseteq 2^\omega$ and the corresponding types $p_f \in [\mathcal{T}_n(T)]$ where

$$p_f = \{ \theta_\alpha(\bar{x}) : \theta_\alpha(\bar{x}) \in \mathcal{T}_n(T) \ \& \ \alpha \subset f \}.$$

(ii) $S_n(T)$ is the set of all n -types of T , with the same topology as in Definition 1.2.1, and it is also called the *Stone Space* (i.e., the dual space of the Boolean algebra $B_n(T)$). The clopen sets of the Cantor space are given by

$$\mathcal{U}_\alpha = \{ f : \alpha \subset f \}.$$

(iii) Hence, $S_0(T)$ is the set of complete extensions of T , i.e., 0-types of T , as in Definition 1.1.6. Since we assume T to be complete theory, $S_0(T)$ consists of a single path. However, $S_0(T_c)$ usually consists of more than one path. Since T_c is Henkinized every such path determines a Henkin model \mathcal{A} off T_c .

(iv) Define $S(T) = \cup_{n \geq 1} S_n(T)$.

(We can also regard $S(T)$ as homeomorphic to a subset of 2^ω as follows. Build a tree $\mathcal{T} \subseteq 2^{<\omega}$ by putting $1^n \hat{\ } 0$ on \mathcal{T} and then putting an isomorphic copy of $\widehat{\mathcal{T}}_n(T)$ above it on \mathcal{T} .)

1.2.4 Atomic Trees and Principal Types

We continue Definition 1.2.1 with properties of trees which will also apply to types.

Definition 1.2.5. [Trees Part 2] Let $\mathcal{T} \subseteq 2^{<\omega}$ be an extendible tree.

(i) Nodes β, γ are *incomparable*, written $\alpha \mid \gamma$, if $(\exists k)[\beta(k) \downarrow \neq \gamma(k)]$.

(ii) Nodes $\beta, \gamma \in \mathcal{T}$ *split* node α on \mathcal{T} if $\alpha \subset \beta$, $\alpha \subset \gamma$, and $\beta \mid \gamma$.

(iii) Node $\alpha \in \mathcal{T}$ is an *atom* if no extensions split α on \mathcal{T} . If α is an atom then the unique extension of $f \supset \alpha$ on \mathcal{T} is an *isolated (principal) path* of $[\mathcal{T}]$, α is a *generator* of f , and we say α *isolates* f . Note that f is a isolated in the topological sense by the basic open set \mathcal{U}_α because

$$\mathcal{U}_\alpha \cap [\mathcal{T}] = \{ f \}.$$

- (iv) \mathcal{T} is *atomic* if for every $\beta \in \mathcal{T}$ there is an atom $\alpha \supseteq \beta$ with $\alpha \in \mathcal{T}$, or equivalently if the isolated points of $[\mathcal{T}]$ are dense in $[\mathcal{T}]$.
- (v) A complete theory T is *atomic* if tree $\mathcal{T}_n(T)$ is atomic for every $n \geq 1$.
- (vi) Node $\beta \in \mathcal{T}$ is *atomless* if there is no atom $\alpha \supset \beta$ with $\alpha \in \mathcal{T}$.

Example 1.2.6. Let T be the theory of the rationals as a dense linear ordering without endpoints $\mathcal{A} = (A, <, A)$ but with all rationals A named. For every rational $a \in A$ any 1-type $p(x)$ must contain exactly one formula of the form $x < \underline{a}$, $x = \underline{a}$, or $x > \underline{a}$, where \underline{a} is the constant symbol naming element $a \in A$. There are countably many isolated types each with a generator of form $x = \underline{a}$, and there are 2^{\aleph_0} nonprincipal types, corresponding to the cuts in the rationals. T is an atomic theory because any finite node on the tree can be extended to an atom.

Proposition 1.2.7. *Let T be a complete theory with $\mathcal{T}_n(T)$ extendible. Clearly,*

$$[\theta_\beta(\bar{x}) \in \mathcal{T}_n(T) \quad \& \quad \alpha \subset \beta] \implies T \vdash (\forall x)[\theta_\beta(\bar{x}) \rightarrow \theta_\alpha(\bar{x})]$$

(i) *If $\theta_\alpha(\bar{x})$ is an atom of $\mathcal{T}_n(T)$ then for all $\theta_\beta \in \mathcal{T}_n(T)$ such that $\beta \supset \alpha$*

$$T \vdash (\forall \bar{x})[\theta_\alpha(\bar{x}) \rightarrow \theta_\beta(\bar{x})].$$

(ii) *If α is split by $\beta \mid \gamma$ and $\theta_\beta(\bar{x}), \theta_\gamma(\bar{x})$ are in $\mathcal{T}_n(T)(\bar{x})$ then they are T -incomparable.*

Proof. (i) If $\theta_\alpha(\bar{x})$ is an atom of $\mathcal{T}_n(T)$ then every $\psi(\bar{x}) \in F_n(T)$ is decided by $\theta_\alpha(\bar{x})$.

(ii) The split $\beta \mid \gamma$ ensures $\beta(i) \neq \gamma(i)$ for some i . Hence, $\theta_i(\bar{x})$ is in one of $\theta_\beta(\bar{x})$ and $\theta_\gamma(\bar{x})$ while $\neg\theta_i(\bar{x})$ is in the other. Therefore, $\theta_\beta(\bar{x})$ and $\theta_\gamma(\bar{x})$ are T -incomparable. \square

Proposition 1.2.8. *Let $\mathcal{T} \subseteq 2^{<\omega}$ be an extendible tree.*

- (i) \mathcal{T} is atomic iff \mathcal{T} contains no atomless element.
- (ii) If \mathcal{T} contains an atomless element, then $\text{card}([\mathcal{T}]) = 2^{\aleph_0}$.
- (iii) If $\text{card}([\mathcal{T}]) \leq \aleph_0$ then \mathcal{T} is an atomic tree.

Proof. (i) is immediate from the definitions. In (ii) if α is atomless, split α into $\beta \mid \gamma$, and keep splitting to produce a perfect subtree $\mathcal{T}' \subseteq \mathcal{T}$. Finally, (iii) follows from (ii). \square

Part (iii) will be used later to show that if T has a countable saturated model, then it has a prime model.

1.3 Complete Decidable Theories

We pause for a moment in the classical development of Vaught's models to consider some computable properties of the definitions so far. For computable model theory we require not only that the theory T be complete, but also *decidable* (written CD), and we explore the Turing degree of certain models \mathcal{A} of T .

Definition 1.3.1. (i) A countable language \mathcal{L} is *decidable* if there is an algorithm to decide for every finite string of symbols ψ of \mathcal{L} whether ψ is a well formed formula of \mathcal{L} . A theory T in a decidable language \mathcal{L} is *decidable* if there is algorithm to decide for a sentence σ of \mathcal{L} whether $T \vdash \sigma$ or not.

(ii) If \mathcal{A} is a model of T ($\mathcal{A} \models T$), then the \mathcal{A} is *decidable* if the elementary diagram $D^e(\mathcal{A})$ is decidable, and \mathcal{A} is *computable* if the atomic diagram $D(\mathcal{A})$ is decidable.

Theorem 1.3.2. [Extendability Theorem]

(i) If T is a complete decidable (CD) theory then all the trees $\mathcal{T}_0(T)$ of sentences and $\mathcal{T}_n(T)$ of types, $n \geq 1$, are computable extendible trees.

(ii) If $\mathcal{T} \subseteq 2^{<\omega}$ is an infinite computable extendible tree, then there is a computable path $f \in [\mathcal{T}]$. The lexicographically least path g is computable.

(iii) If T is a complete decidable (CD) theory, then T has a decidable model.

(iv) If f is an isolated path of a computable extendible tree \mathcal{T} , then f is computable. Hence, in a CD theory any principal type is computable.

Proof. (i) Let T be a decidable theory. Then the Henkin extension T_c is also decidable because of the simple form of the Henkin axioms. We put θ_α on $\mathcal{T}_n(T)$ iff $(\exists \bar{x})\theta(\bar{x}) \in T$ in which case $\theta(\bar{x})$ is consistent and extendible.

(ii) Now $[\mathcal{T}] \neq \emptyset$ by compactness because \mathcal{T} is infinite. We construct the lexicographically least path $g \in [\mathcal{T}]$ as a increasing chain $g = \cup_s g_s$. Let $g_0 = \emptyset$. Given g_s define $g_{s+1} = g_s \hat{\ } 0$ if $g_s \hat{\ } 0 \in \mathcal{T}$ and $g_{s+1} = g_s \hat{\ } 1$ otherwise. This question is decidable because \mathcal{T} is computable.

(iii) The lexicographically least path $g \in [\mathcal{T}_0(T_c)]$ is decidable and produces a decidable Henkin model \mathcal{A}_g .

(iv) Let f be an isolated path of \mathcal{T} with α an atom which isolates f . Then f is the unique path of the tree $\mathcal{T}^\alpha = \{\beta : \alpha \subset \beta \in \mathcal{T}\}$. Apply (ii). \square

1.4 Undecidable Theories: Nonextendible Classes

The main point of the Extendability Theorem 1.3.2 is that all the trees $\mathcal{T}_n(T)$, $n \geq 1$, are *extendible* as well as computable. For an *undecidable* axiomatizable (c.e.) theory T we can still construct a similar tree of complete extensions $\mathcal{T}_0(T)$ which is computable, but it is not extendible.

Fix a computable language \mathcal{L} and let $\{\sigma_i\}_{i \in \omega}$ be a computable enumeration of all \mathcal{L} -sentences. Let T be a computably enumerable (c.e.) theory which is *undecidable*, such as Peano arithmetic (PA). Let $\{T^s\}_{s \in \omega}$ be a c.e. sequence of finite sets of \mathcal{L} -sentences such that $T = \cup_s T^s$. Note that σ_i is consistent with T iff $(\forall s)[-\sigma \notin T_s]$. (This is a Π_1 property.) We define a new computable tree $\mathcal{T}_0(T)$ of complete extensions of T a little more subtly than before.

Definition 1.4.1. For $\alpha \in 2^{<\omega}$ with $|\alpha| = s$ put $\sigma_\alpha \in \mathcal{T}_0(T)$ iff $-\sigma_\alpha \notin T_s$.

This gives a computable tree $\mathcal{T}_0(T)$ whose infinite paths $[\mathcal{T}_0(T)]$ are the complete extensions of T . The main difference is that we may put σ_α on \mathcal{T}_0 at stage s and discover later at stage $t > s$ that no extension σ_β of length t lies on \mathcal{T}_0 , *i.e.*, that node σ_α is not *extendible* on tree \mathcal{T}_0 . The analysis of §1.3 fails, and must be replaced by an more complicated analysis of bases for Π_1^0 -classes as in Chapter ???. Fortunately, in all our work here on the computable content of Vaught's models, the theories T will be CD. Hence, the trees $\mathcal{T}_n(T)$ will be computable and extendible and the Extentability Theorem 1.3.2 will apply.

1.5 Cantor's Back and Forth

Let $\mathcal{A} \equiv \mathcal{B}$ denote elementary equivalence (*i.e.*, that \mathcal{A} and \mathcal{B} satisfy the same \mathcal{L} -sentences, $\mathcal{A} \cong \mathcal{B}$ denote isomorphism. Algebraists classify structure isomorphism up to isomorphisms, and topologists classify spaces up to homomorphism. The idea of using elementary equivalence and types to classify structures is unique to logic.

Cantor introduced the back and forth method to show that any two countable dense linear orderings are isomorphic. The method will be used several times in constructing Vaught's models.

Definition 1.5.1. Let \mathcal{A} and \mathcal{B} be two countable \mathcal{L} -structures. An ω -*back and forth system* for \mathcal{A} and \mathcal{B} is a family \mathcal{F} of finite elementary maps from \mathcal{A} to \mathcal{B} such that the empty map $\emptyset \in \mathcal{F}$ and,

(Back) $(\forall f \in \mathcal{F})(\forall b \in \mathcal{B})(\exists g \in \mathcal{F}) [f \subseteq g \ \& \ b \in \text{range}(g)]$, and

(Forth) $(\forall f \in \mathcal{F})(\forall a \in \mathcal{A})(\exists g \in \mathcal{F}) [f \subseteq g \ \& \ a \in \text{domain}(g)]$.

Theorem 1.5.2. Back and Forth Theorem] *If \mathcal{A} and \mathcal{B} are two countable \mathcal{L} -structures with an ω -back and forth system \mathcal{F} , then $\mathcal{A} \cong \mathcal{B}$.*

Proof. Let the universe of \mathcal{A} be $A = \{a_n\}_{n \in \omega}$ and the universe of \mathcal{B} be $B = \{b_n\}_{n \in \omega}$. We construct the isomorphism f as the union of finite maps $f_n \in \mathcal{F}$ built as follows. Let $f_0 = \emptyset$. Given $f_n \in \mathcal{F}$ choose $g \in \mathcal{F}$ such that $g \supseteq f$ and $a_n \in \text{domain}(g)$. Next choose $h \in \mathcal{F}$ with $h \supseteq g$ and $b_n \in \text{range}(g)$. Now let $f_{n+1} = h$ and $f = \cup_{n \in \omega} f_n$. \square

1.6 Homogeneous Models

Early in the history of model theory Vaught [1958a] and [1961] studied homogeneous models. The idea of homogeneous is very roughly that the elements of \mathcal{A} are distributed uniformly without gaps. Let $\text{Aut}(\mathcal{A})$ denote the group of automorphisms of \mathcal{A} .

1.6.1 Homogeneity and Automorphisms

Definition 1.6.1. A countable \mathcal{L} -structure \mathcal{A} is *homogeneous* if for any two finite sequences $\bar{a} = \{a_0, a_1, \dots, a_{n-1}\}$ and $\bar{b} = \{b_0, b_1, \dots, b_{n-1}\}$ in \mathcal{A} ,

$$(\mathcal{A}, \bar{a}) \equiv (\mathcal{A}, \bar{b}) \implies (\forall c \in \mathcal{A})(\exists d \in \mathcal{A}) [(\mathcal{A}, \bar{a}, c) \equiv (\mathcal{A}, \bar{b}, d)],$$

i.e., for any finite elementary map $f(\bar{a}) = \bar{b}$ and any element $c \in \mathcal{A}$ there is an elementary map $g \supseteq f$ with $c \in \text{domain}(g)$.

Theorem 1.6.2. *A countable \mathcal{L} -structure is homogeneous iff for all tuples \bar{a} and \bar{b} ,*

$$(\mathcal{A}, \bar{a}) \equiv (\mathcal{A}, \bar{b}) \implies (\exists g \in \text{Aut}(\mathcal{A})) [g(\bar{a}) = \bar{b}],$$

i.e., every finite elementary map $f(\bar{a}) = \bar{b}$ can be extended to an automorphism g of \mathcal{A} .

Proof. (\Leftarrow). Obvious.

(\Rightarrow). Let \mathcal{A} and \mathcal{B} be homogeneous and countable. Then the finite maps guaranteed by Definition 1.6.1 constitute an ω -back and forth \mathcal{F} from \mathcal{A} to \mathcal{A} . Therefore, any finite elementary map $g \in \mathcal{F}$ can be extended to an automorphism f of \mathcal{A} by starting with $f_0 = g$ and building $\{f_n\}_{n \in \omega}$ as in the proof of the Back and Forth Theorem 1.5.2. \square

1.6.2 The Type Spectrum $\mathbb{T}(\mathcal{A})$ of a model \mathcal{A}

Definition 1.6.3. Let T be a theory and \mathcal{A} a model of T .

- (i) An n -tuple $\bar{a} \in A$ realizes an n -type $p(\bar{x}) \in S_n(T)$ if $\mathcal{A} \models \theta(\bar{a})$ for all $\theta(\bar{x}) \in p(\bar{x})$. In this case we say that \mathcal{A} realizes p via \bar{a} .
- (ii) Define the *type spectrum* of \mathcal{A}

$$\mathbb{T}(\mathcal{A}) = \{ p : p \in S(T) \quad \& \quad \mathcal{A} \text{ realizes } p \}, \quad \text{and}$$

- (iii) $\mathbb{T}_n(\mathcal{A}) = \mathbb{T}(\mathcal{A}) \cap S_n(T)$, the n -types realized in \mathcal{A} .

1.6.3 Uniqueness of Homogeneous Models

One of the most pleasant properties of homogeneous models is the following result which demonstrates the usefulness of the notion $\mathbb{T}(\mathcal{A})$.

Theorem 1.6.4. [Homogeneous Uniqueness Theorem]

Given a countable complete theory T and countable homogeneous models \mathcal{A} and \mathcal{B} .

$$(1.2) \quad \mathbb{T}(\mathcal{A}) = \mathbb{T}(\mathcal{B}) \quad \implies \quad \mathcal{A} \cong \mathcal{B}.$$

Proof. Let \mathcal{A} and \mathcal{B} be homogeneous, countable² and such that $\mathbb{T}(\mathcal{A}) = \mathbb{T}(\mathcal{B})$. By the Back and Forth Theorem 1.5.2 it suffices to exhibit an ω -back and forth \mathcal{F} between \mathcal{A} and \mathcal{B} . Now T is complete and \mathcal{A} and \mathcal{B} are models of T . Hence, $\mathcal{A} \equiv \mathcal{B}$ so we can add the empty map \emptyset to \mathcal{F} . Given any elementary map $f \in \mathcal{F}$, $f(\bar{a}) = \bar{b}$ for n -tuples \bar{a} and \bar{b} , and any element $c \in A$ let p be the $(n+1)$ -type satisfied by (\bar{a}, c) in \mathcal{A} . There is some $(n+1)$ -tuple (\bar{b}', d') satisfying p in \mathcal{B} because $\mathbb{T}(\mathcal{A}) = \mathbb{T}(\mathcal{B})$. In particular, $(\mathcal{B}, \bar{b}) \equiv (\mathcal{B}, \bar{b}')$. Hence, by homogeneity of \mathcal{B} there is some $d \in B$ such that

$$(\mathcal{A}, \bar{a}, c) \equiv (\mathcal{B}, \bar{b}', d') \equiv (\mathcal{B}, \bar{b}, d).$$

Extend f to $g = f \cup \{(c, d)\}$ and add g to \mathcal{F} . Given g and an element $b \in B$ similarly find an extension $h \supseteq g$ with $b \in \text{range}(h)$ using that \mathcal{A} is homogeneous. \square

²Again this theorem holds in the general case by essentially the same proof as in Marker Theorem 4.3.23 on p. 145. For a structure \mathcal{A} of uncountable cardinality κ we must require that any partial elementary map of cardinality $< \kappa$ can be extended by one point. Here we are interested only in the countable case proved above, which Marker also proves separately as above in his Theorem 4.2.15 on p. 134.

1.7 Prime and Atomic Models

Let the Lindenbaum algebra $F_n(T)$, tree of n -types $\mathcal{T}_n(T)$, and Stone space $S_n(T) = [\mathcal{T}_n(T)]$ be as in Definitions 1.2.2, 1.2.3, and 1.2.4.

Definition 1.7.1. Let T be a complete \mathcal{L} -theory. Recall that a type $p \in S_n(T)$ is principal (isolated) if there is an atom on $\mathcal{T}_n(T)$ which isolates it as in Definition 1.2.5. Define

$$S^P(T) = \{ p : p \text{ is a principal type of } S(T) \}.$$

Proposition 1.7.2. *If T is a complete theory and $p \in S^P(T)$ is a principal type then it is realized in any model of T .*

Proof. If $p(\bar{x})$ is a principal type with generator $\theta(\bar{x})$ then $(\exists \bar{x})\theta(\bar{x})$ is consistent with T , hence in T because T is complete. Therefore, any model $\mathcal{A} \models T$ satisfies $(\exists \bar{x})\theta(\bar{x})$ and realizes p . \square

The next theorem says that a type $p \in S_n(T)$ can be omitted in *some* model \mathcal{A} of T iff p is principle. It was proved by Ehrenfeucht [1957] for a single type and extended by Vaught [1961].

1.7.1 Omitting Types

Theorem 1.7.3. [Omitting Types Theorem] *If T is a countable theory and $\{p_i\}_{i \in \omega}$ is a countable sequence of nonprincipal types, then there is a countable model \mathcal{A} of T which omits every such type p_i , $i \in \omega$.*

Proof. First consider the case of a single type $p(\bar{x}) = p_i(\bar{x})$ on one variable. From T form \mathcal{L}_c and T_c as in Henkin's proof of Theorem 1.1.1. Assume that $C = \{c_i\}_{i \in \omega}$ is a list of all constant symbols of \mathcal{L}_c . We shall construct a path $f \in [\mathcal{T}_0(T_c)]$ by a sequence of finite extensions $f = \cup_s f^s$ where $f^s \subset f^{s+1}$ and form the Henkin model \mathcal{A}_f with $|\mathcal{A}_f| = \{[c_j] : c_j \in C\}$ as in Definition 1.1.4.

For every j to guarantee that $p(x)$ is not realized by $[c_j]$ we must satisfy for the following *requirement*,

$$(1.3) \quad R_j : \quad \text{There exists } \theta(x) \in p(x) \quad \text{with} \quad \neg\theta(c_j) \in f.$$

Stage $s = 0$. Define $f_0 = \emptyset$.

Stage $s + 1$. (Satisfy R_s .)

Let $j = s$. Given $f_s \in [\mathcal{T}_0(T_c)]$ and c_j . Let $f_s(x)$ be the sentence f_s with

all instances of c_j replaced by a new variable x . Type $p(x)$ is nonprincipal so $f_s(x)$ cannot be its generator. Hence, there is some³ extension $\psi(x)$ of $f_s(x)$ with $\neg\psi(x) \in p$. Let $f_{s+1} \in \mathcal{T}_0(T_c)$ of length $> s$ and extending $\psi(c_j)$.

To omit every type p_i the proof is the same. If $s = \langle i, j \rangle$ we meet at stage $s + 1$ the requirement

$$(1.4) \quad R_{\langle i, j \rangle} : \quad \text{There exists } \theta(\bar{x}) \in p_i(\bar{x}) \quad \text{with} \quad \neg\theta(\bar{c}_j) \in f.$$

where $p_i(\bar{x})$ is an n_i -type, $\bar{x} = (x_0, \dots, x_{n_i-1})$ and $\{\bar{c}_j\}_{j \in \omega}$ lists all n_i -tuples of constants of \mathcal{L}_c . \square

1.7.2 Prime and Atomic Models

Definition 1.7.4. [Prime and Atomic Models] Let T be a countable complete theory with infinite models and \mathcal{A} a model of T .

- (i) \mathcal{A} is *prime* if \mathcal{A} can be elementarily embedded in any other model of T .
- (ii) \mathcal{A} is *atomic* if \mathcal{A} if all the types realized in \mathcal{A} are principal, or equivalently

$$(1.5) \quad \mathbb{T}(\mathcal{A}) = S^P(T),$$

by the hypothesis and Proposition 1.7.2.

Theorem 1.7.5. *Let T be a countable complete theory with infinite models. A model \mathcal{A} of T is prime iff \mathcal{A} is countable and atomic.*

Proof. (\implies). Let \mathcal{A} be a prime model of T . By the Downward Skolem Löwenheim theorem T has a countable model \mathcal{C} so \mathcal{A} must be countable. If that \mathcal{A} is not atomic, then it realizes a nonprincipal type p but p . However, p be omitted in some countable model \mathcal{B} of T by the Omitting Types Theorem 1.7.3 and \mathcal{A} cannot be elementarily embedded in \mathcal{B} .

(\impliedby). Let \mathcal{A} be a countable and atomic of T and \mathcal{B} be any other model of T . Let $|\mathcal{A}| = \{a_i\}_{i \in \omega}$. We define an elementary embedding $f : \mathcal{A} \rightarrow \mathcal{B}$ as a union of finite elementary maps $f = \cup f_s$. Let $f_0 = \emptyset$. Given an elementary map f_s such that $f_s(a_i) = b_i, i < s$. Now (a_0, a_1, \dots, a_s) satisfies a principal $(s+1)$ -type with generator say $\theta(x_0, x_1, \dots, x_s)$. Now $(\exists x_s)\theta(\bar{x}) \in T$ because $\mathcal{A} \models (\exists x_s)\theta(a_0, \dots, a_{s-1}, x_s)$ so $\mathcal{B} \models (\exists x_s)\theta(b_0, \dots, b_{s-1}, x_s)$ say via b_s because f_s is elementary. Define $f_s(a_s) = b_s$. \square

³Picture $f_s(x)$ as a branch on the tree $\mathcal{T}_1(T)$. Since it does not generate p there are two extensions $\psi(x)$ and $\rho(x)$ of $f_s(x)$ on $\mathcal{T}_1(T)$ which split $f_s(x)$. At least one must be incompatible with p so we can extend $f_x(x)$ to avoid p .

1.7.3 Homogeneity of and Uniqueness of Prime Models

Theorem 1.7.6. [Prime Uniqueness Theorem] *Let \mathcal{A} and \mathcal{B} be prime models of T . Then*

(i) *If \mathcal{A} is an atomic model, then \mathcal{A} is homogeneous.*

(ii) $\mathcal{A} \cong \mathcal{B}$.

Proof. (i) Suppose $(\mathcal{A}, \bar{a}) \equiv (\mathcal{A}, \bar{b})$ and $c \in |\mathcal{A}|$. The type of (\bar{a}, c) is principal, say with generator $\theta(\bar{x}, y)$. Now $\mathcal{A} \models (\exists y)\theta(\bar{b}, y)$ because $(\mathcal{A}, \bar{a}) \equiv (\mathcal{A}, \bar{b})$. Choose $d \in |\mathcal{A}|$ such that $\mathcal{A} \models \theta(\bar{b}, d)$. Since $\theta(\bar{x}, y)$ is a generator, the type of (\bar{a}, c) is the same as the type of (\bar{b}, d) .

(ii) Let \mathcal{A} and \mathcal{B} be prime (hence countable atomic) models of T . Therefore, but realize exactly the principal types, of T , $\mathbb{T}(\text{scr}\mathcal{A}) = \mathbb{T}(\mathcal{B}) = S^P(T)$. Now by (i) they are homogeneous. Hence, they are isomorphic by the Homogeneous Uniqueness Theorem 1.6.4. \square

The next theorem was proved by Ryll-Nardjewski [1959] and also by Engler, and Svenonius. Later Ehrenfeucht gave a proof using the omitting types theorem.

Corollary 1.7.7. [ω -categoricity Theorem] *Let T be a countable, complete theory with no finite models. Then the following are equivalent.*

(i) *For every n , $S_n(T)$ is finite.*

(ii) *For every n , $B_n(T)$ is finite.*

(iii) *T is ω -categorical (i.e., any two countable models of T are isomorphic).*

Proof. (i) \iff (ii). This follows by the definitions.

(i) \implies (iii). Let T satisfy (i). Then every type in $S(T)$ is principal. Let \mathcal{A} and \mathcal{B} be countable models of T . They are countable and atomic, and hence isomorphic by Theorem ??.

(iii) \implies (i). Let T be ω -categorical. If T has a nonprincipal type p then p can be omitted in one countable model \mathcal{A} , by Theorem 1.7.3, and realized in another countable model \mathcal{B} . But then $\mathcal{A} \not\cong \mathcal{B}$. \square

1.7.4 Atomic Theories and Prime Model Existence

Definition 1.7.8. [Atomic Theory] Let T be a complete theory.

- (i) A theory T is *atomic* if $S_n^P(T)$ is dense in $S_n(T)$ for all $n \geq 1$.
- (ii) The Lindenbaum algebra $B_n(T)$ generated by the tree $\mathcal{T}_n(T)$ is *atomic* if every node $\theta \in \mathcal{T}_n(T)$ can be extended to an atom $\psi \in \mathcal{T}_n(T)$.

Proposition 1.7.9. *In Definition 1.7.8 we have (i) iff (ii).*

Proof. The principal types are dense in $S_n(T)$ iff for every $\theta \in \mathcal{T}_n(T)$ there is a principal type extending θ iff there is an atom $\psi \in \mathcal{T}_n(T)$ extending θ . \square

The main import of the next theorem is the implication (ii) implies (iii) which gives the theorem its name. It is the key fact about constructing atomic models which we shall use over and over.

Theorem 1.7.10. [Prime Model Existence] *Let T be a countable complete theory with infinite models. The following four conditions are equivalent.*

- (i) *The theory T is atomic.*
- (ii) *The Lindenbaum algebra $B_n(T)$ is atomic for every n .*
- (iii) *T has a prime model.*
- (iv) *T has an atomic model.*

Proof. (i) \iff (ii). By Proposition 1.7.9.

(iii) \implies (iv). By Theorem 1.7.5.

(iv) \implies (i). Let $\mathcal{A} \models T$ be atomic. For $\theta(\bar{x}) \in \mathcal{T}_n(T)$ we have $(\exists \bar{x})\theta(\bar{x}) \in T$. Hence, $\mathcal{A} \models \theta(\bar{a})$ for some $\bar{a} \in \mathcal{A}$ but \bar{a} must satisfy a principal type $p(\bar{x})$ which must therefore extend $\theta(\bar{x})$.

(ii) \implies (iii). Assume for every n that $B_n(T)$ is atomic, *i.e.*, that the tree $\mathcal{T}_n(T)$ is atomic in the sense of Definition 1.2.5 (iv) that

$$(\forall \beta \in \mathcal{T}_n(T))(\exists \text{ atom } \alpha \supseteq \beta) [\alpha \in \mathcal{T}_n(T)].$$

From T form \mathcal{L}_c and T_c as in Henkin's proof of Theorem 1.1.1 and the Omitting Types Theorem 1.7.3. Assume that $C = \{c_i\}_{i \in \omega}$ is a list of all constant symbols of \mathcal{L}_c . We shall construct a path $f \in [\mathcal{T}_0(T_c)]$ by a sequence of finite extensions $f = \cup_s f^s$ where $f^s \subset f^{s+1}$ and form the Henkin model \mathcal{A}_f as in Definition 1.1.4 with $|\mathcal{A}_f| = \{ [c_j] : c_j \in C \}$.

We must satisfy the following requirement for every finite tuple $\bar{c} \in C$,

$$(1.6) \quad R_{\bar{c}} : (\exists \text{ an atom } \theta(x) \in \mathcal{T}_n(T)) [\theta(\bar{c}) \in D^e(\mathcal{A})]$$

Stage $s = 0$. Define $f_0 = \emptyset$.

Stage $s + 1$. (Satisfy $R_{\bar{c}_s}$.)

Given f_s let \bar{c}_s be the set of constants occurring in f_s and n minimal so that $c_s \subseteq \{0, \dots, n\}$. Let $\psi(x_0, \dots, x_n)$ be an $\mathcal{L}(T)$ formula such that $\psi(c_0, \dots, c_n)$ is f_s . Clearly $T \cup \psi(\bar{x})$ is consistent and hence is extended by some atom $\beta(\bar{x})$ of $\mathcal{T}_n(T)$, $|\beta| \geq s$, since T is atomic. Let $f_{s+1} = \beta(\bar{c})$. Hence, $f = \cup_s f_s$ produces the elementary diagram of the canonical Henkin model \mathcal{A}_f as in Definition 1.1.4. Clearly, \mathcal{A}_f is atomic. \square

1.7.5 Finite Forcing in Model Theory and Computability

The Prime Model Existence Theorem 1.7.10, the Omitting Types Theorem 1.7.3, and even Henkin's proof of the Completeness Theorem 1.1.1 are examples of *finite forcing* constructions. Finite forcing was used by Kleene-Post [1954] in computability theory, by Ehrenfeucht [1957] and Vaught [1961] in model theory, and by Paul Cohen in set theory [1963].

In a finite forcing argument we build a function $f \in 2^\omega$ by constructing a strictly increasing sequence of finite functions $\{f_s\}_{s \in \omega}$, $f_s \subset f_{s+1}$, to meet a countable sequence of conditions $\{R_e\}_{e \in \omega}$ called *requirements*. At stage $s + 1$ we are given f_s and one or more requirements to be satisfied. Normally, we satisfy a given requirement by making a finite extension of f_s to f_{s+1} to ensure that f_{s+1} lies in some open set \mathcal{U}_σ . We shall continue forcing and generic sets in Chapter ??.

In the Omitting Types Theorem 1.7.3 we made a finite extension to ensure that c_j omits a certain type p_i . Since we can omit countably many nonprincipal types at once we could have used omitting types to prove the Prime Model Existence Theorem 1.7.10 providing that there are only countably many types. However, the proof we gave is that finite forcing is used to *realize* principal types, not omit nonprincipal ones. This applies in more general cases such as Example 1.2.6 with an atomic theory, but uncountably many nonprincipal types.

1.8 Saturated Models

If T is a complete theory, a prime model realizes the smallest set of types and a saturated model the largest.

Definition 1.8.1. Let T be a countable complete theory with infinite models.

- (i) A countable model \mathcal{A} of T is *saturated* if every 1-type $p(\bar{a}, x)$ over a finite set of elements $\bar{a} \in A$ is realized in \mathcal{A} .
- (ii) A countable model \mathcal{A} of T is *weakly saturated* if $\mathbb{T}(\mathcal{A}) = S(T)$, i.e., pure type $p \in S(T)$ is realized in \mathcal{A} .
- (iii) A countable model \mathcal{A} of T is *ω -universal* if $\mathcal{B} \preceq \mathcal{A}$ for every countable model \mathcal{B} of T .

Theorem 1.8.2. (Vaught) *Let \mathcal{A} be a countably infinite model of a complete theory T . Then the following are equivalent.*

- (i) \mathcal{A} is saturated.
- (ii) \mathcal{A} is weakly saturated and homogeneous.
- (iii) \mathcal{A} is ω -universal and homogeneous.

Proof. (i) \implies (ii). Let \mathcal{A} be saturated. Then \mathcal{A} is weakly saturated because if one can realize a 1-type over an n -tuple then one can realize an $(n+1)$ -tuple. For homogeneity of \mathcal{A} , suppose that $(\mathcal{A}, \bar{a}) \equiv (\mathcal{A}, \bar{b})$ and $\bar{a}, \bar{b}, c \in A$. Let $p(\bar{x}, y)$ be the type realized by (\bar{a}, c) . Hence, $(\mathcal{A}, \bar{a}) \models (\exists y)\theta(\bar{a}, y)$ for all $\theta(\bar{x}, y) \in p$. Hence, $(\mathcal{A}, \bar{b}) \models (\exists y)\theta(\bar{b}, y)$ for all $\theta(\bar{x}, y) \in p$ because $(\mathcal{A}, \bar{a}) \equiv (\mathcal{A}, \bar{b})$. Thus, $p(\bar{b}, y)$ is a consistent type over \bar{b} and must be realized in \mathcal{A} . Choose d such that $(\mathcal{A}, \bar{b}, d) \models p(\bar{x}, y)$.

(i) \implies (ii). Let $p(\bar{a}, y)$ be a 1-type over an n -tuple $\bar{a} \in A$. Since $p(\bar{x}, y)$ is consistent it is realized by some $(n+1)$ -tuple (\bar{b}, c) because \mathcal{A} is weakly saturated. Hence, $(\mathcal{A}, \bar{a}) \equiv (\mathcal{A}, \bar{b})$. Now by homogeneity there exists d such that $(\mathcal{A}, \bar{a}, d) \equiv (\mathcal{A}, \bar{b}, c)$. Therefore, d realize $p(\bar{a}, y)$.

(ii) \implies (iii). Clearly, both (ii) and (iii) are equivalent to $\mathbb{T}(\mathcal{A}) = S(T)$. \square

Corollary 1.8.3. [Saturated Uniqueness Theorem] (Vaught) *If \mathcal{A} and \mathcal{B} are countable⁴ saturated models, then $\mathcal{A} \cong \mathcal{B}$.*

Theorem 1.8.4. [Saturated Model Existence] (Vaught) *A theory T has a countable saturated model iff $S_n(T)$ is countable for all n , i.e., iff $S(T)$ is countable.*

⁴Theorem 1.8.2 and Theorem 1.8.3 hold for $\kappa \geq \aleph_0$ by essentially the same proofs, but we are only interested in the countable case.

Proof. If \mathcal{L} -theory T has a countable saturated model then $\mathbb{T}(\mathcal{A})$ is countable, but $\mathbb{T}(\mathcal{A}) = S(T)$ because \mathcal{A} is saturated. Suppose $S(T)$ is countable. Suppose $\cup_n S_n(T) = \{p_i\}_{i \in \omega}$. We build $\mathcal{A} = \cup_s \mathcal{A}_s$ the union of an elementary chain in $\mathcal{L}_c = \mathcal{L} \cup C$, where C is an infinite set of new constants. Given \mathcal{A}_s and a type $p(\bar{x})$ with finitely many new constants $\bar{c} \in C$ we realize type p with fresh constants $\bar{d} \in C$ by adding $\theta(\bar{d})$ for all $\theta(\bar{x}) \in p$ and then taking a model \mathcal{A}_{s+1} . Continuing in this fashion we construct \mathcal{A} which realized any type over a finite set $Y \subset A$. \square

1.9 The Spectrum of Homogeneous Models

For Vaught's results and all our results on their computable content we study *only homogeneous models* \mathcal{A} of a complete theory T . We have, by Theorem 1.8.2,

$$(1.7) \quad \mathcal{A} \text{ is saturated} \quad \iff \quad \mathbb{T}(\mathcal{A}) = S(T) \quad \text{and}$$

$$(1.8) \quad S^P(T) \subseteq \mathbb{T}(\mathcal{A}) \subseteq S(T).$$

If $\mathbb{T}(\mathcal{A})$ coincides with the lefthand endpoint $S^P(T)$, then \mathcal{A} is a prime model by Prime Uniqueness Theorem 1.7.6. If $\mathbb{T}(\mathcal{A})$ coincides with the righthand endpoint $S(T)$, then \mathcal{A} is a saturated model by Theorem 1.8.2. Otherwise, $\mathbb{T}(\mathcal{A})$ will take an intermediate value and the isomorphism type of \mathcal{A} will be completely determined by the type spectrum $\mathbb{T}(\mathcal{A})$ by the Homogeneous Uniqueness Theorem 1.6.4.

This situation is very roughly analogous to the case of the countable models of a theory T which is ω_1 -categorical but not ω -categorical. By the Baldwin-Lachlan theorem [1971] its countable models form an elementary chain,

$$\mathcal{M}_0 \prec \mathcal{M}_1 \prec \dots \mathcal{M}_\omega$$

where \mathcal{M}_0 is the prime model and \mathcal{M}_ω is the countable saturated model of T . For example, if $T = ACF_0$, the theory of algebraically closed fields, then the countable models \mathcal{M}_i are all homogeneous, a special case of (1.8).

In the case of countable homogeneous models of T we may not have a countable saturated model or even a prime model. However, if we *do* have a countable saturated model, then we have a prime model and the countable homogeneous models determined by (1.8) form a partially ordered set

$$\mathcal{A}_0 \subset \mathcal{A}_\alpha \subset \mathcal{A}_\beta \mid \mathcal{A}_\gamma \dots \subset \mathcal{A}_\omega$$

with \mathcal{A}_0 the prime model contained in every member \mathcal{A}_α , and \mathcal{A}_ω the countable saturated model containing every \mathcal{A}_α , and the relationship of inclusion $\mathcal{A}_\alpha \subset \mathcal{A}_\beta$ or incomparability $\mathcal{A}_\beta \mid \mathcal{A}_\gamma$ among the homogeneous models being determined entirely by the inclusion relation of the set of types $\mathbb{T}(\mathcal{A})$ realized in them.

Remark 1.9.1. *Our main tool for studying the computable content of Vaught's models is this. If we are given a complete decidable theory T and a countable homogeneous model \mathcal{B} of T . To construct an isomorphic copy $\mathcal{A} \cong \mathcal{B}$ it suffices to construct \mathcal{A} of that degree such that:*

- (i) \mathcal{A} is homogeneous, and
- (ii) $\mathbb{T}(\mathcal{A}) = \mathbb{T}(\mathcal{B})$.

Part II

Prerequisites in
Computability Theory

Chapter 2

Basic Computability Theory

2.1 Introduction to Computability

Definitions of Turing machine, Turing program P_e , partial computable function φ_e , computably enumerable (c.e.) set W_e and other basic notions can be found in Soare [1987].

2.2 Turing Reducibility

Turing's second fundamental contribution to computability theory was his definition of the information content of a set B being reducible to that of a set A , now known as B being *Turing reducible* to A , abbreviated *T-reducible*, ($B \leq_T A$). Very common is the special case of $B \leq_m A$ where there is a computable function f such that $x \in B$ iff $f(x) \in A$. In the general case, the algorithm, given oracle A and input x , may ask finitely many queries of the form “is $a \in A$ ” during its computation. The maximum element u so queried is the *use* of the computation. In the general case of *T-reducible* we do not put any computable bound $h(x)$ on the size of the use u or the number of queries about A . We define Turing reducible by introducing *oracle Turing machines*. (We identify a set A with its characteristic function χ_A and write $A(x)$ for $\chi_A(x)$.)

2.2.1 Oracle Turing Machine

An *oracle Turing machine* is simply a Turing machine with an extra “read only” tape, called the *oracle tape*, upon which is written the characteristic function of some set A (called the *oracle*), and whose symbols cannot be

printed over. The old tape is called the *work tape* and operates just as before. The reading head moves along both tapes simultaneously. As before, let Q be a finite set of states, Σ_1 the oracle tape alphabet $\{B, 0, 1\}$, Σ_2 the work tape alphabet $\{B, 1\}$, and $\{R, L\}$ the head moving operations right and left. A *Turing program* is now simply a partial map

$$\delta : Q \times \Sigma_1 \times \Sigma_2 \longrightarrow Q \times \Sigma_2 \times \{R, L\},$$

where $\delta(q, a, b) = (p, c, X)$ indicates that the machine in state q reading symbol a on the oracle tape and symbol b on the work tape passes to state p , prints “ c ” over “ b ” on the work tape, and moves one space right (left) on both tapes if $X = R$ ($X = L$). The other details are just as in Chapter ?? §?? except that the cell of the work tape on which the machine starts corresponds to the cell on the oracle tape containing $A(0)$. See Diagram 1.1.

If the oracle machine halts, let u be the maximum cell on the oracle tape scanned during the computation, *i.e.*, the maximum integer tested for membership in A . We define u to be the *use* of the computation and say that the elements $z \leq u$ are *used* in the computation.

These new oracle Turing programs being finite sets of 6-tuples of the above symbols can be effectively coded as in §?? for the coding P_e of ordinary Turing programs. Let \widehat{P}_e denote the e^{th} such oracle program under some effective coding. Note that \widehat{P}_e is *independent* of the oracle A .

2.2.2 Turing Functional and Use Function

Definition 2.2.1. (i) If the oracle Turing program \widehat{P}_e with oracle A on its oracle tape and with input x on its work tape eventually halts with output y , then we write $\Phi_e^A(x) = y$ and we define the corresponding *use function* $\varphi_e^A(x) = u$, where u is the maximum element in the characteristic function of A scanned (*used*) during the computation. We call Φ_e the *Turing functional* (*Turing reduction*) defined by oracle Program \widehat{P}_e . (Do not confuse the A -computable use function $\varphi_e^A(x)$ with the p.c. function φ_e .)

(ii) A partial function θ is *Turing computable in A* (*A -Turing computable*), written $\theta \leq_T A$, if there is an e such that $\Phi_e^A(x) \downarrow = y$ iff $\theta(x) = y$. A set B is *Turing reducible to A* ($B \leq_T A$) if the characteristic function $\chi_B \leq_T A$.

(iii) We also allow (total) functions f as oracles by defining Φ_e^f to be Φ_e^A where $A = \text{graph}(f)$.

(iv) We write $\Phi_{e,s}^A(x) = y$ and write use function $\varphi_{e,s}^A(x) = u$ if e , x , y , and $u < s$, and $\Phi_e^A(x) \downarrow = y$ in $< s$ steps with use u according to program \widehat{P}_e . We

assume that every \widehat{P}_e has been modified so that if $\varphi_{e,s}^A(x) = u$, then $u \geq 1$. This allows us to define $\varphi_{e,s}^A(x) = 0$ in case $\Phi_{e,s}^A(x) \uparrow$.¹ (This definition of $\varphi_{e,s}^A(x) = 0$ allows $\varphi_e^A(x)$ to act as an *indicator function* telling us that $\Phi_{e,s}^A(x) \uparrow$. This will be technically convenient.)

(v) We write $\Phi_{e,s}^\sigma(x) = y$ if for some A such that $\sigma \supseteq A \parallel \varphi_e^A(x)$ we have $\Phi_{e,s}^A(x) = y$ with use function $\varphi_{e,s}^A(x)$. (Recall that $A \parallel z = A \uparrow (z + 1)$.)

2.2.3 A Turing Reduction Φ_e as a C.E. Set of Axioms

We know that from a Turing reduction Φ_e we get a c.e. set

$$V_e = \{ \langle e, \sigma, x, y \rangle : \Phi_e^\sigma(x) \downarrow = y \}.$$

Now in reverse we can think of *defining* a Turing functional by enumerating its corresponding “axioms” into a c.e. set V . Hence, a Turing functional is simply a c.e. set of axioms. The most basic and absolute notions of computability are the interchangeable notions of partial computable function and of computably enumerable set. Now we see that the notion of Turing function is just as basic, being simply a c.e. set, and it can be used to define the p.c. function φ_e by letting the A -oracle be the empty set \emptyset .

2.3 The Limit Lemma and Δ_2^0 Sets

The most effective sets $A \subset \omega$ are the computable ones and the next most effective ones the c.e. sets for which we have a computable sequence $\{A_s\}_{s \in \omega}$ of finite sets (or at least uniformly computable sequence) such that $A = \bigcup_s A_s$. This computable approximation to A is monotonic in the sense that a 0 can change to a 1 but not the reverse. Next we consider the important class of Δ_2 sets. These are “limit computable” in that changes in the approximation can occur finitely often.

The most interesting and important classes are not the class of computable sets, but those sets which can be approximated by computable sequences. There are various levels of effectiveness in these approximations. It is useful to introduce the following additional quantifiers which have become standard.

Definition 2.3.1. (i) $(\exists^\infty x) R(x)$ abbreviates $(\forall y)(\exists z > y) R(z)$, *i.e.*, “there exist infinitely many x such that $R(x)$.”

¹The literature sometimes uses $\Phi(A; e, x, s)$ and $\varphi(A; e, x, s)$ in place of $\Phi_{e,s}^A(x)$ and $\varphi_{e,s}^A(x)$ and also $\{e\}^A(x)$ in place of $\Phi_e^A(x)$.

(ii) $(\forall^\infty x) R(x)$ abbreviates $(\exists y)(\forall z > y) R(z)$, *i.e.*, “for almost every x we have $R(x)$,” sometimes been written as *(a.e. x)*. (These quantifiers are dual to each other because $(\forall^\infty x) R(x)$ holds iff $\neg(\exists^\infty x)\neg R(x)$.)

2.3.1 The Modulus of Convergence

Definition 2.3.2. (i) The set A is Σ_2 if there is computable predicate R with

$$x \in A \iff (\exists y)(\forall z)R(x, y, z).$$

(ii) A set A is Δ_2 if $A \in \Sigma_2$ and $\bar{A} \in \Sigma_2$.

In Post’s Theorem we shall prove that $A \in \Delta_2$ iff $A \leq_T \emptyset'$. The most useful approximation for these classes is the limit computable property in (ii) of the following definition.

Definition 2.3.3. (i) A computable sequence of finite sets $\{A_s\}_{s \in \omega}$ is a Δ_2 -approximating sequence for a set A if $A = \lim_s A_s$, *i.e.*, if there is a function $m(x)$ called a *modulus (of convergence)* of the sequence such that

$$(\forall x)(\forall s \geq m(x)) [A \upharpoonright x = A_s \upharpoonright x].$$

If there is such a sequence we say that A is *limit computable*.

(ii) The *least modulus* is $m(x) = (\mu t)(\forall s \geq t)(\forall y \leq x) [A_t(y) = A(y)]$. Since the sequence is computable, the least modulus is computable in every modulus.

(iii) A special case is that of a c.e. set A . A computable sequence of finite sets $\{A_s\}_{s \in \omega}$ is a Σ_1 -approximating sequence (*c.e. approximating sequence*) to the c.e. set A if $A = \cup_s A_s$. In this case the approximating sequence is monotonically nondecreasing. (Analogously, we could define a Π_1 -approximating sequence (monotonically nonincreasing sequence) for a Π_1 set B but it is equivalent to use a Σ_1 -approximating sequence for the Σ_1 set \bar{B} .)

(iv) For future reference fix a simultaneous computable enumeration $\{W_{e,s}\}_{s \in \omega}$ of the c.e. sets and let $m_e(x)$ be the *least modulus* for $\{W_e\}_{e, s \in \omega}$ also called the *settling function*. Hence,

$$(2.1) \quad m_e(x) = (\mu s) [W_{e,s} \upharpoonright x = W_e \upharpoonright x].$$

Note that $m_e \equiv_T W_e$ because of the monotonicity of the Σ_1 -sequence.

Proposition 2.3.4. *If A is limit computable via $\{A_s\}_{s \in \omega}$ with modulus $m(x)$ then $A \leq_T m$.*

Proof. $A(x) = A_{m(x)}(x)$. □

2.3.2 Limit Computable Functions

Lemma 2.3.5. [Limit Lemma] *f is limit computable iff $f \leq_T \emptyset'$.*

Proof. (\Leftarrow). Fix $B \equiv_T \emptyset'$. Choose e with $W_e = B$. Let $m_e(x)$ be the least modulus as in (2.1). Let $m_B(x) = m_e(x)$ and $B_s = W_{e,s}$. Now $m_B(x) \equiv_T B$ because B is c.e. Suppose $A = \Phi_i^B$. Define the computable sequence

$$A_s(x) = \begin{cases} \Phi_{i,s}^{B_s}(x) & \text{if } \Phi_{i,s}^{B_s}(x) \downarrow = 1 \\ 0 & \text{otherwise.} \end{cases}$$

Define the B -computable function

$$(2.2) \quad m_A(x) = (\mu t) (\exists u \leq t) (\forall y \leq x) [\Phi_{i,t}^{B_t} \upharpoonright^u(y) \downarrow \quad \& \quad t \geq m_B(u)].$$

The first clause in the matrix is computable. The second is B -computable and guarantees that $B_s \upharpoonright^u = B \upharpoonright^u$ for every $s \geq m_A(x)$. Therefore,

$$(\forall s \geq m_A(x)) (\forall y \leq x) [\lim_s A_s(y) = A_{m_A(x)}(y) = A(y)].$$

(\Rightarrow). Let $A = \lim_s A_s(x)$ where $\{A_s\}_{s \in \omega}$ is computable. Define the Σ_1 (and hence c.e.) set of “errors,”

$$(2.3) \quad E = \{ \langle x, s \rangle : (\exists t > s) [A_t(x) \neq A_s(x)] \}.$$

Define $m(x) = (\mu s) [\langle x, s \rangle \notin E]$. Now $A \leq_T m \leq_T E \leq_T K$. \square

Corollary 2.3.6. [Full Limit Lemma] *The following are equivalent,*

- (i) $A \leq_T \emptyset'$.
- (ii) $A \in \Delta_2$.
- (iii) f is limit computable.

Proof. (i) \iff (ii). See Post’s Theorem ?? (iv). \square

In computability theory these three properties are used very often and completely interchangeably. Often an author says one of them (*e.g.*, Δ_2) but immediately uses another (*e.g.*, . limit computable) without explanation.

Imagine the degrees $\mathbf{d} \leq \mathbf{0}'$ as an oval with top $\mathbf{0}'$ and bottom $\mathbf{0}$. The Limit Lemma 2.3.5 exactly characterizes the sets and degrees in this oval. Now imagine a smaller oval inside the first oval with the same top and bottom but containing only the *computably enumerable* degrees.

The next result, building on the first, characterizes the sets and degrees in this inner oval. Let A be limit computable via Δ_2 -approximating sequence $\{A_s\}_{s \in \omega}$ with modulus $m_A(x)$. By Proposition 2.3.4 we always have $A \leq_T m$ for any modulus, but $m \leq_T A$ may fail even for the least modulus. Remarkably, however, if A has c.e. degree, then $m \leq_T A$ holds for *some* modulus m of some particular Δ_2 -approximating sequence.

2.3.3 The Modulus Lemma for C.E. Sets

Lemma 2.3.7. [Modulus Lemma] *If B is c.e. and $A \leq_T B$, then $A = \lim_s A_s$ for some computable sequence $\{A_s\}_{s \in \omega}$ with modulus $m \leq_T B$.*

Proof. Let B be c.e. and $A = \Phi_i^B$. Define A_s and $m_A \leq_T B$ as in the Limit Lemma 2.3.5 and (2.2). \square

Corollary 2.3.8. *A has c.e. degree iff $A = \lim_s A_s$ for some computable sequence $\{A_s\}_{s \in \omega}$ with modulus $m \leq_T f$.*

Proof. (\implies). Let $A \equiv_T B$ with B c.e. Apply the Modulus Lemma to obtain $m \leq_T B \equiv_T A$.

(\impliedby). Let $A = \lim_s A_s$ with modulus $m \leq_T A$. Define the c.e. set $E \geq_T A$ as in the proof of the Limit Lemma. Now $E \leq_T m$ but here $m \leq_T A$. Hence, $A \equiv_T E$ and A has c.e. degree. \square

2.4 Cantor Space Topology and Closed Sets

It is useful to view a Turing reduction Φ_e as a continuous functional on Cantor space 2^ω and Baire space ω^ω . (We give the definitions of the basic open sets for 2^ω since the definitions for ω^ω are similar.)

2.4.1 Representations of Open and Closed Sets

Definition 2.4.1. (i) Using ordinal notation identify the ordinal 2 with the set of smaller ordinals $\{0, 1\}$. Identify the sets $A \subseteq \omega$ with their characteristic functions $f : \omega \rightarrow \{0, 1\}$ and represent this as 2^ω .

(ii) Let $2^{<\omega}$ denote the set of finite strings of 0's and 1's, *i.e.*, finite initial segments of functions $f \in 2^\omega$.

(iii) *Cantor space* is 2^ω with the following topology (class of open sets). For every $\sigma \in 2^{<\omega}$ the *basic open (clopen) set*

$$\mathcal{U}_\sigma = \{ f : f \in 2^\omega \ \& \ \sigma \subset f \}$$

where $\sigma \subset f$ denotes that the function f extends σ . The sets \mathcal{U}_σ are called *clopen* because they are both closed and open. The *open* sets of Cantor space are the countable unions of basic open sets, so the open sets are closed under finite intersection and countable union. (Two other equivalent and well-known definitions of Cantor space are in Exercise ?? and ??.)

(iv) Set $A \subseteq 2^{<\omega}$ is an *open representation* of the open set $\mathcal{U}_A = \bigcup_{\sigma \in A} \mathcal{U}_\sigma$. We may assume A is closed *upwards*, i.e., $\sigma \in A$ and $\sigma \subset \tau$ implies $\tau \in A$.

(v) A set \mathcal{C} is *closed* if its complement \mathcal{U}_A is open, i.e., $\mathcal{C} = \overline{\mathcal{U}_A} = (2^\omega - \mathcal{U}_A)$. In this case $T =_{\text{dfn}} 2^{<\omega} - A$ is a *closed representation* for \mathcal{C} . Now T is closed *downwards* (because A is closed upwards). Hence, T forms a *tree* as in Definition 2.4.2 (i). For a tree $[T]$ we have the closed set $[T] = \overline{\mathcal{U}_T}$, the complement of the open set \mathcal{U}_A for $A = \overline{T}$.

2.4.2 Notation for Trees

Definition 2.4.2. (i) A *tree* $T \subseteq 2^{<\omega}$ is a set closed under initial segments, i.e., $\sigma \in T$ and $\tau \subset \sigma$ imply $\tau \in T$. Fix any tree T .

(ii) The set of *infinite paths* through T is

$$(2.4) \quad [T] = \{ f : (\forall n) [f \upharpoonright n \in T] \}.$$

Note that $[T]$ is always a closed set. If $\mathcal{C} \subseteq 2^\omega$ is any closed set then by Theorem 2.4.3 there is a (nonunique) tree $T \subseteq 2^{<\omega}$ such that $\mathcal{C} = [T]$, which is called a *closed representation* for \mathcal{C} as in Definition 2.4.1 (v).

(iii) For $\sigma \in T$ define the subtree of nodes $\tau \in T$ comparable with σ ,

$$(2.5) \quad T_\sigma = \{ \tau : \sigma \subseteq \tau \quad \text{or} \quad \tau \subset \sigma \}.$$

(iv) Define the subtree of *extendible nodes* of T ,

$$(2.6) \quad T^{\text{ext}} = \{ \sigma \in T : [T_\sigma] \neq \emptyset \}.$$

Note that if T is c.e. then T^{ext} is co-c.e. (See Exercise ??.) Usually for a given tree T there are many other trees T' such that $[T] = [T']$, i.e., many different representations for the same closed set $[T]$. (See Exercise ??.)

To get a concrete example of representations A and T of open and closed sets, define $A = \{ \sigma : (\exists x)[\sigma(x) = 0] \}$. Hence, $\overline{A} = T = \{ 1^n \}_{n \geq 0}$ and $[T] = \{ 1^\infty \}$, the single infinite path of all 1's. As we “climb” the tree from the empty node \emptyset along some finite string σ , we remain on T so long as we

select only 1's, but as soon as we include a 0 in σ we “fall off” the tree T into the open set \mathcal{U}_A .

The closed sets are closed under finite union and countable intersection since the open sets have the dual properties in (iii), closure under finite intersection and countable union. The clopen sets are both open and closed, so any countable union of them is open and any countable intersection is closed.

Theorem 2.4.3. *If T is a tree, then $[T]$ is a closed set, and for every closed set \mathcal{C} there is a tree T such that $\mathcal{C} = [T]$.*

Proof. (\implies). Given tree T let $A = 2^{<\omega} - T$. Then \mathcal{U}_A is open and $[T] = 2^\omega - \mathcal{U}_A$ so $[T]$ is closed.

(\impliedby). Given any closed \mathcal{C} with complement \mathcal{U}_A define T by putting σ in T if $(\forall \tau \subseteq \sigma) [\tau \notin A]$. Then T is closed downward and $[T] = \mathcal{C}$. □

Theorem 2.4.4. [Compactness] *The following very easy and well-known properties hold for Cantor Space 2^ω and are equivalent to compactness (iv).*

- (i) **König's Lemma** *If $T \subseteq 2^{<\omega}$ is an infinite tree, then $[T] \neq \emptyset$.*
- (ii) *If $T_0 \supseteq T_1 \dots$ is a decreasing sequence of trees with $[T_n] \neq \emptyset$ for every n , and intersection $T_\omega = \bigcap_{n \in \omega} T_n$, then $[T_\omega] \neq \emptyset$.*
- (iii) *If $\{\mathcal{C}_i\}_{i \in \omega}$ is a countable family of closed sets such that $\bigcap_{i \in F} \mathcal{C}_i \neq \emptyset$ for every finite set $F \subseteq \omega$, then $\bigcap_{i \in \omega} \mathcal{C}_i \neq \emptyset$ also.*
- (iv) **Finite subcover** *Any open cover $\mathcal{U}_A \supseteq 2^\omega$ has a finite subcover $\mathcal{U}_F \supseteq 2^\omega$ for some finite $F \subseteq A$.*

Proof. (i) Let T be infinite. We construct a sequence of nodes $\sigma_0 \subset \sigma_1 \dots$ such that $f = \bigcup_{n \in \omega} \sigma_n$ and $f \in [T]$. Define a node σ to be *large* if T_σ is infinite. Define $\sigma_0 = \emptyset$ which is large. Given σ_n large, one of $\sigma_n \hat{\ } 0$ or $\sigma_n \hat{\ } 1$ must be large by the Pigeon Hole principle. (This fails for Baire space ω^ω where there are infinitely many possible successors.) Let $\sigma_{n+1} = \sigma_n \hat{\ } 0$ if it is large and $\sigma_{n+1} = \sigma_n \hat{\ } 1$ otherwise.

(ii) Build a new computable tree S by putting σ of length n into S just if $\sigma \in \bigcap_{i \leq n} T_i$ (which is also a tree). Note that S is infinite because $[T_n] \neq \emptyset$ for every n . By König's Lemma (i) there exists $f \in [S]$, but $[S] = [T_\omega]$.

(iii) Define $\widehat{\mathcal{C}}_i = \bigcap_{j \leq i} \mathcal{C}_j$. Hence, $\widehat{\mathcal{C}}_0 \supseteq \widehat{\mathcal{C}}_1 \dots$ is a decreasing sequence of nonempty closed sets. Choose a decreasing sequence of computable trees $T_0 \supseteq T_1 \dots$ such that $[T_i] = \widehat{\mathcal{C}}_i$ and apply (ii).

(iv) Define the closed set $C_S = 2^\omega - U_S$. Fix A and assume for every finite $F \subseteq A$ that $U_F \not\subseteq 2^\omega$, and hence, $C_F \neq \emptyset$. Hence, $C_A \neq \emptyset$ by (iii), so $U_A \neq 2^\omega$. \square

2.4.3 Dense Subsets of Cantor Space

We now introduce the important notion of *dense sets* which we shall further develop in §?? particularly in Definition ??.

Definition 2.4.5. Let \mathcal{S} be Cantor space 2^ω or Baire space ω^ω .

(i) A set $\mathcal{A} \subseteq \mathcal{S}$ is *dense* if $(\forall \sigma)(\exists f \supset \sigma)[f \in \mathcal{A}]$.

(ii) Let $T \subseteq 2^{<\omega}$ be a tree. A point $f \in [T]$ is *isolated* in $[T]$ if

$$(2.7) \quad (\exists \sigma)[[T_\sigma] = \{f\}].$$

We say that σ *isolates* f because $U_\sigma \cap [T] = \{f\}$. If f is not isolated, then f is a *limit point*.

(Isolated points have Cantor-Bendixson (CB) rank 0, and limit points can be classified according to their CB rank. See Exercise ?? and Definition ?? of CB rank and relation to computability properties.)

(ii) A space \mathcal{S} is *separable* if it has a countable base of open sets. (Both Cantor space and Baire space are separable by Definition 2.4.1 of the U_σ .)

(iv) A class $\mathcal{B} \subseteq \mathcal{S}$ is G_δ , *i.e.*, boldface Π_2^0 , if $\mathcal{B} = \bigcap_i \mathcal{A}_i$ a countable intersection of open sets \mathcal{A}_i . (In §?? the \mathcal{A}_i will be *dense* as well as open.)

After open and closed sets, much attention is paid in point set topology to G_δ sets (see Oxtoby[1971]). If the open sets \mathcal{A}_i are also *dense*, then they have special significance. In §?? we shall explore Banach-Mazur games for finding a point $f \in \bigcap \mathcal{A}_i$ where the \mathcal{A}_i are dense and open. This is the paradigm for the *finite extension* constructions in Chapter ??, especially the Finite Extension Paradigm Theorem ??, which we use to construct sets and degrees meeting an infinite sequence of “requirements.” Meeting a given requirement R_i amounts to meeting the corresponding dense open set \mathcal{A}_i .

2.5 Effectively Closed Sets and Π_1^0 -classes

2.5.1 Effectively Open and Closed Set

Definition 2.5.1. (i) If $A \subseteq 2^{<\omega}$ is c.e. and $\mathcal{A} = U_A$, then \mathcal{A} is *effectively (computably) open*.

(ii) If $\mathcal{C} = 2^\omega - \mathcal{U}_A$ for A c.e., or equivalently if $\mathcal{C} = [T]$ for a computable tree $T \subseteq 2^{<\omega}$, then \mathcal{C} is *effectively (computably) closed*.

(iii) $\mathcal{C} \subseteq 2^\omega$ is a Π_1^0 -class if there is a computable relation $R(x)$ such that

$$(2.8) \quad \mathcal{C} = \{f : (\forall x) R(f(x))\}.$$

We call this *lightface* Π_1^0 because it is defined in (2.8) by a universal quantifier outside of a *computable* relation $R(x)$.

(iv) A set $\mathcal{C} \subseteq 2^\omega$ is *boldface* $\mathbf{\Pi}_1$ if there is some set $S \subseteq \omega$ such that

$$\mathcal{C} = \{f : (\forall x) R^S(f(x))\},$$

and we say \mathcal{C} is in Π_1^S . Here R^S denotes a relation computable in the set S which we call the *parameter* determining R^S . A set $\mathcal{A} \subseteq 2^\omega$ is *boldface* Σ_1 or Σ_1^S if its complement $2^\omega - \mathcal{A}$ is boldface $\mathbf{\Pi}_1$ or Π_1^S .

Theorem 2.5.2. *The open (closed) sets of 2^ω are exactly boldface Σ_1 ($\mathbf{\Pi}_1$) sets and any boldface set is lightface in some parameter A .*

Proof. If \mathcal{A} is open, then $\mathcal{A} = \mathcal{U}_A$ for some countable set A . Now A determines exactly which σ contribute in the countable union $\mathcal{A} = \cup_{\sigma \in A} \mathcal{U}_\sigma$. Hence, if we fix A as a parameter, then the definition and properties become *lightface* Σ_1^A , i.e., effectively open *relative* to the oracle A . (But this entire chapter is about working relative to an oracle.) \square

We often convert an open set $\mathcal{A} = \mathcal{U}_A$ into the realm of computability theory as follows. We: (1) usually fix the parameter A and do a construction which is computable *relative* to the parameter A ; (2) often replace the open set \mathcal{U}_A by its complement the closed set $\mathcal{C} = \overline{\mathcal{U}_A}$; and (3) replace the closed set by $\mathcal{C} = [T^A]$ for a A -computable tree T . By fixing the parameter A we can apply all the methods of this chapter on A -computable constructions, such as the Recursion Theorem, construction of an A -c.e. set B , and so forth. (For example, in §?? we are given a game corresponding to to an open set \mathcal{U}_A . Fixing A as a parameter, we define another set B which is c.e. in A and whose complement corresponds to the paths $[T^A]$ for an A -computable tree T^A where we find the winning strategy for the game.)

2.5.2 Effective Continuity of Turing Functionals

Definition 2.5.3. (i) A *partial functional* Φ on Baire space ω^ω is a partial map with $\text{dom } \Phi$ and $\text{rng } \Phi \subseteq \omega^\omega$. If both $\text{dom } \Phi$ and $\text{rng } \Phi \subseteq 2^\omega$, then

Φ is a partial functional on Cantor space 2^ω . (We use the term *functional* for maps whose input and output may be an infinite object $f \in \omega^\omega$ to distinguish from *functions* from ω to ω .)

(ii) Definition 2.2.1 naturally produced a partial functional Φ_e from 2^ω to ω^ω but part (iii) there showed how to view this as a partial map $\tilde{\Phi}$ on ω^ω by defining $\tilde{\Phi}_e^f = \Phi_e^A$ for $A = \text{graph } f$. Likewise, define a partial map $\hat{\Phi}_e : 2^\omega \rightarrow 2^\omega$ by defining $\hat{\Phi}_e^A(x) = y$ if $\Phi_e^A(x) = y$ and $y \leq 1$ and $\hat{\Phi}_e^A(x) \uparrow$ otherwise.

(iii) A partial functional Φ on Cantor space 2^ω or Baire space ω^ω is *effectively continuous* if $\Phi^{-1}(\mathcal{U}_\tau)$ is effectively open for every τ uniformly in τ .

Theorem 2.5.4. [Effective continuity of Turing functionals] *The Turing partial functional $\tilde{\Phi}_e$ is effectively continuous on Baire space ω^ω and the Turing partial functional $\hat{\Phi}_e$ is effectively continuous on Cantor space. From now on both will simply be denoted by Φ_e .*

Proof. The set $\{\langle e, \sigma, x, y \rangle : \Phi_e^\sigma(x) \downarrow = y\}$ is c.e. by the Master Enumeration Theorem ?? and

$$\Phi_e^{-1}(\mathcal{U}_\tau) = \bigcup \{ \mathcal{U}_\sigma : (\exists s) (\forall x < |\tau|) [\Phi_{e,s}^\sigma(x) \downarrow = \tau(x)] \}.$$

The matrix is computable, and the relation remains computable adding the bounded quantifier, so the entire condition is Σ_1 and hence c.e. If A is the c.e. set of σ satisfying this condition then $\Phi_e^{-1}(\mathcal{U}_\tau) = \mathcal{U}_A$ where for $\sigma \in A$ \mathcal{U}_σ is interpreted appropriately as $\{f \in 2^\omega : \sigma \subset f\}$ and $\sigma \in 2^{<\omega}$ for Cantor space and as $\{f \in \omega^\omega : \sigma \subset f\}$ and $\sigma \in \omega^{<\omega}$ for Baire space. \square

Theorem 2.5.5. *A class \mathcal{C} is a Π_1^0 -class iff \mathcal{C} is effectively closed, i.e., $\mathcal{C} = [T]$ for some computable tree T .*

Proof. (\implies). Let $\mathcal{C} = \{f : (\forall x) R(f(x))\}$ for R computable. Define a computable tree $T = \{ \sigma : (\forall \tau \subseteq \sigma) [R(\tau)] \}$. Then $[T] = \mathcal{C}$.

(\impliedby). Let $\mathcal{C} = [T]$ for T a computable tree. Define $R(\sigma)$ iff $\sigma \in T$. Then $\{f : (\forall x) R(f(x))\} = [T]$. \square

2.5.3 The Low Basis Theorem for Π_1^0 -Classes

Theorem 2.5.6. [Low Basis Theorem, Jockusch-Soare] *If $\mathcal{C} \subseteq 2^\omega$ is a nonempty Π_1^0 class, then it contains a member f of low degree ($f' \equiv_T 0'$).*

Proof. Note that \emptyset' can decide whether a computable tree G is finite because

$$(2.9) \quad |G| < \infty \iff (\exists n)(\forall \sigma)_{|\sigma|=n} [\sigma \notin G].$$

The bounded quantifier in front of the computable matrix remains computable, and the $(\exists n)$ quantifier makes the condition Σ_1 and hence computable in \emptyset' . Let T be a computable tree such that $[T] = \mathcal{C}$. Use \emptyset' to define a sequence of infinite computable trees $T = T_0 \supseteq T_1 \supseteq \dots$ as follows. Define

$$(2.10) \quad U_e = \{ \sigma : \Phi_{e,|\sigma|}^\sigma(e) \uparrow \}$$

which is also a computable tree. Given T_e : (1) define $T_{e+1} = T_e \cap U_e$ if $T \cap U_e$ is infinite; and (2) define $T_{e+1} = T_e$ otherwise. If (1) then $\Phi_e^g(e) \uparrow$ for all $g \in [T_{e+1}]$, and if (2) then $\Phi_e^g(e) \downarrow$ for all $g \in [T_{e+1}]$. (Namely, we say that T_{e+1} forces the jump as described in Chapter ??.) Choose $f \in \bigcap_{e \in \omega} [T_e]$ which is an intersection of a descending sequence of nonempty closed sets and hence nonempty, by the Compactness Theorem 2.4.4 (ii). Now \emptyset' can decide using (2.9) which of (1) or (2) holds in the definition of T_{e+1} and hence whether $\Phi_e^f(e) \downarrow$ or not. Therefore, $f' \leq_T \emptyset'$ and f is low. \square

Definition 2.5.7. If A and B are disjoint sets, then S is a *separating set* if $A \subseteq S$ and $B \cap S = \emptyset$.

Theorem 2.5.8. If W_e and W_i are disjoint c.e. sets, then the class of separating sets is a Π_1^0 class.

Proof. Define a computable tree T with $[T]$ the class of separating sets. If $|\sigma| = s$ put σ in T if $\forall x < |\sigma|$

$$[x \in W_{e,s} \implies \sigma(x) = 1] \quad \& \quad [x \in W_{i,s} \implies \sigma(x) = 0].$$

\square

2.6 Domination and Escape

Theorem 2.6.1. [Martin High Domination Theorem] Set A satisfies $\emptyset'' \leq_T A'$ iff there is a dominant function $f \leq_T A$. (A need not be Δ_2^0 .)

Proof. By Theorem ?? we know that $\text{Tot} \equiv_T \emptyset''$. Hence, by the Limit Lemma ??, we have $\emptyset'' \leq_T A'$ iff there is an A -computable $\{0, 1\}$ -valued function $g(x, s)$ such that $\lim_s g(x, s) = \text{Tot}(x)$.

(\implies). Assume $\emptyset'' \leq_T A'$. Given $g(x, s)$ as above we define a dominant function $f \leq_T A$ as follows.

Stage s . (To define $f(s)$). For all $e \leq s$ define

$$t(e) = (\mu t > s) [(\forall x \leq s) [\varphi_{e,t}(x) \downarrow] \text{ or } g(e, t) = 0], \text{ and}$$

$$f(s) = \max\{ t(e) : e \leq s \}.$$

Note that $t(e)$ exists because if φ_e is not total then $g(e, t) = 0$ for almost every (a.e.) t . If φ_e is total then $g(e, t) = 1$ for a.e. t and hence $f(s) > \varphi_e(s)$ for a.e. s .

(\impliedby). Assume $f \leq_T A$ is dominant. Define an A -computable function $g(x, s)$ such that $\lim_s g(x, s) = \text{Tot}(x)$ as follows:

$$g(x, s) = \begin{cases} 1 & \text{if } (\forall z \leq s) [\varphi_{x,f(s)}(z) \downarrow]; \\ 0 & \text{otherwise.} \end{cases}$$

Note that if φ_x is total, then so is

$$\psi_x(y) = (\mu s) (\forall z \leq y) [\varphi_{x,s}(z) \downarrow].$$

Thus, f dominates ψ_x and $g(x, s) = 1$ for a.e. s . If φ_x is not total, then $\varphi_x(y)$ and $\psi_x(y)$ diverge for some y , and $g(x, s) = 0$ for all $s \geq y$. \square

Corollary 2.6.2. *A c.e. degree \mathbf{a} is high ($\mathbf{a} \in \mathbf{H}_1$) iff there is a dominant function f such that $\deg(f) \leq \mathbf{a}$.* \square

2.6.1 Domination on Degrees

We can naturally extend this notion to domination or escape (nondomination) on *degrees*. This will be useful in characterizing nonlow₂ degrees in the next section.

Definition 2.6.3. [Domination on Degrees] Given degrees $\mathbf{a}, \mathbf{b} \leq \mathbf{0}'$.

(i) We say that degree \mathbf{a} *dominates* degree \mathbf{b} if

$$(2.11) \quad (\exists g \leq \mathbf{a}) (\forall f \leq \mathbf{b}) (\forall^\infty x) [f(x) \leq g(x)].$$

(ii) If \mathbf{a} does *not* dominate \mathbf{b} we say that degree \mathbf{b} *escapes* \mathbf{a} , i.e.,

$$(2.12) \quad (\forall g \leq \mathbf{a}) (\exists f \leq \mathbf{b}) (\exists^\infty x) [f(x) > g(x)].$$

Corollary 2.6.4. *If $\mathbf{d} \leq \mathbf{0}'$ then \mathbf{d} dominates $\mathbf{0}$ iff \mathbf{d} is high ($\mathbf{d}' = \mathbf{0}''$).*

Proof. Immediate by Martin's Theorem 2.6.1. \square

2.7 Characterizing Nonlow₂ Sets $A \leq_T \emptyset'$

Now fix any degree $\mathbf{a} \leq \mathbf{0}'$ and relativize the previous proof to the cone of degrees $\{\mathbf{b} : \mathbf{b} \geq \mathbf{a}\}$ with base \mathbf{a} in place of $\mathbf{0}$ and with $\mathbf{0}'$ in place of \mathbf{d} , a degree which may or may not be high in this cone. We obtain the following useful escape type property characterizing nonlow₂ degrees $\mathbf{a} \leq \mathbf{0}'$.

Theorem 2.7.1. [Relativized Martin Domination Theorem] *If $A \leq_T \emptyset'$, then $A'' \leq_T \emptyset''$ (i.e., A is low₂) if and only if there is a function $f \leq \emptyset'$ which dominates every total function $h \leq_T A$.*

Proof. Fix $A \leq_T \emptyset'$. Relativize Martin's Theorem 2.6.1 to the cone of sets $\{X : X \geq_T A\}$. Now A is low₂ ($A'' \equiv_T \emptyset''$) iff \emptyset' , viewed as a member of the cone, is high₁ in the cone, namely with one jump \emptyset' reaches A'' in Turing degree, since A'' is the double jump of the base A of the cone. By Martin's Theorem 2.6.1 this occurs iff there is a function $f \leq_T \emptyset'$ which is dominant above A , namely f dominates every total $h \leq_T A$. \square

Theorem 2.7.2. [Nonlow₂ Escape Theorem] *A degree $\mathbf{a} \leq \mathbf{0}'$ is not low₂ ($\mathbf{a}'' > \mathbf{0}''$) iff $\mathbf{0}'$ does not dominate \mathbf{a} , i.e.,*

$$(2.13) \quad (\forall g \leq \mathbf{0}')(\exists f \leq \mathbf{a})(\exists^\infty x) [g(x) \leq f(x)].$$

Proof. Relativize Martin's Theorem 2.6.1 to \mathbf{a} and consider $\mathbf{0}'$ as playing the role of a degree \mathbf{d} which may be high over \mathbf{a} . In this case $\mathbf{a}'' > \mathbf{0}''$ iff $\mathbf{0}'$ is *not* high over \mathbf{a} iff \mathbf{a} escapes $\mathbf{0}'$ iff (2.13) using (2.12) with \mathbf{b} and \mathbf{a} there replaced here by \mathbf{a} and $\mathbf{0}'$ respectively. \square

2.8 Uniform Enumerations of Functions and Sets

Theorem 2.8.2 relates nicely to Martin's Theorem 2.6.1 on dominant functions and high degrees. Also the notions introduced in Definition 2.8.1 have proved useful in other areas of computability theory, computable model theory, and models of arithmetic.

Definition 2.8.1. (i) If $f(x, y)$ is a binary function then

$$f_y \text{ denotes } \lambda x [f(x, y)].$$

We view $\lambda x, y [f(x, y)]$ as specifying a *matrix* whose rows with entry $f(x, y)$ on the location (x, y) under by usual coordinates. For vertical coordinate $y \in \omega$ we view $\lambda x [f_y(x)]$ as the y^{th} row (viewed horizontally).

(ii) If \mathcal{C} is a class of (unary) functions and \mathbf{a} is a degree, \mathcal{C} is called *\mathbf{a} -uniform* (*\mathbf{a} -subuniform*) if there is a binary function f of degree $\leq \mathbf{a}$ such that

$$\mathcal{C} = \{f_y\}_{y \in \omega} \quad (\text{respectively, } \mathcal{C} \subseteq \{f_y\}_{y \in \omega}).$$

Therefore, f is *uniformly* listing the rows $\{f_y\}_{y \in \omega}$ or in the subuniform case listing a collection of rows containing the given ones.

2.8.1 Limits of Functions

Given $f(x, y)$ as in Definition 2.8.1 we may need to take limits in both the x and y directions. For example, if $\{A_y\}_{y \in \omega}$ is a uniformly computable sequence of computable sets then the vertical limit $B(x) = \lim_y A_y(x)$ is a Δ_2^0 set as in the Limit Lemma 2.3.5. Now suppose that $A_y = W_{f(y)}$ as in the proof of Theorem ??, so that $W_{f(y)}$ is either finite or ω according as W_y is finite or not. Then

$$C(y) =_{\text{dfn}} \lim_x A_y(x) = \text{Tot}(y)$$

so that $C' \geq_T 0''$ a useful fact in many infinite injury constructions such as the Thickness Lemma because any set D thick in C also satisfies $D' \geq_T 0''$.

2.8.2 \mathbf{a} -uniform Enumeration of the Computable Functions

The next useful characterization follows from Martin's Theorem 2.6.1.

Theorem 2.8.2. [Jockusch (1972a)] *If \mathbf{a} is any degree, then statements (i)–(iv) are equivalent:*

- (i) $\mathbf{a}' \geq 0''$ (i.e., \mathbf{a} is high);
- (ii) the computable functions are \mathbf{a} -uniform;
- (iii) the computable functions are \mathbf{a} -subuniform;
- (iv) the computable sets are \mathbf{a} -uniform.

If \mathbf{a} is c.e., then (i)–(iv) are each equivalent to

- (v) the computable sets are \mathbf{a} -subuniform.

Proof. The implications (ii) \implies (iii), (ii) \implies (iv), and (iv) \implies (v) are immediate.

(i) \implies (ii). By Martin's Theorem 2.6.1 choose a dominant function g of degree $\leq \mathbf{a}$. Define $f(\langle e, i \rangle, x) = \varphi_{e, i+g(x)}(x)$ if $\varphi_{e, i+g(y)}(y) \downarrow$ for all $y \leq x$

and $f(\langle e, i \rangle, x) = 0$ otherwise. Now either $f_{\langle e, i \rangle} = \varphi_e$ a total function, or $f_{\langle e, i \rangle}$ is finitely nonzero. In either case $f_{\langle e, i \rangle}$ is computable. If φ_e is total then $g(x)$ dominates $c(x) = (\mu s)[\varphi_{e,s}(x) \downarrow]$, so $\varphi_e = f_{\langle e, i \rangle}$ for some i .

(iii) \implies (i). Let $f(e, x)$ be a function of degree $\leq \mathbf{a}$ such that every computable function is an f_e . Define $g(x) = \max\{f_e(x) : e \leq x\}$. Then g is dominant so $\mathbf{a}' \geq \mathbf{0}''$ by Martin's Theorem 2.6.1.

(iv) \implies (i). By Theorem ?? and Exercise ?? we have

$$(\text{Tot}, \overline{\text{Tot}}) \leq_m (\text{Tot}, \overline{\text{Ext}})$$

via some computable function g . Assume f has degree $\leq \mathbf{a}$ and that the f_e 's are exactly the computable characteristic functions. Then for all e ,

$$\begin{aligned} e \in \text{Tot} &\iff (\exists i) [f_i \text{ extends } \varphi_{g(e)}], \\ &\iff (\exists i) (\forall x) (\forall y) (\forall s) [\varphi_{g(e),s}(x) = y \implies f_i(x) = y]. \end{aligned}$$

Thus, $\text{Tot} \in \Sigma_2^A$. But $\text{Tot} \in \Pi_2 \subseteq \Pi_2^A$. Therefore, $\text{Tot} \in \Delta_2^A$. Hence, $\mathbf{0}'' \leq \mathbf{a}'$ by the Relativized Post's Theorem ??.

(v) \implies (i). (The following resembles the proof that the computable functions are not uniformly computable.) Assume that \mathbf{a} is c.e. but (i) is false and $f(e, x)$ is any function of degree $\leq \mathbf{a}$. We must construct a $\{0, 1\}$ -valued computable function $h \neq f_e$ for all e . Since $\text{deg}(f) \leq \mathbf{0}'$ there is a computable function $\hat{f}(e, x, s)$ such that $f(e, x) = \lim_s \hat{f}(e, x, s)$ and a modulus function $m(e, x)$ for \hat{f} which has degree $\leq \mathbf{a}$ by the Modulus Lemma 2.3.7. Let $p(x) = \max\{m(e, \langle e, x \rangle) : e \leq x\}$. Since $\text{deg}(p) \leq \mathbf{a}$ and (i) fails there is a computable function $q(x)$ which $p(x)$ fails to dominate. Define $h(\langle e, x \rangle) = 1 \div \hat{f}(e, \langle e, x \rangle, q(x))$. Then h is a computable function and $h(\langle e, x \rangle) \neq f_e(\langle e, x \rangle)$ whenever $x \geq e$ and $q(x) \geq p(x)$. (See Exercise ?? that the hypothesis \mathbf{a} c.e. is necessary.) in the proof of (v) \implies (i). □

Corollary 2.8.3 (Jockusch). *If $\mathbf{a} < \mathbf{0}'$ is c.e. then the class of c.e. sets of degree $\leq \mathbf{a}$ is not \mathbf{a} -uniform.*

Proof. Assume \mathbf{a} is a counterexample. Then the computable sets are \mathbf{a} -subuniform so $\mathbf{a}' = \mathbf{0}''$ by (v) \implies (i) of Theorem 2.8.2. However, since the c.e. sets of degree $\leq \mathbf{a}$ are \mathbf{a} -uniform, they are $\mathbf{0}'$ -uniform and so $\mathbf{a}'' = \mathbf{0}''$ by Corollary ??. □

2.9 Inverting the Jump

Note that for any degree \mathbf{a} , $\mathbf{0} \leq \mathbf{a}$ and hence $\mathbf{0}' \leq \mathbf{a}'$, *i.e.*, any jump is above $\mathbf{0}'$. Hence, the jump, viewed as a map on degrees, has range contained in $\{\mathbf{b} : \mathbf{b} \geq \mathbf{0}'\}$. The next theorem asserts that this map is *onto* the set $\{\mathbf{b} : \mathbf{b} \geq \mathbf{0}'\}$. A degree \mathbf{a} is called *complete* if $\mathbf{a} \geq \mathbf{0}'$, so the result also gives a criterion for \mathbf{a} being complete.

Theorem 2.9.1 (Friedberg Completeness Criterion). *For every degree $\mathbf{b} \geq \mathbf{0}'$ there is a degree \mathbf{a} such that $\mathbf{a}' = \mathbf{a} \cup \mathbf{0}' = \mathbf{b}$.*

Proof. Fix $B \in \mathbf{b}$. We shall construct f , the characteristic function of A , by finite initial segments $\{f_s\}_{s \in \omega}$ using a B -computable finite extension construction.

Stage $s = 0$. Set $f_0 = \emptyset$.

Stage $s + 1 = 2e + 1$. (We decide whether $e \in A'$.) We meet the following requirement J_e which is called *forcing the jump* $\Phi_e^A(e)$,

$$(2.14) \quad J_e : (\exists \sigma \subset A) [\Phi_e^\sigma(e) \downarrow \quad \vee \quad (\forall \tau \supseteq \sigma) [\Phi_e^\tau(e) \uparrow]].$$

(If A meets R_e we say that A *forces the jump* on argument e .) Given f_s , use a \emptyset' -oracle to test whether

$$(2.15) \quad (\exists \sigma) (\exists t) [f_s \subset \sigma \quad \& \quad \Phi_{e,t}^\sigma(e) \downarrow].$$

(Note that the matrix is computable, so (2.15) is a Σ_1 condition and thus is computable $\mathbf{0}'$.) If (2.15) is satisfied, let f_{s+1} be the first such σ in the standard enumeration of L of the Master Enumeration Theorem ???. If not, set $f_{s+1} = f_s$.

Stage $s + 1 = 2e + 2$. (We code $B(e)$ into A .) Let $n = lh(f_s)$. Define $f_{s+1}(n) = B(e)$. (This completes the construction.)

Now $f = \cup_s f_s$ is total since $lh(f_{2e}) \geq e$. Let $A = \{x : f(x) = 1\}$, and $\mathbf{a} = \deg(A)$. The construction is B -computable since at odd stages we use a \emptyset' -oracle, at even stages we use a B -oracle, and $\emptyset' \leq_T B$. Since $A \oplus \emptyset' \leq_T A'$ for any A , to prove $A' \equiv_T B \equiv_T A \oplus \emptyset'$ it suffices to prove the following two lemmas.

Lemma 2.9.2. $A' \leq_T B$.

Lemma 2.9.3. $B \leq_T A \oplus \emptyset'$.

Proof of Lemma 2.9.2. Since the construction is B -computable, the sequence $\{f_s\}_{s \in \omega}$ is also B -computable. To decide whether $e \in A'$, we B -computably determine using $\emptyset' \leq_T B$ whether (2.15) holds for f_s , $s = 2e$. If so, $e \in A'$ and otherwise $e \notin A'$ because no $\sigma \supseteq f_s$ has $\Phi_e^\sigma(e)$ defined.

Proof of Lemma 2.9.3. It suffices to show that $\{f_s\}_{s \in \omega}$ is an $(A \oplus \emptyset')$ -computable sequence since $B(e)$ is the last value of f_{2e+2} . The proof is by induction on s . Given $\{f_s : s \leq 2e\}$, use a \emptyset' -oracle to compute f_{2e+1} . If $n = lh(f_{2e+1})$ then $f_{2e+2} = f_{2e+1} \hat{\ } A(n)$, so f_{2e+2} can be computed from f_{2e+1} using an A -oracle. \square

Theorem 2.9.4 (Relativized Friedberg Completeness Criterion). *For every degree \mathbf{c} ,*

$$F_1(\mathbf{c}) : (\forall \mathbf{b}) [\mathbf{b} \geq \mathbf{c}' \implies (\exists \mathbf{a}) [\mathbf{a} \geq \mathbf{c} \ \& \ \mathbf{a}' = \mathbf{a} \cup \mathbf{c}' = \mathbf{b}]].$$

Proof. Do the proof of Theorem 2.9.1 with \mathbf{c} and \mathbf{c}' in place of $\mathbf{0}$ and $\mathbf{0}'$. \square

Corollary 2.9.5. *For every $n \geq 1$, and every degree \mathbf{c} ,*

$$F_n(\mathbf{c}) : (\forall \mathbf{b}) [\mathbf{b} \geq \mathbf{c}^{(n)} \implies (\exists \mathbf{a}) [\mathbf{a} \geq \mathbf{c} \ \& \ \mathbf{a}^{(n)} = \mathbf{a} \cup \mathbf{c}^{(n)} = \mathbf{b}]].$$

Proof. To prove $(\forall \mathbf{c}) F_n(\mathbf{c})$ holds for all $n \geq 1$, use induction on n and the fact that $F_{n+1}(\mathbf{c})$ follows from $F_n(\mathbf{c})$ and $F_1(\mathbf{c}^{(n)})$. \square

Although Theorem 2.9.1 demonstrates a pleasant property of the jump operator, it also demonstrates an unpleasant property, namely that the jump map is not 1:1. To see this, apply Theorem 2.9.1 with $\mathbf{b} = \mathbf{0}''$ to obtain \mathbf{a} such that $\mathbf{a}' = \mathbf{a} \cup \mathbf{0}' = \mathbf{0}''$. Clearly $\mathbf{a} \mid \mathbf{0}'$ yet they have the same jump. It is also possible to have $\mathbf{a} < \mathbf{b}$ and $\mathbf{a}' = \mathbf{b}'$.

2.10 Finite Forcing and Generic Sets

We assume all strings $\sigma \in 2^{<\omega}$ have been effectively numbered. We identify a c.e. set of strings $V_e \subseteq 2^{<\omega}$ with the corresponding c.e. set of integers and use the same notation V_e .

Definition 2.10.1. Let $\mathbb{V} = \{V_e\}_{e \in \omega}$ be a u.c.e. sequence of c.e. sets $V_e \subseteq 2^{<\omega}$.

(i) We say $f \in 2^\omega$ *forces* V_e if for all e we satisfy the following *forcing* requirement F_e ,

$$(2.16) \quad F_e : (\exists \sigma \subset f) [\sigma \in V_e \ \vee \ (\forall \rho \supset \sigma) [\rho \notin V_e]].$$

If σ satisfies the matrix of F_e we say that σ *forces* F_e and any $f \supset \sigma$ also *forces* F_e .

(ii) We say f is *generic with respect to* $\mathbb{V} = \{V_e\}_{e \in \omega}$ (i.e. \mathbb{V} -*generic*) if f forces V_e for every $e \in \omega$.

(iii) We say f is *1-generic* if it is generic with respect to $\{W_e\}_{e \in \omega}$. (The term “1-generic” refers to the fact that f is forcing Σ_1 statements.)

Being generic with respect to V_e , *i.e.*, requirement F_e of (2.16), is more flexible and applicable than the notion of forcing the jump J_e of (2.14). We now prove that the two are equivalent so that we can use the former for most purposes.

Theorem 2.10.2 (Jockusch-Posner). *A set A is 1-generic iff A satisfies every jump requirement $\{J_e\}_{e \in \omega}$ of (2.14) of the Friedberg Completeness Criterion Theorem 2.9.1.*

Proof. (\implies). Define $W_{h(e)} = \{\sigma : \Phi_e^\sigma(e) \downarrow\}$. Now A forces $W_{h(e)}$. Therefore, A forces the jump $\Phi_e^A(e)$, namely satisfies the requirement J_e .

(\impliedby). Define a computable function $f(e)$ by

$$\Phi_{f(e)}^\sigma(z) = \begin{cases} 1 & \text{if } (\exists s \leq |\sigma|) (\exists \tau \subseteq \sigma) [\tau \in W_{e,s}], \\ \text{undefined} & \text{otherwise.} \end{cases}$$

If A satisfies requirement $J_{f(e)}$ of (2.14), then A forces $V_e = \{\sigma : \Phi_{f(e)}^\sigma(f(e)) \downarrow\}$, and A forces W_e . \square

Part III

Computable Content of
Vaughtian Models:
Four Lectures by Robert
Soare

Appendix A

Model Theory References

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Appendix B

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¹Many papers, Kleene [1943, p. 73], 1987, 1987b, Davis [1965, p. 72], Post [1943, p. 200], and others, mistakenly refer to this paper as “[Turing, 1937],” perhaps because the volume **42** is 1936-37 covering 1936 and part of 1937, or perhaps because of the two page minor correction 1937. Others, such as Kleene 1952, 1981, 1981b, Kleene and Post [1954, p. 407], Gandy 1980, Cutland 1980, and others, correctly refer to it as “[1936],” or sometimes “[1936-37].” The journal states that Turing’s manuscript was “Received 28 May, 1936–Read 12 November, 1936.” It appeared in two sections, the first section of pages 230–240 in Volume **42**, Part 3, issued on November 30, 1936, and the second section of pages 241–265 in Volume **42**, Part 4, issued December 23, 1936. No part of Turing’s paper appeared in 1937, but the two page minor correction 1937 did. Determining the correct date of publication of Turing’s work is important to place it chronologically in comparison with Church 1936, Post 1936, and Kleene 1936.

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