Degenerate Delay-Capacity Trade-offs in Ad Hoc Networks with Brownian Mobility

Xiaojun Lin, Gaurav Sharma, Ravi R. Mazumdar, and Ness B. Shroff

***Abstract*— There has been significant recent interest within the networking research community to characterize the impact of mobility on the capacity and delay in mobile ad hoc networks. In this paper, we study the fundamental trade-off between the capacity and delay for a mobile ad hoc network under the Brownian motion model. We show that the 2-hop relaying scheme proposed by Grossglauser and Tse (2001), while capable of achieving $\Theta(1)$ per-node capacity, incurs an expected packet delay of $\Omega(\log n/\sigma_n^2)$, where σ_n^2 is the variance parameter of the Brownian motion model. We then show that in order to reduce the delay by any significant amount, one must be ready to accept a per-node capacity close to static ad hoc networks. In particular, we show that under a large class of scheduling and relaying schemes, if the mean packet delay is $O(n^\alpha/\sigma_n^2)$, for any $\alpha < 0$, then the per-node capacity must be $O(1/\sqrt{n})$. This result is in sharp contrast to other results that have recently been reported in the literature.**

I. Introduction

Since the seminal work of Gupta and Kumar [1], there has been a lot of interest in characterizing the capacity region of ad hoc networks. A major contribution in this direction was made in [2], where the authors show that mobility can significantly increase the capacity of an ad hoc network. In particular, the authors proposed a 2-hop relaying scheme, and showed that it can achieve $\Theta(1)$ per-node capacity[1]. However, the delay related issues were not considered in [2]. In fact, it is pointed out in [2] that the 2-hop relaying scheme could potentially incur an unbounded delay.

There has been substantial recent work on the joint characterization of the delay and capacity in the mobile ad hoc networks [3], [4], [5], [6], [7], [8]. The type of node mobility studied in the literature includes the so-called *i.i.d* mobility [4], [7], [8], random way-point mobility [5], [6], Brownian motion [3], [6], and Markovian mobility [4]. The results in these works are of a similar flavor. They show that it is possible to achieve a bounded expected delay and $\Theta(1)$ per-node capacity, using scheduling schemes that are variants of the Grossglauser-Tse 2-hop relaying scheme. These works then report *trade-offs* between the capacity and delay; i.e., when one is willing to sacrifice the per-node capacity, the delay can be correspondingly reduced. The capacity-delay trade-offs are achieved either by means of introducing some redundancy in the 2-hop relaying scheme [4], [5], [6], or by adjusting the cell size [3], or both [7], [8]. All the previous works report "smooth" capacity-delay trade-offs. For instance, under the random way-point mobility model, the authors of [5] report a scheme that can achieve $\Theta(n^{\alpha-1})$ per-node capacity at $\Theta(n^\alpha)$ delay, for any $\alpha \in [1/2, 1]$. Hence, by reducing α to $\alpha - \epsilon$, $\epsilon > 0$, one can reduce the delay by a factor of $\Theta(n^\epsilon)$, at the cost of reducing the capacity by the same factor. Similar type of smooth trade-offs have been reported under the *i.i.d.* mobility model as well [4], [7], [8].

In this paper, we study the trade-off between the capacity and delay under the Brownian motion model [3], [6]. We note that the results of this paper can easily be extended to other related mobility models such as the random walk mobility model [9] and the Markovian mobility model [4]. This is because the Brownian motion model can be viewed as a limiting case of these other mobility models. Interestingly, our results are in complete disagreement with some recently reported results in the literature for the Brownian motion model or its variants [3], [6]. We show that there is virtually no trade-off between the capacity and delay under the Brownian motion model (see Fig. 1). In particular, we show that under a large class of scheduling and relaying schemes, in order to achieve a delay of $\Theta(n^\alpha/\sigma_n^2)$ for any $\alpha < 0$, where σ_n^2 is the variance parameter of the Brownian motion, the per-node capacity must be $O(1/\sqrt{n})$. Further, we show that the 2-hop relaying scheme proposed by Grossglauser and Tse [2], while capable of achieving $\Theta(1)$ per-node capacity, incurs an expected packet delay of $\Omega(\log n/\sigma_n^2)$. Note that one can achieve $\Theta(1/\sqrt{n\log n})$ per-node capacity for static wireless networks using multi-hop transmission [1]. Thus, in order to achieve any significant capacity gains by exploiting mobility, one must be ready to tolerate huge delays, roughly on the order of $\Theta(1/\sigma_n^2)$, which is close to the delay at $\Theta(1)$ per-node capacity.

We summarize the main contributions of this paper below:

- We rigorously show that, for a large class of scheduling and relaying schemes, the achievable capacity-delay trade-off under the Brownian motion model is *degenerate*. If one attempts to achieve a per-node capacity that is more

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[1]We use the following notation throughout:

$$
\begin{aligned}
f(n) = o(g(n)) \quad &\leftrightarrow \quad \lim_{n\to\infty} \frac{f(n)}{g(n)} = 0,\\
f(n) = O(g(n)) \quad &\leftrightarrow \quad \limsup_{n\to\infty} \frac{f(n)}{g(n)} < \infty,\\
f(n) = \omega(g(n)) \quad &\leftrightarrow \quad g(n) = o(f(n)),\\
f(n) = \Omega(g(n)) \quad &\leftrightarrow \quad g(n) = O(f(n)),\\
f(n) = \Theta(g(n)) \quad &\leftrightarrow \quad f(n) = O(g(n)) \text{ and } g(n) = O(f(n)).
\end{aligned}
$$

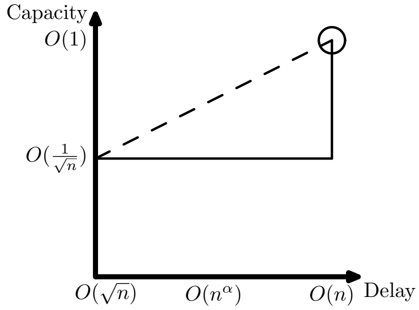

Fig. 1. The degenerate delay-capacity trade-off under the Brownian motion model (the solid line) compared with the "smooth" delay-capacity trade-off under the random way-point mobility model (the dashed line) reported in [5]. We have chosen $\sigma_n^2 = 1/n$ and ignored all logarithmic terms in the figure.

than a logarithmic factor above that of static wireless networks, one must be ready to incur huge delays, roughly on the order of $\Theta(1/\sigma_n^2)$.

- We consider the class of generalized 2-hop relaying schemes and show that they incur a delay of $\Omega(\log(1/a_n)/\sigma_n^2)$, where a_n is related to the concept of the capture neighborhood and the forwarding neighborhood (see Section V). Most scheduling schemes studied in the literature fall into this class with $a_n = O(n^\alpha), \alpha < 0$. Hence, the delay incurred is no less than $\Omega(\log n/\sigma_n^2)$. A special case is the 2-hop relaying scheme of [2] which achieves $\Theta(1)$ per-node capacity and incurs an expected packet delay of $\Omega(\log n/\sigma_n^2)$.
- In related works in the literature, a crucial step in characterizing the delay of the 2-hop relaying scheme is to compute the second moment of the so-called *inter-meeting time* (see Section VI and the Appendix). A technical contribution of this paper is to provide a rigorous analysis of the inter-meeting time under the Brownian motion model, which yields more accurate results than previous works [3].

It is interesting to compare our results for the Brownian motion model with the results in [5] for the random way-point mobility model. Note that both these models are continuous mobility models (i.e., the motion of the nodes is continuous), and both preserve the uniform distribution of the nodes at all times, that is, an initial uniform distribution of the nodes implies that the nodes remain uniformly distributed at all times. However, the capacity-delay relationship under these two models is significantly different. In particular, there exists a smooth trade-off between the delay and capacity under the random way-point mobility model, whereas there is virtually no trade-off under the Brownian motion model. We believe that this difference is a revelation of the fundamental difference in the mobility pattern under these two models. In the random way-point mobility model, nodes move "purposefully," that is, during each trip, a node has some target position in mind (chosen uniformly on the sphere) and it moves along a straight-line path, with no "wandering" at all. Thus the nodes can cover large distances in relatively short time under the random way-point mobility model. This is in contrast to the Brownian motion model, where the nodes always wander around like "drunkards," staying in a local neighborhood for large duration of time. It is therefore intuitive to believe that reducing the mobility delay under the Brownian motion model would be much more difficult.

The rest of the paper is organized as follows. In the next section, we describe our network model and the Brownian motion model, followed by some basic properties of the Brownian motion model in Section III. We then derive our main result in Section IV, showing that there is virtually no trade-off under the Brownian motion model. In Section V, we analyze the delay performance of generalized 2-hop relaying schemes. We provide a discussion of the related works in Section VI, and finally end this paper with some concluding remarks in Section VII.

II. Network and Mobility Model

We consider an ad hoc network with n nodes moving on the surface of a unit sphere[2]. For simplicity, we assume that each node, say i, communicates with a single destination node, say $d(i)$, and that the mapping $i \mapsto d(i)$ is bijective. We assume a uniform traffic pattern, i.e., each *source* generates traffic at the same rate of λ bits per second for its destination. We further assume that the packet arrival processes at each node is independent of the node mobility process. The communication between any *source-destination* pair can possibly be via multiple other nodes acting as relays. That is, the *source* could either send a packet directly to the *destination*, if possible; or, it could forward the packet to one or more *relay* nodes; the relay nodes could themselves forward the packet to other relay nodes; and finally a relay node or the source node itself could deliver the message to the destination.

We assume the following Protocol Model from [1] that governs the radio transmissions between nodes. Let W be the bandwidth of the system in bits per second. Let X_t^i denote the position of the node i, for $i = 1...n$, at time t. Node i can communicate directly with another node j at a rate of W bits per second at time t, if and only if, the following interference constraint is satisfied [1]:

$$d(X_t^k, X_t^j) \geq (1+\Delta) d(X_t^i, X_t^j) \quad (1)$$

for every other node $k \neq i, j$ that is simultaneously transmitting. Here, Δ is some positive number, and $d(x, y)$ denotes the Euclidean distance between points $x, y \in \mathbb{R}^3$. Note that when the unit of information transmitted is a packet, the above interference constraint must be satisfied over the entire duration of the packet transmission from node i to node j.

Let S denote the surface of the unit sphere. We assume that nodes move independently on S according to a Brownian motion model as in [6]. (A similar Brownian motion model on a 2-d torus is also considered in [3].) It is easier to describe the motion of each node using the spherical coordinates. Let θ_t and ϕ_t denote the colatitude and longitude, respectively, of the position of a particular node at time t ($0 \leq \theta_t \leq \pi$ and $0 \leq \phi_t < 2\pi$). When a node moves according to the Brownian motion model on the unit sphere S, the (Itô)

[2]Note that changing the shape of the area from the surface of a sphere to a square or a circle will not change the main results of this paper.

stochastic differential equations for the process (θ_t, ϕ_t) are given by [10]:

$$d\theta_t = \sigma_n dB_t + \frac{\sigma_n^2}{2\tan\theta_t}dt, \tag{2}$$

and

$$d\phi_t = \frac{\sigma_n}{\sin\theta_t}dB'_t, \tag{3}$$

where B_t and B'_t are independent standard one-dimensional Brownian motions (i.e., with variance 1). We call σ_n^2 the *variance* of the Brownian Motion described in (2) and (3). For analysis, it is useful to project each node's position onto the z-axis. Substituting $Y_t = \cos\theta_t$ into (2), and using Itô's Lemma, we obtain

$$dY_t = -\sigma_n^2 Y_t dt - \sigma_n\sqrt{1-Y_t^2}dB_t. \tag{4}$$

Note that Y_t is a diffusion process with *drift coefficient* $-\sigma_n^2 Y_t$ and *diffusion coefficient* $\sigma_n^2(1-Y_t^2)$. We assume that the initial positions of the nodes are *i.i.d.* and uniformly distributed on the unit sphere. This implies that the positions of the nodes will remain uniform at all times.

III. Basic Properties of Brownian Motion on a Sphere

In this section, we summarize some basic properties of Brownian motion on a sphere, which will be used later on. Let us consider the motion of a single node. Let X_t denote its position at time t, which can be represented using the spherical coordinates (θ_t, ϕ_t). Let $Y_t = \cos\theta_t$ be the projection of the node's position on the z-axis, and recall that Y_t is governed by (4).

We first cite the following result from [10] concerning the expected travel time of Y_t:

Lemma 1: Let $-1 < a < x < 1$. Then, in traveling from x to a, Y_t takes an expected time, $V_a(x)$, given by:

$$V_a(x) = \frac{2}{\sigma_n^2}\log\left(\frac{1+x}{1+a}\right).$$

A. The First Hitting Time

The first concept we study is the *first hitting time*. Let A be an arbitrary region on the sphere. We have the following definition:

Definition 1: The first hitting time of A, denoted by T_A, is the first time instant at which X_t enters A; i.e., $T_A = \inf\{t \geq 0 : X_t \in A\}$.

Let Π denote the uniform distribution on the unit sphere S, and denote by $\mathbb{E}_\Pi$ the expectation conditioned on X_0 being distributed according to Π. Let $A = \{x \in S : d_S(x,y) \leq a_n\}$, where y is an arbitrary point on S, d_S denotes the geodesic distance on the sphere, and $a_n > 0$. For $a_n \downarrow 0$, as $n \to \infty$, we have the following result:

Lemma 2: $\mathbb{E}_\Pi[T_A] = \Theta(\log(1/a_n)/\sigma_n^2)$.

Proof: In view of the symmetry of the sphere, taking y to be the south pole (i.e., the bottom most point of S that corresponds to $\theta = \pi$) entails no loss of generality. Now, for $x \in A$, we have $\mathbb{E}[T_A|X_0 = x] = 0$. For $x \notin A$, let z_x denote its z co-ordinate. Note that the radius of S is $\frac{1}{2\sqrt{\pi}}$. The first time that X_t enters A is also the first time that Y_t travels from z_x to $-\cos(2a_n\sqrt{\pi})$. Using Lemma 1, we obtain

$$\mathbb{E}[T_A|X_0 = x] = \frac{2}{\sigma_n^2}\log\left(\frac{1+z_x}{1-\cos(2a_n\sqrt{\pi})}\right).$$

Integrating over all possible positions of the point x on S, and using the fact that x is uniformly distributed on S, we obtain

$$\mathbb{E}_\Pi[T_A] = \int_{\theta=2\sqrt{\pi}a_n}^{\pi}\frac{\sin\theta}{\sigma_n^2}\log\left(\frac{1+\frac{\cos\theta}{2\sqrt{\pi}}}{1-\cos(2a_n\sqrt{\pi})}\right)d\theta,$$

and the result follows after straightforward calculations. ■

Remark 1: If $a_n = n^\alpha, \alpha < 0$, then by Lemma 2, the first hitting time is always $\Theta(\log n/\sigma_n^2)$, regardless of the value of α. Even if we take $a_n = \sqrt{\pi}/4$, which means that the set A covers about half of the sphere, the first hitting time is still $\Theta(1/\sigma_n^2)$. Hence, the first hitting time changes very little when the size of the set A is increased. This result reveals the fundamental difference between the mobility pattern under the Brownian motion model and that under other mobility models (such as the *i.i.d* mobility model [8] and the random way-point mobility model [5]). In these other models, the first hitting time for a set A decreases substantially when the size of the set A is increased. On the other hand, Lemma 2 is not completely surprising given the fact that, under the Brownian motion model, the node always wanders around like a "drunkard." Therefore, it is very difficult for the node to move towards any given destination.

B. The First Exit Time

The second concept that we study is the *first exit time*.

Definition 2: Let $A = \{x \in S : d_S(x,y) \leq a_n\}$. The first exit time for the region A, denoted τ_A, is the first instant of time at which the Brownian motion started at y (the center of A) exits A, i.e.,

$$\tau_A = \inf\{t \geq 0 : X_0 = y, X_t \notin A\}.$$

Assuming $a_n \to 0$ as $n \to \infty$, we have the following result:

Lemma 3: $\mathbb{E}[\tau_A] = \Theta(a_n^2/\sigma_n^2)$.

Proof: Using the symmetry of the sphere, we can set y to be the north pole of S (i.e., the top most point of S that corresponds to $\theta = 0$). It then follows that $\mathbb{E}[\tau_A]$ is the expected travel time of Y_t from 1 to $\cos(2a_n\sqrt{\pi})$. Applying Lemma 1 and performing some straightforward calculations, the result follows. ■

Remark 2: From the above discussion it is clear that under the Brownian motion model a node requires $\Theta(a_n^2/\sigma_n^2)$ time to move a radial distance of a_n. Thus the time a Brownian motion process spends in a region is in proportion to the area of the region. This also points to the well-known result that the path of Brownian motion is nowhere differentiable [11, p380]. Hence, it is inappropriate to define the "velocity" of a node that is moving in accordance with the Brownian motion model.

IV. The Degenerate Capacity-Delay Trade-off

In this section, we show that there is virtually no trade-off between the delay and the capacity under the Brownian motion model. Specifically, we will show that whenever the delay constraint is $O(n^\alpha/\sigma_n^2)$ for any $\alpha < 0$, the per-node capacity is $O(1/\sqrt{n})$. In order to provide the readers with the main insight underlying this result, we use a slightly different network model in this section. We assume that the nodes are executing independent Brownian walks within a *unit square on a plane* (instead of on a *unit sphere*). This change simplifies the exposition substantially. Nonetheless, as we will see later, our results hold for a unit sphere as well.

Consider n nodes on a unit square centered at the origin, executing independent two-dimensional Brownian motions within the square. As will become clear soon, our result does not depend on how the boundary condition is handled: the Brownian motion could either be reflected at the boundary, or wrap around the boundary (like the 2-d torus model in [3]).

In order to prove the main result of this section, namely, that the delay-capacity trade-off under the Brownian motion model is degenerate, we need some supporting results. The main idea behind the proof is that, if the delay is $O(n^\alpha/\sigma_n^2)$ for $\alpha < 0$, then the contribution due to node mobility in the packet delivery is likely very small. Hence, in order to deliver the packet to the destination, relaying over order $\Theta(1)$ distance is required, which results in the throughput capacity being $O(1/\sqrt{n})$.

We start by showing that, if the delay is $O(n^\alpha/\sigma_n^2)$ for $\alpha < 0$, then the contribution due to node mobility in the packet delivery is likely very small.

Let **SQ**(c_n) be the square centered at the origin with length c_n (see Fig. 2). Suppose there are $k_n \le n$ nodes, starting at the origin at time 0. Each node then moves according to a two-dimensional Brownian motion with variance σ_n^2, which can be viewed as the composition of two independent one-dimensional Brownian motion along the x-axis and the y-axis, respectively, each having a variance of $\sigma_n^2/2$. Let $p_{k_n}(c_n, t_n)$ denote the probability of the event that one or more of the k_n nodes ever exit the square **SQ**(c_n) within time t_n. We have the following result concerning $p_{k_n}(c_n, t_n)$:

Lemma 4: If there exists $N_0 < \infty$ such that

$$\frac{c_n^2}{t_n} \ge 8\sigma_n^2 \log n, \text{ for } n \ge N_0, \tag{5}$$

then

$$\lim_{n\to\infty} p_{k_n}(c_n, t_n) = 0.$$

The following corollary is an immediate consequence of Lemma 4.

Corollary 1: If

$$\liminf_{n\to\infty} c_n \log n = c > 0$$

$$\limsup_{n\to\infty} \frac{\sigma_n^2 t_n}{n^\alpha} = c' < +\infty, \text{ for some } \alpha < 0 ,$$

then

$$\lim_{n\to\infty} p_{k_n}(c_n, t_n) = 0.$$

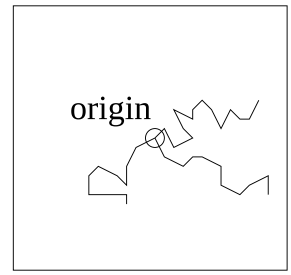

Fig. 2. k_n nodes at the origin

Hence within $O(n^\alpha/\sigma_n^2)$ time ($\alpha < 0$), none out of the k_n nodes can possibly travel an $\Theta(1/\log n)$ distance in *any* direction.

Proof: [Proof of Lemma 4] Consider an arbitrary node. Let X_t be its position at time t. Let B_t^x and B_t^y denote its x-coordinate and y-coordinate, respectively. Then B_t^x and B_t^y are independent one-dimensional Brownian motions with variance $\sigma_n^2/2$. Let $p(c_n, t_n)$ be the probability that this particular node ever exits the square **SQ**(c_n) within time t_n. Let

$$\tau_x^+ \triangleq \inf\{t \ge 0 : B_t^x = c_n/2\},$$
$$\tau_x^- \triangleq \inf\{t \ge 0 : B_t^x = -c_n/2\},$$

and let τ_y^+, τ_y^- be similarly defined with B_t^y in place of B_t^x. Using the union bound, and appealing to the symmetry of the two-dimensional Brownian motion, we obtain

$$p(c_n, t_n) \le \mathbf{P}\{\tau_x^+ \le t_n \text{ or } \tau_x^- \le t_n \text{ or } \tau_y^+ \le t_n \text{ or } \tau_y^- \le t_n\}$$
$$\le 4\mathbf{P}\{\tau_x^+ \le t_n\}.$$

Further, using the Reflection Principle for one-dimensional Brownian motion [11, p394], we have

$$\mathbf{P}\{\tau_x^+ \le t_n\} = 2\mathbf{P}\{B_{t_n}^x \ge c_n/2\}.$$

Since the distribution of $B_{t_n}^x$ is Gaussian with zero mean and variance $\sigma_n^2 t_n/2$, we have,

$$\mathbf{P}\{\tau_x^+ \le t_n\} = 2\int_{\frac{c_n}{\sqrt{2}\sigma_n\sqrt{t_n}}}^{\infty} \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{u^2}{2}\right) du.$$

Using the inequality,

$$\int_x^\infty \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{u^2}{2}\right) du \le \frac{1}{\sqrt{2\pi}} \int_x^\infty \frac{u}{x} \exp\left(-\frac{u^2}{2}\right) du$$
$$= \frac{1}{\sqrt{2\pi}x} \exp\left(-\frac{x^2}{2}\right),$$

we have,

$$\mathbf{P}\{\tau_x^+ \le t_n\} \le 2\sqrt{\frac{\sigma_n^2 t_n}{\pi c_n^2}} \exp\left[-\frac{c_n^2}{4\sigma_n^2 t_n}\right].$$

Using (5), we have

$$\mathbf{P}\{\tau_x^+ \le t_n\} \le \frac{1}{\sqrt{2\pi \log n}} \exp(-2\log n) = \frac{1}{n^2\sqrt{2\pi \log n}}.$$

Hence,

$$p(c_n, t_n) \le \frac{4}{n^2\sqrt{2\pi \log n}}.$$

Finally, since there are k_n nodes, each of them moves according to a two-dimensional Brownian Motion, we have

$$p_{k_n}(c_n, t_n) \leq k_n p(c_n, t_n) \leq \frac{4k_n}{n^2\sqrt{2\pi \log n}}.$$

Noting that $k_n \leq n$, the result then follows. ■

We now show that if each packet is relayed over $\Theta(1)$ distance (on an average), then the throughput capacity would be $O(1/\sqrt{n})$.

Consider a large enough time interval $\mathcal{T}$. The total number of packets communicated end-to-end between all source-destination pairs during the interval is $c_p\lambda n\mathcal{T}$, where $1/c_p$ is the number of bits per packet. Let h_p be the number of times the packet p is relayed, and let l_p^h, for $h = 1, ..., h_b$, denote the transmission range for the h-th relaying. We have the following result:

Lemma 5: Suppose that there exists a constant $c > 0$, such that on average each packet is relayed over a total distance no less than c, i.e.,

$$\frac{\sum_{p=1}^{c_p\lambda nT}\sum_{h=1}^{h_p} l_p^h}{c_p\lambda n\mathcal{T}} \geq c, \tag{6}$$

then

$$\lambda \leq O(1/\sqrt{n}).$$

Proof: We use $d(x, y)$ to denote the Euclidean distance between positions x and y within the unit square. Let X^i denote the position of node i, for $i = 1, ..., n$. Consider nodes i, j transmitting directly to nodes k and l, respectively, at time t. Then, under the Protocol Model, in order for the transmissions to be successful, the following inequalities must hold at the time of transmission:

$$d(X^j, X^k) \geq (1 + \Delta)d(X^i, X^k)$$
$$d(X^i, X^l) \geq (1 + \Delta)d(X^j, X^l).$$

Hence,

$$d(X^j, X^i) \geq d(X^j, X^k) - d(X^i, X^k) \geq \Delta\, d(X^i, X^k).$$

Similarly,

$$d(X^i, X^j) \geq \Delta\, d(X^j, X^l).$$

Therefore,

$$d(X^i, X^j) \geq \frac{\Delta}{2}(d(X^i, X^k) + d(X^j, X^l)).$$

That is, disks of radius $\frac{\Delta}{2}$ times the transmission range centered at the transmitter are disjoint from each other[3]. We can therefore measure the radio resources that each transmission consumes by the areas of these disjoint disks. Note that the total area of the square is 1; for each of these disks, at least $1/4$ of it must lie inside the unit square; and each relaying of a packet lasts $\frac{1}{c_pW}$ amount of time. Thus,

$$\frac{1}{4}\sum_{p=1}^{c_p\lambda nT}\sum_{h=1}^{h_p}\pi\left[\frac{\Delta}{2}l_p^h\right]^2 \leq c_pW\mathcal{T}. \tag{7}$$

[3]A similar observation is used in [1] except that they take a receiver point of view.

By Cauchy-Schwarz Inequality,

$$\left[\sum_{p=1}^{c_p\lambda nT}\sum_{h=1}^{h_p} l_p^h\right]^2 \leq \left[\sum_{p=1}^{c_p\lambda nT}\sum_{h=1}^{h_p}(l_p^h)^2\right]\left[\sum_{p=1}^{c_p\lambda nT}\sum_{h=1}^{h_p} 1\right]. \tag{8}$$

Further, since there are at most n simultaneous transmissions at any given time in the network, we have

$$\sum_{p=1}^{c_p\lambda nT} h_p \leq c_pW\mathcal{T}n. \tag{9}$$

Therefore,

$$\begin{aligned}
\frac{16c_pW\mathcal{T}}{\pi\Delta^2} &\geq \sum_{p=1}^{c_p\lambda nT}\sum_{h=1}^{h_p}(l_p^h)^2 && \text{(using (7))}\\
&\geq \frac{\left[\sum_{p=1}^{c_p\lambda nT}\sum_{h=1}^{h_p} l_p^h\right]^2}{\left[\sum_{p=1}^{c_p\lambda nT} h_p\right]} && \text{(using (8))}\\
&\geq \frac{(c_p\lambda n\mathcal{T}c)^2}{c_pW\mathcal{T}n} && \text{(using (6) and (9)).}
\end{aligned}$$

Hence,

$$\lambda \leq \sqrt{\frac{16W^2}{\pi\Delta^2c^2}}\frac{1}{\sqrt{n}}.$$

■

We are now ready to prove the main result of this section. We first define a general class of scheduling polices that we plan to study. Note that at each time and for each packet p that has not been delivered to its destination yet, a scheduling policy essentially needs to make the following two types of decisions:

- *Replication:* The scheduler needs to decide whether to replicate the packet p to other relay nodes that do not have the packet yet. If yes, the scheduler needs to decide how to schedule radio transmissions to forward the packet p to these new relay nodes. Note that by *replication* we mean packet duplication; i.e., creating redundant copies of the packet. This is different from *capture* (to be defined next) where the number of copies of the packet does not increase.
- *Capture:* The scheduler needs to decide whether to deliver the packet p to the destination immediately, possibly using multi-hop transmission. If yes, the scheduler needs to choose one relay node (possibly the source) that has a copy of packet p and schedule radio transmissions to forward the packet to the destination. When this happens successfully, we say that the chosen relay node has successfully *captured* the destination of the packet p, or a successful capture has occurred for the packet p.

Remark 3: Although our model does allow for other less intuitive alternatives, in a typical scheduling policy a successful *capture* usually occurs when a relay node holding the packet comes within a small area around the destination node, so that fewer resources are needed to forward the packet to the destination. For example, a relay node could enter a disk of a certain radius around the destination, or a relay node could enter the same cell as the destination. We call such an area the *capture neighborhood*. The purpose of *replication* is to reduce the time before a successful *capture* occurs. With more nodes holding the packet p, the likelihood of one of them capturing the destination sooner is higher.

In this paper, we restrict our study to the class of scheduling policies that satisfy the following assumption:

Assumption A:

- Only the source of a packet is allowed to replicate the packet. That is, relay nodes holding a packet are *not* allowed to replicate it further.

Remark 4: Note that almost all scheduling schemes that have been proposed in the literature satisfy Assumption A [2], [3], [4], [5], [6], [7], [8]. If the relay nodes were allowed to replicate, then additional cooperation among the relay nodes would most likely be required (see, for example, the scheme in [12], where the relay nodes know the location of the static destination, and also have some knowledge of the future direction of other nodes' movement, based on which they can *cooperate* to make selective and more efficient replication toward the destination) in order to limit the number of replicas of a packet.

It is worthwhile to elaborate on Assumption A a little bit, since it may seem restrictive at first sight. First of all, we note that the notions of *replication* and *relaying* are different, even though both involve forwarding packets to other relay nodes. For example, when node i decides to *replicate* the packet p to node j, node i can either transmit the packet directly to node j, or use multi-hop transmission; i.e., node i can forward the packet to another node k, and let node k *forward* the packet to node j. (Node k may also keep the copy of the packet p, in which case, both nodes k and j are considered to receive the packet due to the *same* replication decision initiated by node i.) In this example, although both nodes i and k forward the packet p to other nodes, their roles are different. Node i is the one who *initiates* the replication, while node k is just *passively following* the instruction of node i to *relay* the packet to node j. Hence, Assumption A only prohibits relay nodes to *initiate* replications. Multi-hop relaying is still allowed in the replication process. (Multi-hop relaying is also allowed for the relay-to-destination communication, i.e., capture)

If we attempt to develop *distributed* scheduling policies where nodes make replication decisions and capture decisions without any knowledge of the decisions at other nodes, then restricting the replication decisions to the source node is a natural way to *control the number of copies of a packet in the system*. Note that excessive redundancy will reduce the system throughput substantially. The source node of a packet p is in the best position to control both the total number of replications for the packet and the number of relay nodes getting the packet for each replication.

We are now ready to prove the main result of this section:

Proposition 1: Let $\bar{D}$ denote the expected delay averaged over all packets and all source-destination pairs, and let λ denote the throughput of each source-destination pair. For any scheduling policy that satisfies Assumption A, if

$$\bar{D} \le O(n^\alpha/\sigma_n^2), \alpha < 0,$$

then

$$\lambda \le O(1/\sqrt{n}).$$

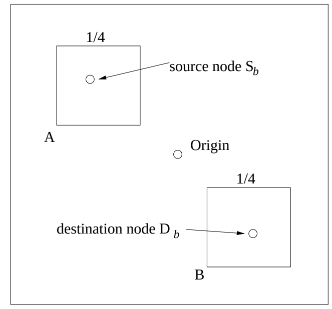

Fig. 3. There exists a constant fraction of packets that originate from nodes in A and are destined to nodes in B.

Proof: Consider squares A and B of length $1/4$, centered at $(-1/4, 1/4)$ and $(1/4, -1/4)$, respectively (see Fig. 3). Since the packet arrivals are independent of the positions of the mobile nodes, there will be a constant fraction f_0 of the packets that have their source nodes in square A and destinations in square B, at the time of arrival. (If the stationary distribution of the positions of the nodes are uniform, then $f_0 = \left(\frac{1}{4}\right)^4 = 1/256$. Otherwise, f_0 is still a positive constant independent of n.) Let Φ_{AB} denote this set of packets. In order to ensure that $\bar{D} \le O(n^\alpha/\sigma_n^2)$, the delay for the packets in Φ_{AB} has to be $O(n^\alpha/\sigma_n^2)$. Precisely, since $\bar{D} \le O(n^\alpha/\sigma_n^2)$, there exists some $N_0 > 0$ and $c_1 > 0$, such that

$$\bar{D} \le c_1 n^\alpha/\sigma_n^2, \text{ when } n \ge N_0. \tag{10}$$

Therefore, the delay of at least half of the packets in Φ_{AB} must be no greater than

$$t_n = \frac{2c_1}{f_0}\frac{n^\alpha}{\sigma_n^2}.$$

(Otherwise, the delay of the other half of the packets in Φ_{AB} must be greater than t_n. Because this other half contributes to at least $f_0/2$ fraction of all packets, the condition (10) will be violated.) Let Φ_{AB}^0 denote the set of packets in Φ_{AB} whose delay is no greater than t_n. Consider an arbitrary packet p which is in Φ_{AB}^0. Let S_p and D_p denote its source node and destination node, respectively. Fig. 4 shows a typical packet delivery. The source nodes S_p moves from position S_0 to U_1, and replicates the packet p to a relay node, say r_1, at position V_1, possibly using multi-hop transmission. The node r_1 then moves independently of S_p. The source node moves on to position U_2, where it replicates the packet p to one more relay node, say r_2, positioned at V_2, and so on. It is also possible to replicate the packet to more than one relay node at the same time (for example, we can take $U_1 = U_2$ if the source node replicates the packet to r_1 and r_2 at the same time). At time $t \le t_n$, a successful capture occurs, as one of the relay nodes holding the packet p (node r_2 in the case shown in Fig. 4) decides to forward the packet to its destination node D_p, which has moved from its initial position D_0 to the position D, at

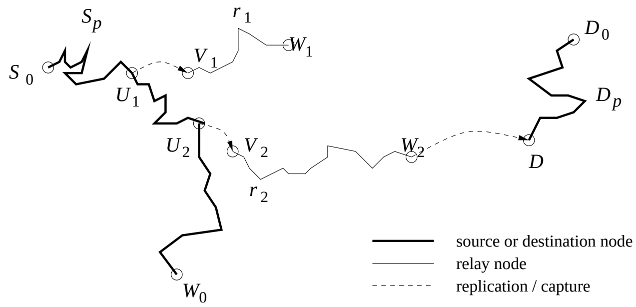

Fig. 4. How a typical packet p is delivered.

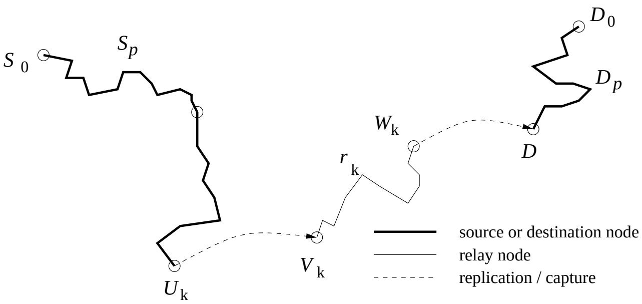

Fig. 5. The relay node r_k

time t. Let k_n denote the total number of relay nodes that hold packet p in this process, and let r_k, for $k = 1, 2, ..., k_n$, denote the k-th relay. Let U_k and V_k denote the position of the source node S_p and the position of the relay node r_k, respectively, at time of *replication*. Let W_k denote the position of the relay node r_k at the time of *capture* (see Fig. 5). Since the direct straight-line path is always the shortest path connecting any two points, we have, for any k,

$$d(S_0, U_k) + d(U_k, V_k) + d(V_k, W_k) + d(W_k, D) + d(D_0, D) \geq d(S_0, D_0).$$

Hence,

$$\begin{aligned} &d(U_k, V_k) + d(W_k, D) \\ &\geq d(S_0, D_0) - d(D_0, D) - d(S_0, U_k) - d(V_k, W_k). \end{aligned} \quad (11)$$

Since S_0 and D_0 are in the squares A and B, respectively,

$$d(S_0, D_0) \geq \frac{\sqrt{2}}{4}.$$

Further, each of the terms $d(D_0, D)$, $d(S_0, U_k)$, and $d(V_k, W_k)$, corresponds to the movement of a different node. There are at most n nodes involved in this process. By setting $c_n = 1/\log n$ in Corollary 1, we can see that, with probability approaching 1 as $n \to \infty$, all of the last three terms in (11) are no greater than $\sqrt{2}/\log n$, for all k. Therefore,

$$d(U_k, V_k) + d(W_k, D) \geq \frac{\sqrt{2}}{4} - 3\frac{\sqrt{2}}{\log n} \geq 1/4$$

for large enough n. Finally, let W_0 denote the position of the source node at time t. Then using a similar argument,

$$d(W_0, D) \geq d(S_0, D_0) - d(D, D_0) - d(S_0, W_0) \geq 1/4.$$

This shows that for each packet p in Φ^0_{AB}, the total distance that the packet p has to be relayed is at least $1/4$. Since Φ^0_{AB} contributes to at least $f_0/2$ fraction of all packets, on an average each packet must be relayed over a distance no less than $f_0/8 > 0$. Hence, by Lemma 5, the per-node throughput λ must be no larger than $O(1/\sqrt{n})$. ■

Remark 5: For the ease of exposition, we have shown the above results for Brownian motion on a plane. However, it is not difficult to see that the argument in Proposition 1 applies to Brownian motion on a unit sphere as well. In particular, in Lemma 4, if we choose $c_n = c/\log n$, the size of the square $\mathbf{SQ}(c_n)$ diminishes to zero as $n \to \infty$. Hence, the difference between such a square on a plane and that on a unit sphere vanishes. Therefore, both Corollary 1 and Proposition 1 hold for Brownian motion on a unit sphere as well.

Proposition 1 shows that the *capacity-delay trade-off under the Brownian motion model is degenerate*: For delay less than $O(n^\alpha/\sigma_n^2), \alpha < 0$, the per-node capacity is at most $O(1/\sqrt{n})$. Since one can achieve $\Theta(1/\sqrt{n \log n})$ per-node capacity for static wireless networks using multi-hop transmission [1], our result shows that whenever the delay is constrained to be less than $O(n^\alpha/\sigma_n^2), \alpha < 1$, Brownian mobility cannot improve the capacity by more than a logarithmic factor. Further, since the packet transmissions are usually carried out at a much faster time-scale than the node mobility, one could view the delay under the multi-hop scheduling (see [1]) as being *almost zero*. Earlier studies have shown that it is possible to achieve $\Theta(1)$ per-node capacity at roughly $\Theta(1/\sigma_n^2)$ delay under the Brownian motion model. Obviously, $\Theta(1)$ is an upper bound on the per-node capacity (under our network model). Hence, if we ignore the logarithmic terms, the capacity-delay trade-off under the Brownian motion model degenerates into two points: one can either achieve $\Theta(1/\sqrt{n \log n})$ per-node capacity at almost no delay, or $\Theta(1)$ per-node capacity at roughly $\Theta(1/\sigma_n^2)$ delay, but nothing in between! Finally, although Proposition 1 is shown under the Brownian motion model, it is not difficult to see that the result also applies to the random walk mobility model [9], or the Markovian mobility model in [4]. This is because as $n \to \infty$, the difference between these mobility models vanishes.

The result of Proposition 1 is in sharp contrast to the results reported in the existing works [3], [6], where it is claimed that certain schemes can provide a smooth trade-off between the capacity and the delay. In section VI, we will point out the most likely reasons for this discrepancy.

V. Delay Under Generalized Two-Hop Relaying Schemes

In Section IV, we have established the fundamental delay-capacity trade-off under the Brownian motion model for a wide class of scheduling schemes. We have shown that, for any scheduling policy that satisfies Assumption A, in order to achieve a per-node capacity greater than $\Omega(1/\sqrt{n})$, the delay must be $\Omega(n^\alpha/\sigma_n^2)$, for all $\alpha < 0$. In this section, we study the delay performance of a more restricted set of scheduling schemes. Our interest in this class of schemes stems from the fact that they have been used in the earlier studies for achieving $\Omega(1/\sqrt{n})$ per-node capacity under various mobility models. Thus, we would like to understand more deeply their delay performance under the Brownian motion model.

These schemes are similar to the 2-hop relaying scheme of Grossglauser and Tse [2]. Hence, we refer to them as *generalized 2-hop relaying schemes*. Compared with the more general class of schemes that we considered in Section IV, these schemes have one additional restriction: For each packet p, the source node is only allowed to replicate the packet p to *one* relay node (denoted by $R(p)$). Other than this restriction, the generalized 2-hop relaying schemes still have substantial flexibility in scheduling packet transmissions. For example, in the *replication phase*, the scheduler still decides when to replicate the packet, and how (e.g., which relay node to replicate the packet p to, and how to schedule the packet transmissions from the source node to the chosen relay node $R(p)$, possibly using multi-hop transmissions). Similarly, in the *capture phase*, the scheduler decides when and how to relay the packet p to the destination, from either the source node or the chosen relay node $R(p)$, possibly using multi-hop transmissions.

To ensure that fewer radio resources are consumed, the replication phase (and the capture phase, correspondingly) typically occurs when the chosen relay node is within a small neighborhood around the source node (and the destination node, correspondingly). For example, a relay node could either enter a disk of a certain radius around the source (or destination), or a relay node could enter the same cell as the source (or destination) in case the network is divided into cells. We call such an area around the source or the destination as the *replication neighborhood* or the *capture neighborhood*, respectively. We further assume that the replication neighborhood and the capture neighborhood are both contained in disks of radius a_n centered at the source node and the destination node, respectively. Again, to ensure that fewer radio resources are consumed, a_n would typically be $o(1)$.

Remark 6: Note that Scheme 2 and Scheme $3(b)$ in [3] are both special cases of the generalized 2-hop relaying schemes that we consider in this section.

We now give a lower bound on the average packet delay of the generalized 2-hop relaying schemes.

Proposition 2: If the replication neighborhood and the capture neighborhood under a generalized 2-hop relaying scheme can be contained inside a disk of radius a_n around the source node and the destination node, respectively, and $a_n = o(1)$, then the average packet delay under the given scheduling scheme must be $\Omega(\log(1/a_n)/\sigma_n^2)$.

Proof: When $a_n = o(1)$, most of the packets will have to be delivered through a relay node. Consider such a random packet that arrives at the source node. Its delay must be no less than the time that it takes for the relay node to move from somewhere within distance a_n around the source, to somewhere within distance a_n around the destination. Since the packet arrivals are independent of the node mobility, the source node and the destination node of the packet are distributed uniformly on the unit sphere at the time of the packet arrival. Therefore, the delay for the packet is no less than the time that it takes for two nodes placed uniformly on the unit sphere to come within a distance of $2a_n$ from each other. Therefore, in view of Lemma 2, the result follows. ■

Remark 7: Note that Proposition 2 holds even if the replication neighborhood and the capture neighborhood differ in shape or size. Further, if $a_n = O(n^\alpha)$, for some $\alpha < 0$, then the delay of the generalized 2-hop relaying scheme is

$$\Omega(\log n/\sigma_n^2). \tag{12}$$

The above result provides a lower bound on the average packet delay under any generalized 2-hop relaying scheme. We have not provided the analysis for the upper bound on the delay. Such an analysis could possibly be carried out using the methodology in [3], under certain additional technical assumptions. We will show in Section VI that, after correcting the derivation of the variance of the so-called *inter-meeting time*, the methodology in [3] will predict the packet delay to be $\Theta(\log n/\sigma_n^2)$, which is consistent with our result in Proposition 2.

VI. Discussion of Related Works

In this paper, we have investigated the capacity-delay tradeoffs of mobile ad hoc networks under the Brownian motion model. Earlier contributions to this problem were made in [3] and [6]. The system model in [6] is the same as in this paper. On the other hand, the system model in [3] is not entirely the same as in this paper, e.g., [3] considers Brownian motion on a 2-d torus, while this paper considers Brownian motion on a sphere. Despite these differences, we would reasonably expect that the main results of this paper still apply for the system model in [3]. In fact, the results in Section IV have been shown directly for planar Brownian motion.

However, the capacity-delay trade-offs obtained in this paper differ substantially from those reported in [3], [6]. In particular, we show that the achievable capacity-delay tradeoff under the Brownian motion model is degenerate, while both [3] and [6] report some sort of smooth trade-offs. We now point out the main errors in the analysis in [3], [6] that, we believe, have led to this difference in the results.

We first look at the results in [3]. It is our understanding that the work in [3] also attempts to study the Brownian motion model (see the first paragraph in [3, Section III.A]). In [3], the nodes are assumed to move according to independent 2-d Brownian motions on a 2-d torus. The Scheduling Scheme 2 and Scheme 3(b) in [3] are of particular interest to us because they both belong to the generalized 2-hop relaying schemes in Section V (and thus satisfy Assumption A in Section IV as well). These two schemes divide the torus into $\frac{1}{\sqrt{a(n)}} \times \frac{1}{\sqrt{a(n)}}$ cells of equal size, where $1/n \le a(n) < 1$. When a packet arrives to the source, it is first replicated to a relay node in the same cell as the source node. After the relay node moves into the same cell as the destination node, the packet is forwarded to the destination. In [3], the authors report that a smooth capacity-delay trade-off can be achieved by varying the cell size $a(n)$. In particular, when $a(n) = 1/n$, Theorem 4 in [3] reports that the delay is

$$\Theta\left(\frac{\sqrt{n}}{v(n)}\right), \tag{13}$$

where $v(n)$ denotes the "speed" of the Brownian motion, which we will discuss in further detail later. On the other hand, when $a(n) = \omega(1/n)$, Theorem 6 in [3] reports that the delay is reduced to

$$\Theta\left(\frac{1}{\sqrt{a(n)}v(n)}\right).$$

The parameter $v(n)$ in [3] is comparable to σ_n in this paper. In the system model in [3], $\frac{1}{\sqrt{n}v(n)}$ is approximately the time taken by a node to move $\Theta(1/\sqrt{n})$ distance, where $\Theta(1/\sqrt{n})$ is roughly the distance among neighboring nodes (see Equation (14) in [3]). In the model in this paper, the time to move $\Theta(1/\sqrt{n})$ distance is $\Theta(\frac{1}{n\sigma_n^2})$ (see Lemma 3). Thus, the parameter $v(n)$ of [3] is equivalent to $\sqrt{n}\sigma_n^2$. If we take this value of $v(n)$, then the results of [3] imply that the schemes there can achieve

$$\Theta(1/\sigma_n^2) \tag{14}$$

delay when $a(n) = 1/n$, and

$$\Theta\left(\frac{1}{\sqrt{na(n)}\sigma_n^2}\right)$$

delay when $a(n) = \omega(1/n)$. On the other hand, our result in Section V reports that the delay must be $\Omega(\log n/\sigma_n^2)$, regardless of the value of $a(n)$. We now identify two main reasons which, we believe, have led to the discrepancy between the results of [3] and that of this paper.

1) The Mapping from Brownian Motion to Random Walk: The delay analysis in [3] is based on the mapping that approximates the Brownian motion of a single node by a 2-d random-walk, where each state of the random walk indicates that the node is in a corresponding cell. With such a mapping, each change of the state of the random walk occurs when the node moves to a neighboring cell. The authors claim that when the area of the cell is changed from $1/n$ to $a(n)$ where $a(n) = \omega(1/n)$, the time taken by a node to move out of a cell, say $t(n)$, changes from $O(\frac{1}{\sqrt{n}v(n)})$ to

$$t(n) = \Theta\left(\frac{\sqrt{a(n)}}{v(n)}\right), \tag{15}$$

where $v(n)$ is the "speed" parameter that we discussed earlier (see the last part of the proof of Theorem 6 in [3]). However, if the underlying mobility model is Brownian motion, then according to Lemma 3, the amount of time that a node takes to move out of a cell is proportional to the *area* of the cell (rather than its *diameter*). Hence, the correct value of $t(n)$ should be

$$t(n) = \Theta\left(\frac{a(n)\sqrt{n}}{v(n)}\right)$$

when the area of the cell is $a(n)$. Using this value of $t(n)$, and following the rest of argument in the proof of Theorem 6 in [3], one can easily see that resizing the cells (i.e., changing $a(n)$ as in Scheme $3(b)$ of [3]) will not significantly affect the delay. In fact, the delay of Scheme $3(b)$ in [3] would remain the same as in (13) with any cell size!

An alternative way of viewing the trade-off result obtained in [3] is that, if one forces $t(n)$ to behave according to (15), one must then assume that the variance of the Brownian motion is $v(n)\sqrt{a(n)}$ when the area of the cell is $a(n)$. This assumption amounts to considering a different mobility model at each cell size $a(n)$. Such an assumption makes the trade-off result less useful because in practice one cannot alter the underlying physical mobility model. Note that the main results of this paper still apply even under this unrealistic assumption. In particular, increasing the size of the cell (as in Scheme $3(b)$ of [3]) still cannot improve the order of the delay. In fact, if the variance of the Brownian motion were increased to $v(n)\sqrt{a(n)}$ (as we pointed out above), then even using $a(n) = 1/n$ would have resulted in roughly the same reduction in delay as that reported in [3] for Scheme $3(b)$. Hence, there is no benefit to increase the cell size.

2) The Derivation of the Variance of the Inter-meeting Time: If we take $v(n) = \sqrt{n}\sigma_n^2$, then based on the above discussion, the methodology in [3] would conclude that the delay of Scheme 2 and Scheme $3(b)$ in [3] should be $\Theta(1/\sigma_n^2)$ for any cell size (see (14)). Thus, there is still a difference of factor $\log n$ compared with our result in Section V (i.e., between (12) and (14)). As we illustrate next, this difference is due to the incorrect calculation of the variance of the so-called "inter-meeting time." For simplicity, consider the case when $a(n) = 1/n$. Recall that the authors of [3] map the Brownian motion on a torus to a random walk on a $\sqrt{n} \times \sqrt{n}$ grid. Each state of the random walk indicates that the node is in a corresponding cell. Each change of the state of the random walk occurs when the node moves to a neighboring cell. The *inter-meeting time* is the time that it takes for the random walk to start from an arbitrary state and return to the same state. Lemma 6 in [3] claims that the variance of the inter-meeting time is $\Theta(n^2)$. In what follows, we will show that the variance of the inter-meeting time is actually $\Theta(n^2 \log n)$, which then accounts for the factor of $\log n$ which is missing in Theorem 4 in [3]. Note that the random walk corresponds to an irreducible, reversible, and temporally homogeneous Markov chain, say s_k, with the state space $S_{\mathbb{Z}} = \{(x, y) : x, y = 0, 1, ..., \sqrt{n} - 1\}$. Consider an arbitrary state $s \in S_{\mathbb{Z}}$. Define the inter-meeting time to be $I_s = \inf\{k > 0 : s_0 = s, s_k = s\}$. Now the following Lemma gives the variance of the inter-meeting time I_s.

Lemma 6: $Var(I_s) = \Theta(n^2 \log n)$.

Proof: We can show Lemma 6 using the following standard results in the Markov chain theory. Let $\Pi_{\mathbb{Z}}$ denote the stationary distribution of s_k. In our case, it is in fact the uniform distribution on $S_{\mathbb{Z}}$. Let T_s denote the *first hitting time* to the state s, i.e.,

$$T_s \triangleq \inf\{k \geq 0; s_k = s\}.$$

We have the following relationship between the variance of I_s and the mean of the *first hitting time* T_s (see [13, Chapter 2, pp. 21]):

$$Var(I_s) = \frac{2\mathbb{E}_{\Pi_{\mathbb{Z}}}[T_s] + 1}{\Pi_s} - \frac{1}{\Pi_s^2}, \tag{16}$$

where the expectation $\mathbb{E}_{\Pi_{\mathbb{Z}}}$ is taken with respect to the stationary distribution of s_k, and Π_s is the stationary probability that

$s_k = s$. Since the total number of states in $S_{\mathbb{Z}}$ is n, we have $\Pi_s = 1/n$. It is well known that the mean of the first hitting time is

$$\mathbb{E}_{\Pi_{\mathbb{Z}}}[T_s] = \Theta(n \log n). \quad (17)$$

(For example, this result is listed without detailed proofs in [13, Chapter 5, pp. 11]. We refer to the row corresponding to 2-dimensional torus and the column corresponding to τ_0 in the table there. The definition of τ_0 in the table is given in [13, Chapter 4, pp. 1] and is the same as $\mathbb{E}_{\Pi_{\mathbb{Z}}}[T_s]$. For the sake of completeness, we have provided a proof for (17) in Appendix A.) Substituting these values in (16), the result follows. ■

In Lemma 6, we have shown that the variance of the inter-meeting time I_s is $\Theta(n^2 \log n)$. If we follow the rest of the analysis in [3], we can conclude that the delay of Scheme 2 and Scheme $3(b)$ there should both be $\Theta(\log n \sqrt{n}/v(n))$, which is then consistent with our result in Section V given the fact that $v(n) = \sqrt{n}\sigma_n^2$. Thus, even if one uses the same mapping from Brownian motion to random walk as in [3], one would obtain the same delay results as those reported in this paper. However, this mapping from the Brownian motion to the random walk is somewhat heuristic. In particular, it is easy to see that future transitions of the state of the induced "random walk" are not entirely independent of the past. We use this mapping here mainly to point out the mistake in the proofs in [3]. It is possible to rigorously define the notion of the inter-meeting time for the Brownian motion and calculate the variance of the inter-meeting time on continuous state-space, thus avoiding the heuristic mapping to a random walk. In Appendix B, we provide such an analysis of the variance of the inter-meeting time for the spherical Brownian motion.

This paper also corrects the previously reported results in [6]. In [6] it is shown that the capacity-delay trade-off is smooth as the number of mobile relays per packet is varied. We now point out the error in [6] that has led to this incorrect conclusion. The derivation in [6] is based on the empirical observation that the *first meeting time* nearly follows an exponential distribution. In [6], the *first meeting time* is defined to be the amount of time that it takes for a node to move, *from an initial position that is uniformly distributed on the unit sphere*, to a neighborhood of the destination. It is then argued that when the source replicates the packet to k_n relays, the delay is reduced roughly by a factor of $1/k_n$, because the mean of the minimum of k_n *i.i.d.* exponentially distributed random variables is $1/k_n$ times the mean of one of them. However, the problem with this argument is that the paths of the relays are not *independent* of each other. In fact, the paths are correlated by the fact that they all receive the packet from the *same* source node. By Lemma 4, even if the source node replicates the packet to $\Theta(n)$ relays, the delay is not reduced much, i.e., it is still close to $\Theta(1/\sigma_n^2)$. This is in contrast to the random way-point mobility model (which is also studied in [6]), where the correlation between the paths of various mobile relays holding the same packet dies out in a very short time, which then allows a smooth trade-off between the delay and the capacity [5].

We believe that this difference in the delay-capacity trade-off under the random way-point mobility model and the Brownian motion model, is a revelation of the fundamental difference in the mobility pattern under these two models. In particular, it appears that the more "diffused" the node motion is, the more difficult it is to reduce the mobility delay.

One possible way of characterizing the *diffused* nature of the node motion could be to look at the distribution of the inter-meeting time: We believe that the more diffused the node motion is, the slower would be the decay of the complementary distribution of the inter-meeting time (for the same mean), thus resulting in larger higher order moments.

One simple measure for quantifying the diffused nature of the node motion[4] could be to look at the coefficient of variation of the inter-meeting time, defined as the ratio of the inter-meeting time variance to the square of the mean inter-meeting time, that is, $Var(I)/\mathbb{E}^2\{I\}$, where I denotes the inter-meeting time. The more diffused the mobility model, the higher we would expect the coefficient of variation of I to be. From the results of this paper and that of [5], it is easy to see that the above ratio is $\Theta(\log n)^5$ in case of the Brownian motion model and $\Theta(1)$ in case of the random way-point mobility model. Thus, at least, in our case, the coefficient of variation of I indeed quantifies how diffused the node motion is under a mobility model.

From the above discussion, we see that the distribution of the inter-meeting time has a strong influence on the delay-capacity relationship under a mobility model. We believe that the future studies, trying to investigate this link between the inter-meeting time distrbution and the delay-capacity relationship, can benefit from the analysis of the inter-meeting time that we provide in Appendix B.

VII. Conclusion

In this paper, we have studied the fundamental trade-off between the delay and the capacity under the Brownian motion model. We have shown that the capacity-delay trade-off under the Brownian motion model is *degenerate*: one can either achieve a per-node capacity of $\Theta(1)$ with $\Omega(\log n/\sigma_n^2)$ delay (using 2-hop relaying), or, once the delay constraint is of the order $O(n^\alpha/\sigma_n^2), \alpha < 0$, one can at most achieve $\Theta(1/\sqrt{n})$ per-node capacity, which is almost the same as that can be achieved for static wireless networks.

As we compare the results in this paper for the Brownian motion model, with the results in [5] for the random way-point mobility model, we find that the delay-capacity relationship of mobile wireless networks is strongly influenced by how directed the node motion is. In particular, it appears that the more "diffused" the node motion is, the more difficult it is to reduce the mobility delay. Our results also indicate that there is some connection between the distribution of the inter-meeting time and the delay-capacity relationship under a mobility model. In particular, it appears that the rate of decay of the inter-meeting time tail is fundamentally linked to the achievable delay and capacity region under a mobility model.

[4]Note that this might not be a good measure of diffusivity in some cases.

[5]Here, we are considering $a_n = o(1)$.

We believe that this paper is a first step toward understanding the impact, the nature of the node mobility has, on the delay-capacity relationship in ad hoc networks.

Appendix

A. Average First Hitting Time Analysis for 2-D Torus

In this section, we show that the mean of the first hitting time on 2-d torus is $\Theta(n \log n)$. Recall the following definitions introduced in Section VI. We have a random walk on a 2-d torus of size $\sqrt{n}\times\sqrt{n}$. The random walk corresponds to an irreducible, reversible, and temporally homogeneous Markov chain, say s_k, with the state space $S_{\mathbb{Z}} = \{(x, y) : x, y = 0, 1, ..., \sqrt{n} - 1\}$. Consider an arbitrary state $s \in S_{\mathbb{Z}}$ and let T_s denote the first hitting time of state s, defined as follows:

$$T_s \triangleq \inf\{k \geq 0; s_k = s\}.$$

Let $\Pi_{\mathbb{Z}}$ denote the uniform distribution on the torus, which is also the stationary distribution of the Markov chain s_k. We are interested in finding the expectation of T_s taken with respect to the uniform distribution of the initial state of the Markov chain, i.e., $\mathbb{E}_{\Pi_{\mathbb{Z}}}\{T_s\}$. Let Q denote the transition matrix of the above Markov chain. It is well known (for example, see [13, Chapter 3, pp. 19]) that

$$\mathbb{E}_{\Pi_{\mathbb{Z}}}\{T_s\} = \sum_{m=2}^{n} (1 - \lambda_m)^{-1}, \tag{18}$$

where $\lambda_m, m = 1, 2.,,, n$ are the eigenvalues of Q, numbered in the descending order, i.e., $1 = \lambda_1 \geq \lambda_2 \geq ... \geq \lambda_n$. The eigenvalues in our case can easily be computed (see [13, Chapter 5, pp. 33]) to be:

$$\lambda_{i,j} = \frac{\cos\left(\frac{2\pi i}{\sqrt{n}}\right) + \cos\left(\frac{2\pi j}{\sqrt{n}}\right)}{2}, \text{ for } i, j = 0, 1, ..., \sqrt{n} - 1.$$

Note that $\lambda_{0,0}$ corresponds to $\lambda_1 = 1$. Substituting this in (18), we obtain

$$\begin{aligned}\mathbb{E}_{\Pi_{\mathbb{Z}}}\{T_s\} &= \sum_{(i,j)\in S_{\mathbb{Z}}\setminus(0,0)} \frac{2}{2 - \cos\left(\frac{2\pi i}{\sqrt{n}}\right) - \cos\left(\frac{2\pi j}{\sqrt{n}}\right)} \\ &= \sum_{(i,j)\in S_{\mathbb{Z}}\setminus(0,0)} \frac{1}{\sin^2\left(\frac{\pi i}{\sqrt{n}}\right) + \sin^2\left(\frac{\pi j}{\sqrt{n}}\right)}.\end{aligned}$$

To obtain a lower bound on $\mathbb{E}_{\Pi_{\mathbb{Z}}}\{T_s\}$, we note that $\sin^2 x \leq x^2$, and therefore

$$\begin{aligned}\mathbb{E}_{\Pi_{\mathbb{Z}}}\{T_s\} &\geq \sum_{(i,j)\in S_{\mathbb{Z}}\setminus(0,0)} \frac{n}{\pi^2(i^2 + j^2)} \\ &\geq \iint_{\{0\leq x,y\leq\sqrt{n}\}\bigcap\{x\geq1 \text{ or } y\geq1\}} \frac{n}{\pi^2(x^2 + y^2)} dxdy \\ &\geq \int_{r=\sqrt{2}}^{\sqrt{n}} \frac{\pi r}{2} \frac{n}{\pi^2 r^2} dr \\ &\qquad \text{(by changing to polar coordinates)} \\ &= \frac{n}{2\pi}[\log\sqrt{n} - \log\sqrt{2}] \\ &= \Theta(n \log n).\end{aligned} \tag{19}$$

To obtain an upper bound on $\mathbb{E}_{\Pi_{\mathbb{Z}}}\{T_s\}$, let A_Z denote the set $\{(i, j) \in S_{\mathbb{Z}} \setminus (0, 0) : i, j \leq \sqrt{n}/4\}$. Using the inequality $\sin x \geq x/2$ for $x \leq \pi/4$, we have

$$\begin{aligned}&\sum_{A_Z} \frac{1}{\sin^2\left(\frac{\pi i}{\sqrt{n}}\right) + \sin^2\left(\frac{\pi j}{\sqrt{n}}\right)} \\ &\leq \sum_{A_Z} \frac{4n}{\pi^2(i^2 + j^2)} \\ &\leq \iint_{\{0\leq x,y\leq\sqrt{n}/4+1\}\bigcap\{x\geq1 \text{ or } y\geq1\}} \frac{4n}{\pi^2(\frac{x^2+y^2}{5})} dxdy \\ &\leq \int_{r=1}^{\sqrt{n}/2} \frac{20n}{\pi^2} \frac{\pi r}{2} \frac{1}{r^2} dr \\ &= \frac{10n}{\pi} \log(\sqrt{n}/2) \\ &= \Theta(n \log n).\end{aligned} \tag{20}$$

Similarly, let B_Z denote the set $\{(i, j) \in S_{\mathbb{Z}} \setminus (0, 0) : i, j \geq 3\sqrt{n}/4\}$. Using $\sin x = \sin(\pi - x)$ and the above techniques, we can show that

$$\sum_{B_Z} \frac{1}{\sin^2\left(\frac{\pi i}{\sqrt{n}}\right) + \sin^2\left(\frac{\pi j}{\sqrt{n}}\right)} \leq \Theta(n \log n). \tag{21}$$

Now consider $(i, j) \in S_{\mathbb{Z}} \setminus A_Z \setminus B_Z$. In this case, we have either

$$\sin(\pi i/\sqrt{n}) \geq \sin(\pi/4) = 1/\sqrt{2},$$

or

$$\sin(\pi j/\sqrt{n}) \geq 1/\sqrt{2}.$$

Since the number of states in $S_Z \setminus A_Z \setminus B_Z$ is less than n, we then have

$$\sum_{S_Z\setminus A_Z\setminus B_Z} \frac{1}{\sin^2\left(\frac{\pi i}{\sqrt{n}}\right) + \sin^2\left(\frac{\pi j}{\sqrt{n}}\right)} \leq \Theta(n). \tag{22}$$

Combining (20-22), we have $\mathbb{E}_{\Pi_{\mathbb{Z}}}\{T_s\} \leq \Theta(n \log n)$. In view of (19), it follows that $\mathbb{E}_{\Pi_{\mathbb{Z}}}\{T_s\} = \Theta(n \log n)$.

B. The Inter-meeting Time Analysis for Spherical Brownian Motion

In this section, we provide a rigorous treatment of the notion of the inter-meeting time. Our treatment does not rely on the mapping from Brownian motion to random walk used in [3]. We start with two related definitions. Consider the motion of an arbitrary node under the Brownian motion model. Let X_t be its position on the unit sphere S at time t. Let $A = \{x \in S : d_S(x, y) \leq a_n\}$, for some $y \in S$. Let $A/2 = \{x \in S : d_S(x, y) \leq a_n/2\}$, i.e., $A/2$ is the circle centered at y with half the radius as A.

Definition 3 (Contact Time): The contact time, denoted by $\tau^A_{A/2}$, is the time it takes for a Brownian motion started at the boundary of $A/2$ to exit A; i.e., $\tau^A_{A/2} = \inf\{t \geq 0 : d_S(X_0, y) = a_n/2, d_S(X_t, y) = a_n\}$.

Definition 4 (Return Time): The return time, denoted by $T^{A/2}_A$, is the time it takes for a Brownian motion started at the boundary of A to enter $A/2$; i.e., $T^{A/2}_A = \inf\{t \geq 0 : d_S(X_0, y) = a_n, d_S(X_t, y) = a_n/2\}$.

We are now ready to define the inter-meeting time:

Definition 5 (Inter-Meeting Time): The inter-meeting time, denoted by I_A, is the time it takes for a Brownian motion started at the boundary of $A/2$ to exit A, and come back to $A/2$; i.e., $I_A = \tau^A_{A/2} + T^{A/2}_A$.

The motivation for the above definitions is the following. Consider two nodes executing independent Brownian walks on the sphere. The return time defined as above, is related to how much time it takes for the two nodes to come close to each other. The contact time is related to how long the two nodes *remain* close to each other. And the inter-meeting time is related to how often the two nodes come in contact with each other. Note that the choice of $A/2$ is not really critical here: one could also use A/ϵ, with any $\epsilon > 1$, instead of $A/2$. The reason why we need $\epsilon > 1$ is that we want most *contacts* to be long enough so as to allow for a packet exchange between the nodes. In the above definitions, we have fixed the position of one of the nodes at y and let the other node move. Alternatively, we can define these terms for the case when both nodes are moving, i.e., we can replace y by a random variable that represents the position of a moving node. Note that when two nodes are executing *i.i.d.* Brownian walks with variance σ_n^2 on the unit sphere, their relative motion is a Brownian walk with variance $2\sigma_n^2$. Hence, for the sake of estimating the order of these times, there will only be a difference of a factor of two between these two types of definitions.

We now estimate the mean and the variance of the inter-meeting time.

Proposition 3: $\mathbb{E}[\tau^A_{A/2}] = \Theta(a_n^2/\sigma_n^2)$, $\mathbb{E}[T^{A/2}_A] = \Theta(1/\sigma_n^2)$, and $\mathbb{E}[I_A] = \Theta(1/\sigma_n^2)$.

Proof: The symmetry of the sphere implies that the choice of $y \in S$, can be arbitrary. In order to estimate $\mathbb{E}[\tau^A_{A/2}]$, we choose y to be the north pole (the top most point of S). It is then clear that $T^A_{A/2}$ is the time it takes for Y_t (see section II) to travel from $\cos(a_n\sqrt{\pi})$ to $\cos(2a_n\sqrt{\pi})$. Appealing to Lemma 1, gives the desired result.

To estimate $\mathbb{E}[T^{A/2}_A]$, we choose y to be south pole (the bottom most point of S)[6]. Thus, $T^{A/2}_A$ is the time it takes for Y_t to travel from $-\cos(2a_n\sqrt{\pi})$ to $-\cos(a_n\sqrt{\pi})$, and once again Lemma 1 gives the desired result.

Now since $\mathbb{E}[I_A] = \mathbb{E}[T^{A/2}_A] + \mathbb{E}[T^A_{A/2}]$, and $a_n = O(1)$, we obtain $\mathbb{E}[I_A] = \Theta(1/\sigma_n^2)$. ■

Now let us look at the variance of the inter-meeting time. Assuming $a_n \to 0$ as $n \to \infty$, we have the following key result:

Proposition 4: $Var(I_A) = \Theta(\log(1/a_n)/\sigma_n^4)$.

Remark 8: This result is consistent with that of Lemma 6. In fact, Lemma 6 reports that the variance of the inter-meeting time I_s for the random walk is $\Theta(n^2 \log n)$. Note that the inter-meeting time I_s for the random walk is defined as the number of *state transitions* before returning to the same state. Since each state transition takes $\Theta\left(\frac{1}{n\sigma_n^2}\right)$ time (when the unit area is divided into $\sqrt{n} \times \sqrt{n}$ cells), the variance of the actual amount of time taken to return back to the same state is

$$\Theta(n^2 \log n)\left[\Theta\left(\frac{1}{n\sigma_n^2}\right)\right]^2 = \log n/\sigma_n^4,$$

which matches with the result in Proposition 4 when $a_n = n^\alpha, \alpha < 0$.

Proof: Recall that $I_A = \tau^A_{A/2} + T^{A/2}_A$, where $\tau^A_{A/2}$ and $T^{A/2}_A$ are as defined earlier. From the Strong Markov property of the Brownian motion, it follows that $\tau^A_{A/2}$ and $T^{A/2}_A$ are independent random variables. Thus, $Var(I_A) = Var(\tau^A_{A/2}) + Var(T^{A/2}_A)$. In order to estimate these variances, we will first establish a more general result, regarding the variance of the travel time for the process Y_t, given by Equation (4). Then, by appropriately choosing the starting and the ending points, we will be able to estimate the variances of $\tau^A_{A/2}$ and $T^{A/2}_A$.

1) Variance of the Travel Time for Y_t: Let $-1 < a < b < 1$, and let $a \le x \le b$. Recall that Y_t is the z-coordinate of the Brownian motion on the unit sphere S, and Y_t is governed by the diffusion process (4) with drift coefficient $\mu(y) = -\sigma_n^2 y$ and diffusion coefficient $\sigma^2(y) = \sigma_n^2(1-y^2)$. Define

$$\tau_{a\wedge b} \triangleq \inf\{t \ge 0 : Y_t = a \text{ or } Y_t = b\}$$

to be the first time that Y_t hits a or b. Let $U_{a\wedge b}(x)$ be the second moment of the travel time for Y_t from x to either a or b, i.e.,

$$\begin{aligned} U_{a\wedge b}(x) &\triangleq \mathbb{E}[{\tau_{a\wedge b}}^2 | Y_0 = x] \\ &= \mathbb{E}\left\{\left[\int_0^{\tau_{a\wedge b}} 1 d\tau\right]^2 | Y_0 = x\right\}. \end{aligned}$$

Using the technique of [14, p202], $U_{a\wedge b}(x)$ must satisfy the following differential equation:

$$\frac{1}{2}\sigma^2(x)\frac{d^2 U_{a\wedge b}(x)}{dx^2} + \mu(x)\frac{dU_{a\wedge b}(x)}{dx} + 2V_{a\wedge b}(x) = 0,$$

where

$$\begin{aligned} V_{a\wedge b}(x) &= \mathbb{E}\left\{\int_0^{\tau_{a\wedge b}} 1 d\tau_{a\wedge b} | Y_0 = x\right\} \\ &= \mathbb{E}[\tau_{a\wedge b} | Y_0 = x], \end{aligned}$$

i.e., $V_{a\wedge b}(x)$ is the mean travel time for Y_t from x to a or b.

The solution to $U_{a\wedge b}(x)$ is given by [14, p197]:

$$\begin{aligned} U_{a\wedge b}(x) = 4\Bigg\{ & u(x)\int_a^b \left[\int_a^\eta V_{a\wedge b}(\xi)m(\xi)d\xi\right] dS(\eta) \\ & - \int_a^x \left[\int_a^\eta V_{a\wedge b}(\xi)m(\xi)d\xi\right] dS(\eta)\Bigg\}, \end{aligned} \quad (23)$$

where $S(\cdot)$ and $m(\cdot)$ are the *scale function* and the *speed measure density* [14, pp.194-195], respectively, defined as follows for the diffusion process (4):

$$\begin{aligned} S(x) &\triangleq \int^x s(\eta)d\eta, \\ m(x) &\triangleq \frac{1}{\sigma^2(x)s(x)}, \end{aligned}$$

[6] We are doing this in order to make sure that Y_t travels in the negative z-axis, so that we can make use of Lemma 1.

with

$$s(x) \triangleq \exp\left\{-\int^x [2\mu(\xi)/\sigma^2(\xi)]d\xi\right\},$$

and $u(x)$ is the probability that Y_t reaches a before b, which is equal to [14, p195]:

$$u(x) = \frac{S(x)-S(a)}{S(b)-S(a)}.$$

For the diffusion process (4), we can calculate these quantities as,

$$\begin{aligned} s(x) &= \exp\left\{-\int^x [2\mu(\xi)/\sigma^2(\xi)]d\xi\right\} \\ &= \exp\left\{\int^x \frac{2\xi}{1-\xi^2}d\xi\right\} \\ &= \exp\left\{-\log(1-x^2)\right\} \\ &= \frac{1}{1-x^2}, \\ S(x) &= \int^x s(\eta)d\eta = \int^x \frac{1}{1-\eta^2}d\eta = \frac{1}{2}\log\frac{1+x}{1-x}, \\ m(x) &= \frac{1}{\sigma^2(x)s(x)} = \frac{1}{\sigma_n^2}. \end{aligned}$$

Note that $b=1$ is an *inaccessible point* of the diffusion process (4). Let $b \to 1$, and define

$$\tau_a \triangleq \inf\{t \geq 0 : Y_t = a\}$$

to be the first time that Y_t hits a. Further, let

$$V_a(x) \triangleq \mathbb{E}[\tau_a | Y_t = x],$$

and

$$U_a(x) \triangleq \mathbb{E}[{\tau_a}^2 | Y_t = x],$$

be the first moment and the second moment, respectively, of the travel time for Y_t from x to a. It is then not difficult to see that, as $b \to 1$,

$$U_{a\wedge b}(x) \to U_a(x), \text{ and } V_{a\wedge b}(x) \to V_a(x).$$

Using Lemma 1, we have

$$V_a(x) = \frac{2}{\sigma_n^2}\log\frac{1+x}{1+a}.$$

We now use the techniques of [15, p422] for treating the *inaccessible point* at $b=1$. Note that

$$S(b) \to \infty, \text{ as } b \to 1,$$

and

$$\int_x^1 V_a(\eta)m(\eta)d\eta < \infty.$$

Letting $b \to 1$ in (23), we obtain

$$\begin{aligned} U_a(x) = 4\Bigg\{&(S(x)-S(a))\left[\int_a^1 V_a(\xi)m(\xi)d\xi\right] \\ &- \int_a^x \left[\int_a^\eta V_a(\xi)m(\xi)d\xi\right] dS(\eta)\Bigg\} \\ = 4&\int_a^x \left[\int_\eta^1 V_a(\xi)m(\xi)d\xi\right] dS(\eta). \end{aligned}$$

Using the values of $V_a(x)$, $m(x)$ and $S(x)$ computed earlier, we have

$$\begin{aligned} &\int_\eta^1 V_a(\xi)m(\xi)d\xi \\ &= \int_\eta^1 \frac{2}{\sigma_n^4}\log\frac{1+\xi}{1+a}d\xi \\ &= \frac{2}{\sigma_n^4}\left[(1+\xi)\log(1+\xi)-(1+\xi)-\xi\log(1+a)\right]_\eta^1 \\ &= \frac{2}{\sigma_n^4}[2\log 2 - 2 - \log(1+a)] \\ &\qquad - \frac{2}{\sigma_n^4}[(1+\eta)\log(1+\eta)-(1+\eta)-\eta\log(1+a)]. \end{aligned}$$

Hence,

$$\begin{aligned} U_a(x) &= 4\int_a^x s(\eta)\left[\int_\eta^1 V_a(\xi)m(\xi)d\xi\right]d\eta \\ &= \frac{8}{\sigma_n^4}\Bigg\{\int_a^x \left[\frac{2\log(2/e)-\log(1+a)}{1-\eta^2}\right]d\eta \\ &\quad - \int_a^x \left[\frac{\log(1+\eta)}{1-\eta} - \frac{1}{1-\eta} - \frac{\eta\log(1+a)}{1-\eta^2}\right]d\eta\Bigg\}. \end{aligned} \tag{24}$$

We can now compute the variances of $\tau_{A/2}^A$ and $T_A^{A/2}$.

2) The Variance of $T_A^{A/2}$: Let

$$a = -\cos(\sqrt{\pi}a_n)\ ;\ x = -\cos(2\sqrt{\pi}a_n).$$

Then,

$$\mathbb{E}[T_A^{A/2}] = V_a(x), \text{ and } Var(T_A^{A/2}) = U_a(x) - V_a^2(x).$$

Using Lemma 1,

$$\mathbb{E}[T_A^{A/2}] = V_a(x) = \Theta(1/\sigma_n^2).$$

We use (24) to compute the order of $U_a(x)$. Note first that

$$\begin{aligned} \int_a^x \frac{1}{1-\eta^2}d\eta &= \frac{1}{2}\log\frac{1+\eta}{1-\eta}\Big|_a^x \\ &= \frac{1}{2}\left\{\log\frac{1+x}{1+a} - \log\frac{1-x}{1-a}\right\}, \\ \int_a^x \frac{1}{1-\eta}d\eta &= -\log(1-\eta)|_a^x \\ &= -\log\frac{1-x}{1-a}, \\ \int_a^x \frac{\eta}{1-\eta^2}d\eta &= -\frac{1}{2}\log(1-\eta^2)\Big|_a^x \\ &= -\frac{1}{2}\log\frac{(1+x)(1-x)}{(1+a)(1-a)}. \end{aligned}$$

As $a_n \to 0$,

$$\begin{aligned} \frac{1+x}{1+a} &= \frac{1-cos(2\sqrt{\pi}a_n)}{1-cos(\sqrt{\pi}a_n)} \to 4, \\ \frac{1-x}{1-a} &\to 1. \end{aligned}$$

Hence,

$$\int_a^x \frac{1}{1-\eta^2}d\eta \to \log 2, \tag{25}$$
$$\int_a^x \frac{1}{1-\eta}d\eta \to 0, \tag{26}$$
$$\int_a^x \frac{\eta}{1-\eta^2}d\eta \to -\log 2. \tag{27}$$

Further, since $-1 \le a \le x \le 0$ when a_n is small, we have,

$$\int_a^x \log(1+\eta)d\eta \le \int_a^x \frac{\log(1+\eta)}{(1-\eta)}d\eta \le \frac{1}{2}\int_a^x \log(1+\eta)d\eta.$$

Note that

$$\int_a^x \log(1+\eta)d\eta = [(1+\eta)\log(1+\eta) - (1+\eta)]_a^x$$
$$= (1+x)\log(1+x) - (1+a)\log(1+a) - (x-a).$$

As $a_n \to 0$, both $(1+x)$ and $(1+a)$ approaches zero. Using the limit that

$$\lim_{x\to 0} x\log x = 0,$$

we have

$$\int_a^x \frac{\log(1+\eta)}{(1-\eta)}d\eta \to 0. \tag{28}$$

Substituting the above limits (25-28) into (24), we have

$$U_a(x) = \frac{8}{\sigma_n^4}\left[-3\log(1+a)(\log 2 + o(1)) \right.$$
$$\left. +2\log(2/e)\log 2 + o(1)\right]$$
$$= \Theta\left(\frac{1}{\sigma_n^4}\log(1/a_n)\right).$$

Therefore,

$$Var(T_A^{A/2}) = U_a(x) - V_a^2(x)$$
$$= \Theta(\log(1/a_n)/\sigma_n^4) - \Theta(1/\sigma_n^4)$$
$$= \Theta(\log(1/a_n)/\sigma_n^4).$$

3) The Variance of $\tau_{A/2}^A$*:* Let

$$a = \cos(2\sqrt{\pi}a_n)\ ;\ x = \cos(\sqrt{\pi}a_n).$$

Then,

$$\mathbb{E}[\tau_{A/2}^A] = V_a(x), \text{ and } Var(\tau_{A/2}^A) = U_a(x) - V_a^2(x).$$

Using Lemma 1, it follows that

$$\mathbb{E}[\tau_{A/2}^A] = V_a(x) = \frac{2}{\sigma_n^2}\log\left(\frac{1+\cos(\sqrt{\pi}a_n)}{1+\cos(2\sqrt{\pi}a_n)}\right).$$

Using

$$\begin{aligned}\lim_{x\to 0}\frac{\log\frac{1+\cos x}{1+\cos 2x}}{3x^2/4} &= \lim_{x\to 0}\frac{\log\left[1+\frac{2\sin(x/2)\sin(3x/2)}{1+\cos 2x}\right]}{3x^2/4}\\ &= \lim_{x\to 0}\frac{\log\left[1+\frac{2\sin(x/2)\sin(3x/2)}{1+\cos 2x}\right]}{\frac{2\sin(x/2)\sin(3x/2)}{1+\cos 2x}}\\ &\quad\times\lim_{x\to 0}\frac{\sin(x/2)\sin(3x/2)}{(x/2)(3x/2)}\\ &\quad\times\lim_{x\to 0}\frac{2}{1+\cos 2x}\\ &= 1,\end{aligned}$$

we have

$$V_a(x) = \frac{3\pi}{2\sigma_n^2}a_n^2 + o(\frac{a_n^2}{\sigma_n^2}).$$

We use (24) again to compute the order of $U_a(x)$. By expanding $1/(1+\eta)$ in its Taylor series around 1, and keeping only the first three dominant terms, we obtain

$$\frac{1}{1+\eta} = \frac{1}{2} + \frac{1-\eta}{4} + \frac{(1-\eta)^2}{8} + o((1-\eta)^2).$$

Similarly, we have

$$\log(1+\eta) = \log 2 - \frac{1-\eta}{2} - \frac{(1-\eta)^2}{8} + o((1-\eta)^2).$$

Using the above approximations, and simplifying, we have

$$\frac{2\log(2/e)}{1-\eta^2} - \frac{\log(1+\eta)}{1-\eta} + \frac{1}{1-\eta}$$
$$= \frac{1}{2}\log 2 + \frac{\log 2}{4}(1-\eta) - \frac{1-\eta}{8} + o(1-\eta). \tag{29}$$

Further, using similar Taylor-series expansions (and noting that $o(1-\eta) = o(1-a)$), we have

$$\begin{aligned}\log(1+a) &= \log 2 - \frac{1-a}{2} + o(1-a),\\ \frac{1}{1+\eta} &= \frac{1}{2} + \frac{1-\eta}{4} + o(1-a),\end{aligned}$$

and thus

$$-\frac{\log(1+a)}{1-\eta^2} + \frac{\eta\log(1+a)}{1-\eta^2}$$
$$= -\frac{\log(1+a)}{1+\eta}$$
$$= -\frac{1}{2}\log 2 + \frac{1-a}{4} - \frac{\log 2}{4}(1-\eta) + o(1-a). \tag{30}$$

Substituting the equalities (29-30) into (24), it follows that

$$\begin{aligned}U_a(x) &= \frac{8}{\sigma_n^4}\left\{\int_a^x \left[\frac{1-a}{4} - \frac{1-\eta}{8}\right]d\eta + \int_a^x o(1-a)d\eta\right\}\\ &= \frac{1}{\sigma_n^4}\left\{2(1-a)(x-a) - (x-a)\right.\\ &\qquad\left. + \frac{x^2-a^2}{2} + o((1-a)(x-a))\right\}\\ &= \frac{1}{\sigma_n^4}\left\{\frac{(x-a)}{2}[3(1-a) - (1-x)]\right.\\ &\qquad\left. + o((1-a)(x-a))\right\}.\end{aligned}$$

Since,

$$1-a = 1-cos(2\sqrt{\pi}a_n) = 2\pi a_n^2 + o(a_n^2),$$
$$1-x = 1-cos(\sqrt{\pi}a_n) = \pi a_n^2/2 + o(a_n^2),$$
$$x-a = cos(\sqrt{\pi}a_n) - cos(2\sqrt{\pi}a_n) = 3\pi a_n^2/2 + o(a_n^2),$$

we have $U_a(x) = 33\pi^2 a_n^4/8\sigma_n^4 + o(a_n^4/\sigma_n^4)$. Thus,

$$\begin{aligned}Var(\tau_{A/2}^A) &= U_a(x) - V_a^2(x)\\ &= 33\pi^2 a_n^4/8\sigma_n^4 - 9\pi^2 a_n^4/4\sigma_n^4 + o(a_n^4/\sigma_n^4)\\ &= 15\pi^2 a_n^4/8\sigma_n^4 + o(a_n^4/\sigma_n^4)\\ &= \Theta(a_n^4/\sigma_n^4).\end{aligned}$$

4) The Variance of I_A: Combining the variances of $\tau^A_{A/2}$ and $T_A^{A/2}$, we obtain

$$\begin{aligned} Var(I_A) &= Var(\tau^A_{A/2}) + Var(T_A^{A/2}) \\ &= \Theta(a_n^4/\sigma_n^4) + \Theta(\log(1/a_n)/\sigma_n^4) \\ &= \Theta(\log(1/a_n)/\sigma_n^4), \end{aligned}$$

proving the Proposition. ■

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