A controversy and the writing of a history The discussion of "small oscillations" (1760-1860) from the standpoint of the controversy between Jordan and Kronecker (1874)

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Camille Jordan and Leopold Kronecker were having a great controversy throughout the whole year of 1874, a controversy originally caused by Jordan's ambition to reorganise the theory of bilinear forms through what he designated as the "simple" notion of "canonical form":

> It is known that any bilinear polynomial $P = \sum A_{\alpha\beta} x_\alpha y_\beta$ ($\alpha = 1, 2, \ldots, n$; $\beta = 1, 2, \ldots, n$) can be reduced to its canonical form $x_1y_1 + \ldots + x_my_m$, by linear transformations applied to the two sets of variables $x_1, \ldots, x_n, y_1, \ldots, y_n$. We now consider the following questions:
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> 1. To reduce a bilinear polynomial P to its canonical form by orthogonal substitutions applied to the two sets of variables $x_1, \ldots, x_n; y_1, \ldots, y_n$.
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> 2. To reduce a bilinear polynomial P to its canonical form by the use of the same substitution on the x's and the y's.
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> 3. To reduce simultaneously two bilinear polynomials P and Q to a canonical form.
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> [...] The third problem has already been solved by M. Weierstrass [...] the solutions given by the geometers from Berlin are, nevertheless, incomplete; we will therefore suggest an extremely simple new method that holds no exception [...]. We will show that the problem of the simultaneous reduction of two functions P and Q is identical to the problem of

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> the reduction of a linear substitution to its canonical form. ([6], 7-11, translation F.B.)

In Jordan's 1873 note to the Parisian academy quoted above, the first problem referred implicitly to geometry and Cauchy's results on the principal axis of conics and quadrics, the second one referred to the arithmetic of quadratic forms and the works of Gauss and Hermite, the third problem referred to analytical mechanic and the solution given by Lagrange to the systems $PY'' + QY = 0$ of linear differential equations with constant coefficients. What was at stake in the 1874 controversy was the organisation of the theory of bilinear form, a theory that was considered as giving a new "homogeneous" and "general" treatment to different problems referring to various theories developed throughout the 19th century. According to Kronecker, the main problem of the theory was the characterisation of the equivalence of pairs of forms (P, Q) (Jordan's third problem in the above quotation). Two theorems stated independently by Weierstrass and Jordan could be used to solve this problem and it was their opposition that generated the 1874 controversy.[1]

On the one hand, Weierstrass had defined in 1868 a complete set of polynomial invariants of non singular pairs of bilinear forms that were computed from the determinant $|P + sQ|$ and designated as the elementary divisors of (P, Q). Looking for invariants was a typical method of the theory of bilinear forms developed in the 1860's as a local field of research limited to a few geometers from Berlin such as Christoffel, Kronecker and Weierstrass. It was in the very different context of group theory that, on the other hand, Jordan had stated in 1870 that a linear substitution could be reduced to a simple canonical form. The two theorems were stated independently and belonged to distinct theories until 1873 when Jordan claimed that his theorem could be used in the theory of forms.

The connection between the two theorems was made thanks to a hundred-year-old mechanical problem. In 1766 Lagrange had devised a method for the integration of a system of n linear equations with constant coefficients for the purpose of describing the small oscillations of a swinging string loaded with an arbitrary number of weights. Lagrange's method was based on an implicit mechanical representation: considering that the oscillations of a swinging string loaded with n weights could be represented as the combination of independent oscillations of n strings loaded with a single weight, one should be able to represent the differential system as a combination of n independent equations $dy_i = \sigma_k y_i$ $(i = 1, \ldots, n)$ (σ_k representing the periods of the independent oscillations). Lagrange proved that $\sigma_1, \ldots, \sigma_n$ were the roots of an algebraic equation of the n^{th} degree computed from the differential system, his method therefore failed when multiple roots appeared. When the method was later applied to describe the secular oscillations of the planets on their orbits, the problem caused by the multiplicity of roots questioned the stability of the solar system and generated a century-long discussion around what was to be known as the equation of secular variations.[2]

The meeting point of Weierstrass's *invariants* and Jordan's *canonical form* was their capacity to give a complete resolution to the above problem, regardless of the

[1]Provided that (P, Q) is non singular that is $|P + sQ|$ does not vanish identically and that the forms coefficients belong to an algebraically closed field.

[2]For a history of this *discussion of small oscillations*, see [3] and [2].

multiplicity of roots. Weierstrass first proved in 1858 that the multiplicity of roots was irrelevant to the subject of mechanical stability.[3] His 1868 theorem then gave a complete characterisation of the systems of linear equations with constant coefficients. Jordan reached the same conclusions in the years 1871-1872, see [5]. They both referred to the *different* problems handled by Lagrange, Laplace, Cauchy, Hermite etc. as a *single* problem of *transformation* of pairs of *forms*: the mathematical notion of *forms* unified various problems that were in the past considered to belong to different theories such as celestial mechanic or geometry.

The question at the heart of the controversy was then to decide which methods should be used to organise the theory of bilinear forms: *reducing to canonical forms or computing invariants*? On the one hand, Jordan argued that the main method of the theory should be the algebraic reduction of forms to their *simplest* canonical forms. On the other hand, Kronecker objected that the theory belonged to the *arithmetic* of forms in the tradition of Gauss and the main problem should thus be characterising *equivalence classes* by the *computation of invariants*. Kronecker argued that Weierstrass' invariants could be effectively computed because they were defined as *g.c.d.*s of the minors extracted from the polynomial determinant $|P + sQ|$ [4]; on the contrary, Jordan's canonical form could not be computed effectively, in general, and should thus be considered a "formal notion" with no "objective meaning." With the aim of criticizing Jordan's canonical form, Kronecker wrote a critical history of the algebraic methods used during the *discussion of small oscillations*. These methods were blamed for their tendency to thinking in terms of the "general" case with little attention given to difficulties that might be caused by assigning specific values to the symbols. Weierstrass' invariants were considered by Kronecker to be exemplary of a "truly general" theorem as opposed to the "so called generality" that consisted in focusing on the "general case" where $|P + sQ| = 0$ had no multiple roots. Kronecker was thus the first to stress a history of what the historian T. Hawkins referred to as the "generic reasoning" in the 18^{th}-19^{th} centuries algebra ([4], 122).

The 1874 controversy opposed two theorems that were two different ends given to a century-long mathematical discussion. The controversy therefore combined mathematical and historical arguments and gave rise to the writing of a history of the methods used by Lagrange, Laplace, Cauchy and Weierstrass. A historical study of how Kronecker and Jordan referred to the past sheds some light on some tacit ideals which led to an opposition on what should be the meanings of "generality," "simplicity," "algebra," "arithmetic" and the notion of "form" in mathematics. Even long after the controversy, the tension between canonical forms and invariants played a major role in the development of the theory of bilinear forms and the creation of the theory of matrices. A careful study of the Jordan-Kronecker controversy was used as a preliminary to a wider historical understanding of the development of the theory of matrices in my doctoral thesis (see [1]).

[3] From a contemporary standpoint, the system $Y'' = AY$ is stable if and only if the matrix A is diagonalizable, that is if and if A's elementary divisors are simple.

[4] From a contemporary standpoint, Kronecker defined the invariant factors of a matrix. These invariants can be defined for any matrix with coefficient belonging to a principal ring while Jordan's canonical form requires an algebraically closed field.

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