

Towards Practical and Synthetical Modelling of Repairable Systems

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Abstract: In this article, we survey the developments with respect to generalized models of repairable systems during the 1990s, particularly for the last five years. In this field, we notice the sharp fundamental problem that voluminous and complicated models are proposed without sufficient evidence (or data) for justifying a success in tackling real engineering problems. Instead of following the myth of using simple models to face complicated reality, and based on our own research experiences, we select and review some *practical* models, in the quickly growing areas: age models, condition monitoring models, and shock and wear models, including the delay-time models. Further, we also notice that there is an attempt to develop *synthetical* models from a different point of view. Therefore, we comment the relevant developments with strong emphasis on stochastic processes reflecting the intrinsic nature of the actual physical dynamics of those repairable system models.

1 Introduction

General speaking, a reliability model should be understood as a *dynamic mechanism* which could capture up the actual patterns of the changes (wear, shocks, adjustments, repair, maintenance etc.) and its relative equilibrium states of any physical (machinery) system during its life time. The comprehensive volume edited by Ushakov and Harrison (1994) is worth mentioning which systematically summarized the achievements in reliability modeling before 1990.

Tracing the developments of recent years on repairable system modeling, there are three tendencies to be mentioned: First, a general tendency to perform reliability modeling by imposing a set of assumptions, particularly by imposing a class of lifetime distributions or the transition probability matrix into the related system working environment. It is undeniable that this modeling fashion in today's scientific community is in nature mathematically backed but commercially oriented a malpractice.

Western science has an excellent tradition: Science reveals the laws and nature of our universe and of society and, therefore, the laws discovered are supposed to represent verifiable facts backed by reproducible experiments or observations. However, industrialization, commercialization and globalization are gradually swallowing our scientific tradition, and inevitably also affect the reliability engineering community which according to the

traditional way should be close to the engineering reality. Nowadays, however, it often appears that model developments do not involve sufficiently engineering information reflecting the underlying physical mechanism and, consequently, reliability engineers and managers cannot apply those models.

The second tendency refers to “pure” statistical or stochastic analysis (discussing issues of estimation and hypothesis testing, particularly on repair improvement effect) on repairable systems which hardly can be found in the reliability engineering literature. In certain sense emphasizing Ascher and Feingold’s (1985) view point, expressed more than twenty-year ago, Scarf (1997) seriously points out that at present too little attention is paid to data collection and considering the usefulness of methods for solving problems through model fitting and validation. Too much attention is paid to the invention of new models, with little thought, it seems, as to their applicability.

Finally, we have noticed that more and more researchers concentrate their efforts in developing synthetical models in order to capture a fuller picture of the engineering system operating reality. Synthetical models in general impose less assumptions and have a wider structure so that they are more engineering oriented and practical.

We must point out that substantial progress has been made in mathematical maintenance optimization, for example, by Beichelt (1993), Dekker (1995, 1996), El-Yaniv and Karp (1997), Garren and Richard (1998), Haurie (1995), Pintelon, Van Puyvelde and Gelders (1995), Righter (1996), Sheu (1993, 1994, 1996), Sheu and Jhang (1996), Singh and Singh (1997), Matsushima, Kaio and Osaki (1996), and Van der Duyn Schouten and Vanneste (1995). It should be admitted that there are two basic categories of research, fundamental research and applied research, although there is an evolution between these two categories. Both of them can be further classified into theory and application. Reliability engineering modeling should be an applied research field using extensively mathematical theory as the universal scientific language. However, it should be data-driven rather than mathematical model-driven. Based on such a view, it should not be regarded as “bias” to comment that many varieties of models appeared in maintenance policy making research just behave like a mathematical model competition. We do accept the fact that any purely mathematical model possesses a certain value and any mathematically defined class of maintenance models and the related optimization processes also generate some useful estimates. However, such estimates do generally not reflect the dynamics in truly physical dimensions. In other words, there are two kind of estimates, those based on the system operating and maintenance performance and those based on the model assumptions. The former reflect the engineering reality and the estimates are statistical (data-based) but the latter reflect only the mathematical modeling ideology. Just as Van Noortwijs, Cooke and Kok (1995) pointed out, “in spite of voluminous theoretical researches, very few applications of maintenance optimization were found”.

During the last five years period two important review papers on maintenance modeling appeared: Pham and Wang (1996) and Scarf (1997). Pham and Wang (1996) reviewed more than 40 models on imperfect maintenance modeling and optimization which were published before 1995 and classified them into eight categories.

Table 1: Pham-Wang's Summary

Pham-Wang's Classification	Model initiated by	Features
(p, q) -rule	Beichelt,1976	Homogeneous hybrid model
$(p(t), q(t))$ -rule	Beichelt,1980	Non-homogeneous hybrid model
Improvement factor method	Malik ,1979	Improvement factor model
Virtual age model	Kijima,1989	Virtual age as the key state parameter
Shock model	Esary,1973	Accumulated shock damages
(α, β) -rule	Wang,1996	Fractional life time model
multiple (p, q) -rule	Shaked, 1986	Multivariate hybrid model
Others	Nakagawa, 1979	

The term *hybrid* here means a probabilistic mixture of *same-as-new* and *same-as-old* as Barlow and Proschan specified and homogeneous refer to the case that the probability “weight” is not time-varying. We use the terms same-as-new and same-as-old instead of the popular but misleading terms good-as-new and bad-as-old which were first used by Ascher as an alternative to minimal repair initiated by Barlow and Hunter (1960) respectively as intuitive images of renewal and nonhomogeneous Poisson Processes (NHPP). For example, if the underlying process distribution of a repairable system is Weibull with shape parameter less than one then the hazard function is monotone decreasing and tends to zero if the operating time tends to ∞ . A renewal type repair would make the system's hazard jumping to $+\infty$. In such a case, the system is not really good after renewal. Similarly, bad-as-old is equally misleading for an NHPP with decreasing RCOF which is an improving system. Good-as-new and bad-as-old only apply to repair-improving systems, but not to systems which improve by non-stopping operation.

In Table 1 we changed some of authors' names compared with the original table of Pham and Wang, because to our knowledge the (p, q) -rule and $(p(t), q(T))$ -rule were first proposed by Beichelt in 1976 and 1980, respectively, i.e., much earlier than Pham and Wang (1996) believed. (Beichelt and Franken's monography (1984) systematically summarizes their results).

Without any doubts, the developments listed in Table 1 have enhanced the repairable system modeling literature greatly, but the endless growth of the number of mathematical models may only partly serve to satisfy the demands of industry and business. Unfortunately, the review of models presented by Pham and Wang (1996) is short of insight into the distinct physical nature of repairable systems and reflects their inclination towards abstract mathematics. This may explain the fact that they ignored in their review paper the developments in conditional monitoring models, say, proportional intensities model,

although they might argue that their attention was focused on optimal maintenance policy making. But without taking into account the operating environment (conditions) information, the maintenance optimization plan is merely partial. Actually, only a few authors, say, Love and Guo (1991) performed some exploration on maintenance planning by including covariate information.

Scarf (1997) passionately promoted the engineering information based models and identified three “special growth areas”:

- condition monitoring technique related models
- complex multi-component system modeling, and
- delay-time distribution models.

However, he ignored the quickly growing area of virtual age models, maybe because he believed that virtual age models are not “simple” enough.

What is a simple model? “Simple” model should not be understood as those which appeared in literature early or are of a simple mathematical form. We believe that there is a common twist between the “simple” models in textbook and the “simple” models in reality. For example, renewal processes (same-as-new) and nonhomogeneous Poisson processes (same-as-old) are simple in mathematical terms, because of the *i.i.d.* assumption and, therefore, appear in the most elementary textbooks on stochastic models or stochastic processes. But in industrial environments the assumptions made are rarely satisfied, as Guo and Love (1995) reported. Both same-as-new and same-as-old are very strong assumptions. Blinding with simplicity seeking, misconceptions and malpractices in reliability engineering literature are persistent symptoms, just as exposed systematically in Ascher and Feingold (1984), Ascher (1992, 1999), Ascher and Kobbacy (1995), Ascher and Hansen (1998, 2000). Therefore, the concept of simple models is a very debatable issue. We would argue that any model must necessarily approximate the engineering reality by capturing up the physical intrinsic underlying mechanisms of the repairable systems. The reliability engineering and maintenance management fields need efficient, reflective, good approximating models, i.e., *synthetical* models.

Engineering practice requires that the models should sufficiently well reflect the impacts of operation and repair or maintenance on the reliability of a repairable system. In other words, we need to take an synthetical look at the reliability system, for example at its design and build-in quality, its working environment, its operating pattern, its working load or intensity, and its repair/maintenance pattern and even at the quality of output (products). Synthetical modeling is based on a thorough examination of the reliability concept from an engineering and management point of view. Engineering speaking, the concept of system reliability can be stated by means of a performance or quality measure associated with the dynamic changes of the *system state space* \mathcal{S} . Such a system performance measure can be represented in a variety of indices, for example, stress, pressure, load, availability, mean time between failures, reliability, intensity function etc. From a holistic point of view, a state description and its dynamic changes should guide to all

the modeling efforts for repairable systems (including non-repairable systems as a special case).

The paper by Lawless and Thiagarajah (1996) contains an example illustrating such state space changes in repairable system modeling. They consider a conditional intensity function (CIF) of the form

$$v(t|\mathcal{A}_t) = e^{\underline{\theta}'\underline{z}(t)}$$

where $\underline{z}(t) = (z_1(t), \dots, z_p(t))'$ is a vector of functions that may depend upon both system operating time t and system operating/failure/repair(corrective maintenance) and system operating history \mathcal{A}_t and $\underline{\theta} = (\theta_1, \dots, \theta_p)'$ is a vector of unknown parameters.

It is fairly clear that the system state is reflected in an aggregate manner by the log-linear form $\underline{\theta}'\underline{z}(t)$. To seek a little more intuitiveness, we can look at the special CIF forms of Cox and Lewis' model as examples:

$$\rho(t) = \exp(\alpha + \beta t),$$

and power law

$$\rho(t) = \alpha t^\beta.$$

Then the same-as-old process can be represented as

$$\underline{z}(t) = (1, t)', \underline{\theta} = (\alpha, \beta)' \text{ and } \underline{z}(t) = (1, \log t)', \underline{\theta} = (\log \alpha, \beta)'$$

and the same-as-new process can be represented as

$$\underline{z}(t) = (1, u(t))', \underline{\theta} = (\alpha, \beta)' \text{ and } \underline{z}(t) = (1, \log u(t))', \underline{\theta} = (\log \alpha, \beta)'$$

where

$$u(t) = t - t_{N(t-)}.$$

We can summarize the above results into a table to seek a better picture.

Table 2: State Spaces for Cox & Lewis and Power Law

Regime	State space	$\rho(t) = \exp(\alpha + \beta t)$	$\rho(t) = \alpha t^\beta$
Same-as-new	$\underline{\theta}$ $\underline{z}(t)$	$(\alpha, \beta)'$ $(1, u(t))'$	$(\log \alpha, \beta)'$ $(1, \log u(t))'$
Same-as-old	$\underline{\theta}$ $\underline{z}(t)$	$(\alpha, \beta)'$ $(1, t)'$	$(\log \alpha, \beta)'$ $(1, \log t)'$

Stimulated by the developments in the reliability engineering literature, the repairable system modeling issue is in essence the study of the law of changes in system states. For translating correctly mathematical language, we need to investigate the stochastic behavior of the state space $\{\mathcal{S}_t, t \in \mathbb{R}^+\}$. It is very intuitive that the developments on repairable system modeling in the literature in essence reflect the different angles of looking

at the system state space or a functional of the state space. For example, the state space for the CIF model proposed by Lawless and Thiagarajah (1996) can be represented as

$$\mathcal{S}_t \triangleq \underline{\theta}' \underline{z}(t), t \in \mathbb{R}^+$$

and therefore the stochastic process

$$\left\{ \left(\underline{\theta}' \underline{z}(t), \mathcal{A}_t \right), t \in \mathbb{R}^+ \right\}$$

specifies the CIF model of Lawless and Thiagarajah (1996).

This paper aims at reviewing the recently quickly growing areas in repairable system modeling, particularly the areas working towards integrated and synthetical models reflecting the *dynamics* of real world engineering mechanisms. Intuitively, a (dynamic) stochastic model

$$\mathbf{X} = \{X_t, t \in \mathbb{R}^+\}$$

is a family of random variables X defined on the same probability space (Ω, \mathcal{A}, P) . Stochastic processes are models capturing simultaneously the probabilistic dynamics both in time and state space. The most commonly mentioned stochastic processes investigated in classical reliability engineering literature are renewal processes (same-as-new), Poisson processes (same-as-new or same-as-old), Markov chains, Semi-Markov processes and point processes. However, most of the mathematically simple and, therefore, most often regarded stochastic processes are too narrowly defined and, thus, not effective enough for a comprehensive reliability modeling.

Point processes are defined with less restrictive assumptions and particularly the multiplicative intensity formality allows the integration of the analysis of system intrinsic characteristics and their environmental impacts. Therefore, point processes deserve a critical attention, although they lack some closed mathematical form and mathematical language describing them is difficult at the first glance (a psychological barrier which should be break down in the perception of engineers and managers). However, in a certain sense, point processes should be regarded as simple and practical models leading to an integrated modeling of repairable systems in a realistic sense. Applications of point processes to repairable system modeling are not uncommon, say, Baxter (1996), Finkelstein (1998, 1999), Guo and Love (1994), Last and Szekli (1995, 1998), and Lawless and Thiagarajar (1996).

A point process $\{N_t, t \in \mathbb{R}^+\}$ is said to admit a stochastic intensity $\{v(t), t \in \mathbb{R}^+\}$ with respect to a filtration (history) $\{\mathcal{F}_t, t \in \mathbb{R}^+\}$ with

$$v(t) = \lim_{h \rightarrow 0} \frac{E[N_{t+h} - N_t | \mathcal{F}_t]}{h}, t \in \mathbb{R}^+.$$

Let T_1, \dots, T_n denote the occurrence times of a point process $\{N_t, t \in \mathbb{R}^+\}$ and $X_n = T_n - T_{n-1}$. Then the stochastic intensity of the point process $\{N_t, t \in \mathbb{R}^+\}$ with respect to its natural filtration $\{\mathcal{F}_t^N \triangleq \sigma(N_s, s \leq t), t \in \mathbb{R}^+\}$ is

$$v(t) = \frac{f^{(n+1)}(t - T_n) \vartheta_{(T_n, T_{n+1})}(t)}{1 - \int_0^{t-T_n} f^{(n+1)}(u) du}, t \in \mathbb{R}^+,$$

where

$$\vartheta_A(\omega) = \begin{cases} 1 & \text{if } \omega \in A \\ 0 & \text{if } \omega \notin A \end{cases}.$$

and $f^{(n+1)}$ is a random function satisfying

$$\begin{aligned} F_{X_{n+1}|T_1, \dots, T_n}(x) &= \Pr[X_{n+1} \leq x | T_1, \dots, T_n] \\ &= \int_0^x f^{(n+1)}(u) du, \forall x \geq 0. \end{aligned}$$

Or equivalently,

$$h_n(x|T_1, \dots, T_n) = \frac{f^{(n+1)}(x - T_n)}{1 - \int_0^{x-T_n} f^{(n+1)}(u) du}, x \in [T_n, T_{n+1}),$$

and

$$v(t) = \sum_{n=1}^{\infty} h_n(t|T_1, \dots, T_n) \vartheta_{[T_n, T_{n+1})}(t), t \in \mathbb{R}^+.$$

We should notice here that the intensity $v(t)$ (ROCOF) should not be mixed up with the hazard function denoted by $h(x)$, which is simply the “*first*” piece of $v(t)$. The total or global accumulated hazard is

$$\begin{aligned} A(t) &= \int_0^t v(u) du \\ &= H_{n+1}(t - T_n | T_1, \dots, T_n) + \sum_{i=1}^n H_i(T_i | T_1, \dots, T_{i-1}), t \in [T_n, T_{n+1}) \end{aligned}$$

where $H_{n+1}(t - T_n | T_1, \dots, T_n)$ is called the partial or local accumulated hazard.

With respect to point counting processes, two fundamental properties are worth to be mentioned which lead to greatly simplified applications. First, we notice that a counting process $\{N_t\}$ can be decomposed into two parts:

- a system term, represented by the compensator $\{A_t\}$, a smoothly varying and predictable process, and

- a pure “noise” term represented by a martingale $\{M_t\}$ with an unpredictable zero-mean

$$E(dM_t|\mathcal{F}_{t-}) = 0,$$

i.e.,

$$M_t = N_t - A_t.$$

Secondly, we notice a local Poisson character of a counting process. In other words, under certain conditions (particularly, the continuity and predictability of the compensator A_t are assumed) the conditional mean and the conditional variance of the increment are equal, i.e.,

$$E(dN_t|\mathcal{F}_{t-}) = \text{var}(dN_t|\mathcal{F}_{t-}) = dA_t,$$

where

$$dN_t = N_{dt} \triangleq N_{(t+dt)-} - N_{t-}.$$

These remarkable simple features will not only help us to understand the basic properties of point counting processes but also provide some clues to estimate the parameters of point counting processes.

In order to have a clear view of the structure on this paper, we list the remaining sections and the models reviewed in Table 3.

Table 3: Model Discussion Allocation.

Section	stochastic models reviewed	basic element of \mathcal{S}_t
2	proportional intensity model	\underline{z}_t
3	virtual age model	v_t
4	compounding model	$X_i = \alpha_0 + \sum_{j=1}^{N_i} Y_j$
5	intrinsic age model	$H_i(t)$
6	synthetical model	

In Section 2 models related to condition monitoring are discussed, particularly proportional covariate models of repairable systems. The original version of proportional hazard model initiated by Cox (1972) was not intended to be applied to repairable systems. However, researchers have noticed that the proportional covariate model constitutes a possibility to capture system state changes during system operations and maintenance. Therefore, modeling imperfect repair should not be regarded as a leisurely discussion after dinner and the idea to model general repair by PI (proportional intensities) covariates is very innovative. Further, we review general covariate models of repairable system as

a modeling option which can be treated as a natural extension of PI models which may better reflect the realistic physical dynamics between system and operating environments.

Section 3 is used to discuss the age related generalized models, for example, the popular *virtual age* model initiated by Kijima, which in nature is a kind (or a transformation) of “accelerated life” model. It can be viewed as a state description of a repairable system or an index of system state (like a “bioclock”). Because of its mathematical simplicity and tractability, the virtual age of a repairable system is probably the deepest research topic and hence created a relatively large portion of the generalized repairable models in reliability engineering, or in a more general sense, in the operations research literature.

Section 4 reviews the compounding models and applications, say, cumulative damage and wear (deterioration) models in recent reliability engineering literature, particularly the delay-time models proposed by Christer.

In Section 5, we briefly review the newly proposed *intrinsic age* model because this model provides some deep insights into the failure mechanism of a repairable system. The intrinsic age model may provide a general frame for repairable system modeling because it captures some locality features under Markovian state assumptions.

Finally, in Section 6, we discuss the possible inter-links among the categories and propose synthetical modeling of repairable system.

2 Proportional Intensities Models

Repairable system modeling should be closely connected to reliability engineering practice rather than to purely mathematical reasoning although without a solid mathematical support no correct engineering results can be obtained. However, no matter how important mathematics is, one should always have in mind that mathematics is only a *tool* and that the system operation reality and the information (including, system operation, its working environment, system repair and maintenance etc.) are fundamental as they reflect reality. Actually without full utilization of the engineering information, mathematical models are in the best case merely non-head ghosts floating around and they will not generate any benefits.

From the engineering point of view, the concept of reliability of a system should be interpreted as the engineering *capacity* of a system to complete its engineering *specified functionality* within the *specified time* under engineering *specified conditions*. Therefore, without exploring the four basic elements underlying reliability, three “specifications” and one “capacity”, it is hard to imagine how we can arrive at a thorough understanding of the concept and accordingly how can we reflect the underlying engineering mechanism in our modeling exercises because today we are very rich in many models with excellent mathematical structures but having no links to the engineering information of the functioning system.

1. *Capacity*

The term of capacity (or operability) should quantify the system functioning performance. It should be emphasized that the quantification of capacity for non-repairable and repairable systems differs. In traditional reliability framework, the capacity is quantified in different ways. For example, by means of the lifetime, which probably is the most frequently used quantity from a physical side. Considering the system from a probabilistic point of view, there are reliability and hazard function for non repairable system, average lifetime and “availability”, mean time to failure (MTFF) etc. from a more physical view and again from probabilistic point of view, intensity, rate of occurrences and conditional hazard function.

Although, the term capacity should be a quantity or index directly linked to the physical mechanism of a functioning system, reliability engineering literature paid only little attention to the physical mechanism side. Because nowadays complex systems are more and more modularized, the failure of a system is caused by the failure of one or more modules and, therefore, the repair (corrective maintenance) or maintenance (preventive maintenance) is often a module replacement. Speaking from the viewpoint of probability, it is important to be aware that the phenomena of system malfunctions are chance events. Any chance event is a break down of the law of causality because of the lack of some conditions under which the event occurs inevitably. Therefore, it is senseless to discuss the capacity of a single system. It only makes sense when evaluating the reliability by observing “repeated experiments” of the same type of the system of interest. Hence, the capacity as a reliability concept has a statistical interpretation and is quantified in terms of probabilities in the traditional reliability theory.

2. *Specified condition*

The external or environmental conditions of a system are referred to as “specified conditions”. These conditions usually include the system applying conditions, maintaining conditions, the environmental conditions and the operating conditions of a system. The reliability of the system under consideration may change substantially if these conditions vary. As a matter of fact, the specified conditions are often vague in nature and their records are full of inaccuracies, a fact which should be taken into account.

3. *Specified functionality*

In traditional reliability treatment, the functionality of a system is described by various (designated) functional specifications. If the functionality of the system acts within its specifications, it is said to be functioning otherwise malfunctioning (or losing its specified functionality). It is quite obvious that the traditional definition of system functionality represents a typical “yes” or “no” two-valued logical thinking. In reality, “evolution” or “transition” states exist between the two extreme states. The evolution is simply the break down of the law of excluding the middle.

4. *Specified time*

Without time, it makes no sense to talk about reliability except for one-shot items. Whenever reliability is discussed, it is required to discuss the reliability within a

specified time. Time can be described as deterministic variable or by a random variable. In classical analysis, time is in general repeated as an explanatory variable with no taste, no color, no direction and a non-reversible uniform flow in the real number system. However, as Guo and Frerichs (2000) pointed out, whenever a human (operator) is involved the behavior of a time variable will become complicated and time will have non-uniformity impacts on the human body system and accordingly their behavior impacts due to time changes should be carefully reflected by the model.

Engineering emphasis in reliability modeling of repairable systems should actually reflect the related engineering or operating information and combine efforts linking the “three specified” and “one capacity” features of system reliability to the formalism.

Conceptually, the hazard function is the propensity to fail shortly after x , given that the non-repairable system is successfully functioning up to time x , or mathematically,

$$h(x) = \lim_{\Delta x \rightarrow 0} \frac{\Pr[x < X < x + \Delta x | X > x]}{\Delta x}.$$

On the one hand, it is fairly clear that the basic definition of hazard function merely reveals the aggregate measure of the four aspects of system reliability, mathematically rather than in an engineering way. On the other hand, associated with the quick advancements in science and technology, equipment becomes more and more complicated and quality and reliability requirements for manufacturing operating performances also become higher and higher. In engineering practices, monitoring some or all operation related indices (parameters, variables) over time and taking necessary engineering actions (e.g., adjustments, corrective maintenance, preventive maintenance etc.) whenever the relevant indices move away from their engineering specified ranges become inevitable in order to keep the system’s functionality at its designed target. During the last two decades, combining monitored operating conditions into system analysis and modeling efforts became evident. And, therefore, it became very clear that the basic definition of reliability and hazard function is not enough to reflect or capture the system operating conditions.

Lawless (1983) pointed out: “hazard function represents an aspect of a distribution that has direct physical meaning, and the information about the nature of the hazard function is helpful in selecting a model”. The introduction of the semi-parametric regression model, called proportional hazard (PH) model,

$$h(t; \underline{z}) = h_0(t)g(\underline{z})$$

by Cox (1972) and its sparkling success with respect to medical applications during the 1980s, made the reliability engineering community shake. But only in the late 1980s industrial applications of the PH model were explored, say, Dale (1985), Jardine, Anderson and Mann (1987) Love and Guo (1991a, 1991b, 1994), Percy, Kobbacy and Ascher (1998) etc.. These early developments can be regarded as an *passive* application of PH modeling. The term of a “passive” application here means simply that covariate information is used

for monitoring the system functioning state changes so that the system performance (or capacity) can be judged and decision on the maintenance action can also be justified accordingly. Such an information is partial and incomplete in the sense that covariates reflecting operating conditions are monitored during the functioning and repair resetting stages, but there is no covariate reflecting repair or maintenance impacts. Kumar and Klefsjö (1994) gave an excellent review on the industrial applications of PH modeling before 1994.

The idea of *active* modeling repairable systems via PI (or PH) models is due to Kumar (1995). The term “active” means that the repair or maintenance actions can be clearly reflected in the proportionality as a part of the covariate structure in the models, for example, Kumar and Westberg (1995, 1997). The proportional intensity (PI) modeling of the repairable system assumes that maintenance impacts do not change the form of the baseline system intensity function but shift the intensity “block” vertically along the intensity axis. To reflect such a vertical shift (both operating conditions and repair impacts) in a proportional formality, we can start from the basic form of PI model

$$v(t; \underline{z}) = v_0(t) \exp(\underline{z}' \underline{\theta}).$$

However, a successful (active) modeling of a repairable system with general repair is due to the delicate application of the stratified PH model. Assume that operational conditions may have J different levels, which can be called *strata*, if the covariate effect is not multiplicative, the proportionality of the hazards corresponding to the strata does not hold. The hazard rate for the system in stratum j can be expressed as

$$v_j(t; \underline{z}) = v_{0j}(t) \exp(\underline{z}' \underline{\theta}).$$

If the baseline hazard rate, $v_{0j}(t)$ is completely unrelated in the different strata, the likelihood function of the stratified PHM is

$$L(\underline{\theta}) = \prod_{j=1}^J \prod_{i=1}^{j_i} \frac{\exp(\underline{z}'_{ij} \underline{\theta})}{\left[\sum_{l,j \in F(t_{ij})} \exp(\underline{z}'_{lj} \underline{\theta}) \right]^{d_i}},$$

where the total time t_{ij} may take the form t or $t - t_{N_t}$ depending upon the circumstance of the operating/repair model. In the case that the baseline hazard rate functions are different in different strata, jumps in hazard rates after repair or the effect of operating and repair history can be effectively captured by a stratified PH models.

As to the covariate structure, proportionality is not the only form. Another possibility is, for example, to select an accelerated life model, say, Devarajan and Ebrahimi (1998), which is in nature the shift by a factor $\psi(\mathbf{z})$ (similar to virtual age modeling). More general covariate structures were examined by Ciampi and Etezadi-Amoli (1985). Love and Guo (1995) summarized the general covariate structure in Table 4.

Table 4: Summary of Covariate Models.

model	hazard	reliability
PH	$\xi(\mathbf{z}) h_0(t)$	$\exp[-\xi(\mathbf{z}) H_0(t)]$
AL	$\psi(\mathbf{z}) h_0(\psi(\mathbf{z}) t)$	$\exp[-H_0(\psi(\mathbf{z}) t)]$
SC	$h_0(\psi(\mathbf{z}) t)$	$\exp\left[-\frac{1}{\psi(\mathbf{z})} H_0(\psi(\mathbf{z}) t)\right]$
Unified	$\xi(\mathbf{z}) h_0(\psi(\mathbf{z}) t)$	$\exp\left[-\frac{\xi(\mathbf{z})}{\psi(\mathbf{z})} H_0(\psi(\mathbf{z}) t)\right]$

The term AL model in Table 4 means accelerate life model, while SC stands for the scale model. Our experiences in statistical analysis have shown that likelihood procedures may impose some favor towards PH model even if the underlying physical process is of other type under a certain class of hazard functions. It is also worth to emphasize the contribution of Lawless and Thiagarajah (1996) with respect to the CIF model development

$$\lambda(t; \mathfrak{A}_t) = e^{\underline{\theta}' \underline{z}(t)},$$

where $\underline{z}(t) = (z_1(t), \dots, z_p(t))'$ is a vector of functions that may depend upon both system operating time t and system operating/failure/repair(maintenance) history \mathfrak{A}_t and $\underline{\theta} = (\theta_1, \dots, \theta_p)'$ is a vector of unknown parameters.

The CIF development constitutes in essence a unification of PI and virtual age models (to be reviewed in Section 3), however, in a very special log-linear form. It is very worthwhile to develop a more general CIF form, say,

$$v(t; \mathfrak{A}_t) = \rho(\underline{z}(t); \underline{\theta}),$$

where ρ is a nonnegative mapping of the real line into the positive real line, i.e., $\rho: \mathbb{R} \rightarrow \mathbb{R}^+$.

Further we notice that the CIF model reflects repair impacts to the system having Lawless and Thiagarajah's form (1996)

$$v(t; H_t) = e^{\underline{\theta}' \underline{z}(t)},$$

In nature, the intensity function is determined by "local" or conditional hazard, thus, the proportionality concerned here is essentially a local property. The current conditional hazard functions can be related to previous one by the formula

$$v_{0,n+1}(t) = v_{0,n}(t) \exp\left(\vartheta_{[T_n, T_{n+1})}(t) \lg(1 - \delta_n)\right), T_n \leq t < T_{n+1},$$

where

$$\vartheta_{[T_n, T_{n+1})}(t) = \begin{cases} 1 & \text{if } t \in [T_n, T_{n+1}) \\ 0 & \text{if } t \notin [T_n, T_{n+1}) \end{cases},$$

and

$$\delta_n \in [0, 1], n \geq 1.$$

Therefore, by adding a term $\vartheta_{[T_n, T_{n+1})}(t) \lg(1 - \delta_n)$ to $\underline{z}'\underline{\theta}$, the new covariate structure can well facilitate a mechanism monitoring both the operating conditions and the maintenance impacts which are reflected by the proportionality, although locally. Therefore, the intensity function can be rewritten as

$$v(t; \underline{z}) = v_{0,n}(t) \exp(\underline{z}'\underline{\theta}),$$

where

$$\underline{z}'\underline{\theta} = \vartheta_{[T_n, T_{n+1})}(t) \lg(1 - \delta_n) + \underline{z}'\underline{\theta}.$$

Notice that any repair or maintenance may cause vertical jumps in the intensity function, but it does not destroy the continuity and the predictability properties of the accumulated intensity and, therefore, the local Poisson property still applies here and provides a foundation for statistical estimation and inference.

Also, a reparametrization of hazard function may help to capture the repair impacts. For a log-linear hazard function

$$\rho(t) = \exp(\alpha + \beta t),$$

it can be formulated as

$$\begin{aligned} \underline{z}(t) &= (1, u(t), t_{N(t-)})' \\ \underline{\theta} &= (\alpha, \beta, \gamma)' \end{aligned}$$

and for the power law intensity model

$$\rho(t) = \alpha t^\gamma,$$

it can be formulated as

$$\begin{aligned} \underline{z}(t) &= (1, \log u(t), \log s(t))' \\ \underline{\theta} &= (\log \alpha, \beta, \gamma)' \end{aligned}$$

where

$$s(t) = \frac{t}{u(t)}.$$

It is quite obvious for both hazard models, if $\beta = \gamma$ we have the same-as-old case but if $\gamma = 0$ we have the same-as-new case. If $\beta \neq \gamma$, the CIF for the Cox and Lewis hazard model can be written as

$$v(t; H_t) = \exp[(\gamma - \beta)t_{N(t-)}] \exp(\alpha + \beta t).$$

It can be seen that if $\beta > \gamma$ the term $\exp[(\gamma - \beta)t_{N(t-)}] < 1$, thus, each repair will decrease the intensity function and the system is improving.

Analogously, for $\beta = \gamma$ the system is neutral (minimal repair) and if $\beta < \gamma$ we have $\exp[(\gamma - \beta)t_{N(t-)}] > 1$ and, thus, each repair will increase the intensity function and the system is damaged. For the power law intensity model the CIF can be expressed as

$$\lambda(t; H_t) = \left(\frac{t}{t - t_{N(t-)}}\right)^{\beta-\gamma} \alpha t^\gamma.$$

Similarly, notice that $\frac{t}{t - t_{N(t-)}} > 1$, if $\beta < \gamma$ implying that $\left(\frac{t}{t - t_{N(t-)}}\right)^{\beta-\gamma} < 1$ holds. Thus, each repair will decrease the intensity function and the system is improving.

Again, for $\beta = \gamma$ the system is neutral (minimum repaired) and if $\beta > \gamma$ then $\left(\frac{t}{t - t_{N(t-)}}\right)^{\beta-\gamma} > 1$ and, thus, each repair will increase the intensity function and the system is damaged.

Because the system operating/environment setting may well affect system reliability and performance (statistical process control), the related covariate information should be a part of the system performance history \mathfrak{A}_t . For illustration consider the CIF,

$$v(t; \mathfrak{A}_t) = e^{\underline{\theta}' \underline{z}(t)},$$

where

$$\underline{z}(t) = (1, u(t), t_{N(t-)}, z_4(t), \dots, z_p(t))'$$

and

$$\underline{z}(t) = (1, \log u(t), \log v(t), z_4(t), \dots, z_p(t))'$$

are vectors of functions and

$$\underline{\theta} = (\alpha, \beta, \gamma, \theta_4, \dots, \theta_p)'$$

and

$$\underline{\theta} = (\log \alpha, \beta, \gamma, \theta_4, \dots, \theta_p)'$$

are vectors of parameters for Cox and Lewis and power law intensities, respectively. For more details on this topic, it is worthwhile to mention the work of Andersen, Borgan, Gill and Keiding (1993) summarizing their experiences with modeling point processes with covariate structure.

3 Virtual Age Models

Virtual age can be understood as an aggregate index of the system state space \mathcal{S} . Long ago researchers have noticed that the chronological age as an element of the state space \mathcal{S} does quite often not reflect the characteristic behaviour of the system. For example, consider three cars of the same type and starting running on road at the same time, say, on January 1, 1998, by the 31st of December, end of year 2000. Two of the cars' mileage meter show 100,000 km, and that of the third car only 7500 km. It is obvious that the three cars have the same chronological age, namely 2 years. But the third car with a

7500 km mileage record is more reliable than the first two cars with 100,000 km mileage record. Even for the cars with same mileage records, different driving behavior and road conditions may cause different performances in the future. In order to recognize such an intrinsic character of a system, a concept similar to the bioclock of human beings has been introduced, called *virtual age*. The following table lists the important virtual age models developed during last decade.

Table 5: Virtual Age Models

Model	Virtual age after repair $v(t_i + 0)$	Authors
Model I	$v(t_{i-1}) + (1 - \delta_i) x_i$	Kijima (1989)
Model II	$\delta_i (v(t_{i-1}) + x_i)$	Kijima (1989)
Improvement factor	$v(t_{i-1}) + x_i - d$	Stadje & Zuckerman (1991)
General	$\phi(v(t_{i-1}) + x_i)$	Finkelstein (1993)
Unified	$(1 - \omega_i)v(t_{i-1}) + (1 - \delta_i)x_i$	Guo & Love (1995)
Generalized	$\psi(v(t_{i-1}), x_i)$	Makis (1995), Dagpunar (1998)
Universal	$\varpi (\varsigma(v(t_{i-1})) + \varphi(x_i))$	

Kijima (1989) argued that if the virtual age of the system immediately after the $(n - 1)^{st}$ repair is $V_{n-1} = y$, then the n^{th} failure time of the system follows the distribution:

$$\Pr [X_n \leq x | V_{n-1} = y] = \frac{R(y) - R(x + y)}{R(y)},$$

with reliability function

$$R(x) = \Pr[X_1 > x].$$

This formula has greatly facilitated the probabilistic foundation for the entire virtual age model developments. As to the forms of the virtual age models laid down by Kijima (1989), the n^{th} repair improvements are basically measured by some functional of the virtual age immediately after the $(n - 1)^{st}$ repair, V_{n-1} , and the n^{th} functional time, x_n .

Kijima Model I is a linear combination of V_{n-1} and x_n with coefficients 1 and $(1 - \delta_n)$, respectively. Model I implies that repair work fixes wear and damages occurred within the time-interval (t_{n-1}, t_n) , with the intention that virtual age increment is proportional to functioning time.

Kijima's virtual age Model II implies that repair work fixes all wear and damages not fixed prior to the $(n - 1)^{st}$ repair and those occurred within the time-interval (t_{n-1}, t_n) , with the intention that virtual age increment is proportional to all the accumulated virtual age, $v(t_{n-1}) + x_i$.

The later virtual age model developments are in a certain sense extensions along these two different ages. Finkelstein's general model (1993) is an extension to Kijima's Model

II, while Makis and Jardine's (1993) model extends Kijima's Model I. The unified model proposed by Guo and Love (1995) can be treated as a linear function version of Makis's generalized model. Definitely, the unified model can facilitate the merging point for Kijima's two models. The unified model gives merely some mathematical conveniences and a non-linear form: $\varpi(\varsigma(v(t_{i-1})) + \varphi(x_i))$, named universal virtual age model which can be further developed.

Virtual age models are physically intuitive and mathematically useful. It is clear that under virtual age model assumptions the maintenance impacts do not change the form of the system intensity function but shift the intensity "block" horizontally along the time axis. Therefore, virtual age models assume that repair does not change the system's distinct probabilistic structure, or more accurately, the parameter structure and system lifetime distribution functions are not altered. Repair only changes the virtual age. This assumption is logically convenient for mathematical manipulations but unrealistic in engineering context, because even adjustments on a brand-new system will inevitable change the system parameter setting and accordingly change the probabilistic structure. Simulation studies carried out by Guo and Love (1995) showed that a small repair improvement leading only to a reduction in virtual age of one percent, eventually would make the lifetime distribution function of a repaired system significantly different from the original one of the same-as-old system.

A virtual age model transforms in essence the repair improvement into system age recovery (or shifts the age of the system to the left along the time axis). Combining such an age recovery idea and the local Poisson property of the general point processes Kijima (1989, 1998), Stadje and Zuckerman (1996), Finkelstein (1993, 1998), Makis and Jardine (1993), Love and Guo (1993, 1996), Love, Zhang, Zitron and Guo (2000), Li and Shaked (1997), Jiang, Cheng and Makis (1998), Chen and Feldman (1997) have investigated optimal maintenance policies.

However, a serious question arises in practice: How large is the improvement by repairs statistically? Our simulation results show that even with one percent improvement at each repair, the system will seriously move away from its baseline intensity function. Whitaker and Samaniego (1989) and Guo and Love (1992), Guo and Bradley (1993) have made a series of attempts to estimate the repair improvement δ . However their experiences indicate that virtual age models often do not allow stable statistical estimation. A chilly fact about the statistical analysis involving the virtual age models is that the estimator of δ (average repair improvement) is not stable and sometimes the variance is very large such that no statistical significance can be obtained. One of the reasons is the nonlinearity in δ which increases its power by one after each repair-piece-wise polynomial in δ . Actually the virtual age structure is externally imposed on the assumed probabilistic structure of the system and it is not necessarily a distinct part of the point process which may cause the semi-positive definite second derivative matrix. Non-linear dynamics may cause a chaotic behavior here. Therefore, explorations of general properties of failure/repair processes have gained more attention in recent years, for example, Boland and El-Newehi (1998), Dagpunar (1998), Gong, Pruettt, Tang (1997), Kijima (1998), Li, Shi and Cao (1997), Love and Guo (1994), Guo and Love (1994), and Stadje and Zuckerman (1996) etc..

At this point the fact should be acknowledged that the most important development in virtual age model research is due to Last and Szekli (1998), who successfully developed a general form of an age model based on the dynamics of marked point processes (Last and Brandt (1995)).

The successive accumulated hazard functions for distributions of the sojourn times (inter-arrival times) of N_t on the filtration (history) $\{\mathcal{F}_t^N, t \in \mathbb{R}^+\}$ is

$$\begin{aligned} & H_{n+1}(t | (t_i, z_i), i = 1, \dots, n) \\ &= \int_0^t \frac{\Pr[X_{n+1} \in ds | (T_i, Z_i) = (t_i, z_i), i = 1, \dots, n]}{\Pr[X_{n+1} \geq s | (T_i, Z_i) = (t_i, z_i), i = 1, \dots, n]}, n \geq 1, \end{aligned}$$

where $\Phi \triangleq \{(T_n, Z_n), n \geq 0\}$ is the marked point process with a sequence of non-negative random variables $\{Z_n, n \geq 0\}$ called marks. For $n = 1$, the accumulated hazard function is given by

$$H_1(t) = \int_0^t \frac{\Pr[X_1 \in ds]}{\Pr[X_1 \geq s]}, t \geq 0.$$

It is obvious that the conditional hazard function of sojourn times is

$$\begin{aligned} & h_{n+1}(t | (t_i, z_i), i = 1, \dots, n) \\ &= \lim_{s \rightarrow 0} \frac{\Pr[X_{n+1} \in ds | (T_i, Z_i) = (t_i, z_i), i = 1, \dots, n]}{\Pr[X_{n+1} \geq s | (T_i, Z_i) = (t_i, z_i), i = 1, \dots, n]} \end{aligned}$$

and thus we obtain the stochastic intensity with respect to the internal filtration $\{\mathcal{F}_t^\Phi, t \in \mathbb{R}^+\}$

$$\lambda(t) = \sum_{n=0}^{\infty} \vartheta_{(T_n, T_{n+1})}(t) h_{n+1}(t - T_n | (T_i, Z_i), i = 1, \dots, n)$$

and the compensators

$$\begin{aligned} A(t) &= H_{n+1}(t - T_n | (T_i, Z_i), i = 1, \dots, n) \\ &\quad + \sum_{i=1}^n H_i(X_i | (T_j, Z_j), j = 1, \dots, i - 1), \end{aligned}$$

which is the total hazard accumulated by the system up to time t .

The virtual age process $\{V_t, t \in \mathbb{R}^+\}$

$$V_t \triangleq V_n + t - T_n, \quad T_n \leq t < T_{n+1}$$

with

$$\begin{aligned} V_n &\triangleq (1 - Z_n)(V_{n-1} + X_n), \\ V_0 &= 0, V_1 = (1 - Z_1)T_1, \end{aligned}$$

where Z_n is the degree of repair received by the system at the time of the n^{th} failure,

$$X_n = T_n - T_{n-1}.$$

The distribution of the time until the next failure admits

$$r_{n+1}(y) \triangleq r(V_n + y), \quad t \geq 0, \quad n \geq 0,$$

where the failure intensity r is a positive real-valued function $r : \mathbb{R}^+ \rightarrow \mathbb{R}^+$.

It is obvious that the virtual age process $\{V_t, t \in \mathbb{R}^+\}$ catches the system state changes during the system functioning and repairing but in general it is not directly observable. The observations of the point process are given by the sequence of pairs $\{(T_n, Z_n), n \geq 1\}$. In general, the degree of repair at the n^{th} failure Z_n depends upon the whole history of functioning and repairs up to the n^{th} failure time T_n , i.e.

$$Z_n = f(T_1, Z_1, \dots, T_{n-1}, Z_{n-1}, T_n).$$

The nature of the random sequence $\{Z_n, n \geq 1\}$ specifies the properties of the virtual age process $\{V_t, t \in \mathbb{R}^+\}$. The following table summaries the impacts of degree of repair sequence $\{Z_n, n \geq 1\}$:

Table 6: Virtual Age Models Covered by Virtual Age Processes.

	Degree of repair $Z_n, n \geq 1$	virtual age V_t ,	Model
i	$Z_n \equiv 1$	t	Same-as-new
ii	$Z_n \equiv 0$	$T_n + t$	Same-as-old
iii	$Z_n = \begin{cases} 1 & \text{with probability } p \\ 0 & \text{with probability } q \end{cases}$	$(1 - Z_n) D_n + t$	Barlow & Proschan
iv	$Z_n = \frac{(1-K_n)X_n}{K_1X_1+\dots+K_{n-1}X_{n-1}+X_n}$	$\sum_{i=1}^{n_t} K_i X_i + t$,	Kijima Model I
v	$Z_n = 1 - K_n$	$\sum_{i=1}^{n_t} \left(\prod_{j=i}^{n_t-j+1} K_j \right) X_i + t$	Kijima Model II
vi	$Z_n = D_n^{-1}(U_n, T_n)$		Block et al.
vii	$Z_n = D_n^{-1}(U_n; (T_i, Z_i), i \leq n-1, T_n)$		Shaked et al.
viii	$Z_n = \frac{g(V(T_n-))}{V(T_n-)}, g(x) \leq x$	$\sum_{i=1}^{n_t} \left(\prod_{j=i}^{n_t-j} (1 - Z_j) \right) X_i + t$	Stadje & Zuckerman

In the above table, the set $\{D_n\}$ denotes a certain class of distributions with support in $[0, 1]$, $\{K_n\}$ denotes a family of random variables with values in $[0, 1]$, and $\{U_n\}$ are i.i.d. uniform distributed random variables in $[0, 1]$.

The development in the theory of marked point processes has unified almost all existing simple virtual age models. Moreover, the difficulties in estimating the repair effects are eased off to a certain extent under this frame. Some non-parametric or semi-parametric estimation on repair effects can be done in terms of semi-martingale inference techniques, say, Rao (1999). Our research experiences have shown that any artificial mathematical

structures imposed on an engineering system always cause some problems in statistical estimation and hypothesis testing and, therefore, we do not discuss further the estimation issue under the Kijima's and other models. We also point out that there are some classes of stochastic processes, say, Lévy processes, including its special cases, stable processes, having attracted particular attention in mathematical finance which could open more space in repairable system modeling because such class itself permit the *jumps* intrinsically, see for example, Bertoin (1996) and Zolotarev (1986).

4 Compound Models

Another way to embed engineering information into system's performance and survivability is shock damages and wear which are often observable or at least partially detectable in terms of an engineering sensors system and treated as accountable causes demanding maintenance attentions. The physical damage measure process $\mathbf{X} = \{X_t, t \in \mathbb{R}^+\}$ is in nature a compound dynamic model

$$X_t = \alpha_0 + \sum_{i=1}^{N_t} Y_i,$$

where $\{N_t, t \in \mathbb{R}^+\}$ is a counting process.

The basic shock damage model assumes that a system survives the first k shocks, $k = 0, 1, 2, \dots$ with probability \bar{P}_k , where the shocks occur according to a Poisson process. The reliability is given by

$$R(t) = \sum_{k=0}^{\infty} \bar{P}_k \frac{(\lambda t)^k}{k!} e^{-\lambda t}, \lambda > 0.$$

Such models were investigated already in the 1950s by Esary and well presented by Esary, Marshall and Proschan (1973). During the last decade the model was extensively used for facilitating determination of maintenance policy as described in the papers by Kijima and Nakagawa (1991), Sheu and Liou (1992), Feng, Adachi and Kowada (1994).

Hopp and Kuo (1998) presented in their paper a very detailed coverage on the cumulative shock model application to the maintenance of an aircraft engine compressor. They pointed out that the model usage is motivated by the fact that it facilitates a "good prediction of the times until failure of components", and "requires relatively few estimated parameters and therefore has relatively simple data requirements", and more importantly, "shocks can be defined so as to be easily monitored using modern aircraft sensors". The engineering observable (sensor) records and accordingly defined shock information provide the foundation for seeking a mathematical model and subsequently for determining an optimal maintenance policy.

The physical process descriptors are identified and quantified in a first step: crack increments and shock occurrences. The crack increments, Y_i , are quantified by a distribution of Polya frequency of order 2 representing the crack growth and the variability in growth as well. Further the characterization of the occurrence of shocks during a flight is balanced between engineering and mathematical consideration. It is assumed that a base rate λ catches “the average number of times the stress on the components exceeds some threshold. . . . because crack growth tends to accelerate with crack size, we must adjust this rate with a non-stationary function $s(\alpha, k)$ ” with α representing crack size found at most recent inspection or replacement and k being the number of shocks recorded since last inspection. With the notation of

$$A(\alpha, k) = \alpha + \sum_{i=1}^k Y_i$$

being the crack size found at inspection when the state is (α, k) . The distribution of $\Delta k(\alpha, k)$, the number of shocks during a flight, is a Poisson $(\lambda s(\alpha, k)) + 1$ random variable. “Since the function $s(\alpha, k)$ offers more degrees of freedom than we are likely to need, there will be considerable latitude in defining this function so as to fit the data.” Let the critical crack size ξ be a random variable so that if the crack size is larger than ξ , then a failure occurs. The probability of failure in flight when the state at the beginning of the flight is (α, k) given by

$$p(\alpha, k) = \Pr \left[A(\alpha, k) + \sum_{i=k+1}^{k+\Delta k(\alpha, k)} Y_i > \xi \right].$$

Compounding models are typically handled in terms of the Poisson occurrence assumption, but it is not necessary to commit ourselves to too many model assumptions. In modeling practices, the general point process occurrence should be assumed but the numerical evaluation can be performed in terms of compound Poisson approximation, see for example, Barbour, Chryssaphinou and Malgorzata (1996).

For a repairable system, the time from the moment a defect can be identified by an inspection to the moment at which the defect causes a system failure is said to be a *delay time* which was initiated by Christer and Wang (1995a, 1995b), and Christer, Wang, Baker and Sharp (1995). Just as Wang (1997) pointed out “the delay time concept defines a two stage stochastic process where the first stage is the initiating phase of a defect, and the second is the stage where the defect leads to a failure.” Assuming that the defects occur and can be identified follow a point process $\{N_t, t \in \mathbb{R}^+\}$ and denoting the i^{th} delay time by $Y_i, i = 0, 1, \dots$ and the inter-arrival time between the i^{th} and $(i + 1)^{st}$ defects by $X_i, \{X_i, i = 1, 2, \dots\}$, then the delay time is given by

$$Y_{N_t} = L - \sum_{i=1}^{N_t} X_i,$$

where L is the lifetime beginning with the start of a defect-free system. The delay time is again a compound dynamics similar to that of cumulative shock damage processes al-

though the current treatments in literature have not touched the compound characteristic so far.

In the work of Scarf (1997), a very simple two-parameter delay time model is mentioned. It assumes Poisson process defect arrivals with rate α ; exponentially distributed delay-time with mean $\frac{1}{\gamma}$; perfect inspection with equal spaced inspections Δ). For a system observed over $(0, T)$, the equation

$$\begin{cases} \widehat{\alpha}T = n \\ \frac{(n-k)\widehat{\gamma}\Delta}{\exp(\widehat{\gamma}\Delta)-1} + \frac{\sum \widehat{\gamma}t_j}{\exp(\widehat{\gamma}t_j)-1} = n - k \end{cases}$$

gives the maximum likelihood estimates, for k failures observed at times t_i ($i = 1, 2, \dots, k$) and $n - k$ defects found at inspections.

Cerone (1993) summarized a modified form of the delay time model for reliability $R_\Delta(t)$ due to a periodic inspection every Δ time units

$$R_\Delta(t) = r_\Delta^{(m)}(t), \quad (m-1)\Delta \leq t \leq m\Delta,$$

where

$$\begin{aligned} r_\Delta^{(m)}(t) &= \sum_{j=1}^{m-1} \kappa_j(\Delta) r_\Delta^{(m-1)}(t - j\Delta) + B_\Delta(t), \\ \kappa_j(\Delta) &= \int_{(j-1)\Delta}^{j\Delta} g(y) M(j\Delta - y) dy, \\ B_\Delta(t) &= \int_t^\infty g(y) dy + \int_{(m-1)\Delta}^t g(y) M(t - y) dy, \\ M(x) &= \int_t^\infty f(u) du = 1 - F(x) = S(x). \end{aligned}$$

Cerone further developed the converse problem in terms of the delay time model, which in its simplest form is as follows: Given the number of inspections, determine the inspection interval Δ that will result in the maximum reliability at some future time-point $t = t^*$. In other words, the converse problem to be addressed here is to find the optimal inspection interval Δ_{\max} given a desired number of inspections $m-1$, by assuming that the system is inspected at fixed periods of length Δ and the system is replaced or repaired as good as new if the system is found to be defect. The optimal length of the inspection interval Δ_{\max} necessarily satisfy the following equation by specifying $t = t^*$:

$$r_\Delta^{(m)}(t^*) = \sum_{j=1}^{m-1} \kappa_j(\Delta) r_\Delta^{(m-1)}(t^* - j\Delta) + B_\Delta(t^*), \quad \frac{t^*}{m} \leq \Delta \leq \frac{t^*}{m-1},$$

where $\kappa_j(\Delta)$ and $B_\Delta(t^*)$ are defined as above.

The work of Van Noortwijk, Cooke and Kok (1995) contains a model of a wear (deteriorating) process for a repairable system which is worth mentioning. Although they aimed at developing a new failure model for hydraulic structures, the spirit of their research points out the direction how to develop a model for a repairable system by utilizing the engineering based information. They note that their “model is based on amounts of deterioration appearing per unit time; on quantities that can be observed, or about which engineers are able to have a subjective opinion.” The fact that there is “a lack of deterioration data is common at the outset” in the field of hydraulic engineering implies that “most maintenance decisions are only based on ideas about average wear-out”.

By assuming a l_1 -isotropic stochastic deterioration process, Van Noortwijk et. al. (1995) obtained the probability of failure without inspection, the probability of failure with inspection and the probability of preventive repair distribution under Bayesian decision theory for the rock-fill top-layer of the bed protection of the Eastern Scheldt Storm-Surge Barrier. Their results clearly indicate that the repairs do change the failure behavior of the wear-out process.

Compound models have gained substantial attention in financial modeling, particularly in insurance industry. Thus, managers and reliability engineers should widen eyes and learn about the computational experiences made in the financial community to arrive at a better engineering modeling practice.

5 Intrinsic Age Models

It is a well-known fact of engineering practice that the task of identifying the formal state space process and subsequently the respective probability law is not trivial for managers and reliability engineers. Therefore, it would be very helpful to have some guideline to have some *prior* forms on hand for repairable system modeling. Such a prior form should not be treated as a frame to force the engineering reality into it, rather it should be treated as a guideline for helping us to grab the dynamic mechanism in the engineering reality.

The *hazard rate* is an intrinsic index of a non-repairable system state in reliability engineering literature. Finkelstein (2000) pointed out that the hazard rate is so important and convenient in lifetime data analysis because it is in essence a *conditional* concept. Furthermore, we notice that for a minimal repair process, the form of the process intensity function enjoys a simple link to the hazard function, a natural function extension. We also notice that the “integrated hazard function” or “accumulated hazard” , $H(x)$, which is a non-decreasing function of x , is actually a quantity measuring the accumulated state change due to non-repairable system operation and wear out, (more accurately speaking accumulated hazard function is a function of the system state space), because the non-repairable system reliability function is

$$R(x) = e^{-H(x)} = \exp \left(- \int_0^x h(u) du \right).$$

For a non-repairable system, the larger the values of the hazard function are over a given time interval, the larger is the accumulated hazard and the lower is the reliability of the system.

For a repairable system, after n failure-repair activities, the system reliability for the first $(n + 1)^{st}$ failures does not depend upon the “total or global” accumulated hazard (compensator), $A(t)$, but upon the “partial or local” accumulated hazard, $H_n(t)$. Notice that maintenance or repair is intended to remove the faults of the system. Sometimes it improves the system so that the system local (conditional) accumulated hazard is decreased and accordingly the system local (conditional) reliability is enhanced. In other words, repair or maintenance actions sometimes shrink the value of system local accumulated hazard in terms of improving the conditional hazard function and hence enhancing system performance. However, we have to admit that repair actions sometimes have the opposite effect, for example, the most likely time a car needs repair is immediately after it has been “repaired”. To emphasize the role of (local or conditional) accumulated hazard, i.e., the cumulated hazard accumulated in time during the operation of the device in the randomly varying environment, Cinlar and Özekici (1989) identified a system intrinsic clock which ticks differently in different environments to measure the functioning age of the device and named it *intrinsic age* of a system to “honor” this unique quantity. The intrinsic lifetime of any device has an exponential distribution with unit parameter and, therefore, Cinlar and Özekici treated it as a critical system lifetime index for reliability modeling.

In the paper of Özekici (1995), the fundamentals of the intrinsic age model are systematically reviewed. The model assumes that $\{Y_t, t \in \mathbb{R}^+\}$ is the minimal process associated with the Markov renewal process $\{(\xi_n, T_n), n \geq 0\}$ on the state space $\mathcal{S} = E \times \mathbb{R}^+$ with semi-Markov Kernel Q :

$$Q(i, j, x) = \Pr [\xi_{n+1} = j, T_{n+1} - T_n \leq x | \xi_n = i], \quad i, j \in E, x \geq 0,$$

where E is the discrete state space of the Markov chain $\{\xi_n, n \geq 0\}$ with transition probability matrix $(p_{ij}) : p_{ij} = \Pr[\xi_{n+1} = j | \xi_n = i]$. Furthermore, $\{(\xi_n, T_n), n \geq 0\}$ is assumed to have an infinite lifetime so that $\sup_n T_n = +\infty$ and $p_{ij} = Q(i, j, +\infty)$.

Let τ represent the lifetime of the system and Υ_t be the intrinsic age at time $t \geq 0$. Then the reliability of the system for state $i \in E$ is given by

$$R_i(t) = \Pr [L > t | \xi = i], \quad i \in E, t \geq 0,$$

with conditional hazard function $h_i(t)$ and conditional accumulated hazard

$$H_i(t) = \int_0^t h_i(u) du$$

and

$$R_i(t) = e^{-H_i(t)}.$$

From the above arguments, an intrinsic clock of the system is defined as

$$\Upsilon_t = H_i(t).$$

In terms of the system intrinsic age, the real lifetime is obtained as

$$\tau = \inf \{s \geq 0; \Upsilon_t > \tilde{\tau}\},$$

where $\tilde{\tau}$ is a random variable representing the intrinsic lifetime. Furthermore, for the system reliability we get:

$$\begin{aligned} R_i(t) &= \Pr[\tau > t | Y = i] \\ &= \Pr[\tilde{\tau} > \Upsilon_t | Y = i] \\ &= e^{-\Upsilon_t}. \end{aligned}$$

Accordingly, the reliability of a system can be described by the intrinsic age process $\{\tilde{\tau}, t \in \mathbb{R}^+\}$ and its probability law governing the stochastic process. Research on the intrinsic age process is facilitated by the circumstance that for any given environment state $i \in E$, the system has an exponential distributed intrinsic lifetime with parameter 1. As a matter of fact, the intrinsic age is simply the transformed (by conditional accumulated hazard $H_i(t)$) real lifetime under environment state $i \in E$, similar as in the theory of Poisson processes where the transformed sojourn lifetime in a non-homogeneous Poisson process under compensator is a homogenous Poisson process with unit parameter.

Let

$$H_i^{-1}(x) = \inf \{t \geq 0; H_i(t) > x, i \in E\}.$$

Then the intrinsic age process $\{\tilde{\tau}_t, t \in \mathbb{R}^+\}$ is the following non-decreasing continuous stochastic process

$$\tilde{\tau}_{T_n+s} = \begin{cases} H_{X_n}(H_{X_n}^{-1}(\Upsilon_{T_n}) + s) & \text{if } \Upsilon_{T_n} < a_{X_n}, \\ \Upsilon_{T_n} & \text{if } \Upsilon_{T_n} \geq a_{X_n}, \end{cases} \text{ with } X_n \in E$$

where

$$\begin{aligned} a_i &= \lim_{t \rightarrow \infty} H_i(t), i \in E, \\ s &\leq T_{n+1} - T_n, n \geq 0. \end{aligned}$$

It is clear that for given failure rate function $\{r_i(\cdot); i \in E\}$ and a realization of the environment process Y , the intrinsic age process is completely defined. Therefore, the intrinsic age model can be used as a convenient tool for extracting the system dynamics information (system intrinsic age). As soon as the intrinsic age exceeds a random threshold which is also exponentially distributed with unit parameter, the new deterioration state of the system is observed and a repair or maintenance action is carried out and thus the system state is changed. If a repair improvement of the system is treated as a change in the system's "environmental state" from i to j , $i, j \in E$, (with the assumption that the state space E is still discrete, although far from reality), and a state-related lifetime distribution

as well as an intrinsic age is assumed for every state, then the intrinsic age process can be used as a model for a repairable system. Özekici (1995) just followed the idea to pursue “the optimal repair problem” in terms of a dynamic programming approach because the imposed Markovian structure fits particularly well for such an optimization scheme. A further advantage of the intrinsic age model is that under such a modeling frame we may estimate n separate marginal lifetime distributions, each for one component, rather than to pursue the complex joint n -dimensional distribution for the whole multi-component system.

Finally, we do feel that the assumption of a discrete state space E of the intrinsic age model may not be realistic in certain situations. However, semi-Markov models are widely used for modeling repairable systems. Thus, a main problem in the intrinsic age process modeling is the specification of the state space E , because of the crisp set definition which excludes a possible state evolution during a device’s operation. It would be desirable to have less rigid state space requirements. However, this would demand for the development of a different approach.

6 Concluding Remarks

As illustrated, both the models of the repairable system based on the intrinsic age as well as those based on the virtual age address the state space issue from a mathematical point of view as the state parameters are abstract and invisible (mathematical) descriptors. It is an unfortunate situation that the development of models in reliability engineering sailed away from the physical features emerging from engineering practice. Just as Scarf (1997) pointed out “the modelers themselves take a low priority than understanding the process of interest of decision maker. This has not often been the case in the past where, even with the best intentions, so called applications and case studies appear to have been motivated by the need to find an application for a particular model, rather than by the solution of the problem of the interest to the engineer or manager.”

In this paper, we have tried to cover most generalized models in repairable system modeling, particularly concentrating on “hot” spots as well as on practical models with the exceptions of the multi-component repairable system modeling and software reliability modeling during the recent five years’ period. The manner we reviewed the models is not exhausting, but we tried to seek their common underlying physical and engineering mechanism, i.e., the stochastic models approximating those operating/idling (even cold idling)/failing/maintaining life paths. We mentioned some of our own works before 1995 to emphasize our own backgrounds and experiences for our criticisms and suggestions in repairable system model developments. Some master pieces of work may not be covered here because of the authors’ limited knowledge and we apologize for it.

The reason why we emphasized the importance of synthetical and practical models rather than mathematical simple models is two fold: The first one is the need of engineers and

managers to get models able to reflect their process information or data. The second point is that the theory of stochastic models is often not readily available to engineers and managers. We review the models and their underlying mechanism in an interpreting and informal manner and not tied to the rigorous mathematical developments.

The intrinsic age process with a Markov renewal process foundation is an extremely simple tool linked to hazard and accumulative hazard concepts and thus will be an useful frame for a wide range of repairable system modeling. The physical role of intrinsic age is different from virtual age which is more global (similar to a bioclock in human body). Intrinsic age is a more local concept and it may better illustrate the instantaneous failure behavior. On the other hand the virtual age is invisible but does function within every system. Virtual age models enjoy great attention among researchers, but the model structures imposed for virtual age may cause some problems as we pointed out in this paper. In certain sense, the introduction of marked point processes provides virtual age models with more insight. Particularly, the marked point process approach may be reasonably justify all the statistical inference issues, say, estimation, stability of the estimation etc. The condition monitoring models for repairable systems are an excellent hybrid of mathematical theory and engineering information.

Currently, as a part of system design, a lot of system with sensors, self-checking mechanism is built-in but those information are largely ignored by system reliability modeler. Practical experiences of using proportional hazards (intensities), accelerating life models and possible more general modeling structures not only won the heart of engineers and managers but also promoted more interest on theoretical research. In this aspect, the book of Andersen, Borgan, Gill and Keiding (1993) provides a very general guideline, although their models are not specific for reliability modeling and their examples are all taken from medical research fields. As to the criticism mentioned in Scarf (1997), in practice intuitive engineering knowledge always can help to overcome difficulties. For example, a properly combined virtual age model and a proportional intensity mixture may lead to a more efficient model, just as Guo and Love (1994, 1995) in their simulation paper demonstrated. The cumulative damage and wear processes and their close cousins, the delay-time related processes, are in nature compound stochastic models. Some aspects of them were fully explored but it would be worthwhile to devote more efforts to investigate delay-time related compound processes. Although some progress has been made in statistical analysis, for example, Silver and Fiechter (1995), Guo and Love (1996), Tang, Tang and Moskowitz (1996), Costantini and Spizzichino (1997), Gasmyr (1998), and Xie and Lai (1998) etc., much further work has to be done in order to arrive at satisfactory procedures.

It is reasonable to say that the two extreme cases: same-as-new and same-as-old are best explored because of some mathematical simplicity. However, those two extremes are often not realistic. One of the reason why they were popular is the fact that they well reflect the yes or no classic Cantor set theory logic. Practice cannot rule out the states evolving between the perfect repair and the minimum one, therefore, new concepts must be introduced to reflect reality although some further mathematical complexity has to be induced.

It is worth mentioning that the four categories of stochastic models reviewed here are evolving into the synthetical modeling direction for better approximating the operating/idle/maintain (repair) dynamics of repairable systems. But we are afraid to say that restricting attention to any specific category may cause the modeling practice forcing reality to fit into a model frame - just like putting an adult's body into a child's coffin! Therefore, it would be better to merge the models into some hybrid versions. Say, the point process with multiplicative stochastic intensity developed by Andersen, Borgan, Gill and Keiding (1993) and that of Last and Brandt (1995) can be merged. A counting process $\{N_t, t \in \mathbb{R}^+\}$ is said satisfying multiplicative intensity model if

$$\lambda_{ij}(t; \underline{\theta}) = \alpha_{ij}(t; \underline{\theta})Y_{ij}(t), i = 1, \dots, m, j = 1, 2, \dots, l$$

where $\alpha_{ij}(t; \underline{\theta})$ is deterministic and $Y_{ij}(t)$ is a predictable process which is observable in the sense it does not depend upon the parameter $\underline{\theta}$. On the other hand, the marked process setting $\{(\tau_n, Z_n), n \geq 0\}$ with $\{\tau_n\}$ as stopping times associated with the counting process $\{N_t, t \in \mathbb{R}^+\}$, and mark sequence $\{Z_n\}$ record the marks related to the history of the point process $\{\mathcal{F}_n\}$ and current changes to the system. Then, intuitively, modeling of the system dynamic changes includes its operating behavior, its functioning conditions (by covariates) and the repair/maintenance impacts (by marks) and therefore it would approximate the engineering reality better and in nature it is a synthetical model. The only question left is how to obtain the marks $\{Z_n\}$ and how to quantify them. Practically, $\{Z_n\}$ may be obtained indirectly by information from system sensors and the repair/maintenance cost structure. Our past modeling experiences have exercised partial merging models and more results will be reported soon.

The last point we would like to rise is that all the above mentioned stochastic models for repairable systems are often evaluated under the reliability and related cost *standing alone* environments. We use the term, standing alone, to indicate that the repairable system modeling does generally not include the environmental issue, i.e. the location of the system and the aim the system is installed for. Actually reliability is a part of the total quality measures of product or service, while the quality issue itself is subject to the total business environment. Over-emphasizing the optimization of the reliability index and its related cost function is questionable in practice. Researchers in recent years begin to realize that the current outlook of reliability modeling may lead to nowhere. For example, Sonin (1996) reveals that increasing the reliability of a machine reduces the period of its work. Therefore, a holistic view on reliability and quality improvement research should put repairable system modeling under the overall business and production environments and interested reader can reference the work by Ushakov and Harrison (1994), Göb, Beichelt, Dräger, Ramalhoto and Schneideman (1994), Wang, Gary and Scott (1996) and Diederid, Jan and Hontelez (1997).

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