On the number of edges of quadrilateral-free graphs

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Abstract

If a graph has \( q^2 + q + 1 \) vertices \( (q > 13) \), \( e \) edges and no 4-cycles then \( e \leq \frac{1}{2}q(q+1)^2 \). Equality holds for graphs obtained from finite projective planes with polarities. This partly answers a question of Erdős from the 1930’s.

1 Results

Let \( f(n) \) denote the maximum number of edges in a (simple) graph on \( n \) vertices without four-cycles, (i.e., quadrilateral-free). Erdős [6] proposed the problem of determining \( f(n) \) more than 50 years ago, and still no formula appears to be known. McCuaig [15] calculated \( f(n) \) for \( n \leq 21 \). Clapham, Flockart and Sheehan [4] determined all the extremal graphs for \( n \leq 21 \). This analysis was extended to \( n \leq 31 \) by Yuanzheng and Rowlinson [18] by an extensive computer search. Asymptotically \( f(n) \sim \frac{3}{2}n^{3/2} \) (see Brown [2] and Erdős, Rényi and T. Sós [10]).

If \( q \) is a prime power and \( n = q^2 + q + 1 \), then a graph with \( n \) vertices and \( \frac{1}{2}q(q+1)^2 \) edges and no 4-cycles can be constructed from a projective plane of order \( q \) (the polarity graph, defined first by Erdős and Rényi [9], see below in Section 2). Erdős [7], [8] conjectured that the polarity graph is optimal for large \( q \). In [11] it was proved that

\[
f(q^2 + q + 1) \leq \frac{1}{2}q(q+1)^2
\]

(1)

for all even \( q \). It follows that equality holds in (1) for \( q = 2^\alpha \) \( (\alpha \geq 1) \).

In a previous version of this paper [12] it was shown that for large enough \( q \), not only is Erdős’ conjecture valid but also the only extremal graphs are the polarity graphs. For \( q = 2 \) there are 5, and for \( q = 3 \) there are 2 graphs with the maximum number of edges and so the lower bound on \( q \) is essential. (The obvious condition, \( q \geq q_0 \), was left out from the

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first announcement of the result in [11]). It seems there are no further exceptional cases for $q > 5$. That proof in [12] is rather involved and lengthy and uses the machinery of the theory of finite linear spaces and quasi-designs. The aim of this note is to give a short, simplified proof that (1) is valid for all $q \neq 1, 7, 9, 11, 13$. The description of the extremal graphs will appear in [12].

**Theorem 1** Let $G$ be a quadrilateral-free graph with $e$ edges on $q^2 + q + 1$ vertices, and suppose that $q \geq 15$. Then $e \leq \frac{1}{2}q(q + 1)^2$.

**Corollary 1** Let $q$ be a prime power greater than 13, $n = q^2 + q + 1$. Then $f(n) = \frac{1}{2}q(q+1)^2$.

## 2 Quasi-designs and finite linear spaces

In this section we recall a few results we use in the proof. There is a deep connection between 0–1 intersecting families, (i.e., any two sets have at most one common element), linear spaces (definition below), and quadrilateral-free graphs. First of all, the family of neighborhoods, $\{N(x) : x \in V\}$, of a $C_4$-free graph, $G = (V, E)$, is 0–1 intersecting.

Consider a 0–1 intersecting family, $\mathcal{F}$, of $(q+1)$-element sets on $q^2 + q + 1$ elements and suppose that $\mathcal{F}$ has two disjoint members. Metsch [16] proved that for $q \geq 15$

$$|\mathcal{F}| \leq q^2 + 1. \quad (2)$$

Consider a family of $(q+1)$-element sets, $\mathcal{R}$, on $q^2 + q + 1$ elements and suppose that $|\mathcal{R}| \geq q^2$ and it is 1-intersecting (i.e., $|R \cap R'| = 1$ holds for each pair of distinct $R, R' \in \mathcal{R}$). Vanstone [17] proved that $\mathcal{R}$ is actually a partial projective plane, i.e., one can find a family $\mathcal{P}$ such that

$$\mathcal{R} \cup \mathcal{P} \quad (3)$$

forms (the line system of) a projective plane of order $q$. Dow [5] proved that for such an extension

$$\mathcal{P} \text{ is unique.} \quad (4)$$

A **linear space** is a pair $(P, \mathcal{L})$ consisting of a set $P$ of points and a family of subsets of $P$, $\mathcal{L}$, called lines, such that any two distinct points $x$ and $y$ are contained in a unique line and each line has at least 2 points. The linear space is called **trivial** if it has only one line, $\mathcal{L} = \{P\}$. de Bruijn and Erdős [3] proved that for every nontrivial linear space

$$|\mathcal{L}| \geq |P|. \quad (5)$$

A **polarity** $\pi$ of a projective plane $(P, \mathcal{L})$ is a bijection $\pi : P \leftrightarrow \mathcal{L}$ which preserves incidences. A point $x$ is called **absolute** with respect to $\pi$ if $x \in \pi(x)$. The number of absolute points is denoted by $a(\pi)$. A bijection $x_i \leftrightarrow L_i$ is a polarity if and only if the corresponding incidence matrix, $M$, of the projective plane is symmetric. Moreover, $a(\pi) = \text{trace}(M)$. A theorem of Baer [1] states that for every polarity $\pi$

$$a(\pi) \geq q + 1. \quad (6)$$
The polarity graph. Consider a projective plane, $H$, of order $q$, with polarity $\pi$. Let $M$ be a symmetric incidence matrix of $H$ defined by $\pi$. Replace the 1’s on the main diagonal by 0’s. The matrix $A$ obtained in this way is an adjacency matrix of a graph $G$, called the polarity graph; $G$ is quadrilateral free. More properties of this and other symmetric graphs can be found in [13].

If $H$ is Desarguesian then a polarity $\pi$ can be defined by $(x, y, z) \leftrightarrow [x, y, z]$. Two distinct points $(x, y, z)$ and $(x', y', z')$ are joined in $G$ if and only if $xx' + yy' + zz' = 0$. A point not on the conic $x^2 + y^2 + z^2 = 0$ is joined to exactly $q + 1$ points and each of the $q + 1$ points on this conic is joined to exactly $q$ points, so $G$ has $\frac{1}{2}q(q+1)^2$ edges.

3 The proof of Theorem 1

Let $G = (V, E)$ be a four-cycle free graph on $n$ vertices with $e$ edges. The set of vertices adjacent to the vertex $x \in V$ is called the neighborhood, and it is denoted by $N(x) := \{y \in V \setminus \{x\} : xy \in E\}$. The size of $N(x)$ is called the degree of $G$ at $x$, and it is denoted by $\deg(x)$. Suppose that $n = q^2 + q + 1$, where $q > 1$ is an integer.

**Lemma 1** Let $G$ be a quadrilateral-free graph on $n = q^2 + q + 1$ vertices, with $q > 1$. Suppose that the maximum degree, $\Delta(G)$, satisfies $\Delta(G) \geq q + 2$. Then $e \leq \frac{1}{2}q(q+1)^2$.

This Lemma 1 comes from [11]. Its proof is based on the following inequalities where $x_0$ is any vertex of degree $\Delta$:

$$\binom{n - \Delta}{2} \geq \text{the number of paths of length 2 in } G \text{ with endpoints in } V \setminus N(x_0)$$

$$\geq \sum_{x \neq x_0} \binom{\deg(x) - 1}{2} \geq (n - 1) \left(\frac{(2e - \Delta - n + 1)}{2}\right).$$

From now on, we suppose that the maximum degree of $G$ is at most $q + 1$. We may also suppose that $e \geq \frac{1}{2}q(q+1)^2$. This implies that the number of vertices of degree $q + 1$ is at least $q^2$. Let $R = \{N(x) : x \in V, |N(x)| = q + 1\}$, $R = \{x \in V : |N(x)| = q + 1\}$.

We may suppose that each vertex has degree at least 2. Indeed, $\deg(x) \leq 1$ implies $2e = \sum_{v \in V} \deg(v) \leq 1 + (n - 1)(q + 1) = q(q + 1)^2 + 1$. Since $2e$ is even, we get the desired upper bound. (Let us note, that in [4] it was proved that each vertex has degree at least 2 for every extremal graph for all $n \geq 7$.)

We may even suppose that $|N(x) \cap R| \geq 2$ for each $x \in V$. Suppose, on the contrary, that for some vertex $x_0$ the neighborhood $N_0 = N(x_0)$ contains at least $|N_0| - 1$ vertices of $G$ of degree less than $q + 1$. The degree of $x_0$ is exactly $|N_0|$. We obtain

$$\sum_{x \in V(G)} (q + 1 - \deg(x)) \geq (q + 1 - |N_0|) + (|N_0| - 1) = q.$$

This implies $e \leq \left\lfloor \frac{1}{2}(nq + n - q) \right\rfloor$, the desired upper bound.
Case 1: Suppose that $\mathcal{R}$ contains two disjoint sets. Then, by (2), $|\mathcal{R}| \leq q^2 + 1$, so $G$ contains at least $q$ vertices of degree at most $q$. Therefore $2e \leq n(q + 1) - q = q(q + 1)^2 + 1$ and we get the desired upper bound.

Case 2: Suppose $\mathcal{R}$ contains no disjoint sets, i.e., $\mathcal{R}$ is a 1-intersecting family of size at least $q^2$. Then (3) implies that there exists a family $\mathcal{P}$ such that $\mathcal{R} \cup \mathcal{P}$ form a projective plane. For every $N = N(x)$, $N \notin \mathcal{R}$, the restricted hypergraph $\mathcal{N} := \mathcal{P} N$ is a linear space (not considering the hyperedges of size less than 2), i.e., $\mathcal{N} := \{N \cap P : P \in \mathcal{P}, |P \cap N| \geq 2\}$.

Suppose that there exists a neighborhood $N_0 = N(x_0)$ such that $N_0 = \mathcal{N}(x_0)$ is not a trivial space. The inequality (5) gives that $|N_0| \geq |N_0|$, which implies $|V \setminus R| = |\mathcal{P}| \geq |N_0| \geq |N_0|$. Hence there are at least $|N_0| - 1$ vertices of $G$ of degree less than $q + 1$ distinct from $x_0$. The degree of $x_0$ is exactly $|N_0|$. Then (7) holds, implying the desired upper bound for $e$.

From now on, we may suppose that for each neighborhood $N$ with $|N| \leq q$ there exists a unique $P = P(N) \in \mathcal{P}$, such that $N \subseteq P$. Then the incidence matrix, $M$, of $\mathcal{R} \cup \mathcal{P}$ majorizes the adjacency matrix, $A$, of $G$, i.e., $M$ is obtained from $A$ by changing a few 0's to 1. Here we suppose that the ordering of the vertex sets and $\mathcal{R}$ in both matrices are the same, and for the row $N \notin \mathcal{R}$ we associate the row $P(N)$ in $M$. We also suppose that the first $|R|$ rows (and columns) of $A$ correspond to the vertices of $R$. The extra entries of $M$ must be in the rows corresponding to $\mathcal{P}$, and in the columns corresponding to $V \setminus R$, i.e., $M$ and $A$ coincide outside the lower right corner.

The matrix $A$ is symmetric, and we claim that the matrix $M$ is symmetric, too. If not, then $M$ and its transpose $M^T$ give two different extensions of the partial projective plane $\mathcal{R}$. However, by (4) these two extensions must be the same, apart from the ordering of the rows. But every row contains at least two 1's from the first $|R|$ columns, so the ordering of the rows is also determined.

Finally, (6) implies that $M$ has at least $q + 1$ nonzero elements on its main diagonal. However, $\text{trace}(A) = 0$, so $M$ was obtained by adding $q + 1$ new elements to the main diagonal of $A$, i.e., $G$ is the polarity graph. □

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