

# On the Visualisation of Differential Forms

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Some people are a little nervous of differential forms. They're used to vectors having 3 components but how do you interpret an object that has 6, 9 or even hundreds of components. In fact there are those who take the attitude that tensors and differential forms are simply to be tolerated or got used to. This is wrong! If you draw the right pictures then differential forms and exterior calculus are the most natural thing in the world. If I manage to make myself clear in this document you'll be able to draw simple pictures of magnetic monopoles, understand the Maxwell field equations, find Stokes' equation so trivial you'll wonder why anyone even bothered stating it and never be frightened by differential geometry again. Hmm...maybe it won't be quite that good an experience but it helped me a lot to work this stuff out.

First of all I must credit Misner, Thorne and Wheeler (MTW) [1] for writing their fantastic book *Gravitation*. They introduce some superbly drawn diagrams that give great insight into differential geometry. However I make an elementary change to the principle behind their diagrams and find that suddenly things become even clearer. I'm sure they have reasons for drawing their diagrams the way they do - as I never finished their book (it's not exactly what you'd call a quick read) I don't know what other benefits their diagrams may bring. I'm also sure that my diagrams are not unique - just something that everyone in 'the industry' already knows but don't bother writing in textbooks.

## 1 Vectors

Before getting onto forms I'll review vectors. Here my pictures are no different to anyone else's but as forms are defined by their action on vectors they make a good place to start. I must point out that none of this stuff is rigorous. Not

that it's wrong either! That simply isn't my point. These are just intended as little mental pictures to use as intuition pumps.

So let's start with vector fields on a manifold. A vector, at a point, is simply a little arrow tangent to the surface at that point. In some sense it is an infinitesimal arrow but as we want pictures we can think about we imagine little arrows instead. A vector field is simply an assignment of a vector to each point in the manifold. Again this is difficult to draw so we can be lazy and draw a vector field as a whole load of arrows liberally sprinkled over the manifold. This is probably the picture you already have in your head anyway! Now the rigorous definition of a vector is that it is a derivation on the set of smooth functions on the manifold. This means that you ought to think of the little arrows as acting on functions by differentiating along the line along which they point.

Let's work with the plane  $\mathbf{R}^2$  with the usual co-ordinates  $x$  and  $y$ . At each point the basis for the space of vectors is given by the unit vectors along the axes written as  $\frac{\partial}{\partial x}$  and  $\frac{\partial}{\partial y}$ . You can draw  $\frac{\partial}{\partial x}$  as a little horizontal arrow and  $\frac{\partial}{\partial y}$  as a little vertical arrow.

## 2 Multivectors

So what can you do with vector fields? Well one thing you can do is use the arrows to define a way to deform the manifold. Just imagine the manifold is made of a sheet of rubber and push each point along the arrow. This process can be called 'Lie' dragging. This is a very general-purpose operation. If you draw a picture on a manifold and then Lie drag it you get a deformed version of that picture. By taking the difference between the objects represented by the dragged and the non-dragged versions of the picture you can define the Lie derivative of that object. This is a very general-purpose definition - if you can draw a little picture of something that can be dragged then there's a good chance you can find that something's Lie derivative. Another thing you can do with vector fields is take their exterior product. Suppose  $a$  and  $b$  are vectors, then  $a \wedge b$  is  $a \otimes b - b \otimes a$ . You can think of  $a \wedge b$  as the matrix  $c_{ij}$  with components  $a_i b_j - a_j b_i$ . Visualising  $a \wedge b$  is easy: it's the little parallelogram that has the vectors  $a$  and  $b$  as its sides. We call this a bivector. Strictly speaking we should choose an orientation for this parallelogram i.e. we should make a choice of normal vector to it.  $b \wedge a$  can then be thought of as the same parallelogram but with the normal pointing the opposite way. This talk of normals will lead onto another operation in a moment but now we consider generalisations to exterior products of more

vectors - multivectors.

To take the product of two multivectors we simply draw the little volume that is a parallelepiped whose faces are the original multivectors. For example  $\frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y}$  is a little square. In  $\mathbf{R}^3$  the multivector  $\frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y} \wedge \frac{\partial}{\partial z}$  extends the square into the third dimension making a cube. Note that what is important is the generalised area and orientation of the parallelepiped. This means, say, that

$$\frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y} = \frac{1}{2} \left( \frac{\partial}{\partial x} - \frac{\partial}{\partial y} \right) \wedge \left( \frac{\partial}{\partial x} + \frac{\partial}{\partial y} \right)$$

even though the parallelograms look different. When we go to more than 3 dimensions there is a catch. If we add two bivectors, say, the result isn't always writable in the form  $a \wedge b$  with  $a$  and  $b$  vectors. We may have to write the result as  $a \wedge b + c \wedge d$  and in higher dimensions things become more complex. We can still visualise  $a \wedge b + c \wedge d$  as a pair of parallelograms. On the other hand it's not immediately obvious, just by looking at our pictures, when two bivectors are equivalent. In other words it's hard to tell when

$$a \wedge b + c \wedge d = a' \wedge b' + c' \wedge d'$$

Going back to 2D again we see that starting with a vector we can pick a vector that is normal to it but with the same length. (We are assuming a metric now to give us an idea of length or orthogonality.) Actually there are two such vectors so take the one corresponding to a 90 degree anti-clockwise rotation. In some sense this is a dual to the original vector and we can generalise this. For example in 3D there is a vector perpendicular to  $a \wedge b$  whose length is the area of the parallelogram. This should be looking familiar - the dual of  $a \wedge b$  is the usual cross product  $a \times b$ . This generalises to higher dimensions: to get the dual we take a maximal parallelepiped perpendicular to the original one but with the same generalised area. In 4D we find the dual of  $\frac{\partial}{\partial x} \wedge \frac{\partial}{\partial y}$  is  $\frac{\partial}{\partial z} \wedge \frac{\partial}{\partial t}$  for example.

Notice how I'm getting a bit lazy about specifying the signs of things. Well it's going to get worse. Forgetting about minus signs will probably halve the amount of typing I have to do and leave you with an exercise to try afterwards. Incidentally, a good place to find out more about multivectors is Lounesto's book [2].

### 3 Forms

We will use the same picture as MTW for one-forms. Think of a vector as a pin and a one-form as an onion. You evaluate a one-form on a vector by counting

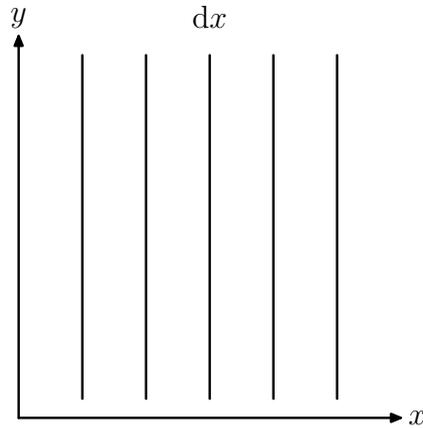


Figure 1: A 1-form

how many onion layers it goes through. More generally we represent a one-form on an  $n$ -manifold by drawing  $(n - 1)$ -dimensional surfaces on it. We call these surfaces leaves. In the general case we can still count how many of these leaves each vector goes through. A differential form This picture makes it easy to integrate a one-form.

A 1-form is integrated along a curve. All we do is take the directed count of the number of leaves the curve goes through. Note that this view also gives a nice picture of the exterior derivative applied to a scalar function  $f$ .  $df$  is represented as the contour lines of  $f$  with the contours corresponding to equally spaced 'heights'. We will naturally get contour lines more bunched up where  $df$  is larger. Note that integrating  $df$  along a curve counts how many contours we go through i.e. tells you how much  $f$  has varied along the length of the curve. This is simply a restatement of

$$\int_a^b df = f(b) - f(a).$$

Here's we I start differing from MTW. In order to represent the exterior product of two  $n$ -forms you find the intersection of the diagrams representing the forms. For example if we draw  $dx$  as vertical lines and  $dy$  as horizontal lines then  $dx \wedge dy$  is the set of intersection points which will form a uniform grid. Now we can integrate this form over an area. Draw the area and count the number of points inside it. Easy!

The exterior product of two forms In general, on an  $n$ -dimensional manifold,

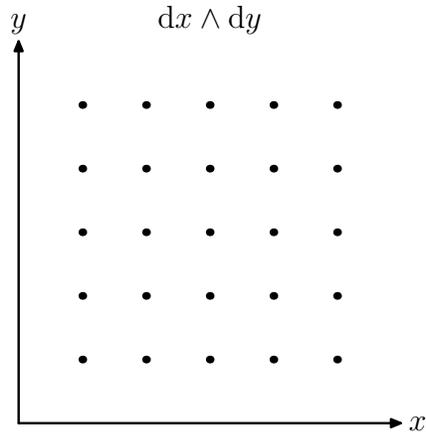


Figure 2: A 2-form

we represent an  $r$ -form by using  $(n - r)$ -dimensional surfaces. The exterior product of an  $r$ -form and an  $s$ -form gives an  $(r + s)$ -form, corresponding to the fact that in general intersecting an  $(n - r)$ -dimensional surface with an  $(n - s)$ -dimensional one gives an  $(n - r - s)$ -dimensional result. An  $n$ -form on an  $n$ -manifold is also known as a top form. These are represented by 0-leaves - i.e. points. A top-dimensional form is essentially a volume density on the manifold - and this is nicely pictured by the density of the 0-leaves.

## 4 The Exterior Derivative

So far we've considered some pretty simple forms. Here's a slightly more complex one:  $xdy$ . This is represented by horizontal lines but as we move along the positive  $x$ -axis the form is getting larger and so the 'density' of lines must be increased. In other words as we move right there must be more horizontal lines. So we have a picture with lots of horizontal lines but more towards the right than towards the left. In other words some of the horizontal lines on the right hand side must stop short as we travel leftwards. So these lines actually have endpoints. This suggests that if we have a picture of a form a natural operation to perform on it is to find the 'boundary' of its diagram. In the case of 1-forms on a 2-manifold this means finding the endpoints. What could this mean in more conventional terms? Consider the case  $xdy$ . The number of horizontal lines is roughly proportional to  $y$ . In other words the number of lines that 'come

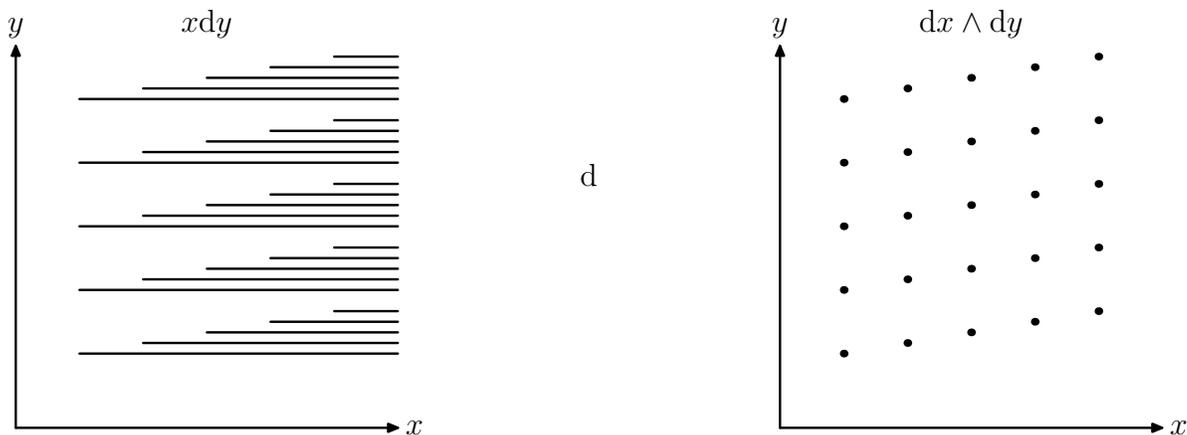


Figure 3: The Exterior Derivative

to an end' in any area is roughly the same wherever we look.

This is just like the example  $dx \wedge dy$  above. You may have guessed what is going on here:  $d(xdy) = dx \wedge dy$ . In other words exterior derivative is none other than finding the boundary of the picture. We can now deduce that  $dd\omega = 0$  from the topological fact that the boundary of a boundary is empty.

## 5 Stokes' Theorem

Now we can look at Green's Theorem - the 2D version of Stokes' Theorem. We have

$$\int_C d\omega = \int_{\partial C} \omega$$

The left hand side is exactly as above: we count the number of 1-leaves we  $C$  through as we go around the closed loop that makes the boundary of a region  $C$ . The right hand side is the integral of a 2-form. We represent a 2-form using a cloud of 0-leaves (points) and the integral over a region is simply a count of the number of points within the region. The above theorem simply says that the number of 1-leaves that come to an end within  $C$  is the number of 1-leaves that cross into  $C$  but don't come back out again. What could be simpler? Integrals in higher dimensions are similar. For example consider 2-forms in 3 dimensions. These are represented by 1-leaves. These are integrable over 2-surfaces. The integral is easy to visualise - it is the number of 1-leaves that

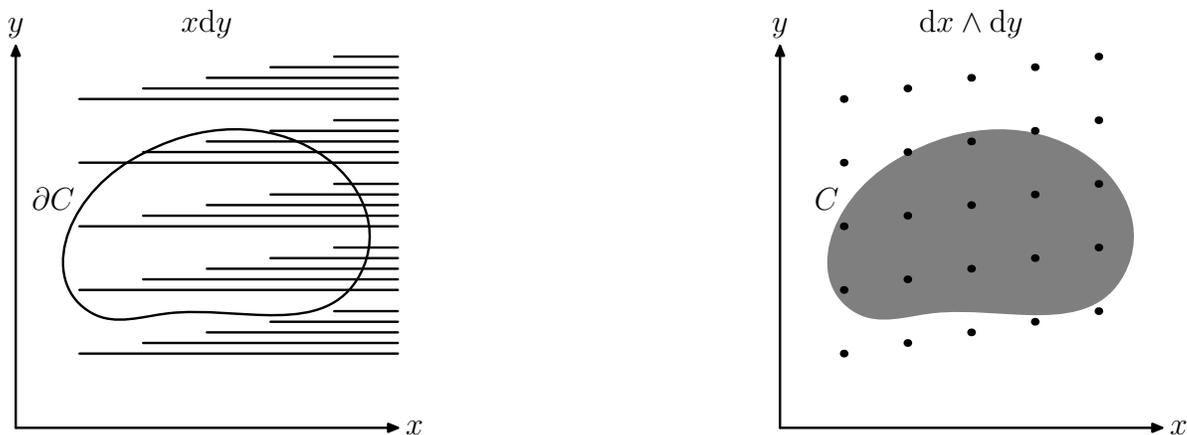


Figure 4: Stokes's Theorem. The (signed) number of times the loop on the left crosses the 1-leaves of the integrand is the same as the number of 0-leaves contained in the area on the right.

pierce the 2-surface. If the 2-form represents magnetic field lines then you can view the integral of it as the amount of flux going through the surface.

## 6 Introduction

Unfortunately, as we mentioned above, bivectors are more difficult to visualise in greater than three dimensions. This problem also causes difficulties with forms. By choosing to use these pictures we make the operator (among others) easy to visualise but we lose the ability to easily work with the most obvious of operators - addition. One possibility is to represent the form  $a + b$  by drawing the union of the two diagrams for  $a$  and  $b$ . This works - for example you can use the leaf counting approach to integration with these diagrams - but it becomes harder to tell when two forms are equivalent. For example consider  $dx + dy$ . We could represent this either by diagonal lines or by both horizontal and vertical lines. Both are fine for many purposes but it's not clear, just by looking at the diagram, that both represent the same form.

## 7 Hodge Dual

Just as we can define duals for vectors we can also define duals for forms. In this case it is called the Hodge dual and is denoted by  $*$ . Again I'm going to be lazy about minus signs - the full treatment of Hodge dual requires quite a bit of care with these. The idea is this - at any point in a leaf we can draw the maximal dimension leaf that is perpendicular to it. We then fill the whole manifold with such perpendicular surfaces and arrange that their density is that same as that for the original surfaces. The form defined by these surfaces is the Hodge dual. For example the picture of  $dx$  is equally spaced vertical lines. Orthogonal to these are horizontal lines and these must also be equally spaced. So  $*dx = dy$ . Consider now  $dx$  in three dimensions. This is represented by uniformly spaced  $y$ - $z$  planes. Normal to these are the lines parallel to the  $x$ -axis. These must also be equally spaced and so represent  $dy \wedge dz$ . So  $*dx = dy \wedge dz$ .

## 8 Maxwell's Equations

We now have enough mathematical equipment to start doing something for real. Let's try Maxwell's Equations in a vacuum in 2D and try for a radially symmetric solution. The equations are

$$dF = 0 \text{ and } d * F = 0$$

with  $F$  a 1-form. A 1-form in two dimensions is represented by 1-leaves i.e. curves. We'll try for a rotationally symmetric solution. The fact that  $dF = 0$  means that we are looking either for closed curves or curves going off to infinity or the singularity. We'll try the obvious thing - a form represented by closed circles with origin at zero. The only thing to determine now is how well spaced these circles must be. Well consider  $d * F$ . This is represented by orthogonal curves and this can only mean radial lines. As  $d * F = 0$  these lines must stretch all the way from the origin to infinity. Draw lots of such lines equally spaced around the origin. As you move away from the origin these lines become further apart. In fact their density drops off as  $1/r$ . Now the definition of Hodge dual means that  $*F$  and  $F$  have the same density at each point. This means that the spacing between circles must increase as  $1/r$ . It shouldn't be hard to see we

have just drawn  $dr/r$ . Solutions to Maxwell's Equations in two dimensions are essentially pairs of sets of curves so that (1) curves in one set are perpendicular

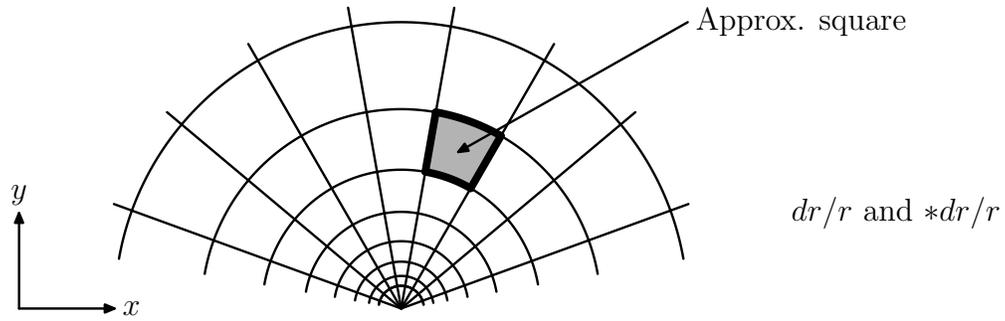


Figure 5: Solving Maxwell's Equations

to the other and (2) the spaces between the curves form areas that are roughly square.

## References

- [1] Charles W. Misner, John A. Wheeler, Kip S. Thorne, Gravitation, WH Freeman & Co., 1973
- [2] Pertti Lounesto, Clifford Algebras and Spinors, Cambridge University Press, 1997