On the number of quasi-kernels in digraphs

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Abstract

A vertex set $X$ of a digraph $D = (V, A)$ is a kernel if $X$ is independent (i.e., all pairs of distinct vertices of $X$ are non-adjacent) and for every $v \in V - X$ there exists $x \in X$ such that $vx \in A$. A vertex set $X$ of a digraph $D = (V, A)$ is a quasi-kernel if $X$ is independent and for every $v \in V - X$ there exist $w \in V - X, x \in X$ such that either $vx \in A$ or $vw, wx \in A$. In 1994, Chvátal and Lovász proved that every digraph has a quasi-kernel. In 1996, Jacob and Meyniel proved that, if a digraph $D$ has no kernel, then $D$ contains at least three quasi-kernels. We characterize digraphs with exactly one and two quasi-kernels, and, thus, provide necessary and sufficient conditions for a digraph to have at least three quasi-kernels. In particular, we prove that every strong digraph of order at least three, which is not a 4-cycle, has at least three quasi-kernels. We conjecture that every digraph with no sink has a pair of disjoint quasi-kernels and provide some support to this conjecture.
1 Introduction, terminology and notation

A vertex set $X$ of a digraph $D = (V, A)$ is a kernel if $X$ is independent (i.e., all pairs of distinct vertices of $X$ are non-adjacent) and for every $v \in V - X$ there exists $x \in X$ such that $vx \in A$. A vertex set $X$ of a digraph $D = (V, A)$ is a quasi-kernel if $X$ is independent and for every $v \in V - X$ there exist $w \in V - X, x \in X$ such that either $vx \in A$ or $vw, wx \in A$. A digraph $T = (V, A)$ is a tournament if for every pair $x, y$ of distinct vertices in $V$, either $xy \in A$ or $yx \in A$, but not both. A vertex of out-degree zero is called a sink.

While not every digraph has a kernel (e.g., a directed cycle $\vec{C}_n$ has a kernel if and only if $n$ is even), Chvátal and Lovász [2] (see also Chapter 12 in [1]) proved that every digraph has a quasi-kernel. Jacob and Meyniel [3] proved that, if a digraph $D$ has no kernel, then $D$ contains at least three quasi-kernels. While the assertion of Chvátal and Lovász generalizes the fact that every tournament has a 2-serf, i.e., a quasi-kernel of cardinality 1, the Jacob-Meyniel theorem extends the result of Moon [4] that every tournament with no sink has at least three 2-serfs.

While the Jacob-Meyniel theorem provides sufficient conditions for a digraph to have at least three quasi-kernels, in Section 2, we characterize digraphs with exactly one and two quasi-kernels, and, thus, provide necessary and sufficient conditions for a digraph to have at least three quasi-kernels (see Theorem 2.6). In particular, we prove that every strong digraph, of order at least three, different from the 4-cycle $\vec{C}_4$ has at least three quasi-kernels. Note that, in our proofs, we naturally use the Chvátal-Lovász theorem, but not the more powerful Jacob-Meyniel theorem.

In Section 3, we pose a conjecture that every digraph with no sink has a pair of disjoint quasi-kernels. We show that the conjecture is true for every digraph which possesses a quasi-kernel of cardinality at most two and every kernel-perfect digraph, i.e., a digraph for which every induced subdigraph has a kernel. It was proved by von Neumann and Morgenstern [5] that every acyclic digraph is kernel-perfect. Richardson’s theorem was later improved in a number of papers, cf. Section 12.3 in [1].

We use the standard terminology and notation on digraphs as given in [1]. We still provide most of the necessary definitions for the convenience of the reader.

For a digraph $D$, the vertex (arc) set is denoted by $V(D)$ ($A(D)$). Let $x, y$ be a pair of vertices in $D$. If $xy \in A(D)$, we say $x$ dominates $y$, and $y$ is dominated by $x$, and denote it by $x \rightarrow y$. A digraph $D$ is strong if, for every ordered pair $x, y$ of distinct vertices in $D$, there is a path from $x$ to $y$. An orientation of a digraph $D$ is an oriented graph obtained from $D$ by deleting exactly one arc from each 2-cycle in $D$. A biorientation of $D$ is a digraph, which is a subdigraph of $D$ and superdigraph of an orientation of $D$. The closed in-neighbourhood (closed out-neighbourhood) of a set $X$ of vertices of a digraph $D = (V, A)$
is defined as follows.

\[ N_D[X] = X \cup \{ y \in V : \exists x \in X, y \rightarrow x \} \quad \text{and} \quad N_D^+[X] = X \cup \{ y \in V : \exists x \in X, x \rightarrow y \} \]

For disjoint subsets \( X \) and \( Y \) of \( V(D) \), let \( X \times Y = \{ xy : x \in X, y \in Y \} \), \( (X, Y)_D = (X \times Y) \cap A(D) \); \( D[X] \) is the subdigraph of \( D \) induced by \( X \). If the digraph under consideration is clear from the context, then we will omit the subscript \( D \).

\section{Digraphs with exactly one and two quasi-kernels}

We start with the following:

\textbf{Lemma 2.1} Let \( x \) be a vertex in a digraph \( D \). If \( x \) is a non-sink, then \( D \) has a quasi-kernel not including \( x \).

\textbf{Proof}: Let \( y \in N^+[x] - \{ x \} \) be arbitrary. If \( N^-[y] = V(D) \), then \( y \) is the required quasi-kernel. If \( N^-[y] \neq V(D) \), let \( Q' \) be a quasi-kernel in \( D - N^-[y] \). If \( y \) dominates a vertex in \( Q' \), then \( Q' \) is a quasi-kernel in \( D \), which does not contain \( x \). If \( y \) does not dominate a vertex in \( Q' \), then \( Q' \cup \{ y \} \) is a quasi-kernel in \( D \), which does not include \( x \).

The following is an easy characterization of digraphs with merely one quasi-kernel.

\textbf{Theorem 2.2} A digraph \( D \) has only one quasi-kernel if and only if \( D \) has a sink and every non-sink of \( D \) dominates a sink of \( D \). If a digraph \( D \) has only one quasi-kernel \( Q \), then \( Q \) is a kernel and consists of the sinks of \( D \).

\textbf{Proof}: Assume that \( D \) has a sink and every non-sink of \( D \) dominates a sink of \( D \). Let \( S \) be the set of sinks in \( D \). To see that \( S \) is a unique quasi-kernel of \( D \), it is enough to observe that every sink must be in a quasi-kernel.

Let \( D \) have only one quasi-kernel \( Q \). To see that \( Q \) is the set of sinks in \( D \), observe that \( Q \) contains all sinks in \( D \) and, by Lemma 2.1, \( Q \) does not have non-sinks. If \( x \) is a non-sink and \( x \) does not dominate a vertex in \( Q \), then \( Q \cup \{ x \} \) is another quasi-kernel of \( D \), a contradiction. Thus, we have proved that \( D \) has a sink and every non-sink of \( D \) dominates a sink of \( D \).

In view of Theorem 2.2, the following assertion is a strengthening of the Jacob-Meyniel theorem for the case of digraphs with no sinks.

\textbf{Theorem 2.3} Let \( D \) be a digraph with no sink. Then \( D \) has precisely two quasi-kernels if and only if \( D \) has an induced 4-cycle or 2-cycle, \( C \), such that no vertex of \( C \) dominates a vertex in \( D - V(C) \) and every vertex in \( D - V(C) \) dominates at least two adjacent vertices in \( C \).
To prove Theorem 2.3, we will extensively use the following:

**Lemma 2.4** Let a digraph $D$ have exactly two quasi-kernels, $R$ and $Q$. Then the following claims hold:

(i) If a vertex $x$ in $R$ dominates some vertex $y$ such that $V(D) \neq N^{-}[y]$, then $Q - y$ is the only quasi-kernel in $D - N^{-}[y]$;

(ii) $\{R, Q\}$ is the set of quasi-kernels of every biorientation of $D$, in which both $R$ and $Q$ contain non-sinks.

**Proof:** Let $R_1, R_2, \ldots, R_k$ be the quasi-kernels in $D - N^{-}[y]$. Then $R_1', R_2', \ldots, R_k'$ are quasi-kernels in $D$, where $R_i' = R_i$ if $(y, R_i) \neq \emptyset$ and $R_i' = R_i \cup \{y\}$, otherwise, $i = 1, 2, \ldots, k$. Since $D$ has only two quasi-kernels, $k \leq 2$. Since $x \in N^{-}[y]$ and $x \in R$, we conclude that $R - y$ is not a quasi-kernel in $D - N^{-}[y]$. By the Chvátal-Lovász theorem, every digraph has a quasi-kernel, so $Q - y$ is the unique quasi-kernel in $D - N^{-}[y]$.

Let $D'$ be a biorientation of $D$, in which both $R$ and $Q$ contain non-sinks. Clearly, every quasi-kernel in $D'$ is a quasi-kernel in $D$. However, by Theorem 2.2, neither $R$ nor $Q$ can be the only quasi-kernel in $D'$. Thus $\{R, Q\}$ is the set of quasi-kernels of $D'$.

**Proof of Theorem 2.3:** We first show that, if $D$ has precisely two quasi-kernels, then $D$ has the above-described structure. We will prove this assertion by induction on $|V(D)|$. The assertion is clearly true when $|V(D)| \leq 2$, so we may assume that it is true for all digraphs, $D^*$, with $|V(D^*)| < |V(D)|$. Let $Q_1$ and $Q_2$ be the only two quasi-kernels in $D$.

Note that by Lemma 2.1, $Q_1$ and $Q_2$ must be disjoint (if $x \in Q_1 \cap Q_2$ then use Lemma 2.1 for $x$). We now prove the following claims.

**Claim A:** If $(Q_i, Q_j) \neq \emptyset$ ($\{i, j\} = \{1, 2\}$), then for every $w \in Q_i$, $(w, Q_j) \neq \emptyset$.

**Proof of Claim A:** Let $xy \in (Q_i, Q_j)$ and let $w$ be a vertex in $Q_i$ which has no arc into $Q_j$. By Lemma 2.4(i), $Q_j - y$ is the unique kernel in $D - N^{-}[y]$ and, thus, by Theorem 2.2, we must have an arc from $w$ to $Q_j - y$ since $w \in V(D) - N^{-}[y]$, a contradiction.

**Claim B:** Both $(Q_1, Q_2)$ and $(Q_2, Q_1)$ are non-empty.

**Proof of Claim B:** Clearly $Q_1 \cup Q_2$ is not an independent set, as then it would be a quasi-kernel. Hence, without loss of generality we may assume that $(Q_1, Q_2) \neq \emptyset$. Suppose that $(Q_2, Q_1) = \emptyset$. Since $Q_1$ is a quasi-kernel, there exists a 2-path from any given $x \in Q_2$ to $Q_1$, say $xyz$ ($z \notin Q_1 \cup Q_2$ and $y \in Q_1$).

We now show that every vertex in $Q_2$ must dominate $z$. Suppose that this is not the case, and let $w$ be a vertex not dominating $z$. By Lemma 2.4, $Q_1$ is the only quasi-kernel in $D - N^{-}[z]$. However, by Theorem 2.2, this is a contradiction against the fact that $w$ dominates no vertex in $Q_1$ ($w \in V(D) - N^{-}[z]$). Thus, $Q_2 \subseteq N^{-}[z]$. 


Let $D'$ be any orientation of $D$ for which $(z, Q_2)_{D'} = \emptyset$, and let $ab$ be an arc in $(Q_1, Q_2)_{D'}$. Since $z \in V(D') - N_{D'}^{-}[b]$, we have $V(D') \neq N_{D'}^{-}[b]$. By Lemma 2.4, $Q_2 - b$ is the only quasi-kernel in $D' - N_{D'}^{-}[b]$. By Theorem 2.2, $Q_2 - b$ is a kernel in $V(D') - N_{D'}^{-}[b]$. However, $Q_2 - b$ is not a kernel in $D' - N_{D'}^{-}[b]$ as $z$ dominates no vertex in $Q_2 - b$, a contradiction.

Claim C: Let $\{a, b\}$ be a set of two distinct vertices from $Q_1$ and let $\{c, d\}$ be a set of two distinct vertices from $Q_2$. Then we cannot have both $a \rightarrow c$ and $d \rightarrow b$.

Proof of Claim C: Assume that $a \rightarrow c$ and $d \rightarrow b$. Suppose first that $c \not\rightarrow b$. By Lemma 2.4, $Q_1 - b$ is the only quasi-kernel in $V(D) - N^{-}[b]$. However, since the arc $ac \in D - N^{-}[b]$ we see that $Q_1 - b$ contains a non-sink in $V(D) - N^{-}[b]$ in contradiction with Theorem 2.2.

Suppose now that $c \rightarrow b$, and let $D'$ equal $D - bc$ (if $bc \notin D$, then $D' = D$). By Lemma 2.4, $Q_2 - c$ is the only quasi-kernel in $V(D') - N^{-}[c]$. However, since the arc $db \in D' - N_{D'}^{-}[c]$ we see that $Q_2 - c$ contains a non-sink in contradiction with Theorem 2.2.

Claim D: Either $D[Q_1 \cup Q_2]$ is a 2-cycle or $D[Q_1 \cup Q_2]$ contains an induced 4-cycle.

Proof of Claim D: If either $Q_1$ or $Q_2$ has only one vertex, then without loss of generality we may assume that $|Q_1| = 1$. If $|Q_2| = 1$ then by Claim B, $D(Q_1 \cup Q_2)$ is a 2-cycle, so assume that $|Q_2| \geq 2$. Let $Q_1 = \{x\}$ and observe that by Claims A and B there exists a pair $a, b$ of distinct vertices in $Q_2$ such that $ax, xb \in A(D)$. Let $D'$ be any orientation of $D$ with $ax, xb \in A(D')$. By Lemma 2.4, $Q_1 - x$ is the only quasi-kernel in the non-empty digraph $D' - N_{D'}^{-}[x]$, which contradicts the fact that $Q_1 = \{x\}$.

Therefore, we may now assume that both $Q_1$ and $Q_2$ have cardinality at least two. By Claim B, there exists an arc $x_2x_1$ in $(Q_2, Q_1)_{D'}$. Let $y_1 \in Q_1 - \{x_1\}$ be arbitrary, and observe that $(y_1, Q_2) \neq \emptyset$, by Claims A and B. By Claim C, $y_1x_2 \in (y_1, Q_2)$. Let $y_2 \in Q_2 - \{x_2\}$ be arbitrary. Analogously, we have $y_2y_1 \in A(D)$. Finally, Claims A and C imply that $x_1y_2 \in A(D)$. Therefore, $C = x_2x_1y_2y_1x_2$ is a 4-cycle. Observe that $C$ is an induced 4-cycle, by Claim C and the fact that $\{x_1, y_1\}$ and $\{x_2, y_2\}$ are independent sets (they are subsets of quasi-kernels).

Claim E: If $abdec$ is a 4-cycle such that $\{a, c\} \subseteq Q_1$ and $\{b, d\} \subseteq Q_2$, then there is no arc from $\{a, b, c, d\}$ to any vertex in $D - \{a, b, c, d\}$.

Proof of Claim E: Assume that the claim is false and that there exists a vertex $z \in V(D) - \{a, b, c, d\}$ such that there is an arc from $\{a, b, c, d\}$ to $z$. Without loss of generality, assume that $az \in A(D)$, and consider the following two cases.

Case 1: $z \rightarrow c$. Let $D'$ be any orientation of $D$ with $zc, az \in A(D')$. By Lemma 2.4, $Q_2 - z$ is the only quasi-kernel in $D' - N_{D'}^{-}[z]$. However, the existence of the arc $bc \in D'$ contradicts Theorem 2.2.
Case 2: \( z \neq c \). By Lemma 2.4(i), \( Q_1 - c \) is the only quasi-kernel in \( N_D - [c] \). However, the existence of the arc \( az \in D - N^-[c] \) contradicts Theorem 2.2.

**Claim F:** If \( abcd \) is a 4-cycle such that \( \{a, c\} \subseteq Q_1 \) and \( \{b, d\} \subseteq Q_2 \), then every vertex in \( D - \{a, b, c, d\} \) dominates two adjacent vertices on \( abcd \).

**Proof of Claim F:** Let \( x \in V(D) - \{a, b, c, d\} \) be arbitrary. If \( x \) has no arc into \( \{a, b, c, d\} \), then consider the digraph \( D^* = D - N^-[x] \). Clearly, \( Q_1 - N^-[x] \) and \( Q_2 - N^-[x] \) are distinct quasi-kernels in \( D^* \); \( D^* \) cannot have another quasi-kernel as \( D \) has only two quasi-kernels. Therefore there are exactly two quasi-kernels in \( D^* \), and by our induction hypothesis, these quasi-kernels are precisely \( \{a, c\} \) and \( \{b, d\} \). Observe that, by Claim E, \( x \) is adjacent to no vertex from the set \( \{a, b, c, d\} \). However, this means that both \( \{x, a, c\} \) and \( \{x, b, d\} \) are quasi-kernels in \( D \), contradicting the fact that \( Q_1 \) and \( Q_2 \) are disjoint. Therefore, \( x \) must have an arc into \( \{a, b, c, d\} \). Observe that since \( x \) is arbitrary, this implies that \( \{a, c\} \) and \( \{b, d\} \) are quasi-kernels in \( D \).

Without loss of generality, assume that \( x \rightarrow a \) in \( D \). Suppose also that \( x \neq b \) and \( x \neq d \), as otherwise we would be done. However, these assumptions imply that \( \{x, b, d\} \) also is a quasi-kernel, along with \( \{a, c\} \) and \( \{b, d\} \), a contradiction.

**Claim G:** If \( C = D[Q_1 \cup Q_2] \) is a 2-cycle, then no vertex of \( C \) dominates a vertex in \( D - V(C) \) and every vertex in \( D - V(C) \) dominates both vertices in \( C \).

**Proof of Claim G:** Let \( C = xzy. \) Assume there exists an arc \( xz \), \( z \neq y \). Consider an orientation, \( D' \), of \( D \) such that \( D' - N_D[y] \) contains \( z \) and does not contain \( y \). On one hand, \( D' \) has no quasi-kernels other than \( \{x\} \) and \( \{y\} \); on the other hand, either \( Q \) or \( Q \cup \{x\} \) is a quasi-kernel in \( D' \), where \( Q \) is a quasi-kernel in \( D' - N_D[y] \). We have arrived at a contradiction. Therefore \( (V(C), V(D) - V(C)) = \emptyset \). Furthermore, every vertex \( v \in V(D) - V(C) \) must dominate both vertices on \( C \) since otherwise there would be a quasi-kernel containing \( v \).

Claims D,E, F and G prove the assertion on the structure of \( D \).

Now assume that \( D \) has the structure described in this theorem, and \( C \) is the cycle in \( D \). If \( C \) is a 2-cycle, then it is easy to see that each of the two vertices on \( C \) is a quasi-kernel (and kernel) in \( D \), and that there are no other quasi-kernels in \( D \). So now assume that \( C = abcd \) is an induced 4-cycle in \( D \). Observe that \( \{a, c\} \) and \( \{b, d\} \) are quasi-kernels in \( D \). Since \( (\{a, b, c, d\}, V(D) - \{a, b, c, d\}) = \emptyset \), any quasi-kernel in \( D \) must contain a vertex, \( x \), in \( C \). Since the successor \( x^+ \) of \( x \) in \( C \) has to be able to reach the quasi-kernel with a path of length at most two, \( (x^+)^+ \) must also belong to the quasi-kernel. Since all other vertices are adjacent to one of these vertices, the only quasi-kernels are \( \{a, c\} \) and \( \{b, d\} \).

\( \square \)
As corollaries we obtain the following two theorems.

**Theorem 2.5** A strong digraph $D$ of order at least three has at least three quasi-kernels, unless $D$ is $\vec{C}_4$.

**Proof:** Immediate from the previous theorems, Theorems 2.2 and 2.3.

**Theorem 2.6** Let $D$ be a digraph, $S$ the set of sinks in $D$, $R$ the set of vertices that have an arc into $S$, and $H = D - S - R$. Then $D$ has precisely two quasi-kernels, if and only if one of the following holds:

(a) There is a 2-cycle $C$ in $H$ such that at most one of the vertices in $C$ has an arc into $R$, no vertex of $C$ dominates a vertex in $H - V(C)$, and every vertex in $H - V(C)$ dominates both vertices in $C$.

(b) There is an induced 4-cycle, $C$, in $H$ such that no vertex of $C$ dominates a vertex in $D - V(C)$ and every vertex in $H - V(C)$ dominates two adjacent vertices in $C$.

(c) The digraph $H$ has at least two vertices. There is a vertex $x$ in $H$ such that no vertex of $H$ is dominated by $x$, all the vertices of $H - x$ dominate $x$, i.e., $(V(H) - \{x\}, x) = (V(H) - \{x\}) \times \{x\}$, and there is a kernel $Q$ in $H - x$, consisting only of sinks in $H - x$. Moreover, there is no arc from $Q$ to $R$.

(d) The digraph $H$ consists of a single vertex.

**Proof:** We first show that, if $D$ has precisely two quasi-kernels, then $D$ has the above-described structure. Let $D$ be a digraph with exactly two quasi-kernels. If $D$ has no sinks, then by Theorem 2.3, $D$ has the structure described in part (a) or (b) with $R \cup S = \emptyset$. Hence, we may assume that $D$ contains some sinks, and let $S$, $R$ and $H$ be as defined in the formulation of this theorem. Let us first prove that $H$ has at most one sink.

Suppose that there are at least two sinks in $H$. Let $x$ and $y$ be two distinct sinks in $H$. Note that both $x$ and $y$ have arcs into $R$, since otherwise they would belong to $S$ or $R$. Let $Q_1$ be a quasi-kernel in $H$, $Q_2$ a quasi-kernel in $H - x$, and $Q_3$ a quasi-kernel in $H - y$. Since $\{x, y\} \subseteq Q_1$, $\{x, y\} \cap Q_2 = \{y\}$ and $\{x, y\} \cap Q_3 = \{x\}$ we see that $Q_1 \cup S$, $Q_2 \cup S$ and $Q_3 \cup S$ are 3 different quasi-kernels in $D$, a contradiction. Hence, $H$ has at most one sink.

Suppose that there is exactly one sink $x$ in $H$. Since the case of $H$ having exactly one vertex is trivial, we may assume that $H$ contains at least two vertices. Let $Q_1$ be a quasi-kernel in $H$, and let $Q_2$ be a quasi-kernel in $H - x$. Note that $S \cup Q_1$ and $S \cup Q_2$ are different quasi-kernels in $D$ (as $x \in Q_1$ and $x$ has an arc into $R$). Therefore, $Q_2$ must be the unique quasi-kernel in $H - x$, and, by Theorem 2.2, $Q_2$ is a kernel in $H - x$ consisting only of sinks in $H - x$. Since $x$ is the only sink in $H$, every vertex in $Q_2$ dominates $x$. 


Therefore, \( \{x\} \) is a quasi-kernel in \( H \). Since \( x \) must be the unique quasi-kernel in \( H \) and \( x \) is a sink, we must have \( (V(H) - \{x\}, x) = (V(H) - \{x\}) \times \{x\} \). Thus, \( S \cup \{x\} \) and \( S \cup Q_2 \) are quasi-kernels in \( D \). If there is a vertex \( w \in Q_2 \) which dominates a vertex in \( R \), then let \( Q_3 \) be a quasi-kernel in \( H - w - x \), and observe that \( Q_3 \cup S \) is a third quasi-kernel, a contradiction. Therefore, \( D \) has the structure described in part (c).

Suppose now that \( H \) has no sink. (Since \( D \) has more than one quasi-kernel, \( H \) is non-empty.) By Theorem 2.2, there are at least two quasi-kernels, \( Q_1 \) and \( Q_2 \), in \( H \). If \( Q \) is a quasi-kernel in \( H \), then \( S \cup Q \) is a quasi-kernel in \( D \). Hence, \( Q_1 \) and \( Q_2 \) are the only quasi-kernels in \( H \), and, thus, the structure of \( H \) is provided by Theorem 2.3. Let \( C \) be the 2-cycle or induced 4-cycle given in Theorem 2.3.

If \( C \) is a 2-cycle, \( xyx \), then, by Theorem 2.3, to show that \( D \) has the structure described in part (a) it suffices to prove that at most one of the vertices \( x \) and \( y \) has an arc into \( R \). Assume that both \( x \) and \( y \) have arcs into \( R \). Let \( Q_3 \) be a quasi-kernel in \( H - x - y \), if \( V(H) \neq \{x, y\} \), and the empty set, otherwise. However, \( S \cup x \), \( S \cup y \) and \( S \cup Q_3 \) are three different quasi-kernels in \( D \), a contradiction.

If \( C \) is an induced 4-cycle, \( abcd \), then, by Theorem 2.3, to show that \( D \) has the structure described in part (b) it suffices to prove that no vertex in \( V(C) \) dominates a vertex in \( R \). Without loss of generality, assume that \( a \) dominates a vertex in \( R \). By Lemma 2.1, there exists a quasi-kernel, \( Q \), in \( H - a \), which does not contain \( b \), as \( b \) is not a sink in \( H - a \). However, \( Q \cup S \), \( \{a, c\} \cup S \) and \( \{b, d\} \cup S \) are three different quasi-kernels in \( D \), a contradiction.

This proves that, if \( D \) has exactly two quasi-kernels, then \( D \) has the structure described in the formulation of this theorem. If \( D \) has the structure provided in part (a), (b), (c) or (d), then it is not too difficult to check that there are exactly two quasi-kernels in \( D \). □

3 Disjoint quasi-kernels

If a digraph \( D \) has a sink \( x \), then every quasi-kernel in \( D \) must contain \( x \). Hence, a digraph with sinks has no disjoint quasi-kernels. However, we suspect that the following holds.

**Conjecture 3.1** Every digraph with no sink has a pair of disjoint quasi-kernels.

By Lemma 2.1, this conjecture holds for digraphs with exactly two quasi-kernels: see the first paragraph in the proof of Theorem 2.3. We will show that the conjecture is also true for every digraph which possesses a quasi-kernel of cardinality at most two and every kernel-perfect digraph. We recall that a digraph \( D \) is kernel-perfect if every induced subdigraph of \( D \) has a kernel.

We start with the following useful lemma.
Lemma 3.2 Let $D$ be a digraph and let $Y$ be a set of vertices in $D$ such that $D[Y]$ is kernel-perfect. Then there exists a quasi-kernel, $Q$, in $D$, such that $Q \subseteq V(D) \setminus (N^-[Y] \setminus Y)$.

Proof: Let $H = D - N^-[Y]$ and let $Q_1$ be a quasi-kernel in $H$ (if $H = \emptyset$ then $Q_1 = \emptyset$). Let $Y'$ contain all vertices from $Y$, which have no arc into $Q_1$. Since $D[Y]$ is kernel-perfect, there is a kernel, $K'$, in $D[Y']$. We claim that $Q = Q_1 \cup K'$ is the desired quasi-kernel in $D$.

Clearly, $Q$ is an independent set as $Q_1$ and $K'$ are independent sets, there is no arc from $K'$ to $Q_1$ (by the definition of $Y'$) and there is no arc from $Q_1$ to $K'$ (by the definition of $H$). By the definition of $Q_1$, every vertex in $H$ can reach $Q$ with a path of length at most 2. Observe that every vertex in $Y - K'$ dominates a vertex in $Q$ (each $y \in Y$ either has an arc into $Q_1$ or is a vertex in $Y'$ and has an arc into $K'$ or is in $K'$). Therefore, every vertex in $N^-(Y)$ can reach $Q$ with a path of length at most 2. □

Corollary 3.3 Every kernel-perfect digraph with no sink has a pair of disjoint quasi-kernels.


Corollary 3.4 Let $D$ be a digraph, and let $S = \{x, y\}$ be a set of distinct vertices in $D$ such that $N^+[x] - S \neq \emptyset$ and $N^+[y] - S \neq \emptyset$. Then there exists a quasi-kernel in $D$, which is disjoint from $S$.

Proof: Let $u \in N^+[x] - S$ and $v \in N^+[y] - S$ be arbitrary (possibly $u = v$). Since $D[\{u, v\}]$ is obviously kernel-perfect and $S \subseteq N^-[\{u, v\}] - \{u, v\}$, the desired result follows from Lemma 3.2. □

It follows from Corollary 3.4 that Conjecture 3.1 is true for every digraph with a quasi-kernel of cardinality at most 2.

Corollary 3.4 cannot be improved to sets of size 3, by the following example. Let $V(D) = \{x_1, x_2, x_3, y_1, y_2, y_3, z_1, z_2, z_3\}$ and let the arc set of $D$ contain the 3-cycles $x_iy_iz_ix_i$ for $i = 1, 2, 3$, $z_1z_2z_3z_1$ and $y_1y_2y_3y_1$ as well as the arcs $\{z_1y_2, z_1y_3, z_2y_1, z_2y_3, z_3y_1, z_3y_2\}$. By the definition of $D$, $X = \{x_1, x_2, x_3\}$ is a quasi-kernel of size 3. To see that $X$ intersects any other quasi-kernel in $D$, observe that every pair of vertices in $D - X$ is adjacent and none of the vertices in $D - X$ is a quasi-kernel (for example, the shortest path from $x_2$ to $y_1$ is of length 3). At the same time, $\{x_1, y_3\}$ and $\{y_1, x_2\}$ are disjoint quasi-kernels.
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References


