Approximation of distributions by convolutions in the Hausdorff metric*

Alexander Petukhov†

August 28, 2000

Abstract

The paper deals with finding criteria for the Hausdorff convergence of sequences of convolution operators on quasi-Banach spaces of periodic real-valued distributions (generalized functions). In particular, the criteria for convergence on the Hardy classes, on the class of regular Borel measures, and on the class of pseudomeasures are found. These assertions are special cases of the general result obtained for rather wide collection of spaces. The given result relies essentially on the explicit description of the set of bounded convolution operators, acting from the fixed space of the mentioned collection to the space $\mathbb{L}^\infty$. Solution to these problems became possible due to the introduction of the notion a the canonical graph of an arbitrary distribution.

It is shown that the Hausdorff convergence on the class of distributions, containing the space $\mathbb{C}^\infty$, is equivalent to absence for this class the (generalized) Gibbs phenomenon.

The obtained results are applied to the study of the Hausdorff convergence of processes of the summation of Fourier series by the Cesàro and Vallée-Poussin methods.

AMS subject classification: 41A35, 42A24, 26E25, 42A45, 46B10, 46F20
Key Words: the Hausdorff approximation, distributions, operators of convolution, summation of Fourier series, bounded operators, duality and reflexivity

0.1 Introduction

The concept of the Hausdorff distance between functions as the Hausdorff distance between their (complemented) graphs was generated in the sixties mainly due to works of Bulgarian mathematicians. The state of art of this subject in the end of the seventies is reflected in the book by B.Sendov [1]. The Hausdorff metrics has a number of attractive properties. In our opinion, the main of these properties is its naturalness for human eyes. Roughly speaking, functions are considered as close if their graphs are "visually" close (almost coincide). From this point of view the uniform metric was the predecessor of the Hausdorff metric. However, natural area of its usage is restricted by the space of continuous functions. The Hausdorff distance is used for measuring deviations between arbitrary bounded functions defined on

*The results of the paper were partly published without proofs in [6].

†This research was partially supported under Grant NSF KDI 578AO45

a metric compactum. Thus, as it was mentioned in [1], in some sense the uniform distance is a special, or to be more exact a limit case of the Hausdorff distance. On the space of continuous functions the Hausdorff and uniform convergence are equivalent. This is an essential distinction of the Hausdorff convergence and, for example, the convergence in the integral metrics. It gives one more argument that the Hausdorff metrics is the nearest relative of the uniform metric.

At the same time, absence of linearity is essential lack of function spaces with the Hausdorff metrics. In such spaces, if they contain discontinuous functions, it is essentially impossible to introduce operations of a linear space which are continuous with respect to the metric. Moreover, the accepted in [1] approach assumes identifying the concept of a function and its complemented graph. It makes problematic the introduction of a natural linear structure because objects of the operations are multi-valued functions (more precisely, their graphs). The lack of the linear structure does not create any difficulties for the problems related to the best approximations. However, usually such approach does not allow to use linear methods of approximations, at least for the spaces of functions which can have rather massive sets of discontinuity points. In this connection it is necessary to mention paper [2], where the problem of the approximation of, so-called, H-continuous functions by positive operators in the Hausdorff metrics was considered.

In our paper [3], we used other approach. We suggested to consider approximations by linear (integral) operators of the form

$$J_n(f) = \int_a^b f(t)K_n(x,t)dt$$

on the classical normed space of the functions essentially bounded on the segment $[a,b]$. However, we studied the convergence in the Hausdorff metrics (not in the uniform metric). In [3], necessary and sufficient conditions for the kernels K_n, providing convergence $J_n(f)$ to f in the Hausdorff metric for all essentially bounded functions, were found.

To discuss the convergence in the Hausdorff metric it is necessary to introduce the concept of a function graph properly. Such a definition of canonical graphs of the bounded function, invariant with respect to the change of a function on a set of measure zero, was introduced in [3]. The same definition without any change was used in [4] and [5] for integrable functions. Canonical graphs of unbounded functions are not compact sets. Therefore, to measure distance between unbounded functions it is necessary to compactify the axis of values by means of contracting with a bounded strictly monotone function.

In [4] and [5], the criteria for convergence of convolution sequences

$$I_n(f) = f * \mu_n \tag{0.1}$$

on the classes of 2π-periodic functions with summable pth power ($1 \le p \le \infty$) and on the class $\mathbb{BV}$ (functions of bounded variation) were found.

In [7], the convergence of Fourier series of functions of the Besov—Lizorkin—Triebel classes was studied.

The goal of this work is studding the convergence of operator sequences (0.1) on quasi-Banach spaces of periodic real-valued distributions (generalized functions). Section 1 is devoted to the problem of a reasonable definition of the (canonical) graph of a distribution.

In particular, it is shown that the Abel — Poisson sums of any distribution converge in the Hausdorff metric. In Section 2, a special class of quasi-Banach spaces of distributions is introduced. These spaces have a very important property. Their Fourier multipliers, acting to the space of essentially bounded functions, are determined explicitly. In Sections 3, some general results for spaces of this class about the convergence of operator sequences of the form (0.1) are obtained. In Section 3, the criteria for the convergence of sequence (0.1) on the real Hardy spaces $\Re\mathbf{H}^p$ $(0 < p < 1)$, on the space of regular Borel measures and on some other classes, containing the Dirac δ-function, are obtained. In Section 4, the approximations on certain spaces which do not satisfy the requirements of Section 3 are considered. In particular, a criterion for the convergence on the class $\Re\mathbf{H}^1$ is found. Finally, in Section 6, relying on the general results, the convergence of various summation methods of Fourier series on these spaces are studied.

We denote by $\mathbb{T}$ the unit circle with the natural metric. The distance between $\theta_1, \theta_2 \in \mathbb{T}$ is defined to be the length of the shortest arc connecting θ_1 and θ_2. We denote it by $\mid \theta_1, \theta_2 \mid$. In what follows all function spaces (spaces of distributions) are real-valued, but for convenience sometimes we shall use the complex notation. As usual we denote by $\mathbb{C} = \mathbb{C}(\mathbb{T})$, $\mathbb{C}^\infty = \mathbb{C}^\infty(\mathbb{T})$ respectively the spaces of continuous and infinitely differentiable functions. Recall that a sequence $f_n \in \mathbb{C}^\infty$ is called converging to $f \in \mathbb{C}^\infty$ in the topology of the space $\mathbb{C}^\infty$ if $\|f_n^{(k)} - f^{(k)}\|_\infty \to \infty$ for all $k = 0, 1, 2, \ldots$, where $\|\cdot\|_\infty$ is the uniform norm. The space $\mathbb{C}^\infty$ consists of functions, whose Fourier coefficients decrease faster any power.

The space of *distributions* (generalized functions) $\mathbb{D}$ on $\mathbb{T}$ is defined to be the set of all bounded linear functionals on $\mathbb{C}^\infty$. Principal facts, concerning periodic distributions, can be found in the book by R.Edwards ([8], Chapter 12). Here we recall some of them.

The space $\mathbb{D}$ can be identify with the set of formal trigonometric series $\sum_{n\in\mathbb{Z}} c_n e^{inx}$, where $\mathbb{Z}$ is a set of integer numbers with coefficients of temperate growth, i.e., $\mid c_n \mid = O(|n|^k)$ as $|n| \to \infty$ for some natural k which can be different for different distributions. This correspondence is set by the relations $c_n := \hat{f}(n) := f[e^{inx}]$, where $f \in \mathbb{D}$. The requirement of real-valuedness of a distribution coincides with the condition $c_{-n} = \bar{c}_n$. We say that a sequence $f_n \in \mathbb{D}$ is convergent to $f \in \mathbb{D}$ in $\mathbb{D}$ if $f_n[\phi] \to f[\phi]$ for any function $\phi \in \mathbb{C}^\infty$, or (it is the same) if f_n converges to f coefficientwise, i.e., $\hat{f}_n(k) \to \hat{f}(k)$ for all $k \in \mathbb{Z}$, and there exists p such that $\lim_{n\to\infty} \sup_{k\in\mathbb{Z}} |\hat{f}_n(k)|/|k|^p < \infty$. It means that the Fourier coefficients of the sequence f_n are majorized by coefficients of some distribution. In view of the generalized convergence, we put the sign "=" in the formula

$$f(x) = \sum_{n\in\mathbb{Z}} c_n e^{inx} \equiv \sum_{n\in\mathbb{Z}} \hat{f}(n) e^{inx}$$

instead of "$\sim$", as it is accepted in the classical theory of Fourier series.

We denote by $D[f]$ the generalized derivative of the distribution f,

$$D[f](x) = \sum_{n\in\mathbb{Z}} in\hat{f}(n)e^{inx}.$$

Indefinite integral of a distribution is a distribution only if $\hat{f}(0) = 0$. We denote by $S[f]$ the primitive of such distribution,

$$S[f] = \sum_{\substack{n \in \mathbb{Z} \\ n \neq 0}} \hat{f}(n) e^{inx}.$$

A convolution of two distributions f_1 and f_2 is defined to be the distribution

$$f(x) = f_1 * f_2\,(x) := \sum_{n \in \mathbb{Z}} \hat{f}_1(n) \hat{f}_2(n) e^{inx}.$$

Let

$$P_r(x) := \sum_{n \in \mathbb{Z}} r^{|n|} e^{inx}, \quad 0 \le r < 1,$$

be the Poisson kernel, then it is natural to reserve the notation P_1 for the δ-function (the Dirac measure). Any distribution $f(x) = \sum_{n \in \mathbb{Z}} a_n e^{inx}$ can be harmonically extended into the unit disc of the complex plane by means of the formula

$$F(z) := \sum_{n \in \mathbb{Z}} a_n r^{|n|} e^{inx} = f * P_r(x) =: f^r(x),$$

where z is an arbitrary complex number of the unit disc, r and x are its absolute value and argument correspondingly.

Let us recall that two distributions coincide on an open arc $\gamma \subset \mathbb{T}$ if they coincide as functionals on any function $\phi \in \mathbb{C}^\infty$, $\operatorname{supp} \phi \subset \gamma$.

For any $\phi \in \mathbb{C}^\infty$ and $f \in \mathbb{D}$ the operation of pointwise multiplication is defined by the formula

$$(\widehat{f\phi})(n) := \sum_{m \in \mathbb{Z}} \hat{\phi}(m) \hat{f}(n-m)$$

or by equivalent representation of f as the functional $(\phi f)[u] = f[\phi u]$ for any $u \in \mathbb{C}^\infty$.

We denote by $\{\chi_\delta\}_{\delta > 0}$ and fix an arbitrary set of non-negative functions of $\mathbb{C}^\infty$ for which $\|\chi_\delta\|_\infty = 1$, $\chi_\delta(\theta) = 1$ when $|\theta, 0| < \delta$, and $\chi_\delta(\theta) = 0$ when $|\theta, 0| > 2\delta$.

Recall main definitions related to the Hausdorff distance between functions. Let A and B be two subsets of a metric space (X, ρ). Then the Hausdorff distance between these sets is defined to be

$$H(A, B) = \inf\{\varepsilon \mid A \subset U_\varepsilon(B), B \subset U_\varepsilon(A)\},$$

where $U_\varepsilon(\cdot)$ is an ε-neighbourhood of a set with respect to the metrics ρ. The compact subsets (X, ρ) form a metric space with the metric H.

The complemented graph $F(f)$ of, generally speaking, a multi-valued function $f : \mathbb{T} \to \mathbb{R}^1$ is the smallest closed set convex with respect to the y-axis, containing the graph of the function $y = f(x)$. We recall that a plane set is called convex with respect to the y-axixs if together with any pair of points (x, y_1), (x, y_2) it contains whole segment connecting these points. The Hausdorff distance between functions f and g is defined to be the Hausdorff distance between their complemented graphs: $H(f, g) = H(F(f), F(g))$, where a metrics on the set $\Gamma = \mathbb{T} \times \mathbb{R}^1$, inducing the metrics H depends on the task. If we deal only with the bounded functions, the Minkowski metric

$$\rho((x_1, y_1), (x_2, y_2)) = \max\{|x_1, x_2|, |y_1 - y_2|\}$$

is the most suitable. In this case, the set of all bounded periodic functions, whose graphs are complemented, forms a metric space. If there is necessity to consider unbounded functions, their complemented graphs are not compact sets. In [5] we proposed to replace the natural metric on $\mathbb{R}^1$ with the metric $d(y_1, y_2) = |\eta(y_1) - \eta(y_2)|$, where η is an arbitrary bounded strictly increasing function. The topology of the real line $\mathbb{R}^1$ does not depend on the choice of the function η. Therefore, it is possible to take, for example, $\eta(y) = 2\arctan y/\pi$. By adding the points $\pm\infty$ to $\mathbb{R}^1$, we get the compact set $\bar{\mathbb{R}}^1$ with the metrics d. Then we define the metrics on Γ as follows

$$\rho((x_1, y_1), (x_2, y_2)) = \max\{|x_1, x_2|, d(y_1, y_2)\}, \tag{0.2}$$

In this case, the set of all complemented graphs (or functions whose graphs are complemented) forms a metric space.

In what follows we often shall use the same notation $F(f)$ for the graph and the multivalued function $F(f)(x)$. The current sense will be clear from a context.

1 Canonical graphs of distributions

In [3] we introduced the concept of a canonical graph of an arbitrary measurable and finite almost everywhere function. Let us recall that the canonical graph of such a function f is defined to be the complemented graph of the restriction of f to the set of its points of approximate continuity. The main argument for the benefit of such definition is its invariancy with respect to the change of the function f on a set of Lebesgue measure zero. This argument is really crucial for integrable functions because functions of $\mathbb{L}^1$, differing on a null set, coincide as distributions. At the same time, for arbitrary distributions such an approach does not work for two reasons. At first there is no the initial concept of a graph of an arbitrary distribution, using which it would be possible to construct a complemented graph. Secondly, even if it is possible with a reasonable manner to introduce the concept of a distribution graph, changing it on a null set we can get the graph of other distribution. The Dirac measure and the identical zero give such an elementary example.

Despite of the mentioned above difficulties it is possible to introduce the reasonable (at least, from the point of view of the considered tasks) definition of the canonical graph of a distribution. The main idea of such a definition bases on the possibility of harmonic extension of any distribution from $\mathbb{T}$ into the unit disc. In this case, we have the regular graph of such extension. It remains to take its closure and to extract the trace of the closure on $\mathbb{T}$. Now we implement this reasoning accurately.

Let $f \in \mathbb{D}$, $F(z) = f * P_r(\theta)$. Let us recall that the cluster set of the function F at the point $\theta \in \mathbb{T}$ (see, for example, [9]) is defined to be the set $C(F, \theta)$, consisting of those points $\alpha \in \bar{\mathbb{R}}^1$ for which the sequence of complex numbers z_n, satisfying conditions $|z_n| < 1$, $z_n \to z_0$ ($|z_0| = 1$, $\arg z_0 = \theta$), and $\lim_{z_n \to z_0} F(z_n) = \alpha$, exists.

Definition 1.1. *The canonical graph $F^*(f)$ of a distribution f is define to be the graph of, generally speaking, the multi-valued function $C(F, \theta)$.*

Naturally, we define the Hausdorff distance between distributions as the Hausdorff distance between their canonical graphs, $H^*(f, g) = H(F^*(f), F^*(g))$. We abbreviate $f_n \stackrel{H^*}{\longrightarrow} f$

that fact that $\lim_{n\to\infty} H^*(f_n, f) = 0$. Obviously, the set $F^*(f)$ is closed and convex with respect to the y-axis, i.e., it has all properties of a complemented graph. Therefore, it can be approximated in the Hausdorff metric by graphs of continuous functions with an arbitrary accuracy. The convergence of the Abel — Poisson sums on the class $\mathbb{L}^1$ (see [5]) implies the equivalence of Definition 1.1 and the corresponding definition in [3].

Despite of its attraction, the distance H^* has one essential lack. This distance is not a metric on the space of distributions. A reason is the possibility of coincidence of canonical graphs of two different distributions. For example, all derivatives of the function P_1 have identical canonical graphs. It is easy also to construct examples of bounded functions with the same canonical graphs. Indeed, we take a set $E \subset \mathbb{T}$ such that for any arc $\gamma \subset \mathbb{T}$ we have $\operatorname{meas}(\gamma \bigcap E) > 0$ and $\operatorname{meas}(\gamma \setminus E) > 0$. Then the functions χ_E and $\chi_{\mathbb{T}\setminus E}$ have identical canonical graphs. We note that on the set of canonical graphs the distance H^* is a metric.

Thus, the distance H^* does not "feel" the difference between some of distributions. Nevertheless, it turns out (see Theorem 5.4 below), that all assertions which are proved in Sections 3 and 4 for the distance H^* remains in force for the stronger distance $H^\Box$ (see Section 5) of Hausdorff's type. The distance $H^\Box$ is a metric on $\mathbb{D}$. The convergence in this metric means the Hausdorff convergence $f_n^r \xrightarrow{H^*} f^r$ uniformly with respect to r, $0 \le r \le 1$. Observe that the convergence in the metric $H^\Box$ implies the Hausdorff convergence of a sequence of the corresponding functions of two variables harmonic in the unit disc.

In Definition 1.1, it is unimportant that the distribution f is real-valued and belongs to $\mathbb{D}$. We need only the possibility of harmonic extension into the unit disc. So, we can consider formal trigonometric series $\sum_{n\in\mathbb{Z}} c_n e^{in\theta}$ whose coefficients satisfy the inequality $\lim_{n\to\infty} \sqrt[n]{|c_n|} \le 1$, as above f. However, in what follows, we restrict ourselves to considering only quasi-Banach spaces with elements from $\mathbb{D}$.

The values of the function $C(F, x)$ are segments of the extended real line $\bar{\mathbb{R}}^1$. We denote by $\operatorname{ess\,sup} f(x)$ the maximum value of $C(F, x)$ at the point $x \in \mathbb{T}$. In particular, this value can be equal to $+\infty$ or $-\infty$. The value $\operatorname{ess\,inf} f(x)$ is introduced similarly. Thus, it is convenient to give the definitions

$$\operatorname*{ess\,sup}_{x\in E} f(x) := \sup_{x\in E} \operatorname{ess\,sup} f(x),$$

$$\operatorname*{ess\,inf}_{x\in E} f(x) := \sup_{x\in E} \operatorname{ess\,inf} f(x),$$

for $E \subset \mathbb{T}$ and $f \in \mathbb{D}$. When E is an open arc and f is an essentially bounded function, these definitions coincide with the classical definitions of essential upper and essential lower bounds.

Theorem 1.1. *For any $f \in \mathbb{D}$ we have $f^r \xrightarrow{H^*} f$ as $r \to 1$.*

Before to proceed to the proof of Theorem 1.1, we prove several elementary lemmas.

Lemma 1.2. *If a distribution f vanishes on an open arc γ, then $f * P_r\ (\theta) \to 0$ uniformly on any closed arc $\gamma_0 \subset \gamma$ as $r \to 1$.*

Proof of Lemma 1.2. Let δ be the Hausdorff distance between arcs γ and γ_0. We define the distributions $\nu_r := \chi_{\delta/4} \cdot P_r$, $\mu_r := (1 - \chi_{\delta/4}) \cdot P_r$. Obviously, $P_r = \nu_r + \mu_r$. For any $\theta \in \gamma_0$, since

f vanishes on γ, and the support of $\chi_{\delta/4}$ belongs to the $(\delta/2)$-neighborhood of zero, we have $f * \nu_r(\theta) = 0$. As $f \in \mathbb{D}$, there exist $k \geq 1$ and $M > 0$ such that $\|S^k[f - \hat{f}(0)]\|_1 = M < \infty$. At the same time, for any natural l and $\theta \neq 0$ we have $D^l P_r(\theta) \to 0$ as $r \to 1$ uniformly outside of any neighbourhood of zero. Besides, $((1-\chi_{\delta/4}) \cdot P_r)\widehat{\ }(0) \to 0$. It means that for any $\varepsilon > 0$ there exists $r_0 < 1$ such that for any $r_0 \leq r < 1$ we have

$$\|D^k[(1-\chi_{\delta/4}) \cdot P_r]\|_\infty < \varepsilon/2M, \quad \hat{f}(0) \cdot ((1-\chi_{\delta/4}) \cdot P_r)\widehat{\ }(0) < \varepsilon/2.$$

Hence,

$$\begin{aligned}\|f * \mu_r\|_\infty &= \|\hat{f}(0) \cdot ((1-\chi_{\delta/4}) \cdot P_r)\widehat{\ }(0) + S^k[f - \hat{f}(0)] * D^k[\mu_r]\|_\infty \\ &\leq \varepsilon/2 + \|S^k[f - \hat{f}(0)]\|_1 \cdot \|D^k[\mu_r]\|_\infty \leq \varepsilon/2 + M \cdot \varepsilon/2M = \varepsilon.\end{aligned}$$

Therefore, for such r and $\theta \in \gamma_0$ we have

$$|f * P_r(\theta)| = |f * \mu_r(\theta)| < \varepsilon,$$

As was to be shown. □

Lemma 1.3. *Let $f \in \mathbb{D}$, then for any $\theta_0 \in \mathbb{T}$ and $\delta > 0$ we have*

$$\overline{\lim_{r\to 1}} \inf_{|\theta,\theta_0|<\delta} f^r(\theta) \leq \operatorname{ess\,inf} f(\theta_0). \tag{1.1}$$

Proof of Lemma 1.3. Validity of Lemma 1.3 when $\operatorname{ess\,inf} f(\theta) = +\infty$ as well as the left-hand part of (1.1) is equal to $-\infty$ is obvious. We conduct the proof by contradiction. Let inequality (1.1) do not hold for some $\theta_0 \in \mathbb{T}$ and $\delta > 0$. Without loss of generality we can suppose $\theta_0 = 0$, $\operatorname{ess\,inf} f(0) < 0$. We also assume that there is a sequence $r_n \nearrow 1$ such that

$$A_n := \inf_{|\theta,0|<\delta} f^{r_n}(\theta) > 0, \quad A_n \to A > 0 \quad (n \to \infty).$$

The sequence f^{r_n} converges to f in $\mathbb{D}$. Therefore, in view of continuity (in the topology of $\mathbb{D}$) of the operation of pointwise multiplication by a function of $\mathbb{C}^\infty$, the sequences $\phi_n = \chi_{\delta/2} f^{r_n}$ and $\psi_n = (1-\chi_{\delta/2}) f^{r_n}$ converge in $\mathbb{D}$ respectively to the distributions $\phi = \chi_{\delta/2} f$ and $\psi = (1-\chi_{\delta/2}) f$. According to a maximum principle for harmonic functions, we have

$$\inf_{\substack{0\leq r\leq 1\\ \theta\in\mathbb{T}}} \phi_n * P_r(\theta) = \inf_{\theta\in\mathbb{T}} \phi_n(\theta) \geq 0.$$

From the generalized convergence of the sequence ϕ_n to ϕ as $n \to \infty$ we have the uniform convergence of the sequence $\phi_n * P_r(\theta)$ to $\phi * P_r(\theta)$ on compact sets of the open unit disc. Hence, $\phi^r(\theta)$ is a non-negative harmonic function, i.e., ϕ is a positive measure. At the same time, according to Lemma 1.2, for any $\varepsilon > 0$ there exists r_0 such that for $r_0 \leq r < 1$ and $|\theta, 0| < \delta/4$ we have $|\psi\ (\theta) * P_r(\theta)| < \varepsilon$. Therefore,

$$\operatorname{ess\,inf} f(0) \geq \inf_{\substack{0\leq r<1\\ \theta\in\mathbb{T}}} \phi^r(\theta) + \inf_{\substack{0\leq r<1\\ |\theta,0|<\delta/4}} \psi^r(\theta) \geq 0 - \varepsilon.$$

It follows from here that in view of arbitraryness of the choice of ε we obtain $\operatorname{ess\,inf} f(0) \geq 0$ that contradicts our initial assumption. Lemma 1.3 is proved. □

Proof of Theorem 1.1. We need to prove that for any $\varepsilon > 0$ there exists $0 \le r_0 < 1$ such that for $r_0 \le r < 1$ the inclusions

$$F^*(f) \subset U_\varepsilon(F(f * P_r)), \tag{1.2}$$

$$F(f * P_r) \subset U_\varepsilon(F^*(f)),$$

where $U_\varepsilon(\cdot)$ is the ε-neighbourhood of a set with respect to metric (0.2), hold. Obviously, validity of the last inclusion for r sufficiently close to 1 follows immediately from Definition 1.1. We show the possibility of fulfillment of inclusion (1.2).

Let us assume the contrary. Let there exist $\varepsilon > 0$ and a sequence $r_i \nearrow 1$ such that

$$F^*(f) \setminus U_\varepsilon(F(f^{r_i})) \neq \varnothing.$$

Then there exists the sequence $\{(x_i, y_i)\} \subset \mathbb{T}\times\bar{\mathbb{R}}^1$ such that $(x_i, y_i) \in F^*(f), (x_i, y_i) \notin U_\varepsilon(F(f^{r_i}))$. In view of the fact that F^* (f) is a compact set with respect to metric (0.2), without loss of generality we suppose that this sequence converges to the point $(x_0, y_0) \in F^*(f)$ which locates below graphs of the functions f^{r_i}. Then for sufficiently big i and $|\theta, x_0| < \varepsilon/2$ we have $f^{r_i}(\theta) > y_0 + \varepsilon/2$. Hence,

$$\varlimsup_{i\to\infty} \inf_{|\theta,x_0|<\varepsilon/2} f^{r_i}(\theta) > y_0 \ge \operatorname{ess\,inf} f(x_0)$$

that contradicts Lemma 1.3. Theorem 1.1 is proved. □

2 Bounded operators of convolution and linear functionals

Here we consider a class of quasi-Banach spaces for which the general approach to studding the convergence of operators of the form (0.1) in the Hausdorff metric is possible. The main feature of these spaces is the simplicity of description of bounded operators of convolution (or, in other terminology, Fourier multipliers), acting from these spaces to the space $\mathbb{L}^\infty$ (see Lemma 2.2 below). We shall see in what follows that it is useful for our tasks.

The majority of the statements of this section is very simple. We give their full proofs, though it is probable that all or many of them were already considered in mathematical literature.

Let X be a quasi-Banach space, $X \subset \mathbb{D}$. For its quasi-norm, except for norms of the spaces $\mathbb{C}$, $\mathbb{L}^p$ $(1 \le p \le \infty)$ and $\mathbf{M}$, we use the notation $\|\cdot|X\|$. If the opposite is not stipulated, the symbol $\subset$, used with quasi-Banach spaces, will designate topologically continuous embedding.

Remind that the sum of quasi-Banach spaces X_1 and X_2 is defined to be the quasi-Banach space

$$Y = X_1 + X_2 := \{f + g \mid f \in X_1, g \in X_2\}$$

with the quasi-norm

$$\|h \mid Y\| = \inf\{\|f \mid X_1\| + \|g \mid X_2\| \mid f \in X_1,\ g \in X_2,\ h = f + g\};$$

and the intersection of them is the space

$$Z = X_1 \cap X_2 := \{f \mid f \in X_1, f \in X_2\}$$

with the quasi-norm $\|f \mid Z\| = \max\{\|f \mid X_1\|, \|f \mid X_2\|\}$.

Definition 2.1. *We denote by $\mathcal{H}$ a collection of quasi-Banach spaces X for which (at least, after equivalent renormalization) the following conditions are fulfilled:*
(a) $\mathbb{C}^\infty \subset X \subset \mathbb{D}$;
(b) For any distribution $f \in X$

$$\|f \mid X\| = \sup_{r<1} \|f * P_r \mid X\|; \tag{2.1}$$

(c) The space X is invariant with respect to the shift of the argument, and for any $f \in X$ we have $\|f(t) \mid X\| = \|f(-t) \mid X\|$;
(d) There are $k > 0$ and $C > 0$ such that for any distribution $f \in X$

$$\|f \cdot \sin nx \mid X\| \leq Cn^k \|f \mid X\|.$$

Condition (d) guarantees the continuity in X of the operator of multiplying by a function of $\mathbb{C}^\infty$. This is necessary in the following sections. It will be seen, if we omit condition (d) in Definition 2.1, all results of this section stay valid.

The renormalization does not influence validness of conditions (a) and (c). We give an elementary example when the renormalization can be necessary to satisfy property (b). Let X be the space of essentially bounded functions. The norm of $f \in X$ in this space is set as the sum of a regular uniform norm and exact upper bound of the values of discontinuities of the function f. Obviously, such norm does not satisfy condition (b). However, it is equivalent to the uniform norm.

Let's remark, that the class $\mathcal{H}$ contains overwhelming majority of classical quasi-Banach real-valued spaces of periodic functions and distributions. So, for example, it contains the Besov — Lizorkin — Triebel spaces $\mathbf{B}^s_{p,q}$ and $\mathbf{F}^s_{p,q}$ (in particular, the Lebesgue spaces $\mathbb{L}^p$ $(1 < p \leq \infty)$ and the Hardy spaces $\Re\mathbf{H}^p$ $(0 < p \leq 1)$), **BMO**, $\mathbb{BV}$ (the functions of bounded variation), **M** (regular Borel measures), real parts of the analytical Bergman and Bloch classes, the classes of distributions ℓ^p $(0 < p \leq \infty)$ whose Fourier coefficients satisfy the inequality $\sqrt[p]{|c_n|^p}$, and so on.

Let $X \in \mathcal{H}$, then we denote by $\overset{\circ}{X}$ (or X°) the completion of the space $\mathbb{C}^\infty$ in the quasi-norm of the space X;

$$\overset{\square}{X} := \{f \in \mathbb{D} \mid \sup_{r<1} \|f * P_r \mid X\| < \infty\}.$$

Obviously, the embeddings $\mathbb{C}^\infty \subset \overset{\circ}{X} \subset X \subset \overset{\square}{X} \subset \mathbb{D}$ are valid. The space X is separable only in the case when $X = \overset{\circ}{X}$. Besides, if $f \in \overset{\circ}{X}$, then $\|f - f^r \mid X\| \to 0$ as $r \to 1$ (the last is explained in the proof of Lemma 2.3).

Let $X \in \mathcal{H}$, then we denote by $[X]$ or $[\overset{\circ}{X}, \overset{\square}{X}]$ a collection of all spaces $Y \in \mathcal{H}$ such that

$$\overset{\circ}{X} \subset Y \subset \overset{\square}{X} \tag{2.2}$$

We note that embeddings (2.2) themselves do not guarantee that Y belongs to $\mathcal{H}$. Actually, let $Y = \mathbb{C} + \mathbb{BV}$. Let us partition $\mathbb{T}$ into $2n$ identical arcs γ_i, enumerated for definiteness clockwise. Let function $f \in Y$ be equal to the identity on the arcs with even numbers and is equal to zero on the arcs with odd numbers. Then $\|f \mid \mathbb{C}+\mathbb{BV}\| = 2n$, $\|f * P_r \mid \mathbb{C}+\mathbb{BV}\| \to 1$ as $r \to 1$. It means, $\mathbb{C}+\mathbb{BV} \notin \mathcal{H}$, though, obviously, $\mathbb{C} \subset \mathbb{C}+\mathbb{BV} \subset \mathbb{L}^\infty$.

Lemma 2.1. *The operation* $[\cdot]$ *breaks up* $\mathcal{H}$ *into equivalence classes.*

Proof of Lemma 2.1. Obviously, it suffices to prove that for any X, $Y \in \mathcal{H}$ it follows from belongness Y to $[X]$ that

$$\overset{\circ}{Y} = \overset{\circ}{X}, \tag{2.3}$$

$$\overset{\square}{Y} = \overset{\square}{X}, \tag{2.4}$$

where the equality means the coincidence of quasi-normed spaces up to equivalent renormalization.

We show that equality (2.3) holds. If $f \in \overset{\circ}{Y}$, then $\|f - f^r \mid Y\| \to 0$. It follows from the embedding $Y \subset \overset{\square}{X}$ that there is a constant $C_1 > 0$ such that $\|f \mid X\| \le C_1\|f \mid Y\|$ and $\|f - f^r \mid X\| \le C_1\|f - f^r \mid Y\| \to 0$ as $r \to 1$. Therefore, $\overset{\circ}{Y} \subset \overset{\circ}{X}$.

Let $f \in \overset{\circ}{X}$. Then the embedding $\overset{\circ}{X} \subset Y$ implies the existence of a constant $C_2 > 0$ such that $\|f \mid Y\| \le C_2\|f \mid X\|$. Hence, $\|f - f^r \mid Y\| \le C_2\|f - f^r \mid X\| \to 0$ as $r \to 1$. Thus, $\overset{\circ}{X} \subset \overset{\circ}{Y}$.

Let us prove equality (2.4). Let $f \in \overset{\square}{Y}$, then

$$C_1\|f \mid Y\| = C_1 \sup_{r<1} \|f^r \mid Y\| \ge \sup_{r<1} \|f^r \mid X\| = \|f \mid X\|.$$

Consequently, $f \in \overset{\square}{X}$ and $\overset{\square}{Y} \subset \overset{\square}{X}$. Since $f \in \overset{\square}{X}$, in similar way, we obtain $\|f \mid X\| \ge C_1\|f \mid Y\|$. Lemma 2.1 is proved. □

In what follows, if the other is not stipulated, we consider only spaces from $\mathcal{H}$.

We denote by $M(X, \mathbb{L}^\infty)$ the set of bounded convolution operators (in the terms of [8], Chapter 12, multiplier operators), acting from the quasi-Banach space $X \subset \mathbb{D}$ to $\mathbb{L}^\infty$, and denote by X^* the space of continuous linear functionals in X.

Let us prove the main statement of this section.

Lemma 2.2. $M(X, \mathbb{L}^\infty) = \left(\overset{\circ}{X}\right)^*$.

Remark. It is easy to prove that a set of compact convolution operators, acting from $X \in \mathcal{H}$ to $\mathbb{L}^\infty$, coincides with the space $\left(\overset{\circ}{X}\right)^{*\circ}$.

Proof of Lemma 2.2. We denote by $T_\tau[\cdot]$ the operator $T_\tau : f(t) \to f(\tau - t)$. Let $f \in X$, $g \in X^*$, then $\langle g, f\rangle$ is defined to be the value of the linear functional g on the distribution f.

Let $\mu \in \left(\overset{\circ}{X}\right)^*$ and norms of shift operators are bounded uniformly by the number C. For any function $\phi \in \mathbb{C}^\infty$

$$|\langle\mu, \phi\rangle| = \left|\sum_{n\in\mathbb{Z}} \hat{\mu}(n)\hat{\phi}(-n)\right| \le \left\|\mu \mid \left(\overset{\circ}{X}\right)^*\right\| \cdot \|\phi \mid X\|,$$

where the series converges absolutely. From here for any distribution $f \in X$ we have

$$\|\mu * f\|_\infty = \sup_{r<1} \|\mu * f * P_r\|_\infty = \sup_{r<1}\sup_{\tau\in\mathbb{T}} |\langle\mu, T_\tau[f^r]\rangle| \le$$
$$\sup_{r<1}\sup_{\tau\in\mathbb{T}} \left\|\mu \mid \left(\overset{\circ}{X}\right)^*\right\| \cdot \|T_\tau[f^r] \mid X\| \le C \sup_{r<1} \left\|\mu \mid \left(\overset{\circ}{X}\right)^*\right\| \cdot \|f^r \mid X\| =$$
$$= C \left\|\mu \mid \left(\overset{\circ}{X}\right)^*\right\| \cdot \|f \mid X\|.$$

Conversly, let $\mu \in M(X, \mathbb{L}^\infty)$ and $\phi \in \mathbb{C}^\infty$, then

$$|\langle\mu, \phi\rangle| = \left|\sum_{n\in\mathbb{Z}} \hat{\mu}(n)\hat{\phi}(-n)\right| = |\mu * T_0[\phi](0)| \le \|\mu * T_0[\phi]\|_\infty \le$$
$$\le \|\mu \mid M(X, \mathbb{L}^\infty)\| \cdot \|\phi \mid X\|.$$

In view of the density of $\mathbb{C}^\infty$ in $\overset{\circ}{X}$ we obtain

$$\left\|\mu \mid \left(\overset{\circ}{X}\right)^*\right\| \le \|\mu \mid M(X, \mathbb{L}^\infty)\|.$$

We note that for the validity of Lemma 2.2 properties (b) and (c) of Definition 2.1 are essential and it is impossible to omit them. Indeed, let X be a sum of the space of continuously differentiable functions $\mathbb{C}^{(1)}$ and the singular component of the space $\mathbb{BV}$, i.e., those functions of $\mathbb{BV}$ whose regular derivative are equal to zero almost everywhere. Obviously, such space does not satisfy property (b). Then $\overset{\circ}{X} = \mathbb{C}^{(1)}$ and X is invariant with respect to the shift of the argument and the operator of derivation is a bounded functional on $\mathbb{C}^{(1)}$ and is unbounded as an operator acting from X in $\mathbb{L}^\infty$. The weighted space with the uniform norm, whose continuous weight vanishes at some point, gives an example of space, non-invariant with respect to the shift, satisfying (2.1), for which the statement of Lemma 2.2 is not valid. □

Lemma 2.3. *Let* $X = \overset{\circ}{X}$, $\mu \in \left(\overset{\circ}{X}\right)^*$, *then* $f * \mu \in \mathbb{C}$.

Proof of Lemma 2.3. At first, we show that $\|f - f^r \mid X\| \to 0$ as $r \to 1$. Let $\varepsilon > 0$ be an arbitrary number. Then there exists $\phi \in \mathbb{C}^\infty$ such that $\|f - \phi \mid X\| < \varepsilon/3$. ϕ^r converges to ϕ in the topology of the space $\mathbb{C}^\infty$. Therefore, for r sufficiently close to 1 we have $\|\phi^r - \phi \mid X\| < \varepsilon/3$. Hence,

$$\|f - f^r \mid X\| \le C_0(\|f - \phi \mid X\| + \|\phi - \phi^r \mid X\| + \|\phi^r - f^r \mid X\|) \le$$
$$C_0(\varepsilon/3 + \varepsilon/3 + \sup_{r<1} \|(\phi - f) * P_r \mid X\|) = C_0\varepsilon,$$

where C_0 is a constant, generally speaking, not equal to the identity because X is a quasi-normed space.

Let $g = f * \mu$, then $g^r = f^r * \mu$ and

$$\|g - g^r\|_\infty = \|(f - f^r) * \mu\|_\infty \le \|f - f^r \mid X\| \cdot \|\mu \mid M(X, \mathbb{L}^\infty)\|.$$

Since the first multiplicand goes to zero as $r \to 1$, g^r uniformly converges to g. Therefore, $g \in \mathbb{C}$. □

Lemma 2.4. *$M(X, \mathbb{L}^\infty) \in \mathcal{H}$.*

Proof of Lemma 2.4. Fulfillment of the embeddings $\mathbb{C}^\infty \subset M(X, \mathbb{L}^\infty) \subset \mathbb{D}$ follows immediately from the embeddings $\mathbb{C}^\infty \subset X \subset \mathbb{D}$, from Lemma 2.2, and from that fact $\mathbb{D} = (\mathbb{C}^\infty)^*$, $\mathbb{D} = M(\mathbb{C}^\infty, \mathbb{L}^\infty)$.

We show the validity of condition (b) of Definition 2.1. On the one hand,

$$\|\mu \mid M(X, \mathbb{L}^\infty)\| = \sup\left\{\|\mu * f\|_\infty \;\Big|\; f \in \overset{\circ}{X}, \|f \mid X\| \le 1\right\} \ge \sup_{r<1} \sup\left\{\|\mu * f^r\|_\infty \;\Big|\; f \in \overset{\circ}{X}, \|f \mid X\| \le 1\right\} = \sup_{r<1} \sup\left\{\|\mu^r * f\|_\infty \;\Big|\; f \in \overset{\circ}{X}, \|f \mid X\| \le 1\right\} = \sup_{r<1} \|\mu^r \mid M(X, \mathbb{L}^\infty)\|. \quad (2.5)$$

On the other hand, for any $\varepsilon > 0$ there exists a distribution $f \in \overset{\circ}{X}$, $\|f \mid X\| = 1$ such that $\|\mu * f\|_\infty \ge \|\mu \mid M(X, \mathbb{L}^\infty)\| - \varepsilon$ and for r close enough to 1 we have

$$\|\mu * f\|_\infty \le \|\mu * f^r\|_\infty + \|\mu * (f - f^r)\|_\infty \le \|\mu^r * f\|_\infty + \varepsilon \le \|\mu^r \mid M(X, \mathbb{L}^\infty)\| + \varepsilon \le \sup_{r<1} \|\mu^r \mid M(X, \mathbb{L}^\infty)\| + \varepsilon.$$

Combining the both inequalities, we obtain

$$\|\mu \mid M(X, \mathbb{L}^\infty)\| \le \sup_{r<1} \|\mu^r \mid M(X, \mathbb{L}^\infty)\| + 2\varepsilon.$$

In view of arbitraryness of $\varepsilon > 0$ it follows from here and from (2.5) that condition (b) of Definition 2.1 holds.

Now we show that the class $M(X, \mathbb{L}^\infty)$ is invariant with respect to the shift operator $T_\tau[\cdot]$. Let $\mu \in M(X, \mathbb{L}^\infty)$, $\tau \in \mathbb{T}$. Then

$$\|T_\tau[\mu] \mid M(X, \mathbb{L}^\infty)\| = \sup\left\{\|\mu * T_\tau[f]\|_\infty \;\Big|\; f \in \overset{\circ}{X}, \|f \mid X\| = 1\right\} \le \|\mu \mid M(X, \mathbb{L}^\infty)\| \cdot \|T_\tau[f] \mid X\| \le C\|\mu \mid M(X, \mathbb{L}^\infty)\| \cdot \|f \mid X\|.$$

Invariance with respect to the change of a variable $t \to t + \tau$ is shown similarly.

Let us show fulfillment of property (d) for $\mu \in M(X, \mathbb{L}^\infty)$. It is convenient to conduct the proof in the complex form. We note that, in view of property (c), property (d) remains in force (with the same k but other C) if, in its statement, we replace the function $\sin nx$ by the function $\cos nx$. Thus, there is a constant $C_0 > 0$ such that for any $f \in X$ we have

$$\|f * (\mu \cdot \sin nx)\|_\infty \le \|f * (\mu \cdot z^n)\|_\infty = \|z^n((f \cdot z^{-n}) * \mu)\|_\infty \le$$
$$\le \|f \cdot z^{-n} \mid X\| \cdot \|\mu \mid M(X), \mathbb{L}^\infty)\| \le C_0 \|f \mid X\| \cdot \|\mu \mid M(X, \mathbb{L}^\infty)\|,$$

as was to be shown. □

Banach spaces for which $X = \overset{\square}{X} \in \mathcal{H}$ have a remarkable property which is close as a matter of fact to the reflexivity. Let us formulate this property.

Lemma 2.5. *Let $X \in \mathcal{H}$ and X is a Banach space. Then*

$$M(M(X, \mathbb{L}^\infty), \mathbb{L}^\infty) = M(M(\overset{\square}{X}, \mathbb{L}^\infty), \mathbb{L}^\infty) = X^{\circ * \circ *} = \overset{\square}{X}.$$

Remark. Obviously, if X is a quasi-Banach (not Banach) space, the statement of Lemma 2.5 ceases to be valid. It follows from the fact that the space $X^{\circ * \circ *}$ is a Banach space.

Proof of Lemma 2.5. In view of Lemma 2.2 only the last equality requires the proof. Let $Y = X^{\circ *}$, $Z = Y^{\circ *}$, $h \in \mathbb{D}$. Bearing in mind that for any Banach space B and $b \in B$ we have

$$\|b \mid B\| = \sup\{\langle c, b\rangle \mid \|c \mid B^*\| \le 1\} \tag{2.6}$$

(see, for instance, [10], Chapter 5) and by Lemma 2.4, we obtain a chain of the equalities

$$\|h \mid Z\| = \sup_{r<1} \|h^r \mid Z\| = \sup_{r<1} \sup\{\langle h^r, g\rangle \mid g \in \overset{\circ}{Y}, \|g \mid Y\| \le 1\} =$$
$$\sup_{r<1} \sup\{\langle h, g^r\rangle \mid g \in \overset{\circ}{Y}, \|g \mid Y\| \le 1\} =$$
$$\sup_{r<1} \sup\{\langle h^r, g\rangle \mid g \in Y, \|g \mid Y\| \le 1\} = \sup_{r<1} \|h^r \mid X\| = \|h \mid X\|.$$

Obviously, the assertion of Lemma 2.5 follows from this. We note that we used (2.6) (which is not valid for quasi-Banach spaces) in the penultimate equality. □

Lemma 2.6. *Let $X, Y \in \mathcal{H}$. Then $X \cap Y \in \mathcal{H}$ and $(X \cap Y)^\circ = \overset{\circ}{X} \cap \overset{\circ}{Y}$.*

Proof of Lemma 2.6. Let $z \in X \cap Y$. Then

$$\|z \mid X \cap Y\| = \max\{\|z \mid X\|, \|z \mid Y\|\} = \max\{\sup_{r<1} \|z^r \mid X\|, \sup_{r<1} \|z^r \mid Y\|.$$

Thus, condition (b) of Definition 2.1 is fulfilled. The validity of conditions (a), (c), (d) is obvious.

The mentioned above example $(\mathbb{C} + \mathbb{BV})$ shows that the sum of two spaces from $\mathcal{H}$ can do not belong any more to $\mathcal{H}$.

Let $h \in (X \bigcap Y)^\circ$. Then $\|h^r - h \mid X \cap Y\| \to 0$ as $r \to 1$.

Hence, $\|h^r - h \mid X\| \to 0$ and $\|h^r - h \mid Y\| \to 0$ as $r \to 1$. Therefore, $h \in \overset{\circ}{X} \bigcap \overset{\circ}{Y}$.

Conversely, let $g \in \overset{\circ}{X} \bigcap \overset{\circ}{Y}$. Then

$$\|g^r - g \mid X \cap Y\| = \max\{\|g^r - g \mid X\|, \|g^r - g \mid Y\|\} \to 0 \text{ as } r \to 1$$

because each of expressions in braces tends to zero. Hence, $g \in (X \cap Y)^\circ$. □

In what follows we need the following modification of the Banach — Steinhaus theorem.

Lemma 2.7. *Let $X, Y \in \mathcal{H}$, $\{\mu_n\}_{n=1}^{\infty} \subset \mathbb{D}$ and a sequence of the operators $I_n(f) = \mu_n * f$ is unbounded as a sequence of operators acting from X to Y. Then there exists a distribution $f_0 \in X$ such that*

$$\overline{\lim_{n\to\infty}} \|I_n(f_0) \mid Y\| = \infty.$$

Remark. If each of operators of the sequence $I_n(\cdot)$ is bounded, or there is only a finite number of unbounded operators, the statement of Lemma 2.7 is a special case of the Banach — Steinhaus theorem.

The statement of Lemma 2.7 follows from the following lemma.

Lemma 2.8. *Let $X, Y \in \mathcal{H}$, $\mu \in \mathbb{D}$ and the operator $I(f) = \mu * f$ is unbounded as an operator acting from X to Y. Then a set of distributions $f \in X$ for which $\|\mu * f \mid Y\| = \infty$, i.e., $\mu * f \notin \overset{\square}{Y}$) is complementary to a set of the first category (or in the terms of [10], Chapter 1, is a residue of X).*

Proof of Lemma 2.8. Fix an arbitrary sequence $r_n \to 1$. We consider the sequence $p_n(\mu) := P_{r_n} * \mu$ of the Abel — Poisson sums of the distribution μ. Each of the operators $J_n(f) = p_n(f) * f$ is bounded as an operator acting from X to Y. On the other hand, these operators are not bounded uniformly with respect to n, otherwise, they would converge to the bounded operator. Therefore, by the Banach — Steinhaus theorem there exists a set E, being a residue of X, on which the sequence of operators $J_n(\cdot)$, acting from X to Y, is unbounded. For any $f \in E \subset X$ the sequence $J_n(f)$ converges to $I(f)$ in the topology of the space $\mathbb{D}$. As $\overline{\lim_{n\to\infty}} \|J_n(f) \mid Y\| = \infty$, we have

$$\|I(f) \mid Y\| = \sup_{r<1} \|I(f) * P_r \mid Y\| \geq \overline{\lim_{n\to\infty}} \|J_n(f) \mid Y\| = \infty$$

(or $I(f) \notin \overset{\square}{Y}$). As was to be shown. □

From Lemmas 2.6 and 2.7 we obtain an elementary necessary condition for the Hausdorff convergence of operators (0.1) which gives some initial understanding about sequences of operators which have pretension to be convergent.

Corollary 2.1. *If we have $\mu_n * f \xrightarrow{H^*} f$ for any distribution $f \in X \in \mathcal{H}$, then*

$$\overline{\lim_{n\to\infty}} \|\mu_n \mid \mathbf{M} + \left(\overset{\circ}{X}\right)^* \| < \infty.$$

Proof of Corollary 2.1. Indeed, if X, Y are Banach spaces, according to the duality theorem (sf.,[11], [12]), if $X \cap Y$ is dense in the both spaces X and Y, then $(X \bigcap Y)^* = X^* + Y^*$. From here and Lemmas 2.2 and 2.6, we obtain

$$M(X \cap \mathbb{L}^{\infty}, \mathbb{L}^{\infty}) = (X \cap \mathbb{L}^{\infty})^{\circ *} = X^{\circ *} + \mathbf{M}.$$

Hence, if $\overline{\lim_{n\to\infty}} \|\mu_n \mid \mathbf{M} + (\overset{\circ}{X})^*\| = \infty$, then, according to Lemma 2.7, there exists $f \in X \cap \mathbb{L}^{\infty}$ for which $\overline{\lim_{n\to\infty}} \|f * \mu_n\|_{\infty} = \infty$ that contradicts the Hausdorff convergence of $f * \mu_n$ to f.

We note that it follows from the mentioned duality theorem that if $X_1, X_2 \in \mathcal{H}$ and $X_1 = \overset{\square}{X}_1$, $X_2 = \overset{\square}{X}_2$, then $X_1 + X_2 \in \mathcal{H}$. Actually, from Lemma 2.4 we obtain $Y_1 := X_1^{\circ *} \in \mathcal{H}$, $Y_2 := X_2^{\circ *} \in \mathcal{H}$. Then from Lemmas 2.4 – 2.6 we have $X_1 = Y_1^{\circ *}$, $X_2 = Y_2^{\circ *}$. Therefore, $X_1 + X_2 = \left(\overset{\circ}{Y}_1 \cap \overset{\circ}{Y}_2\right)^* \in \mathcal{H}$. $\square$

Let γ be a closed arc of $\mathbb{T}$, then denote by X_γ the set of those distributions of X whose supports belong to γ. We note that X_γ with the norm of the space X itself is a Banach space.

We denote by $\mathbb{L}_\sigma^\infty$ the set of distributions bounded on the arc $(-\sigma, \sigma)$. It is possible to introduce the seminorm

$$\|f \mid \mathbb{L}_\sigma^\infty\| = \operatorname*{ess\,sup}_{|\theta,0|<\sigma} |f(\theta)| := \max\left\{ |\operatorname*{ess\,sup}_{|\theta,0|<\sigma} f(\theta)|, |\operatorname*{ess\,inf}_{|\theta,0|<\sigma} f(\theta)| \right\}$$

in $\mathbb{L}_\sigma^\infty$

In what follows we need one more modification of the Banach — Steinhaus theorem.

Lemma 2.9. *Let $X \in \mathcal{H}$, $\gamma \subset \mathbb{T}$ is a closed arc, $\sigma > 0$, $\{\mu_n\}_{n=1}^\infty \subset \mathbb{D}$, and a sequence of the operators $I_n(f) = \mu_n * f$ is unbounded as a sequence of operators acting from X_γ to semi-normed spaces $\mathbb{L}_\sigma^\infty$. Then there exists a distribution $f_0 \in X_\gamma$ such that*

$$\overline{\lim_{n\to\infty}} \|I_n(f_0) \mid \mathbb{L}_{2\sigma}^\infty\| \to \infty.$$

Proof of Lemma 2.9. It is unessential for the Banach — Steinhaus theorem that the space $\mathbb{L}_\sigma^\infty$ is seminormed. Therefore, in the case when all (for the exception of a finite number) operators $I_n(f)$ are bounded the statement of the lemma follows directly from the Banach — Steinhaus theorem.

Let infinitely many operators in the sequence $I_n(\cdot)$ be unbounded. We obtain an analogue of Lemma 2.8. Let the operator $I(f) = \mu * f$ be unbounded as an operator acting from X_γ to $\mathbb{L}_\sigma^\infty$. We show that in this case for all f of some residue X_γ we have

$$\|I(f) \mid \mathbb{L}_{2\sigma}^\infty\| = \infty.$$

Fix an arbitrary sequence $r_n \nearrow 1$. We consider the sequence $p_n(\mu) := P_{r_n} * \mu$ of the Abel — Poisson sums of the distribution μ. Then each of the operators $J_n(f) = P_n(f) * f$ is bounded as an operator acting from X to $\mathbb{L}^\infty$, the more so as operator acting from X_γ to $\mathbb{L}_\sigma^\infty$. On the other hand, these operators are unbounded in totality. The last follows, for example, from Lemma 1.3. Therefore, by the Banach — Steinhaus theorem, there is a set E being a residue of X_γ, on which a sequence of the operators $J_n(\cdot)$, acting from X_γ to $\mathbb{L}_\sigma^\infty$, is unbounded. On the other hand, the sequence $J_n(f)$ converges to $I(f)$ in the topology of the space $\mathbb{D}$. Thus, we have

$$\operatorname*{ess\,sup}_{|\theta,0|<2\sigma} |I(f)(\theta)| \geq \operatorname*{ess\,sup}_{|\theta,0|\leq\sigma} |I(f)(\theta)| \geq \overline{\lim_{n\to\infty}} \operatorname*{ess\,sup}_{|\theta,0|<2\sigma} |J_n(f)(\theta)| = \infty.$$

for any $f \in E$.

The assertion of Lemma 2.9 follows from this. $\square$

Now we give one more statement about operators acting from X_γ to $\mathbb{L}_\sigma^\infty$.

Lemma 2.10. *Let $Y \subset \mathbb{D}$ be the space of distributions specifying the bounded operators of convolution acting from X_γ to $\mathbb{L}_\sigma^\infty$ with the norm*

$$\|\mu \mid Y\| = \|\mu \mid M(X_\gamma, \mathbb{L}_\sigma^\infty)\|.$$

Then functions from $\mathbb{C}^\infty$ are pointwise multipliers in Y.

Proof of Lemma 2.10. Let $\phi \in \mathbb{C}^\infty$. It is convenient to carry out the proof in the complex form. Obviously,

$$\|(e^{in\theta} \cdot \mu) * f(\theta) \mid \mathbb{L}_\sigma^\infty\| = \|e^{-in\theta} \cdot (\mu * (e^{in\theta} f))(\theta) \mid \mathbb{L}_\sigma^\infty\| = \|\mu * (e^{in\theta} f)(\theta) \mid \mathbb{L}_\sigma^\infty\| \leq \|\mu \mid Y\| \cdot \|e^{in\theta} f \mid X_\gamma\| \leq Cn^k \|\mu \mid Y\| \cdot \|f \mid X_\gamma\|,$$

where C and k are constants from condition (d) of Definition 2.1. From here in view of the fact that $\hat{\phi}(n)$ decrease faster than any power, we have

$$\|(\phi \cdot \mu) * (\theta) \mid \mathbb{L}_\sigma^\infty\| \leq \left(\sum_{n \in \mathbb{Z}} |\hat{\phi}(n)| \cdot Cn^k \right) \|\mu \mid Y\| \cdot \|f \mid X_\gamma\|.$$

Thus, the norm of the operator of multiplication by the function of $\mathbb{C}^\infty$ is bounded by the number $C \sum_{n \in \mathbb{Z}} |\hat{\phi}(n)| n^k$, as was to be shown. □

3 Convergence on classes of distributions, containing the Dirac δ-measure

Definition 3.1. *Let say that a sequence of operators of the form (0.1) converges:*
(a) on $g \in \mathbb{D}$ if $I_n(g) \xrightarrow{H^*} g$;
(b) on the two distributions if it converges on the identity and on P_1 (the Dirac measure);
(c) on the set $X \subset \mathbb{D}$ if it converges for any $f \in X$.

Theorem 3.1. *The sequence of operators (0.1) converges on the class $X = \overset{\circ}{X} \subset \mathbb{C}$ (or on the class $X = \mathbb{C}^\infty$) if and only if the sequence μ_n converges to P_1 in the topology of the space X^*.*

Theorem 3.2. *The sequence of operators (0.1) converges on the class* **M** *if and only if it converges on the two distributions.*

Remark. The assertion of Theorem 3.2 almost literally repeats the criterion for convergence of sequence (0.1) on the class $\mathbb{BV}$ (see [4], [5]). The difference is that instead of the convergence on P_1 the convergence on the function $S[P_1 - 1]$ or, easier speaking, on the step-functions was required.

Theorem 3.3. *Sequence (0.1) converges on the space of distributions X, $P_1 \in X$, if and only if the following two conditions hold simultaneously:*

(a) Sequence (0.1) converges on the two distributions;

(b) For any $\delta > 0$ and $f \in X$

$$\lim_{n\to\infty} \|((1-\chi_\delta)\mu_n) * f\|_\infty = 0.$$

Theorem 3.4. *Sequence (0.1) converges on the space of distributions $X = \overset{\circ}{X}$, $P_1 \in X$, if and only if the following two conditions hold simultaneously :*

(a) Sequence (0.1) converges on the two distributions;

(b) For any $\delta > 0$

$$\overline{\lim_{n\to\infty}} \|((1-\chi_\delta)\mu_n) \mid X^*\| < \infty.$$

The following statement follows immediately from Theorem 3.4 and from the fact that ($\Re\mathbf{H}^p)^* = \mathbf{B}^{1/p-1}_{\infty,\infty}$ (sf., [18], Chapter 2).

Corollary 3.1. *Sequence (0.1) converges on the space $\Re\mathbf{H}^p$ $(0 < p < 1)$ if and only if the following two conditions hold simultaneously:*

(a) Sequence (0.1) converges on the two distributions;

(b) For any $\delta > 0$

$$\overline{\lim_{n\to\infty}} \|((1-\chi_\delta)\mu_n) \mid \mathbf{B}^{1/p-1}_{\infty,\infty}\| < \infty.$$

The next statement is a little bit less obvious.

Corollary 3.2. *Sequence (0.1) converges on the space of pseudomeasures ℓ^∞ if and only if the following two conditions hold simultaneously:*

(a) The sequence (0.1) converges on the two distributions;

(b) For anyone $\delta > 0$

$$\lim_{n\to\infty} \|((1-\chi_\delta)\mu_n) \mid \ell^1\| = 0.$$

The sufficiency of the conditions of Corollary 3.2 is obvious because condition (b) implies condition (b) of Theorem 3.3. The necessity of condition (a) is obvious, and the necessity of condition (b) is a consequence of the Schur theorem (sf, [10], Chapter 8), according to which, in ℓ^1, the convergence in norm is equivalent to the weak convergence. Indeed, we initially can suppose that for any $\delta > 0$

$$\overline{\lim_{n\to\infty}} \|((1-\chi_\delta)\mu_n) \mid \ell^1\| < \infty.$$

Otherwise, we would come to the contradiction with condition (b) of Theorem 3.4. Therefore, at least for big n, $(1-\chi_\delta)\mu_n \in \ell^1$. But then, if there is no convergence of this sequence to zero in the norm of ℓ^1, i.e., condition (b) is not fulfilled, there exists a continuous functional on ℓ^1 (pseudomeasure f) whose values on the given sequence do not tend to zero. In the language of convolutions it means that

$((1-\chi_\delta)\mu_n) * f(0) \not\to 0$. It contradicts condition (b) of Theorem 3.3.

Theorem 3.5. *Sequence (0.1) converges on the class $\mathbb{D}$ if and only if the following two conditions hold simultaneously:*

(a) Sequence (0.1) converges on the two distributions;

(b) For any $\delta > 0$ the sequence $(1-\chi_\delta)\mu_n$ converges to zero in the topology of the space $\mathbb{C}^\infty$.

Validity of Theorem 3.1 is obvious and we shall not conduct its proof here. The proofs of Theorems 3.2 – 3.2 are based on the following auxiliary statements.

Lemma 3.6. *Let μ_n converge in $\mathbb{D}$ to P_1. Then for any $\varepsilon > 0$ and $f \in \mathbb{D}$ and sufficiently large n we have*

$$F^*(f) \subset U_\varepsilon(F^*(f * \mu_n)).$$

Lemma 3.7. *Let sequence (0.1) converge on the two distributions and $\operatorname{supp}\mu_n \subset [a_n, b_n]$, $a_n \to 0$, $b_n \to 0$ as $n \to \infty$, then it converges on $\mathbb{D}$.*

We remind the definition of the Steklov functions of a distribution f. Let

$$V_h(\theta) = \begin{cases} 1/(2h), & |\theta, 0| \leq h, \\ 0 & |\theta, 0| > h. \end{cases}$$

Then the distribution

$$S_{h,1}[f](\theta) := v_h * f(\theta) = \sum_{n\in\mathbb{Z}} \frac{\sin nh}{nh} \hat{f}(n) e^{in\theta}$$

is called the Steklov function of the first order of the distribution f. The Steklov functions of the kth order are defined as follows:

$$S_{h,k}[f](\theta) := S_{h,1}[S_{h,k-1}[f]](\theta) = \sum_{n\in\mathbb{Z}} \left(\frac{\sin nh}{nh}\right)^k \hat{f}(n) e^{in\theta}.$$

From Lemma 3.7, we have an obvious corollary.

Corollary 3.3. *The Steklov functions of any order converge on $\mathbb{D}$ as $h \to 0$.*

Remark. For any distribution f and sufficiently large k its Steklov functions are continuous. The given corollary means that the Steklov functions could be used for an equivalent definition of canonical graphs.

Lemma 3.8. *Let $f \in \mathbb{D}$, $\theta_0 \in \mathbb{T}$, $\delta > 0$. Then*

$$\lim_{\varepsilon\to 0} \inf\{\operatorname{ess\,sup}\ f * \mu(\theta) \mid \mu > 0, \|\mu\|_1 = 1, \operatorname{supp}\mu \subset (-\varepsilon, \varepsilon)\} \geq \operatorname{ess\,sup}\ f(\theta_0).$$

Lemma 3.9. *Let sequence (0.1) converge on the class X. Then for any $\delta > 0$ and $f \in X$ we have*

$$\lim_{n\to\infty} \|(1-\chi_\delta)\mu_n * f\|_\infty = 0.$$

Lemma 3.10. *Let $\varkappa_n \in \mathbb{D}$ and $X = \overset{\circ}{X}$. Then $\|f * \varkappa_n\|_\infty \to 0$ for any $f \in X$ if and only if the following two conditions hold:*

(a) $\|P * \varkappa_n\|_\infty \to 0$ for any trigonometric polynomial P;

(b) $\overline{\lim\limits_{n\to\infty}} \|\varkappa_n \mid X^*\| < \infty$.

Proof of Lemma 3.6. We assume the contrary. Let the statement of Lemma 3.6 be not valid for some $f \in \mathbb{D}$. Then, it is easy to see, there exist $\delta > 0$ and $\theta_0 \in \mathbb{T}$ such that either

(a) $\underline{\lim\limits_{n\to\infty}}$ $\operatorname{ess\,sup}_{|\theta,\theta_0|<\delta} f * \mu_n(\theta) < \operatorname{ess\,sup} f(\theta_0)$
or

(b) $\overline{\lim\limits_{n\to\infty}}$ $\operatorname{ess\,inf}_{|\theta,\theta_0|<\delta} f * \mu_n(\theta) > \operatorname{ess\,inf} f(\theta_0)$

holds. For definiteness we suppose that condition (a) holds.

Obviously, without loss of generality it is possible to assume that $\theta_0 = 0$ and for all n

$$\operatorname*{ess\,sup}_{|\theta,0|<\delta} f * \mu_n(\theta) \le 0, \tag{3.1}$$

$$\operatorname{ess\,sup} f(0) > 0 \tag{3.2}$$

Otherwise, we extract an appropriate subsequence, subtract the constant from f, and translate the argument.

The sequences $\chi_{\delta/2}(f * \mu_n)$ and $(1 - \chi_{\delta/2})(f * \mu_n)$ converge in $\mathbb{D}$ respectively to $\chi_{\delta/2} f$ and $(1 - \chi_{\delta/2}) f$. In view of (3.1) and the uniform convergence on compact sets of the open unit disc we obtain that $(\chi_{\delta/2} f) * P_r(\theta) \le 0$ for any $0 \le r < 1$, $\theta \in \mathbb{T}$. Therefore, $\chi_{\delta/2} f$ is a negative measure. At the same time, the distribution $(1 - \chi_{\delta/2}) f$ vanishes on $(\delta/2)$-neighborhood of zero. Consequently, from Lemma 1.2, we obtain $\operatorname{ess\,sup} f(0) \le 0$ that contradicts (3.2). Lemma 3.6 is proved. □

Proof of Lemma 3.7. Let $f \in \mathbb{D}$ and ε is an arbitrary positive number. We need to show that for sufficiently large n the inclusions

$$F^*(f) \subset U_\varepsilon(F^*(I_n(f))), \tag{3.3}$$

$$F^*(I_n(f)) \subset U_\varepsilon(F^*(f)) \tag{3.4}$$

hold. The fulfillment of inclusion (3.3) follows from Lemma 3.6. Let us show validity of (3.4). We assume the contrary. Let there be $\varepsilon > 0$ and somehow large n for which inclusion (3.4) is not valid. Then it is easy to see that there are a point θ_0 and a sequence θ_n for which either

(a) $\overline{\lim\limits_{n\to\infty}}$ $\operatorname{ess\,sup} (f * \mu_n)(\theta_n) > \operatorname{ess\,sup} f(\theta_0)$,
or

(b) $\underline{\lim\limits_{n\to\infty}}$ $\operatorname{ess\,inf} (f * \mu_n)(\theta_n) < \operatorname{ess\,inf} f(\theta_0)$

is valid. For definiteness we assume that condition (a) is fulfilled. Then, naturally,

$$\operatorname{ess\,sup} f(0) < \infty.$$

We also suppose that $\theta_0 = 0$ and $\operatorname{ess\,sup}\, f(0) > 0$. In view of the fact that the set $F^*(f)$ is closed and convex with respect to the y-axis, for any $\sigma > 0$ there exists $\delta > 0$ such that for the case $|\theta, 0| < \delta$ we have

$$\operatorname{ess\,sup}\, f(\theta) < \operatorname{ess\,sup}\, f(0) + \sigma.$$

We assume that

$$\overline{\lim_{n\to\infty}} \operatorname{ess\,sup}\,(f * \mu_n)(\theta_n) > \operatorname{ess\,sup}\, f(0) + 2\sigma. \tag{3.5}$$

Obviously, μ_n can be represented in the form $\mu_n = \xi_n + \zeta_n$, where $\{\xi_n\}$ are positive measures, satisfying the conditions Lemma 3.7, and $\{\zeta_n\} \subset \mathbb{C}^\infty$ is a sequence of negative functions, converging to zero in the topology of the space $\mathbb{C}^\infty$. Therefore, there are N_1 such that for $n > N_1$ we have

$$\|f * \zeta_n\|_\infty < \sigma, \tag{3.6}$$

and N_2 such that for $n > N_2$ we have $\operatorname{supp} \xi_n \subseteq (-\delta/8, \delta/8)$ and

$$\|\xi_n\|_1 < 1 + \sigma/(2(\operatorname{ess\,sup}\, f(0) + \sigma)). \tag{3.7}$$

We represent f as the sum $f = \chi_{\delta/2} f + (1 - \chi_{\delta/2}) f =: f_1 + f_2$. It turns out that $f_1 \in \mathbf{M}$ because this distribution is bounded from above. Consequently, the measure f_1 can be represented as a sum of positive and negative components ϕ and ψ, $\phi > 0$, $\psi < 0$, and $\|\phi\|_\infty \le \operatorname{ess\,sup}\, f(0) + \sigma$. From here and from (3.6) – (3.7) for $n > \max\{N_1, N_2\}$ we obtain

$$\begin{aligned}
&\operatorname*{ess\,sup}_{|\theta,0|<\delta/8} f * \mu_n(\theta) \le f * \xi_n + \|f * \zeta_n\|_\infty < \\
&\qquad \phi * \xi_n + \psi * \xi_n + f_2 * \xi_n + \sigma = \phi * \xi_n + \psi * \xi_n + \sigma \le \\
&\qquad \|\phi\|_\infty \cdot \|\xi_n\|_1 + 0 + \sigma/2 \le (\operatorname{ess\,sup}\, f(0) + \sigma)(1 + \sigma/(2(\operatorname{ess\,sup}\, f(0) + \sigma))) = \\
&\qquad\qquad \operatorname{ess\,sup}\, f(0) + 2\sigma.
\end{aligned}$$

Thus, for sufficiently large n inequality (3.5) does not hold. The obtained contradiction completes the proof of Lemma 3.7. □

Proof of Lemma 3.8. Obviously, the expression under the sign of a limit is a monotonically decreasing function of ε. Therefore, the limit in the assertion of Lemma 3.8 exists. In the case of the opposite inequality, there exists a sequence μ_n, satisfying conditions of Lemma 3.7 and not converging on f. It contradicts Lemma 3.7. Moreover, it is easy to see that it contradicts Lemma 3.6 simultaneously. □

Proof of Lemma 3.9. Let us assume the contrary. We assume that there are $\delta > 0$ and $f \in X$, for which

$$\overline{\lim_{n\to\infty}} \|\nu_n * f\|_\infty > 0,$$

where $\nu_n = (1 - \chi_\delta)\mu_n$. Obviously, by shifting the argument of f if it is necessary, without loss of generality we can initially consider that for an arbitrary beforehand given $\sigma > 0$ we have

$$\overline{\lim_{n\to\infty}} \operatorname*{ess\,sup}_{|\theta,0|<\sigma} |\nu_n * f(\theta)| > 0.$$

We shall assume also that $\sigma < \delta/32$. Obviously, taking an arbitrary infinitely differentiable resolution of the identity $\{\phi_i\}$ on $\mathbb{T}$ whose supports $\operatorname{supp}\phi_i$ belong to arcs of length not exceeding $\delta/16$, we obtain that at least for one of the functions $g = f \cdot \phi_i$, as well as for f, the inequality

$$\overline{\lim_{n\to\infty}} \operatorname*{ess\,sup}_{|\theta,0|<\sigma} |\nu_n * g(\theta)| > 0. \tag{3.8}$$

holds.

If

$$\operatorname{supp} g \cap [-2\delta, 2\delta] = \varnothing, \tag{3.9}$$

then g is the required function on which the contradictory is achieved, i.e., $\mu_n * g \overset{H^*}{\not\to} g$. It follows directly from the fact that $\operatorname{supp}\chi_\delta\mu_n \cap \operatorname{supp} g = \varnothing$ and, hence, $\chi_\delta\mu_n * g(\theta) = 0$ for sufficiently small θ. Therefore, (3.9) is not valid. Obviously, in this case $\operatorname{supp} g \cap [-15\delta/16, 15\delta/16] = \varnothing$. Let γ is an arc, length of which does not exceed $\delta/16$, which contains the support of g.

Let $\varkappa_n = (1 - \chi_{\delta/4}) \cdot \mu_n$. In view of the fact that $\operatorname{supp}(\mu_n - \varkappa_n) \cap \gamma = \varnothing$ we obtain that if $\overline{\lim}_{n\to\infty} \operatorname{ess\,sup}_{|\theta,0|<\sigma} |\varkappa_n * g(\theta)| > 0$, then by analogy with the previous reasoning g is the required function, leading us to the contradiction. Therefore,

$$\lim_{n\to\infty} \operatorname*{ess\,sup}_{|\theta,0|<\sigma} |\varkappa_n * g(\theta)| = 0. \tag{3.10}$$

According to Lemma 2.9 the sequences of operators $J_n(f) = \varkappa_n * g$ is bounded as a sequence acting from X_γ to $\mathbb{L}^\infty_\sigma$ because otherwise there is a distribution $h \in X_\gamma$ for which

$$\overline{\lim_{n\to\infty}} \operatorname*{ess\,sup}_{|\theta,0|<2\sigma} |\varkappa_n * h(\theta)| = \infty,$$

that leads to contradiction.

At last, recalling that $(1 - \chi_\delta) \cdot \varkappa_n = \nu_n$, from (3.10) and from Lemma 2.10 we obtain

$$\lim_{n\to\infty} \operatorname*{ess\,sup}_{|\theta,0|<\sigma} |\nu_n * g(\theta)| = 0$$

that contradicts (3.8). The proof of Lemma 3.9 is completed. □

Proof of Lemma 3.10. The proof of the sufficiency of conditions (a) and (b) is standard. We omit it. The necessity of condition (a) is obvious. The necessity of condition (b) is consequence of Lemmas 2.2 and 2.7. □

Proof of Theorem 3.2. The necessity of the convergence on the two distributions is obvious. We show the sufficiency. From the convergence on the δ-function we obtain that μ_n can be represent in the form of the sum $\mu_n = \varkappa_n + \nu_n$, where $\|\nu_n\|_\infty \to 0$. It follows from the convergence on the identity that the sequence $\varkappa_n$ can be chosen satisfying the conditions of Lemma 3.7. Therefore, for any measure f we have $f * \varkappa_n \overset{H^*}{\longrightarrow} f$ as $n \to \infty$. Since

$$\|f * \mu_n - f * \varkappa_n\|_\infty = \|f * \nu_n\|_\infty \le \|f\|_1 \cdot \|\nu_n\|_\infty \to 0$$

as $n \to \infty$, then $f * \mu_n \overset{H^*}{\longrightarrow} f$. Theorem 3.2 is proved. □

Proof of Theorem 3.3. We show the sufficiency. Let $f \in X$. Fix an arbitrary sequence of numbers $\varepsilon_i \searrow 0$. We choose a sequence δ_n by induction. We suppose $\delta_0 = \varepsilon_0$. If we already chose the numbers $\{\delta_k\}_{k=0}^n$ and $\delta_n = \varepsilon_l$, we set $\delta_{n+1} = \varepsilon_l$ if

$$\sup_{m>n} \|((1-\chi_{\varepsilon_{l+1}})\mu_m) * f\|_\infty \geq 2^{-l}$$

or

$$\inf_{m>n} \|\chi_{\varepsilon_{l+1}}\mu_m\|_1 \leq 1 - 2^{-l};$$

and we set $\delta_{n+1} = \varepsilon_{l+1}$ otherwise.

Let $\mu_n = \nu_n + \varkappa_n$, where $\nu_n = \chi_{\delta_n}\mu_n$. Then the sequence ν_n satisfies the conditions of Lemma 3.6 and, consequently, $\nu_n * f \xrightarrow{H^*} f$. At the same time, according to the choice of δ_n we have

$$\|\varkappa_n * f\|_\infty = \|((1-\chi_{\delta_n}) \cdot \mu_n) * f\|_\infty \to 0.$$

Therefore, $\mu_n * f \xrightarrow{H^*} f$. As was to be shown.

The necessity of condition (a) is obvious, and the necessity of condition (b) was proved in Lemma 3.9. □

Proof of Theorem 3.4. Let conditions (a) and (b) of Theorem 3.4 hold. From condition (a) we have fulfillment of condition (a) of Lemma 3.10. This lemma implies fulfillment of condition (b) of Theorem 3.3. Thus, the sufficiency of the conditions of Theorem 3.4 is proved.

The necessity of condition (b) is obvious. The necessity of condition (b) follows from Lemma 3.10 and Theorem 3.3. □

Proof of Theorem 3.5. We show the sufficiency. For positive integers r we denote by $D^k\mathbf{M}$ the space of distributions for which $S^k[f - \hat{f}(0)] \in \mathbf{M}$, and we denote by $S^k\mathbb{L}^\infty$ the space of functions for which $D^k[f] \in \mathbb{L}^\infty$ (i.e., in other notations this is the Sobolev space W_∞^k). Then $(D^k\mathbf{M})^{\circ *} = S^k\mathbb{L}^\infty$.

Let f_0 be an arbitrary distribution. Then there is k such that $f_0 \in D^k\mathbf{M}$. The convergence of the sequence $\varkappa_n = (1-\chi_\delta)\mu_n$ to zero in the topology of the space $\mathbb{C}^\infty$ means, in particular, that the norms $\|\varkappa_n \mid S^k\mathbf{L}^\infty\|$ tend to zero for any k. It follows from this that condition (b) of Theorem 3.3 for $X = D^k\mathbf{M}$ holds. It means the sequence (0.1) converges on f.

The proof of the necessity we conduct by contradiction. If for some $\delta > 0$ the sequence $\varkappa_n = (1-\chi_\delta)\mu_n$ does not converge to zero in the topology of the space $\mathbb{C}^\infty$ it means, that there exists a positive integer k_0 such that a sequence of the norms $\|\varkappa_n \mid S^k\mathbb{L}^\infty\|$ does not converge to zero. We take the minimum possible k_0. Then taking the distribution $D^k[P_1]$ as f, we obtain $\|f * \varkappa_n\|_\infty \not\to 0$. That contradicts condition (b) of Theorem 3.3. Thus, the sequence can not be converging on the class $D^k\mathbf{M}$ and, especially, on $\mathbb{D}$. □

4 Convergence on classes $\Re\mathbf{H}^1$ and $\mathbf{c}_0$

The spaces $\Re\mathbf{H}^1$ and $\mathbf{c}_0 := (\ell^\infty)^\circ$ (besides, obviously, $\overset{\square}{\mathbf{c}}_0 = \ell^\infty$, $\mathbf{c}_0^* = \ell^1$) do not contain δ-function and, consequently, do not satisfy the conditions of Theorems 3.3 and 3.4. Therefore, they require separate consideration. We explain the reasons why we consider here these spaces.

The cases of the spaces $\Re\mathbf{H}^p = \mathbb{L}^p$ $(1 < p < \infty)$ were considered in [4] and [5] (see Theorem C below), and the case $0 < p < 1$ we considered in Section 3. The space $\Re\mathbf{H}^1$ remains the only blank in the scale of the spaces $\Re\mathbf{H}^p$ $(0 < p < \infty)$ and we, naturally, would like to fill in this blank.

As to the space $\mathbf{c}_0$, it has the same meaning for the space ℓ^∞ as the space $\mathbb{L}^1$ for the space $\mathbf{M}$ or $\mathbb{C}$ for $\mathbb{L}^\infty$.

Besides, the assertions of criteria for the convergence on these spaces have more complex form than assertions for the spaces containing P_1, and, it seems, they could serve as a sample for obtaining general results for quasi-Banach spaces which were not under consideration in Section 3.

We recall that $\mathbb{L}\log\mathbb{L} \subset \mathbb{L}$ is the space of (real-valued) functions on $\mathbb{T}$ for which the function $|f|\cdot\max\{0,\log|f|\}$ is integrable. We denote by $\exp\mathbb{L}$ the space of continuous linear functionals on $\mathbb{L}\log\mathbb{L}$. More detail information about these spaces can be found, for example, in [14] and [15].

Let ϑ is a 2π-periodic function, $\vartheta(x) = \pi - x$, $x \in [0, 2\pi)$. Obviously, $\vartheta = S[P_1 - 1]$. The sequence (0.1) is called *converging on the two functions* if it converges in the Hausdorff metric on the identity and on the function ϑ (see [4] and [5]).

Let μ be a regular Borel measure on $\mathbb{T}$. Then we denote by μ^+ and μ^- respectively the positive and and negative components in the Jordan decomposition $\mu = \mu^+ + \mu^-$;

$$\mu^\delta(E) = \mu^+(E \setminus [-\delta, \delta]) - \mu^-(E),$$

where $E \subset \mathbb{T}$.

Theorem A ([5]). *The sequence of operators (0.1) converges on the class* $\mathbb{BV}$ *if and only if it converges on the two functions.*

Theorem B ([5]). *The sequence of operators (0.1) converges on the class* $\mathbb{L}^\infty$ *if and only if the following two conditions hold:*

(a) The sequence (0.1) converges on the two functions;

(b) For any $\delta > 0$ *we have* $\lim\limits_{\substack{n\to\infty \\ \operatorname{meas} E \to 0}} \mu_n^\delta(E) = 0$.

Theorem C ([5]). *The sequence of operators (0.1) converges on the class* $\mathbb{L}^p$ $(1 \le p < \infty)$ *if and only if the following two conditions hold:*

(a) The sequence (0.1) converges on the two functions;

(b) For any $\delta > 0$ *we have* $\overline{\lim\limits_{n\to\infty}} \|\mu_n^\delta\|_q < \infty$, *where* $q = p/(p-1)$ $(q = \infty$ *for* $p = 1)$.

Theorem 4.1. *The sequence of operators (0.1) converges on the class* $\Re\mathbf{H}^1$ *if and only if the following three conditions hold:*

(a) The sequence (0.1) converges on the two functions;

(b) For any $\delta > 0$ *we have* $\overline{\lim\limits_{n\to\infty}} \|(1-\chi_\delta)\mu_n \mid \mathbf{BMO}\| < \infty$;

(c) $\overline{\lim\limits_{n\to\infty}} \|\mu_n^- \mid \exp\mathbb{L}\| < \infty$.

Theorem 4.2. *The sequence of operators (0.1) converges on the class* $\mathbf{c}_0$ *if and only if the following three conditions hold:*

(a) Sequence (0.1) converges on the two functions;

(b) For any $\delta > 0$ *we have* $\overline{\lim_{n\to\infty}} \|(1-\chi_\delta)\mu_n \mid \ell^1\| < \infty$*;*

(c) $\|\overline{\lim_{n\to\infty}} \mu_n^- \|_\infty < \infty$.

The appearance of condition (c) selects these two assertions from all considered above and in [5]. As we shall see in proofs, the reason of the appearance of condition (c) is that, in these spaces, there are elements which cannot be represented as the sum of positive and negative distributions, belonging to the same space.

At the same time, maximum of spaces embedded in $\Re\mathbf{H}^1$, possessing this property, is the space $\mathbb{L}\log\mathbb{L}$ (sf., [16], Chapter 5). Thus, there is an understandable reason for the appearance of the space $\exp\mathbb{L} = \mathbf{M}(\mathbb{L}\log\mathbb{L}, \mathbb{L}^\infty)$ in the statement of Theorem 4.1. Similarly, such the space for $\mathbf{c}_0$ is the space of continuous measures $\mathbf{M}_0$, i.e., measures without a discrete component. Then $\mathbb{L}^1 \subset \mathbf{M}_0 \subset \mathbf{M}$. Consequently, $\mathbb{L}^\infty = M(\mathbf{M}_0, \mathbb{L}^\infty)$.

We note that the condition such as conditions (c) of Theorems 4.1 and 4.2 could appear in the assertions of Section 3. However, in Theorems 3.3 – 3.5 it, actually, appeared "hidden" in condition (a).

Proof of Theorem 4.1. The generalized convergence of the sequence μ_n to P_1 follows from a condition a). Hence, according to Lemma 3.6, for any fixed $\varepsilon > 0$ when n is sufficiently large, we have

$$F^*(f) \subset U_\varepsilon(F^*(I_n(f))).$$

We show that for sufficiently large n the converse inclusion

$$F^*(I_n(f)) \subset U_\varepsilon(F^*(f))$$

takes place. Let us partition the circle $\mathbb{T}$ into k equal arcs γ_i of length $\varepsilon_0 = 2\pi/k < \varepsilon/3$. We denote by m_i and M_i the essential upper and lower bounds of f on the union of the ith and adjacent with it arcs, and f_i is the restriction of f to this set; $M := \max\{\mid m_i \mid, \mid M_i \mid\}$, where the last maximum is taken only over finite values. Let d is a number such that the inequalities

$$m_i - d \le g(x) \le M_i + d$$

on all the arcs γ_i implies the inclusion

$$F^*(g) \subset U_\varepsilon(F^*(f)).$$

We choose numbers δ_0 and N_0 so that for $n > N_0$, $0 < \delta < \delta_0$ and $x \in \gamma_i$ $(i = 1, 2, \dots, k)$ we have

$$M_i - d/3 \le (\chi_\delta \cdot \mu_n)^+ * f(x) \le M_i + d/3. \tag{4.1}$$

For those i for which at least one of the numbers m_i and M_i is finite, f_i belongs to the space $\mathbb{L}\log\mathbb{L}$. The norm of this space is absolutely continuous with respect to Lebesgue measure, i.e., if $f \in \mathbb{L}\log\mathbb{L}$, then for any $\varepsilon > 0$ there is $\delta > 0$ such that the inequality meas $E < \delta$

$(E \subset \mathbb{T})$ implies $\|\chi_E \cdot f \mid \mathbb{L}\log\mathbb{L}\| < \varepsilon$. Therefore, there exist δ_1 and N_1 such that for any f_i the inequality meas $E < \delta_1$ implies

$$\|\chi_E \cdot f_i \mid \mathbb{L}\log\mathbb{L}\| < d/3K, \tag{4.2}$$

where $K = \sup_{n>N_1} \|\mu_n^- \mid \exp\mathbb{L}\| < \infty$. At the same time, it is easy to show from conditions (a) and (b) that for any $\delta > 0$ and $f \in \Re H^1$ the sequence $\|(1-\chi_\delta)\mu_n * f\|_\infty$ converges to zero. Consequently, there is N_2 such that for $n > N_2$ we have

$$\|(1-\chi_{\delta_2})\mu_n * f\|_\infty \le d/3, \tag{4.3}$$

where $\delta_2 = \min\{\delta_0, \delta_1, \varepsilon/6\}$.

Thus, if i is such that at least one of the numbers m_i and M_i is finite, according to (4.1) – (4.3), for all $N > \max\{N_0, N_1, N_2\}$ and $x \in \gamma_i$ we have

$$\begin{aligned}\mu_n * f(x)) = (\chi_{\delta_2}\cdot\mu_n)^+ * f(x) + (\chi_{\delta_2}\cdot\mu_n)^- * f(x) + (1-\chi_{\delta_2})\cdot\mu_n * f(x) \le \\ M_i + d/3 + \|(\chi_{\delta_2}\cdot\mu_n)^- \mid \exp\mathbb{L}\|\cdot\|f_i \mid \mathbb{L}\log\mathbb{L}\| + \|(1-\chi_{\delta_2})\cdot\mu_n * f\|_\infty \le \\ M_i + d/3 + (d/3K)\cdot K + d/3 = M_i + d.\end{aligned}$$

Similarly, we obtain $\mu_n * f(x) > m_i - d$. It completes the proof of the sufficiency of conditions (a) – (c).

The necessity of condition (a) is obvious, and the necessity of condition (b) follows from Lemma 3.10. Let us show the necessity of condition (c).

We assume the contrary. Let condition (c) of Theorem 4.1 does not hold. Then, obviously, for any $\delta > 0$ we have

$$\overline{\lim_{n\to\infty}} \|(\chi_\delta \cdot \mu_n)^- \mid \exp\mathbb{L}\| = \infty. \tag{4.4}$$

Indeed, if it is not so, condition (b), the necessity of which is already proved, would be not valid.

Let $\Omega = \sup \|\mu_n\|_1$. Obviously, it is possible without loss of generality to assume that $\Omega < \infty$. We construct the required function f which leads to contradiction in the form of a sum of series $f = \sum_{n=0}^\infty g_n$. We introduce the notation $f_m := \sum_{n=0}^m g_n$, $h_m := \sum_{n=m}^\infty g_n$. Let M be an arbitrary positive number, $\delta_n = 2^{-n-1}$. We construct a sequence of the functions g_n and a sequence of natural numbers $l_n \nearrow \infty$, proceeding from the following conditions:

1) g_n is a continuous non-positive function supported on the interval $(a_n,\ b_n)$, where $a_n = c_n - \delta_n$, $b_n = c_n + \delta_n$, $c_n = 2\sum_{k=0}^{n-1} \delta_n$, $n = 0,1,2,\dots$; $c_0 = 0$;

2) $g_n * \mu_{l_n}(c_n) \ge 2M$, $n = 0,1,2,\dots$;

3) $\max_{x\in[a_n,b_n]} \mid f_{n-1} * \mu_{l_n}(x)) \mid < M/4$, $n = 0,1,2,\dots$;

4) $\|g_n \mid \mathbb{L}\log\mathbb{L}\| \le 2^{-n-1}$, $n = 0,1,2,\dots$;

5) $\|g_n \mid \mathbb{L}\log\mathbb{L}\| \le \dfrac{\sigma_k M}{16\pi}\dfrac{2^{k-n}}{\Omega C}$, where σ_k is Lebesgue measure of the set

$$A_k := \{x \in [a_k, b_k] \mid g_k * \mu_{l_k}(x))M\},$$

$\sigma_k > 0$, $k = 0,1,\dots,n-1$; $n = 1,2,\dots$; C is a constant of embedding of the space $\mathbb{L}\log\mathbb{L}$ in $\mathbb{L}^1$.

We need the following assertion.

Lemma 4.3. *Let μ be a regular Borel measure on $\mathbb{T}$ and $\|\chi_{(-\delta,\delta)}\cdot\mu^- \mid \exp\mathbb{L}\| > K$. Then there exists a continuous non-positive function ψ, equal to zero outside the interval $(-\delta,\delta)$, for which $\|\psi \mid \mathbb{L}\log\mathbb{L}\| \le 1$, $\psi * \mu(0) > K$.*

We omit the proof of this lemma because it repeats the proof of Lemma 5 of [5] almost literally.

We construct the functions g_n and the numbers l_n by induction. We choose the number l_0 satisfying the condition $\|\chi_{[-\delta_0,\delta_0]}\cdot\mu^-_{l_0} \mid \exp\mathbb{L}\| \ge 4M$. Then, according to Lemma 4.3, there exists a continuous function g_0, satisfying conditions 1), 2) and 4). As $g_0 * \mu_{l_n}$ is a continuous function, we have $\sigma_0 > 0$,

Let's assume we have conducted constructing for $n = 0, 1, \dots, m-1$. Then the function g_m and the number l_m are chosen as follows.

In view of continuity of the function f_{m-1}, the sequence of operators (0.1) converges on it uniformly and, therefore, for sufficiently large l_m condition 3) holds. By (4.4), for sufficiently large l_m the inequality

$$\|\chi_{[-\delta_m,\delta_m]}\cdot\mu^-_{l_m} \mid \exp\mathbb{L}\| \ge 2M\cdot\max\left\{2^{m+1};\ \frac{16\pi}{M\sigma_k}\cdot\frac{\Omega C}{2^{m-k}},\ k = 0,1,\dots,m-1\right\}.$$

holds too. Then, according to Lemma 4.3, there is a continuous non-positive function g'_m which is equal to zero outside the interval $(-\delta_m, \delta_m)$ and satisfies the inequalities $g'_m * \mu_{l_m}(0) \ge 2M$ and

$$\|g'_m \mid \mathbb{L}\log\mathbb{L}\| \le \min\left\{2^{-m-1};\ \frac{\sigma_k M}{16\pi}\cdot\frac{2^{k-m}}{\Omega C},\ k = 0,1,\dots,m-1\right\}.$$

We take the function $g'_m(x - c_m)$ as above $g_m(x)$. Thus, the possibility of construction of the sequences of the functions g_n and the numbers l_n, satisfying conditions 1) – 5), is proved.

According to condition 4), the function f belongs to the space $\mathbb{L}\log\mathbb{L}$, $\|f \mid \mathbb{L}\log\mathbb{L}\| \le 1$ and $f \le 0$. We study behavior of the function $f * \mu_{l_n}$ on the set A_n.

According to condition 5),

$$\|h_{n+1}\|_1 \le C\|h_{n+1} \mid \mathbb{L}\log\mathbb{L}\| \le \frac{\sigma_n M}{16\pi}\cdot\frac{1}{\Omega}.$$

Therefore, from the Chebyshev inequality we obtain

$$\operatorname{meas}\{x \in \mathbb{T} \mid |h_{n+1} * \mu_{l_n}(x))| \ge M/4\} \le \frac{2\pi\|h_{n+1} * \mu_{l_n}\|_1}{M/4} \le$$
$$\le 8\pi\cdot\|h_{n+1}\|_1\cdot\|\mu_{l_n}\|_1/M \le \sigma_n/2.$$

It follows from here and from 3) and 5) that there is a set of the Lebesgue measure $\sigma_n/2$ on the interval (a_n, b_n) such that we have

$$f * \mu_{l_n}(x) = f_{n-1} * \mu_{l_n}(x) + g_n * \mu_{l_n}(x) + h_{n+1} * \mu_{l_n}(x)) \ge -M/4 + M - M/4 = M/2$$

at any point x of this set. Therefore, $H^*(f * \mu_{l_n}, f) \ge M/2$. That contradicts the convergence of sequence (0.1) on the class $\Re\mathbf{H}^1$. Theorem 4.1 is proved. □

Similarly it is possible to obtain the following statement which we give without the proof.

Theorem 4.4. *The sequence of operators (0.1) converges on the class* $\mathbb{L}\log\mathbb{L}$ *if and only if the following two conditions hold:*

(a) Sequence (0.1) converges on the two functions;

(b) For any $\delta > 0$ *we have* $\overline{\lim\limits_{n\to\infty}}\|\mu_n^- + (1-\chi_\delta)\mu_n^+ \mid \exp\mathbb{L}\| < \infty$.

Remark. Obviously, in the assertion of condition (b) of Theorem 4.4 the function $\chi_\delta \in \mathbb{C}^\infty$ can be replaced with the characteristic function of the interval $\chi_{(-\delta,\delta)}$.

Proof of Theorem 4.2. Validity of the inclusion $F^*(f) \subset U_\varepsilon(F^*(f))$ for large n can be proved in the same manner as the proof of the previous theorem. We show the converse inclusion

$$F^*(I_n(f)) \subset U_\varepsilon(F^*(f)). \tag{4.5}$$

According to condition (c), there are a positive number K and a positive integer n_0 such that for $n > n_0$ we have

$$\mu_n^-(x)) > -K. \tag{4.6}$$

As usually, we partition $\mathbb{T}$ into k arcs γ_i of length $\varepsilon_0 \le \varepsilon/3$. We denote by M_i and m_i the essential upper and lower bounds of the function f on the union of γ_i with the adjacent arcs. We choose $d > 0$ such that fulfillment of the inequalities $m_i - d \le I(f)(x) \le M_i + d$ on all the arcs γ_i implies (4.5). It follows from conditions (a) and (b) that for sufficiently large n, μ_n are measures and there exist the numbers δ_0 and N_0 such that for $n > N_0$, $0 < \delta < \delta_0$, $x \in \gamma_i \quad (i = 1, 2, \dots, k)$ we have

$$m_i - d/3 < (\chi_\delta \cdot \mu_n)^+ * f(x) < M_i + d/3. \tag{4.7}$$

Distributions of $\mathbf{c}_0$ bounded from above or from below are continuous measures. Therefore, for those i for which either m_i or M_i are finite the restriction of f to the union γ_i with adjacent arcs is a continuous measure. Consequently, according to (4.6), there is δ_1 such that for all δ, $0 < \delta < \delta_1$, and $x \in \gamma_i$ we have

$$|f * (\chi_\delta\mu_n)^-(x))| < d/3 \tag{4.8}$$

for n bigger certain N_1.

From conditions (a) and (b), according to Lemma 2.10 we obtain that by fixing $\delta_2 = \min\{\delta_0, \delta_1\}$, for n greater certain N_2 and when $0 < \delta < \delta_2$ we have

$$\|(1-\chi_\delta)\mu_n * f\|_\infty < d/3. \tag{4.9}$$

Collecting (4.7) – (4.9), we obtain (4.5). The sufficiency is proved.

The necessity of condition (a) is obvious and the necessity of condition (b) follows from Lemmas 3.9 and 3.10. In the case of invalidity of condition (c) there exists a function $f \in \mathbb{L}^1 \subset \mathbf{c}_0$ on which sequence (0.1) diverges. This fact is an immediate consequence of the criterion for the convergence on the class $\mathbb{L}^1$ (see [4], [5]). □

5 The Gibbs phenomenon and convergence

As it was already mentioned above, in [5] we obtained results for the spaces $\mathbb{BV}$ and $\mathbb{L}^p$ which were similar to the results of Sections 3 – 4. Besides, for these spaces two other kinds of results were obtained. At first, it was shown that the convergence takes place if and only if the generalized Gibbs phenomenon does not occur. The definition of the generalized Gibbs phenomenon will be given below. Secondly, it turns out that not only the sequence of the convoluting kernels is "suitable" for the given spaces (if they satisfy the appropriate conditions) but the converse fact also takes place. If we fix a set of the kernel sequences, satisfying conditions of the criterion, then the spaces, on which the given sequences of operators converge, are determined, in fact, uniquely. Here we show that the given assertions are, somewhat, universal even in a little bit strengthened form.

Definition 5.1. *A distribution f is called essentially unbounded everywhere, if*

$$\operatorname{ess\,sup} f(x) \equiv +\infty, \qquad \operatorname{ess\,inf} f(x)) \equiv -\infty,$$

i.e., the distribution f is essentially unbounded both from above and from below on any arc.

We note that the similar concept was introduced in [5] only for summable functions.

Theorem 5.1. *Let $X \in \mathcal{H}$ be a Banach space and either $X = \overset{\circ}{X}$ or $X = \overset{\square}{X}$. A distribution f is not essentially unbounded everywhere and $f \notin X$. Then there exists a sequence of operators (0.1), converging on X and not converging on f.*

Remark. Generally speaking, Theorem 5.1 ceases to be valid for quasi-Banach spaces. It is caused by the fact that Lemma 2.5 is not valid for them.

This theorem strengthens the statements from [5] because in that paper we imposed the initial limitations on f. So, for example, in the assertion for $X = \mathbb{BV}$ we assumed that $f \in \mathbb{L}^\infty$, and in the case $X = \mathbb{L}^p$ $(1 < p \le \infty)$ we assumed that f is a summable function.

Proof of Theorem 5.1. Let $f \notin X$ and be not essentially unbounded everywhere. Then it is easy to see that there is a point $\theta_0 \in \mathbb{T}$ at which either $\infty < \operatorname{ess\,sup} f(\theta_0) < \infty$ or $-\infty < \operatorname{ess\,inf} f(\theta_0) < \infty$. For determinancy we assume $\theta_0 = 0$ and the former holds.

If $f \notin \overset{\square}{X}$ (in particular, it happens if $X = \overset{\square}{X}$), according to Lemmas 2.2 and 2.5, $f \notin M(X^{\circ *}, \mathbb{L}^\infty)$. Hence, there exists a sequence of distributions $\varkappa_n$, $\|\varkappa_n \mid X^{\circ *}\| = 1$, such that $f * \varkappa_n \in \mathbb{C}$ and $f * \varkappa_n(0) \to +\infty$ as $n \to \infty$. We take an arbitrary sequence $\varepsilon_n \to 0$ such that $\varepsilon_n(f * \varkappa_n(0)) \to +\infty$. Then the sequence of operators (0.1) with the kernels $\mu_n = P_1 + \varepsilon_n \varkappa_n$ converges on the class X because for any distribution $h \in X$, according to Lemma 2.2, we have

$$\|h * (\varepsilon_n \varkappa_n)\|_\infty \le \varepsilon_n \|h \mid X\| \cdot \|\varkappa_n \mid X^{\circ *}\| = \varepsilon_n \|h \mid X\| \to 0.$$

At the same time, obviously, there is no convergence on f.

Let $f \in \overset{\square}{X}$ and $f \notin \overset{\circ}{X}$. In view of the fact that the distribution f cannot be approximated with an arbitrary accuracy in the norm of the space X by functions of $\mathbb{C}^\infty$, there is $C > 0$ such that

$$\|f - f^r \mid X\| > C. \tag{5.1}$$

Let $r_k = 1 - 1/k$, $k = 1, 2, \dots$ Then in view of (5.1) and Lemma 2.5, for any k there exists a distribution $\varkappa_k$,

$$\|\varkappa_k \mid M(X, \mathbb{L}^\infty)\| = 1, \tag{5.2}$$

such that $\|(f - f^{r_k}) * \varkappa_k\|_\infty = \|f * (P_1 - P_{r_k}) * \varkappa_k\|_\infty > C$. By (5.2), we obtain that there is a distribution $\varkappa_0$ such that $|\hat{\varkappa}_k(n)| \le |\hat{\varkappa}_0(n)|$. Hence, the sequence $(P_1 - P_{r_k}) * \varkappa_k$ converges in $\mathbb{D}$ to zero, and in particular, it converges to zero coefficientwise.

On the other hand, according to the definition of spaces of $\mathcal{H}$ and by Lemma 2.4, we have

$$\|(P_1 - P_{r_k}) * \varkappa_k \mid M(X, \mathbb{L}^\infty)\| \le \|\varkappa_k \mid M(X, \mathbb{L}^\infty)\| + \|P_{r_k} * \varkappa_k \mid M(X, \mathbb{L}^\infty)\| \le 2. \tag{5.3}$$

Thus, denoting by ξ_k the sequence $(P_1 - P_{r_k}) * \varkappa_k$, we obtain $\|f * \xi_k\|_\infty > C$. For every k we can choose an appropriate r, $0 < r < 1$, so that for the function $\zeta_k = \xi_k * P_{r_k}$ the inequality $\|f * \zeta_k\|_\infty > C$ holds. In this case the function $f * \zeta_k$ becomes continuous. We denote by ν_k the functions obtained of ζ_k by the shift of an argument and, may be, by the change of a sign. We choose the distributions ν_k so that the inequality $f * \nu_k(0) > C$ holds. Then, obviously, $\mu_k = P_1 + \nu_k$ is the required sequence of kernels of the operators (0.1). Indeed, the divergence on f is obvious. The convergence on X $(=\overset{\circ}{X})$ follows from the convergence of a sequence of the operators $J_n(g) = g * \nu_k$ to zero on polynomials and from boundedness (according to (5.3)) of the norms $\|\nu_k \mid M(X, \mathbb{L}^\infty)\|$. Theorem 5.1 is proved. □

The comprehensive answer to a question what happens if a distribution f is essentially unbounded everywhere gives the following assertion.

Theorem 5.2. *Let a sequence μ_n converge to P_1 in $\mathbb{D}$ and f is a distribution essentially unbounded everywhere. Then*

$$f * \mu_n \xrightarrow{H^*} f.$$

Remark. Theorem 5.2 contains a certain paradox, consisting in the fact that the generalized convergence to P_1 guarantees the Hausdorff convergence on "very good" distributions of $\mathbb{C}^\infty$ (see Theorem 3.1) and on "very bad" functions essentially unbounded everywhere. There is no guarantee for all remaining distributions because always there exists a space $X = \overset{\circ}{X} \ne \mathbb{C}^\infty$, $X \in \mathcal{H}$, which does not contain the given distribution. For example, we can take as above X the space of k times continuously differentiable functions for sufficiently large k.

Corollary 5.1. *For any essentially unbounded everywhere distribution the conclusion of Theorem 5.1 ceases to be valid.*

Indeed, if sequence (0.1) converges on the class $X \in \mathcal{H}$, it converges on the class $\mathbb{C}^\infty$. Hence, by Theorem 3.1, it converges to P_1 in $\mathbb{D}$. Therefore, it follows from Theorem 5.2 that it converges on any distribution essentially unbounded everywhere.

Theorem 5.2 follows immediately from Lemma 3.6.

We proceed now to considering the Gibbs phenomenon on classes $X \in \mathcal{H}$. Following [5], we give a definition of the (generalized) Gibbs phenomenon. We say that for the sequence of operators (0.1) the Gibbs phenomenon occurs on the function f if there exists a point $\theta_0 \in \mathbb{T}$ such that one of the inequalities

$$\lim_{\delta\to 0} \overline{\lim_{n\to\infty}} \operatorname*{ess\,sup}_{|\theta,\theta_0|<\delta} I_n(f)(\theta) > \operatorname{ess\,sup} f(\theta_0),$$

$$\lim_{\delta\to 0} \underline{\lim_{n\to\infty}} \operatorname*{ess\,inf}_{|\theta,\theta_0|<\delta} I_n(f)(\theta) < \operatorname{ess\,inf} f(\theta_0)$$

holds. The difference with the corresponding definition in [5] consists in the absence of requirement of summability of the distributions f and $I_n(f)$. It is not necessary here because we gave the definitions of essential upper and lower bounds for an arbitrary distribution.

The canonical graphs of distributions can have non-zero area and even contain whole domains, so it would be natural to call the given phenomenon as the external Gibbs phenomenon because in this case the approximating sequence of the canonical graphs comes out of the canonical graph of the approximated function on the non-zero distance. We say that for the sequence of operators (0.1) *the internal Gibbs phenomenon* occurs on the function f if there exist $\delta > 0$ and $\theta_0 \in \mathbb{T}$ such that one of the inequalities

$$\underline{\lim_{n\to\infty}} \operatorname*{ess\,sup}_{|\theta,\theta_0|<\delta} I_n(f)(\theta) < \operatorname{ess\,sup} f(\theta_0),$$

$$\overline{\lim_{n\to\infty}} \operatorname*{ess\,inf}_{|\theta,\theta_0|>\delta} I_n(f)(\theta) < \operatorname{ess\,inf} f(\theta_0),$$

hold.

Obviously, sequence (0.1) converges on an arbitrary fixed distribution if and only if both internal and external Gibbs phenomena are absent. It is natural to say that the corresponding Gibbs phenomenon is absent on the class of distributions X if it is absent on every $f \in X$.

It was shown in [5] that sequence (0.1) converges on the classes $\mathbb{BV}$ and $\mathbb{L}^p$ $(1 \le p \le \infty)$ if and only if the (external) Gibbs phenomenon is absent, i.e., for these classes the absence of the external Gibbs phenomenon implies the absence of the internal one. It turns out, this property is universal and does not depend on topological properties of the space.

Theorem 5.3. *Let X be an arbitrary set of distributions, $X \supset \mathbb{C}^\infty$. Then the sequence of operators (0.1) converges on the set X if and only if the (external) Gibbs phenomenon does not occur on it.*

Proof of Theorem 5.3. Obviously, the convergence on the set X implies the absence of the external Gibbs phenomenon.

We show the converse fact. Let the external Gibbs phenomenon for sequence (0.1) on set X does not occur. The convergence of sequence (0.1) on $\mathbb{C}^\infty$ follows immediately from here. Hence, according to Theorem 3.1, the sequence μ_n converges to P_1 in the generalized sense. It follows from here and Lemma 3.6 that the internal Gibbs phenomenon also does not occur. Thus, sequence (0.1) converges on the set X. As it was required. □

We introduce a new distance of Hausdorff's type mentioned in Section 1. Let $f, g \in \mathbb{D}$, then

$$H^{\square}(f,g) := \sup_{r<1} H(f^r, g^r) \quad (= \sup_{r\le 1} H^*(f^r, g^r)).$$

The distance $H^{\square}$ is a metrics. Obviously, the convergence of a sequence of distributions with respect to the distance $H^{\square}$ implies the convergence in the sense of the distance H^*.

It turns out, despite of distinctions of the distances H^* and $H^{\square}$, the convergence of sequence (0.1) on a class $X \in \mathcal{H}$ takes or does not take place simultaneously.

Theorem 5.4. *Let $X \in \mathcal{H}$. Then sequence (0.1) converges on the class X with respect to the distance H^* if and only if it converges on this class with respect to the distance $H^{\square}$.*

Proof of Theorem 5.4. Obviously, it suffices to show that the convergence of (0.1) with respect to the distance H^* implies the convergence in the metric $H^{\square}$. We assume the contrary. Let sequence (0.1) converge on the space $X \in \mathcal{H}$ in H^* and do not converge in the metrics $H^{\square}$. $f \in X$ is a function on which there is no convergence in $H^{\square}$. Then there exist a sequence of natural numbers $n_k \nearrow \infty$ and a sequence of real numbers r_k, $0 \le r_k < 1$, such that for some $\varepsilon_0 > 0$ we have

$$H(f^{r_k}, I_{n_k}(f^{r_k})) > \varepsilon_0. \tag{5.4}$$

The sequence μ_n converges to P_1 in $\mathbb{D}$. It follows from here that $I_n(f)$ (more precisely, the sequence of harmonic extensions into the unit disc) uniformly converges to f on compact sets. Therefore, the sequence r_k has a unique limit point which is equal to 1.

It follows from Theorem 1.1 and from (5.4) that for sufficiently large k we have

$$H(f, I_{n_k}(f^{r_k})) > \varepsilon_0. \tag{5.5}$$

Hence, for every such k either

$$F(I_{n_k}(f^{r_k})) \bigcap U_\varepsilon(F^*(f)) \neq \varnothing \tag{5.6}$$

or

$$F^*(f) \bigcap U_\varepsilon(F(I_{n_k}(f^{r_k}))) \neq \varnothing \tag{5.7}$$

takes place. According to Theorem 3.1, the sequences μ_n converges to P_1 in $\mathbb{D}$. The same can be told about the sequence $\mu_{n_k} * P_{r_k}$. Hence, it follows from Lemma 3.6 that (5.7) cannot be satisfied for infinite number of the indexes k.

In the case when (5.6) is valid for infinite number of the indexes k, there exist $\theta_0 \in \mathbb{T}$ and a sequence $\theta_k \to \theta_0$ such that we have either

$$\overline{\lim_{k\to\infty}} \operatorname{ess\,sup} \ I_{n_k}(f^{r_k}(\theta_k)) > \operatorname{ess\,sup} \ f(\theta_0) \tag{5.8}$$

or

$$\underline{\lim_{k\to\infty}} \operatorname{ess\,inf} I_{n_k}(f^{r_k}(\theta_k)) < \operatorname{ess\,inf} f(\theta_0).$$

For definiteness we assume that (5.8) is satisfied. In view of the convergence of the sequence of operators (0.1) on constants without loss of generality we can suppose that the left-hand part of inequality (5.8) is positive and the right-hand part is negative. Moreover, there is $\delta > 0$ such that $\operatorname{ess\,sup}_{|\theta,\theta_0|<\delta} f(\theta) < 0$.

We represent the Poisson kernel P_{r_k} in the form $P_{r_k} = \nu_k + \varkappa_k$, where $\nu_k = \chi_{\delta/8} \cdot P_{r_k}$. As the sequence $f * \mu_{n_k}$ converges in $\mathbb{D}$ and $\varkappa_k$ converges to zero in the topology of the space $\mathbb{C}^\infty$, we get the uniform convergence of the sequence $F * \mu_{n_k} * \varkappa_k$ to zero. Hence, it follows from (5.8) that we have

$$\varlimsup_{k\to\infty} \operatorname{ess\,sup} \mu_{n_k} * f * \nu_k(\theta_k) > 0. \tag{5.9}$$

In view of the convergence of the sequence of operators (0.1) on f the inequality

$$\operatorname*{ess\,sup}_{|\theta,\theta_0|<\delta/2} \mu_{n_k} * f(\theta) < 0$$

is valid for sufficiently large k. In view of positivity of the function ν_k it contradicts (5.9). Theorem 5.4 is proved. □

6 Applications to summation of Fourier series

We begin with the consideration of summation processes by the Cesàro methods (C, α). Recall main definitions. For $\alpha \neq -1, -2, \ldots$ and positive integers n the numbers

$$A_n^\alpha := \frac{(\alpha+1)\cdot\ldots\cdot(\alpha+n)}{n!} = C_n^{(n+\alpha)} \simeq \frac{n^\alpha}{\Gamma(\alpha+1)}, \tag{6.1}$$

where $\Gamma(\cdot)$ is the gamma-function, is called the Cesàro numbers (sf., [14]). The Cesàro sums for a function f is defined to be trigonometric polynomials $\sigma_n^\alpha(f) = f * K_n^\alpha$, where

$$K_n^\alpha(t) = \sum_{\nu=0}^{n} A_{n-\nu}^{\alpha-1} D_\nu(t)/A_n^\alpha, \tag{6.2}$$

and $D_n(t)$ is the Dirichlet kernel,

$$D_n(t) := \left(\frac{1}{2} + \sum_{\nu=1}^{n} \cos \nu t\right) = \frac{\sin(n+1/2)t}{2\sin t/2}.$$

By applying the Abel transform to (6.2) several times, we obtain the well-known Kogbetliantz formula (see [13] and [14], ChapterIII)

$$K_n^\alpha(t) =$$
$$\Im\left(\frac{e^{it/2}}{2A_n^\alpha \sin t/2}\left(\frac{e^{int}}{(1-e^{-it})^m} + \sum_{k=1}^{m} A_{n+k}^{\alpha-k}\frac{1}{(1-e^{-it})^k} + \sum_{\nu=1}^{\infty} A_{n+m+\nu}^{\alpha-m+1}\frac{e^{-\nu t}}{(1-e^{it})^m}\right)\right), \tag{6.3}$$

where $\Im(z)$ is an imaginary part of the complex number z. In view of (6.1) the series in the last expression converges only if $M > \alpha$. Nevertheless, formula (6.3) makes a sense even in the case $m \leq \alpha$ because for $t \neq 0$ this series is summable to a finite number by the Abel — Poisson method.

Theorem 6.1. *Let X is either $\mathbf{M}$ or ℓ^∞. Then the Cesàro method (C,α) converges on X if and only if $\alpha \geq 1$.*

Proof of Theorem 6.1. The necessity of the condition $\alpha \geq 1$ follows from the results of papers [3] – [5], where it was shown that for $\alpha < 1$ the convergence is absent even on the class $\mathbb{L}^\infty$.

The convergence of the method (C,α) for $\alpha \geq 1$ on the class $\mathbf{M}$ follows from Theorem 3.2 due to the convergence of the method (C,α) on the two distributions.

The sufficiency of the condition $\alpha \geq 1$ for $X = \ell^\infty$ we deduce from relation (6.3) for $m > \alpha$. Let δ be an arbitrary positive number. We show that $\|(1-\chi_\delta)K_n^\alpha \mid \ell^1\| \to 0$ as $n \to \infty$. Let us consider three addends of formula (6.3).

Let $\phi_n^1(t) = e^{int}/A_n^\alpha$, $\psi^1(t) = (1-\chi_\delta(t))e^{it/2}(1-e^{it})^{-\alpha}/(2\sin t/2)$. Obviously, $\|\phi_n^1 \mid \ell^1\| \to 0$ as $n \to \infty$, and since $\psi \in \mathbb{C}^\infty$, we have

$$\|\psi^1\phi_n^1 \mid \ell^1\| \to 0 \text{ as } n \to \infty. \tag{6.4}$$

Let

$$\phi_n^{2,k} = A_{n+k}^{\alpha-k}/A_n^\alpha, \quad \psi^{2,k}(t) = (1-\chi_\delta(t))e^{it/2}(1-e^{it})^{-k}/(2\sin t/2), \quad k = 1,2,\ldots,m.$$

Then, obviously,

$$\phi_n^{2,k} \to 0 \text{ as } n \to \infty. \tag{6.5}$$

Therefore, $\|\phi_n^{2,k} \mid \ell^1\| \to 0$ as $n \to \infty$. In view of the fact that all $\psi^{2,k}$ belong to $\mathbb{C}^\infty$ we have

$$\left\|\sum_{k=1}^m \phi_n^{2,k}\cdot\psi^{2,k} \mid \ell^1\right\| \to 0 \text{ as } n \to \infty. \tag{6.6}$$

As it was mentioned above, for $m > \alpha$ the series $\phi_n(t) = \sum_{\nu=1}^\infty A_{n+m+\nu}^{\alpha-m-1}e^{-i\nu t}$ converges absolutely. Obviously, the norms $\|\phi_n \mid \ell^1\|$ are bounded in the aggregate. Therefore, denoting by ϕ_n^3 the function ϕ_n/A_n^α, we obtain $\|\phi_n^3 \mid \ell^1\| \to 0$ as $n \to \infty$. Since $\psi^3(t) = (1-\chi_\delta(t))e^{it/2}(1-e^{it})^{-m}/(2\sin t/2) \in \mathbb{C}^\infty$, then

$$\|\phi_n^3\cdot\psi^3 \mid \ell^1\| \to 0 \text{ as } n \to \infty. \tag{6.7}$$

Joining (6.4), (6.6) and (6.7), we obtain

$$\|(1-\chi_\delta)K_n^\alpha \mid \ell^1\| \leq \|\phi_n^1\cdot\psi^1 \mid \ell^1\| + \sum_{k=1}^m \|\phi_n^{2,k}\cdot\psi^2 \mid \ell^1\| + \|\phi_n^3\cdot\psi^3 \mid \ell^1\| \to 0$$

as $n \to \infty$. As was to be shown.

□

Corollary 6.1. *The Cesàro method (C,α) converges on the class ℓ^p $(2 \leq p \leq \infty)$ if and only if $\alpha \geq 1$.*

Obviously, the necessity of the condition $\alpha \geq 1$ follows from the embeddings $\mathbb{L}^\infty \subset \mathbb{L}^2 \subset \ell^p$, and its sufficiency follows from Theorem 6.1.

Theorem 6.2. *The Cesàro method converges on the class* $\Re\mathbf{H}^p$ $(0 < p \le 1)$ *if and only if*

(a) $\alpha \ge 1,\ 1/2 \le p \le 1$;

(b) $\alpha \ge 1/p - 1,\ 0 < p < 1/2$.

Proof of Theorem 6.2. According to Corollary 3.1 of Theorem 3.4, we just need to estimate the norms $\|(1-\chi_\delta)K_n^\alpha \mid \mathbf{B}^{1/p-1}_{\infty,\infty}\|$ for the fixed $\delta > 0$. We show that for this estimation of the norms only the first term of relation (6.3) is important. Acting by analogy with the proof of Theorem 6.1, we obtain from (6.5) that

$$\lim_{n\to\infty} \|\phi_n^{2,k} \cdot \psi^2 \mid \mathbf{B}^{1/p-1}_{\infty,\infty}\| = 0$$

for any $k = 1, 2, \dots, m$. Obviously, for any positive integer l and sufficiently large m the series $\sum_{\nu=1}^\infty A^{\alpha-m-1}_{n+m+\nu} e^{-i\nu t}$ is the Fourier series of a function which is continuously differentiable l times. It follows from here that $\overline{\lim\limits_{n\to\infty}} \|\phi_n^3 \cdot \psi^3 \mid \mathbf{B}^{1/p-1}_{\infty,\infty}\| < \infty$ for sufficiently large m.

We estimate the first term. Let the number $1/p$ is fractional and $[1/p]$ is its whole part. We differentiate the function $\phi_n(t) = e^{int}/A_n^\alpha$ $[1/p] - 1$ times. Then the derivative $D^{([1/p]-1)}[\phi_n(t)] = (in)^{[1/p]-1} e^{int}/A_n^\alpha$ is bounded in the norm of the space $\mathbf{B}^{1/p-[1/p]-1}_{\infty,\infty}$ if conditions (a) or (b) are valid. It is unbounded if conditions (a) and (b) are not valid. In the case when $1/p$ is integer greater than 1, we need to differentiate $[1/p] - 2$ times the function ϕ_n and to estimate the Zygmund norm of the obtained function or (that is the same), the norm in $\mathbf{B}^1_{\infty,\infty}$.

The case $p = 1$ follows from the embeddings $\mathbb{L}^\infty \subset \Re\mathbf{H}^1 \subset \mathbb{L}^1$ in view of the fact that the condition $\alpha \ge 1$ is necessary and sufficient for the convergence on the classes $\mathbb{L}^\infty$ and $\mathbb{L}^1$. □

We proceed to the consideration of summation of Fourier series by the Vallée-Poussin methods. We recall that the Vallée-Poussin kernel with integer parameters m and k, $0 \le m < k$, is defined to be the trigonometric polynomial

$$V_m^k(t) = \sum_{i=m}^{k-1} D_i(t)/(k-m).$$

In [3] – [5], the following theorem was proved.

Theorem D. *The Vallée-Poussin method with the kernels* $\{V_{k_n}^{m_n}\}_{n=1}^\infty$, *where* $k_n \to \infty$, *converges on the class* $\mathbb{L}^p$ *if and only if:*

(a) $m_n = o(k_n)$ *if* $p = \infty$;

(b) $m_n = O(k_n^{p/(p+1)})$ *if* $1 \le p < \infty$.

Theorem 6.3. *The Vallée-Poussin method converges on the classes* $\mathbf{M}$, $\mathbb{L}^\infty$ *and* $\Re\mathbf{H}^p(1/2 \le p < 1)$ *if and only if* $m_n = o(\sqrt{k_n})$.

We recall two useful formulae for the representation of the Vallée-Poussin kernels:

$$V_{k_n}^{m_n}(t) = \frac{1}{k_n - m_n} \cdot \frac{\sin^2 m_n t/2 - \sin^2 k_n t/2}{\sin^2 t/2}, \tag{6.8}$$

$$V_{k_n}^{m_n}(t) = \frac{1}{2(k_n - m_n)} \cdot \frac{\cos m_n t - \cos k_n t}{\sin^2 t/2}, \tag{6.9}$$

Proof of Theorem 6.3. To prove the necessity we show that in the case when the condition $m_n = o(\sqrt{k_n})$ does not hold, the sequence of the Vallée-Poussin operators does not converge on the Dirac δ-measure. Let there exist $C > 0$ and a sequence of positive integers $n_l \neq \infty$ such that $m_{n_l} > \sqrt{k_l}$. Obviously, we can assume that $m_{n_l} < k_{n_l}/2$ because in the case of fulfillment of the converse inequality for infinite number of the indexes l it follows from the result by G.Natanson [17] that for the operator sequence the classical Gibbs phenomenon occurs. For simplification of the notation we omit the index l.

By substituting $t = t_n = 2\pi/k_n$ in formula (6.8), we obtain

$$V_{k_n}^{m_n}(t_n) = -\frac{\sin^2(\pi m_n/k_n)}{(k_n - m_n)\sin^2 \pi/k_n} < -\frac{((\pi m_n/k_n)(2/\pi))^2}{k_n(\pi/k_n)^2} = -\frac{m_n^2}{\pi^2 k_n} < -\left(\frac{C}{\pi}\right)^2.$$

It contradicts the convergence of the sequence of operators (0.1) on P_1.

To prove the sufficiency, at first, we show the convergence on the two distributions. The convergence on the identity is obvious. We show the convergence on P_1. Indeed, it is easy to see from (6.8) that outside an arbitrary neighborhood of zero the sequence $\left(V_{m_n}^{k_n}\right)^+$ uniformly converges to zero. On the other hand, according to (6.8), we have

$$\left(V_{m_n}^{k_n}\right)^- \geq -\frac{\sin^2(m_n t/2)}{(k_n - m_n)\sin^2 t/2} \geq -\frac{m_n^2}{k_n - m_n} \geq -\frac{(o(\sqrt{k_n}))^2}{k_n - o(\sqrt{k_n})} = o(1)$$

as $n \to \infty$. Consequently, the convergence on the class $\mathbf{M}$ is proved.

Let us show the convergence on the class ℓ^∞. Indeed, we have

$$\|(\cos m_n t - \cos k_n t)/(2(k_n - m_n)) \mid \ell^1\| \to 0, \text{ as } n \to \infty.$$

At the same time, for any $\delta > 0$ we have $(1 - \chi_\delta)/(\sin^2 t/2) \in \mathbb{C}^\infty$. Thus, according to (6.9) and from the boundedness on ℓ^∞ of the operator of multiplication by a function from $\mathbb{C}^\infty$, we have $\|(1 - \chi_\delta)V_{m_n}^{k_n} \mid \ell^1\| \to 0$ as $n \to \infty$. Now the convergence on the class l^∞ follows from Corollary 3.2 of Theorem 3.4.

Let us show the sufficiency of the condition $m_n = o(\sqrt{k_n})$ for the convergence on the class $\Re\mathbf{H}^{1/2}$. To prove it we need to show that for any $\delta > 0$ we have

$$\varlimsup_{n\to\infty} \|(1 - \chi_\delta)V_{m_n}^{k_n} \mid \mathbf{B}_{\infty,\infty}^1\| < \infty. \tag{6.10}$$

The space $\mathbf{B}_{\infty,\infty}^1$ coincides with the Zygmund space. We need to prove the uniform boundedness of derivatives of the function $(1 - \chi_\delta)V_{m_n}^{k_n}$. Indeed,

$$\mid (\cos m_n t - \cos k_n t)'/(2(k_n - m_n)) \mid \leq (m_n + k_n)/(2(k_n - m_n)) \leq 1$$

and $((1 - \chi_\delta)/(\sin^2 t/2)) \in \mathbf{C}^\infty$. Inequality (6.10) follows from here and from (6.9). Theorem 6.3 is proved. □

Theorem 6.4. *The Vallée-Poussin method does not converge on any class* $\Re\mathbf{H}^p(0 < p < 1/2)$ *for any set of the parameters* $\{k_n\}$ *and* $\{m_n\}$.

Remark. Due to already proved Theorem 6.3, the given result is not unexpected. Indeed, from our point of view "the best" of the Vallée-Poussin methods is the Fejer method. However, according to Theorem 6.3, it can provide the convergence only on the class $\Re\mathbf{H}^{1/2}$.

Proof of Theorem 6.4. The fulfillment of the inequality

$$\overline{\lim_{n\to\infty}} \|\chi_\delta V_{m_n}^{k_n} \mid \mathbf{B}_{\infty,\infty}^{1/p-1}\| < \infty$$

for any $\delta > 0$ is necessary for the convergence. For $p < 1/2$ it means that derivatives of the functions $\chi_\delta V_{m_n}^{k_n}$ have to be uniformly bounded with respect to the norm of one of the Zygmund —Lipshitz classes. However, obviously, the function

$$(\cos m_n t - \cos k_n t)'/(2(k_n - m_n)) =$$

$$= (k_n \sin k_n t - m_n \sin m_n t)/(2(k_n - m_n))$$

is not bounded as a sequence of any of the Lipschitz classes. Multiplication by the function $(1 - \chi_\delta)/\sin^2 t/2$ can not correct this property. Theorem 6.4 is proved. □

References

[1] B.Sendov, "Hausdorff approximations", Kluwer, Dortrecht, 1990.

[2] P.P. Korovkin, Experiment of axiomatic construction of certain problems of the approximation theory of functions one variable, *Uchenye Zapiski Kalininskogo gos. ped. inst.* **69** (1969), 91 – 109.

[3] A.P. Petukhov, On approximation of functions by singular integrals in the Hausdorff metric, *Mat. Sb.* **135 (177)** (1988), 235 – 252 (in Russian); English transl. in *Math. USSR - Sb.* **63** (1989).

[4] A.P. Petukhov, On approximation of functions by convolution operators in the Hausdorff metric, *Dokl. Akad. Nauk SSSR* **327** (1992), 187 – 190 (in Russian); English transl. in *Russian Acad. Sci. Dokl. Math.* **46** (1993), 439 – 442.

[5] A.P. Petukhov, On the convergence of sequences of convolution operators in the Hausdorff metric. *Algebra i Analiz,* **5** (1993), 87 – 117 (in Russian); English transl. in *St. Petersburg Math. J.* **5** (1994), 943 – 969.

[6] A.P. Petukhov, On approximation of periodic distributions in the Hausdorff metric, *Rossiisk. Akad. Nauk Dokl.* **336** (1994), 25 – 28 (in Russian); English transl. in *Russian Acad. Sci. Dokl. Math.* **49** (1994), 460 – 464.

[7] A.P. Petukhov, Convergence of Fourier series for functions in the classes of Besov—Lizorkin—Triebel, *Mat. Zametki,* **56** (1994), 63 – 70 (in Russian); English transl. in *Math. Notes* **56** (1994), 694 – 698.

[8] R.Edwards, "Fourier series: a modern introduction", 2nd ed., Springer-Verlag, Berlin, Heidelberg, and New York, 1982.

[9] E.F.Collingwood and A.J.Lohwatter, "The Theory of Cluster Sets," Cambridge Univ. Press, Cambridge, 1966.

[10] L.V. Kantorovich, G.P.Akilov, "Functional analysis," Pergamon Press, Oxford, 1982.

[11] J.L.Lions and J.Peetre, Sur une classe d'espaces d'interpolation. *Inst. Hautes Etudes Sci. Publ.* **19** (1964), 5 – 68.

[12] Jöran Bergh and Jörgen Löfström, "Interpolation spaces," Springer-Verlag, Berlin, Heidelberg, and New York, 1976.

[13] Kogbetliantz E. Analogies entre les s éries trigonométriques et les séries sphériques. *Ann. de l'Ecole Nor. Sup.* 1923. **40** (1923), 259 – 323.

[14] A.Zygmund, "Trigonometric series," 2nd ed., Cambridge Univ. Press, Cambridge, 1959.

[15] M.A.Krasnosel'skij and Ya.B.Rutickij, "Convex Functions and Orlicz spases," Groningen. Neterland, P.Noordhoff Ltd. 1961

[16] Paul Koosis, Introduction to $\mathbf{H}^p$ spaces. Cambridge Univ. Press, Cambridge, 1980.

[17] G.I.Natanson, The Gibbs phenomenon for Vallée-Poussin sums, *Studies of Modern Problems in the Constructive Theory of Functions* (V.I.Smirnov, ed.), Fizmatgiz, Moscow, 1961, 206 – 213.

[18] Hans Triebel, "Theory of function spaces," Akademische Verlag, Geest&Portig, Leipzig, 1983.