CR-TANGENT MESHES

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SUMMARY

This paper extracts some properties from the field of Computational Geometry and applies them to the design of Spatial Meshes. By means of two transformations: one inversive, the other projective, a way for making up polyhedra approximating spheres and other rotated second order surfaces (ellipsoids, paraboloids and hyperboloids), starting from a simple Voronoi Diagram, is shown. The procedure and its theoretical basis are related. CR-TANGENT is a name proposed from its Spanish meaning. CR: "Cuádricas de Revolución".

1. INTRODUCTION: GEOTANGENT MESHES

It is well known that the design of the Geodesic Dome [3] has received some optimisations since it was proposed; a really interesting alternative procedure of design appeared twelve years ago under the name of "Geotangent Domes" or Geotangent Meshes. Under the title "Polyhedral structures that approximate a sphere", Yacoe [7] defined a polyhedrical body that approximates this surface with these summarised features:

F1. It has polygonal faces, the edges of which are tangent to circles; these circles are the intersection between the plane holding the polygon and the sphere.
F2. Each vertex joins 3 or 4 polygons.
F3. The starting step considers a central horizontal ring of hexagons (six or more) surrounding the equator circle. The next steps in the construction fill different horizontal rings towards the north pole of the sphere; each one of these rings holds new polygons (non regular) of 5, 6, 7 or 8 edges.
F4. The inscribed circle within each polygon is tangent to the inscribed circles of every one of their neighbours (figure 1).
F5. The size of any polygon is chosen in such a way that it is the most similar possible to its neighbours in the ring immediately below.
F6. In a ring, the number of polygons can be:
- The same as the ring immediately below.
- Half that of the ring immediately below.
- Only one polygon (it only happens on the north pole).

F7. When a ring reduces the number of polygons to half that of its previous one, it is very possible that a vertex joins four edges. As it is not desirable, it is then suggested that some filler polygons be introduced at these vertexes (figure 2).

A polyhedron made up from these propositions is presented in figures 1 and 2, where we have shown in some way the meaning of each one of the features listed above. Dimensioning such a body involves the solution of a non-linear system of

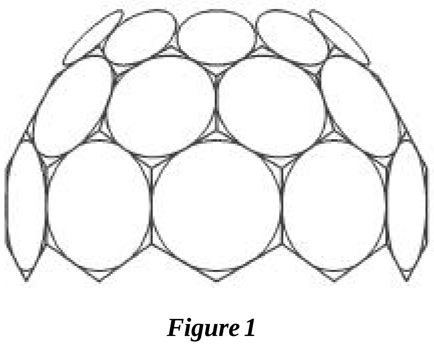


Figure 1

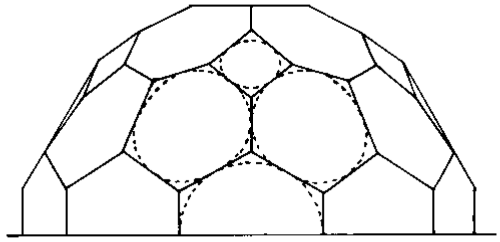


Figure 2

equations arising from the analytical expression of its characteristics (F1 to F7). In general, we cannot say that the solution of this system is always easy, neither geometrically intuitive.

Two years later, Yacoe proposed, under the title "Polyhedral structures that approximate an ellipsoid" [8] the adaptation of the related ideas and procedure to this new shape. In Figure 3, a solution is shown, where it can be seen that most of the features F1 to F7 remain and each polygon has a circumscribed ellipse, the points of contact of which with their neighbours make the edges of the approximating polyhedron appear. In other words, the edges of the polyhedron are tangent to the ellipsoid and this latter intersects the former in the collection of the drawn ellipses. Again, the method for dimensioning the body involves the solution of a non-linear system of equations neither easy, nor geometrically intuitive.

2. INTRODUCTION: DUALITY AND DUAL METHODS

As contrasted with the lattice structures, Wester [6], has proposed an interesting alternative suggesting that a plate structure, where plates (plane elements which are able to resist forces in their own plane only) are stabilized by shear forces, constitutes a new type and concept of structure with an applicability that is, perhaps, deeper than the former.

From the point of view of Statics, lattice and plate designs are equivalent: the equilibrium of force vectors on a node of a lattice structure is formulated in the same way as the equilibrium of moments (created by forces around a reference point) on a face of a plate structure. This equivalence is known as Duality.

From the point of view of Morphology, there can be found a relationship between a polihedrical

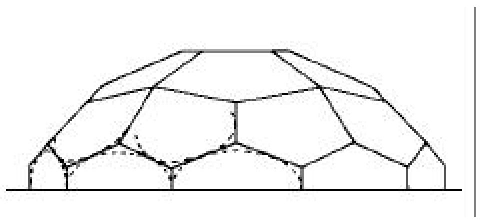


Figure 3

approach obtained via lattice and its corresponding one via plate (and vice versa). Wester proposes this correspondence in terms of Duality; three types of dual transformation, DuT, DuM and DiT [5] are, basically, able to supply the mentioned equivalence lattice-plate. It is remarkable that some of these dual transformations give, from a spherical lattice, dual solutions such as double curved shapes different from the sphere (the reciprocal one is possible too).

Besides Wester, some other researchers have studied structural dualism. Concerning the procedure to obtain the shape of the structure, it can be assumed that its geometrical support is the transformation known as polarity.

3. VORONOI DIAGRAM AND DELAUNAY TRIANGULATION ON THE PLANE

The topological similarity between the Geotangent Models (spherical or elliptical) or Plate Structures and the Voronoi Diagram cannot pass unnoticed. The same happens when someone considers a lattice mesh (Fuller, Shweddler, etc.) and contemplates the tesselation arising from a Delaunay Triangulation. Voronoi Diagrams and Delaunay triangulations belong to Computational Geometry, a modern field of study that can supply some new ideas to the structural design. We will make a brief introduction of Voronoi Diagram and Delaunay Triangulation in plane 2D; some useful references of Computational Geometry are offered at the end of this paper [2], [4].

Let us consider a set S of points in the plane, $S = \{S_1, S_2, . . . S_n\}$, where $S_i = (x_i, y_i)$. Given a member S_i of this set, the polygon of Voronoi of S_i, $V(S_i)$, is defined as the set of points P_i on the plane that are closer to S_i than to any other point S_j of S. A Voronoi polygon is shown in figure 4a.

Each point S_i of S defines a Voronoi polygon; the n regions arising from the set S partition the plane into a convex net which is known as the Voronoi Diagram of S (see figure 4a). Each line segment of the diagram is a Voronoi edge, and its end points are called Voronoi Vertices. Notice that the S_i points are not vertices of the diagram: we will refer to them as Generator Points.

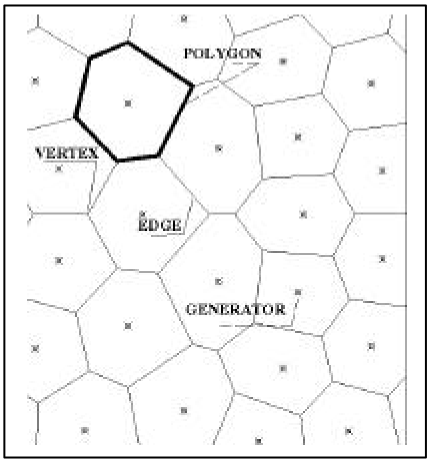


Figure 4a. *Voronoi Diagram*

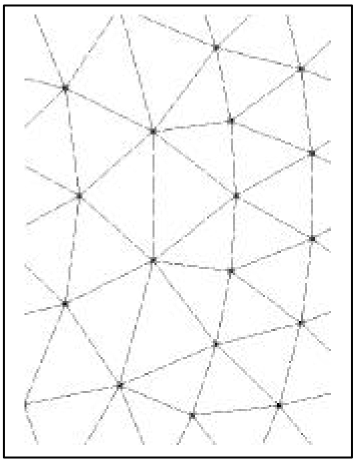


Figure 4b. *Delaunay Triangulation*

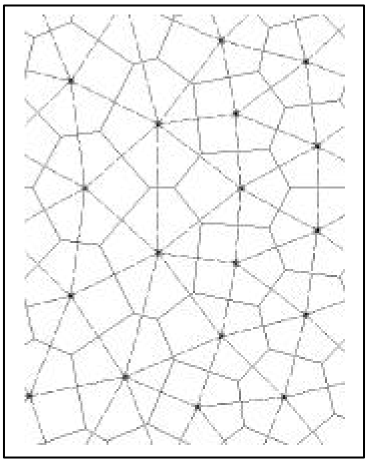


Figure 4c. *Duality between Voronoi Diagram and Delaunay Triangulation*

Next, we list a number of important properties of the Voronoi Diagram:

?? Every Voronoi Polygon $V(S_i)$ contains only one S_i generator point .
?? Every $V(S_i)$ is a convex polygon.
?? Every edge of the Voronoi Diagram is the perpendicular bisector of two generator points S_i and S_j.
?? Every vertex of Voronoi is the common intersection of exactly three edges of the diagram. Only when four generator points are cocircular, does the related Voronoi vertex join 4 edges.
?? The straight-line dual of the Voronoi Diagram is a triangulation of S. (This means a planar subdivision of the plane, where every polygon is a triangle and where each vertex is a generator point; see fig. 4b, 4c). This last property is due to Delaunay (1934).

This collection of properties does not depend on the situation of the generator points on the plane, so it is possible to make up regular, semi-regular, periodical, or any other type of tesselations via Voronoi if the set S is suitably proposed. In the following pages, we will show that it is possible and easy to transform a flat Voronoi Diagram into a polyhedron approximating a double curved surface. We remark on that we do not start our work from a lattice design but a simple set of points on the plane: our design effort is to select the most adequate one. Wherever these points are placed, we find a polyhedron as the solution; as designers, we need only study the position of the generators in order to obtain a "good design".

4. SOME USEFUL PROPERTIES OF THE VORONOI DIAGRAM

Assisted by Figure 5, consider the stereographic projection which describes the restriction of the

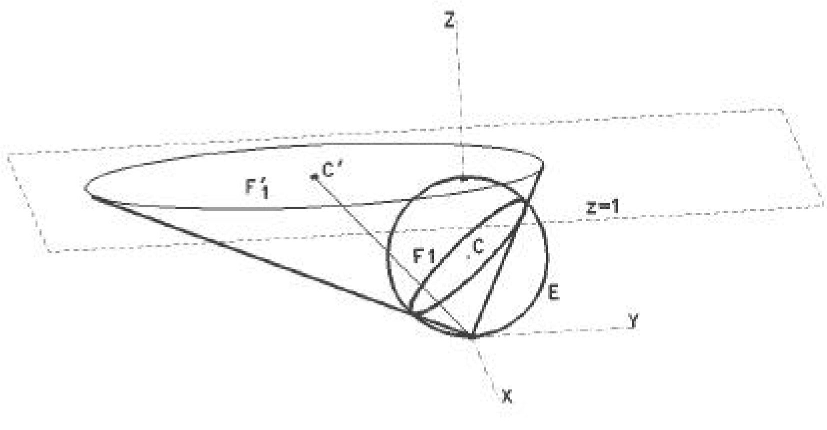


Figure 5

inversion on E^3 to the plane *z=1* (or equivalently, to the sphere E, inversive image of this plane). Suppose then a plane P holding C (the center of E). Whatever the angle that this plane forms with the OXY, it is well known that the inversive shape of the circumference F1 (arising from the intersection of P with E) is another circumference F'1, the center of which is C´, resulting from the intersection between the plane z=1 and the line perpendicular to P from O (pole of the inversive transformation).

Suppose, then, two points P1 and P2 on the sphere (figure 6) and consider also the planes ?1 and ?2, tangent to E at the points P1 and P2. Consider, at last, the line n of intersection of ?1 and ?2.

PROP. 1. *The projection of n on the plane z=1 from O is the perpendicular bisector of the pair of points P'1 and P'2 (transformed from P1 and P2 in the inversion).*

Indeed, with C, the points P1 and P2 define the circumference F1, the transformed inverse of which is F´1. We then draw r and s, tangent to F1 at P1 and P2 respectively. From point I, where r and s cut each other, we consider n, perpendicular to the plane of the circumference F1. The straight line n is the intersection of the planes ?1 and ?2.

The line n' is projected by transforming two selected points of n: its improper one and I (intersection of r and s). In accordance with the property written above, the inverse of the improper point of n is C' (center of F´1) because OC' is perpendicular to the plane of F1. On the other hand, I' results from the intersection of r' and s', tangent to F´1 in P'1 and P'2 because the inversive transformation keeps the tangency. F´1 being a circumference, it follows (figure 7) that n' must be the radius perpendicular to the segment P'1P'2 and, consequently, its perpendicular bisector.

Let then S'={P'1, P'2, . . . , P'n} be a set of points in the plane z=1. It results that:

PROP. 2. *The inversive image of the Voronoi Diagram of S' is a polyhedron that approximates the sphere E* $[x^2+y^2+(z-1/2)^2=1/4]$ *in such a way that each one of its faces is tangential to the sphere. There is a symmetric correspondence between each Voronoi polygon and each face of the polyhedron.*

This is true because the inversive transformation has the symmetric property and PROP. 1 is being

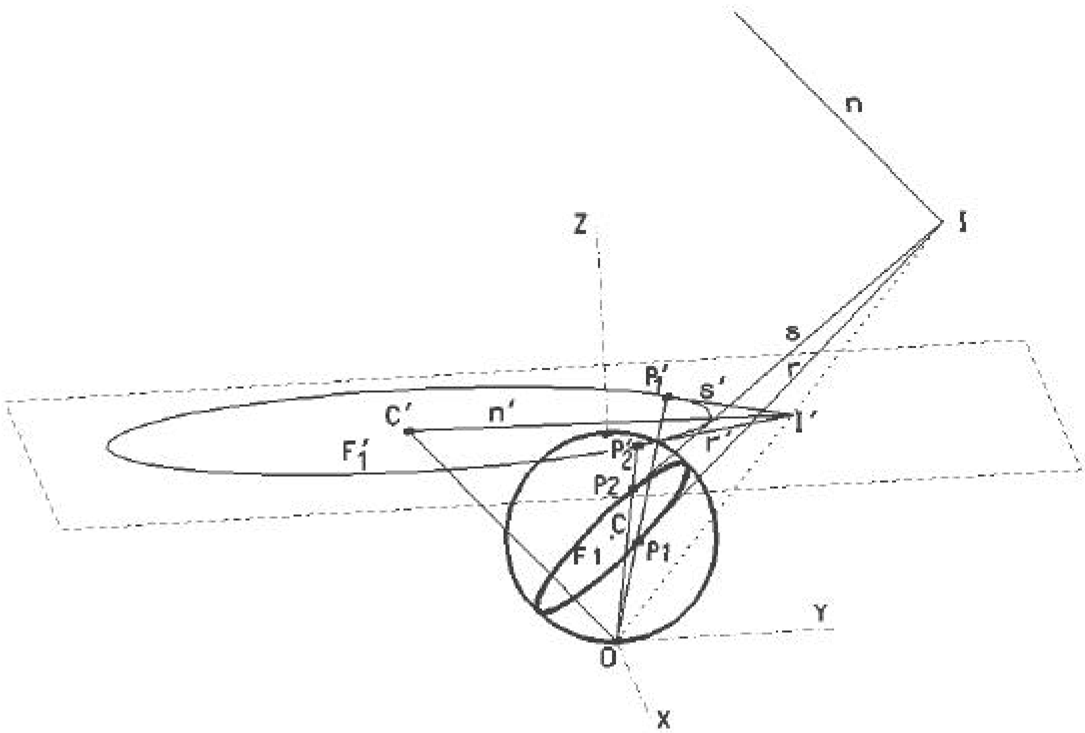


Figure 6

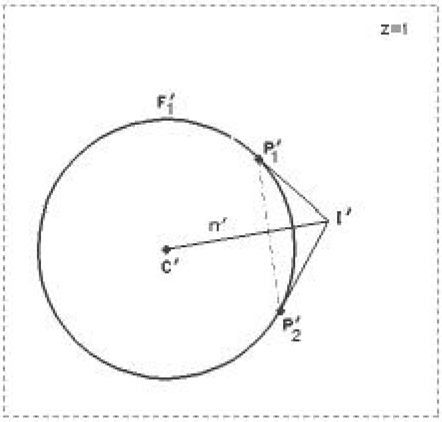


Figure 7

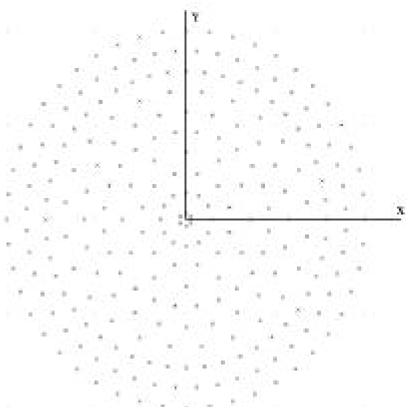


Figure 9

applied from the plane to the sphere. Figure 8 shows a polyhedron made up following all of these considerations.

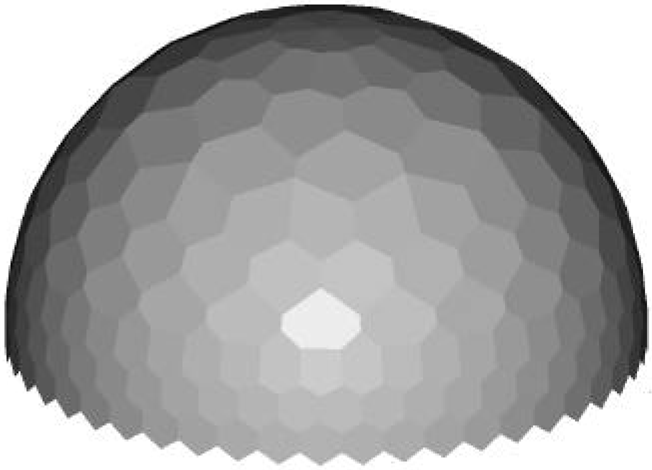


Figure 8

5. A 2D PROCEDURE FOR CREATING MESHES MADE UP BY NON-TRIANGULAR FACES

We are proposing the contribution as it came about: believing we had a geometrical explanation for an alternative way for designing Geotangent domes, we have arrived at a very different family of approaches to the sphere, where the property of

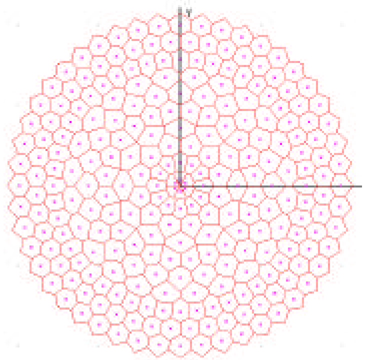


Figure 10

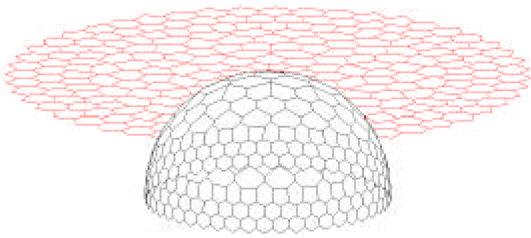


Figure 11

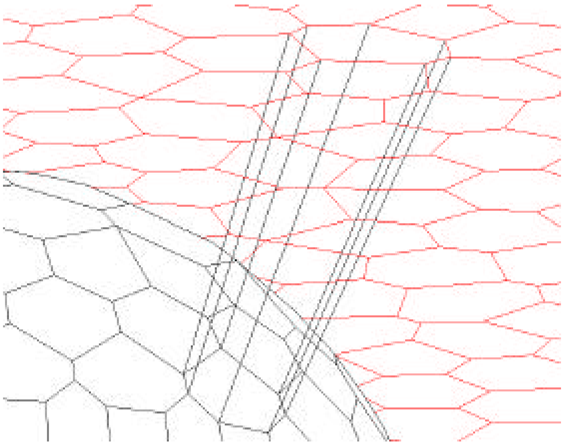


Figure 12

tangency is not produced on the edges of the polyhedron but in their faces.

On the other hand, this supposes a more intuitive way for choosing the final shape of the body because its topology can be proposed by means of a 2D Voronoi Diagram at the plane z=1. We can handle different hypotheses easily, as we illustrate in this sample case:

- Figure 9: a set of points belonging to the plane z=1, (viewed from the point of the infinite of the axis OZ) is defined.

- Figure 10: the Voronoi Diagram associated to this set is constructed.

- Figure 11: the polyhedron derived from this diagram, once property 2 is applied, is obtained. Figure 12 describes the relation between the edge of Voronoi and the edge of the polyhedron: the south pole of the sphere is the centre of the inversive transformation.

Even though the set of points (figure 9) does not need to follow any rule to be created, we have proposed a distribution by means of concentric rings in order to have different horizontal ones in the sphere: each of them is made by identical polygons (1 or 2 per ring). A realistic image of the obtained body has been displayed at the end of the previous point (figure 8).

6. A POLYHEDRON APPROXIMATING A ROTATION PARABOLOID

Suppose [2] that we now apply a suitable projective

$$\begin{pmatrix} X \\ Y \\ Z \end{pmatrix} \quad \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad \begin{pmatrix} x \\ y \\ z \end{pmatrix}$$

transformation, {E1}, to E^3 and let (X, Y, Z, T) be the coordinates in the transformed space to the point (x, y, z, t). Specifically, this relation is proposed:

where t and T are, respectively, the homogeneous coordinates of the related points. Such a transformation:

- Is a rotation in E^4.
- Maps the plane z=0 to the plane at infinit {E1}
- Converts the sphere E $[x^2+y^2+(z\text{-}1/2)^2=1/4]$ into the rotation paraboloid P $[X^2+Y^2+T^2=Z.T]$ that, written in inhomogeneous coordinates, results:
  $Z = X^2+Y^2+1$
- Maps the plane z=1 to the plane Z=1.
- Converts the sphere E_2 $[x^2+y^2+z^2=1]$ into a rotation hyperboloid H_2 that, written in inhomogeneous coordinates, results:
  $X^2+Y^2\text{-}Z^2 + 1 = 0$

Then, if we consider the set of points S' ={P'1, P'2, . . . , P'n} (see point 4), on the plane z=1, it follows that:

PROP 3. *The projection from the point (0, 0, -|) of the Voronoi Polygons of S' to the paraboloid P [Z = X^2+Y^2+1] makes up a polyhedron the faces of which are tangent to P. Each point of the set S' is transformed to the point of contact between the face of the polyhedron and the surface. The edges of the polyhedron are those transformed from the edges of the Voronoi Diagram of S'.*

Indeed, the transformation keeps all the points of tangency and intersection. So, the same relation proposed in PROP. 2 between the sphere and the plane z=1 remains but, now, the pole of projection has gone to the infinite (0, 0,-|) and the sphere has became a paraboloid. See figures 13 and 14.

7. CR-TANGENT MESHES.

This last property relating the edges of the Voronoi Diagram and the edges of the polyhedron approximating a paraboloid was proposed in [4] and has been, in short, the clue needed to have arrived at the proposals shown in this paper. However, there

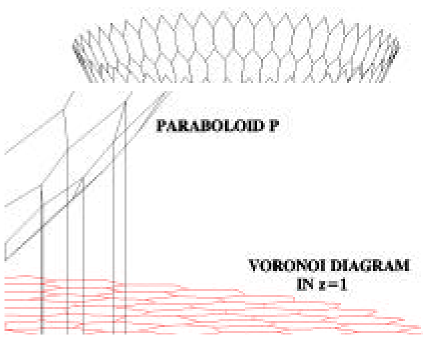


Figure 14

are some more, which can be checked:

PROP 4. *The projection from the point (0,0,-1) of the Voronoi Polygons of S' to the hyperboloid H [$X^2+Y^2-Z^2+1 = 0$] makes up a polyhedron the faces of which are tangent to H. Each point of the set S' is transformed to the point of contact between the face of the polyhedron and the surface. The edges of the polyhedron are those transformed from the edges of the Voronoi Diagram of S'.*

Figure 13 shows the relationship between the Voronoi Diagram on z=1 and the related paraboloid

the same way, figure 15 displays the polyhedron approximating the hyperboloid H while figure 16 does it with the used projection.

Let us consider any circumference the center of which is placed on a point of the Z axis (0, 0, Zc); if we apply an inversive transformation of the Voronoi Diagram of a Set of points belonging to the plane z=1, we know that a polyhedron approximating the sphere [$x^2 + y^2 + (z - z_c)^2 = (1 - z_c)^2$] is obtained, whenever the pole of inversion is the south pole of the surface: (0, 0, $2z_c$-1). When this sphere is transformed according to the expression {E1}, it results that:

- The transformed shape of the sphere: [$x^2 + y^2 + (z - z_c)^2 = (1 - z_c)^2$] is the second order surface: $X^2 + Y^2 - Z^2 (1 - 2z_c) - 2z_cZ + 1 = 0$
- The transformed shape of the south pole of the sphere, (0, 0, $2z_c$-1), is the point (0, 0, $1/2z_c$-1).

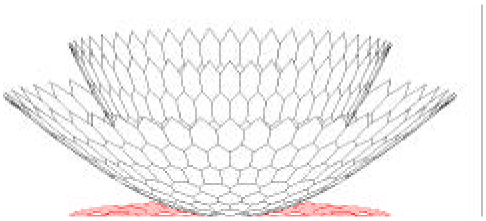


Figure 15

So, properties 3 and 4 are, simply, particular cases of:

PROP 5. *The coordinates (0,0 of S' to the seco SF [$X^2 + Y^2$ - up an approxim are tangent to transformed to t of the polyhedro polyhedron are the Voronoi Dia*

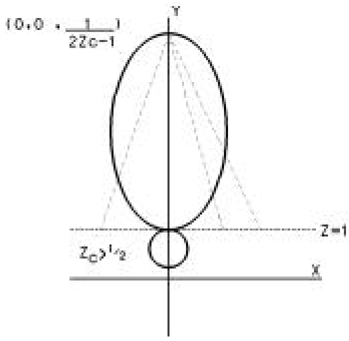


Figure 17

- *When z_c =1/2, SF is a rotated paraboloid.*
- *When z_c <1/2, SF is a rotated (two folds)*
- *When z_c >1/2, SF is a rotated ellipsoid.*

In figure 17 an illustrative picture of each of this cases is offered.

8. SUMMARISING

Once this paper is read, it becomes clear that one of the deepest sets of Geometry, the Voronoi Diagrams, enters the field of the design of Spatial Meshes. **A simple 2D procedure**, with a very easy computer implementation, can be used for making up new models shaped as rotated ellipsoid, paraboloid or hyperboloid. The approximating polyhedron holds non-triangular polygons tangent to the surface it approaches. It is not difficult to notice that, following the related procedure, **local**

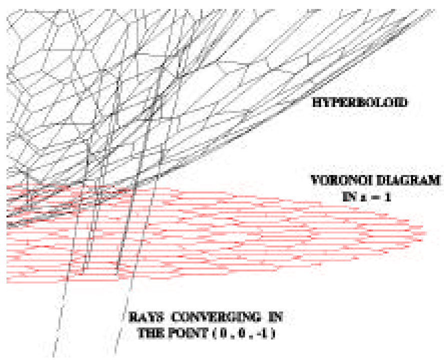
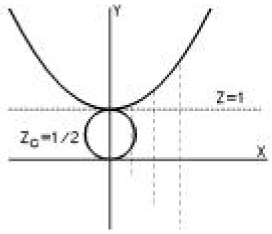
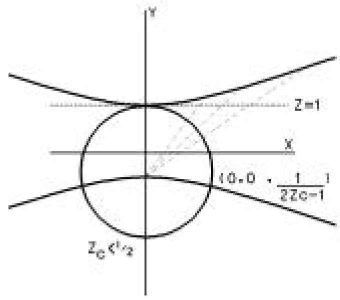


Figure 16

changes in this resulting polyhedron are really easy to carry out. On the other hand, the properties of Voronoi Diagram (see point 4 of this paper) **guarantee the stability** of the body obtained.





Returning to Computational Geometry: does the lattice type of structure not suggest the Delaunay Triangulation? Is it possible to revisit this kind of design as related to a 2D triangulation in the plane z=1? See again property 2 and consider not its Voronoi Diagram, but only the mesh of points S': yes, it is! The only thing needed is to construct the Delaunay Grid and to apply the inversive transformation.

As it was stated before (see point 3), we remark that we do not start our work from a lattice design (to be transformed) but **a simple set of points on the plane**: our design effort is to select the most adequate tesselation, derived from this set via Voronoi Diagram. Wherever these points are placed, we find a polyhedron as the solution; as designers, we need only to study the position of the generators (or locally move some of them) in order to obtain a "good design".

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