Dürer's Magic Square, Cardano's Rings, Prince Rupert's Cube, and Other Neat Things

V. Frederick Rickey

Abstract: Recreational mathematics is as old as mathematics itself, so a survey of its history is out of the question. Instead we discuss a few neat things, setting each in its historical context and explaining their significance. As a benchmark for looking forward and back we shall take Charles Hutton's *Recreations in Mathematics and Natural Philosophy*, which in turn is based on works of Ozanam and Montucla on recreational mathematics.

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Introduction

If you will pardon me, I will begin immodestly, with a personal story. Some years ago I was working in my office when the phone rang. The caller said that he had learned that I was interested in old mathematics books and that he had one that he thought would interest me. We talked for a while and he offered to send it to me. Needless to say, I was skeptical, for one does not just send seventeenth-century books to strangers. I asked if he was a book dealer and he said that he was not, he was just an individual interested in old books. Eventually we agreed that he would send the book to me so that I could enjoy reading it. It was an early edition of Ozanam's work on recreational mathematics. It was in French so my reading was slow, but I did enjoy it. Every couple of weeks I would call him and report on what I had read. I asked if I should send it back, but he was clarly in no hurry to have it back. After several months he called me and asked if the Bowling Green library would be interested in the book. I responded that it would be a valuable addition to the collection but that, honestly, I would likely be the only reader. It would be better if it were in a collection that was better known to scholars. I suggested the University of Michigan because I was well aware that they had a substantial collection of rare mathematics works [16]. Several weeks later he called again and asked if I would take the book up to the rare book room at Michigan to see if they were interested in purchasing it. I agreed and contacted the rare book librarian. Since I was a known user of the collection, the librarian, Peggy Daub, asked me to give my opinion as to whether the library should purchase it. So I asked her to pull all of the works by Ozanam in the collection. I took the book up and spent an afternoon examining all of their copies. It was clear that it would be a nice addition to the collection, but not an essential addition. I suggested they buy it if it was not too expensive. As instructed by the owner, I left the book there and thought the story was over. But it was not.

Several years later two undergraduate classics majors asked if I would direct a readings course in Euclid for them, as they saw no merit in taking an algebra class to fulfill their liberal arts requirement (they were right). I agreed and said that our library had an 1984 edition of Gerard of Cremona's twelfth-century Latin translation of Euclid and we could read that. They agreed and it was a wonderful experience. I also suggested that we could go up to the University of Michigan and look at some of their early Euclids. In preparation for this trip, I went up to make the arrangements. One thing I wanted to show them was the first printed edition of Euclid, the Ratdolt printing of 1482, but I could not find it in the computer catalog and I knew they had one as I had previously examined it. When I asked, the librarian, Peggy Daub, said that it was an incunabula (a book published in the sixteenth-century, literally 'in the cradle' of printing) and was catalogued by publisher, not author. She also remarked that the library had recently received a gift of a collection of mathematical works and asked if I would like to see the list. Of course I wanted to see what new items they had. A few minutes later she appeared with a sheepish grin on her face, saying that my name was in the file. I was puzzled, but she said that I had suggested to the donor that he give his collection to Michigan. There were about fifty books in the gift including the Ozanam that I had read earlier. Also there was a copy of the 1670 edition of the *Arithmetica* of Diophantus, the one where Fermat's last theorem is first stated [7]. Needless to say, I am now always a welcome reader in the rare book room.

When I joined the West Point faculty in 1998, there was a copy of the four volumes of Charles Hutton's English translation of Ozanam's work on recreational mathematics in my outer office, which contains three bookcases full of mostly nineteenth-century works. Markings in it indicated that it was an 'old text' that was used at the Academy, but it did not take much investigation to learn that this was an error made by an earlier bibliographer. What was used as a text was Hutton's *A Course of Mathematics*; this has been confirmed by comparing the mathematics in several copy books carefully written by cadets with Hutton's *Course*. Nonetheless, I enjoyed reading Hutton's *Recreations in Mathematics and Natural Philosophy* (1803). It is the contents of this work that I shall describe to you here.

Jacques Ozanam published his *Récréations* in 1694 [1], Montucla revised it in 1778 [4] and Hutton translated it into English in 1803 [5]. For more detail on the complicated publishing history of this work, see [1, 2, 3, 4, 5, 6]. Unless otherwise stated I will be using the 1803 translation of the *Recreations* by Hutton. Since these three individuals are hardly household names in the mathematical community, I will begin with some biographical information.

Ozanam, Montucla, and Hutton

Jacques Ozanam (1640–1717) studied to be a Catholic priest, but after his father died, devoted himself to mathematics, being almost entirely self taught. In Lyons he began a career of teaching individuals, mostly for free. "He was addicted to gaming; his private pecuniary resources were limited; and the stern realities of distress would speedily dissipate all illusions about the dignity of teaching science for its own sake." [6, p. v]. When he loaned money to two men without any bond they showed their gratitude by arranging for him to come to Paris where there were more opportunities. He was young, gallant, and handsome, but his penchant for gambling kept him impoverished. He married a woman of modest means and they had twelve children, most of whom died young. His wife's death ended a happy marriage and he lived the rest of his life in a melancholy state.

The first volume of Hutton's *Recreations* contains a short note "On the life and writings of Ozanam, the first author of these *Mathematical Recreations*," which is the source of an oft-told tale about him.

> Ozanam possessed a mild and calm disposition, a cheerful and pleasant temper, an inventive genius, and a generosity almost unparalleled. After marriage, his conduct was irreproachable; and, at the same time that he was sincerely pious, he had a great aversion to disputes about theology. On this subject he used to say, that it was the business of the Sorbonne doctors to discuss, of the Pope to decide, and of a *Mathematician to go straight to heaven in a perpendicular line.* [5, pp. xiii–xv]

Ozanam was not a creative mathematician but he was an excellent expositor. He wrote on algebra, trigonometry, fortifications, perspective, cartography, and higher geometry. But his fame rests on three expository works, the two volume *Dictionnaire mathématique* (1691), the five volume *Cours de mathématiques* (1693) and the two (later four) volume *Récréations mathématiques et physiques* (1694). All of these books sold well and ran to many editions, especially the last two. It is certainly for his work on recreational mathematics that Ozanam will be most remembered. His book was the precursor of books on recreational mathematics which followed in the next 200 years.

Jean Étienne Montucla (1725-1799) received a solid education in ancient languages and mathematics at the Jesuit *collège* in his native Lyons. The curriculum was principally directed to the ancient languages, "but having a natural taste for philological studies, and a powerful memory, he was enabled to acquire an accurate knowledge of several modern languages; among which Italian, German, Dutch, and English are mentioned." [6, p. vi] After legal studies at Toulouse he moved to Paris, where he supported himself with a variety of government positions, and where he took up a serious interest in the history of mathematics. His *Histoire des recherches sur la quadrature du cercle* (1754; English 1873) earned him a corresponding membership in the Berlin Academy. After publishing a sourcebook on smallpox (Daniel Bernoulli was the first to model the disease with a differential equation) he published the work which was to bring him lasting fame, his *Histoire de Mathématiques* (1758, 2 volumes; second edition 1799–1802, 4 volumes). This is a work that is still valuable to scholars (who have the knowledge to separate the still valuable material from the dated).

Then in 1778 Montucla produced the work of interest to us here, a new edition of Ozanam's *Récréations* [4]. The work was published anonymously under the initials 'C. g. f.' which abbreviate 'Chanla géomètrie forézien,' where the word 'forézien' means 'from Feurs' or 'of the Forez region.' "So carefully had he concealed his connection with the work, that on its completion, a copy was sent to him, in his capacity of censor, for examination and approval." [6, p. vi]. It is said that he corrected several items, made some additions, and sent it back for publication.

> Besides expunging from the work of Ozanam much that was absurd, puerile, and obsolete, he enriched the edition with dissertations upon almost every branch of practical science; and much of what he added is valuable even at the present day. [6, p. vi].

Charles Hutton (1737 – 1823) was born in Newcastle, the youngest son of an overviewer (supervisor) of a coal mine. When he was seven, Hutton was involved in a street-brawl and severely dislocated his left elbow. He hid this injury from his parents and by the time they learned of it, it was too late to treat it properly, so the injury became permanent. Since Hutton was unable to join his older brothers in the mine, he was sent to school to learn to read. After several years the teacher left and Hutton replaced him, thus beginning a habit of teaching by day and learning by night.

One pupil that Hutton attracted was Robert Shafto. He made his private library available to Hutton and then encouraged him to publish. Hutton's first work, *The Schoolmasters Guide, or a Complete System of Practical Arithmetic*, appeared in 1764 and became the standard school text for half a century. During the Christmas holiday of 1666, Hutton advertised that "Any schoolmaster, in town or country, who are desirous of improvement in any branch of the mathematics, by applying to Mr Hutton" [13, p. 63]. This in-service training was repeated the next year. That there was ample audience is attested to by the 59 schoolmasters from the Newcastle area who were subscribers to his next book, *A Treatise on Mensuration* (1767). Besides its mathematical interest this work is noted for the woodcuts by the young Thomas Bewick, who became one of the great masters of the art. Alas, this just makes the book more expensive for the historian of mathematics to acquire.

In 1760, Hutton opened his own school in Newcastle. This became a success and he became known as an excellent teacher. His patron, Shafto, suggested that he should move to London and apply for a vacancy at the Royal Military Academy [RMA] in Woolwich [12]. The position was to be filled by competitive examination which lasted several days. Col. Watson, Bishop Horsley, the editor of Newton's works, and Nevel Maskelyn, the Astronomer Royal, examined the eleven candidates. Half were judged satisfactory for the post, but Hutton stood out, so he obtained this professorship in 1773. He remained at Woolwich for 34 years.

Howson so nicely tells one event in Hutton's career that I shall quote the passage in its entirety:

> In 1786 Hutton began to suffer from pulmonary disorders. The RMA was situated near the river [Thames] and dampness began to affect his chest; his predecessor Simpson [of Simpson's Rule fame] had in fact died from a chest complaint. Hutton decided then to move, and bought land on the hill south of the river overlooking Woolwich. There he built himself a house and also others for letting. No sooner had he done this than it was decided to move the Academy from the damp riverside to the hilltop. A magnificent new building was erected, but, in the eyes of George III, its attractiveness was spoiled by the presence of Huttons houses. These were therefore sold to the crown who promptly demolished them, leaving Hutton with a hefty profit from his speculation, sufficient to guarantee his financial future. Thus a physical disability turned him to mathematics and ill-health made him rich. [13, pp. 66-67.]

Huttons most important and best known work was his *Mathematical and Philosophical Dictionary*. This appeared in two volumes in 1795. The USMA library has it, but the first volume is not the first printing. There is an 1813 letter from Joseph G. Swift, the first graduate of West Point and later Superintendent, saying that he read it before it was sent on to USMA library. Hutton worked on this for 10 or 12 years. It is an excellent survey of mathematics, includes biographies of many mathematicians, and is a pioneer contribution to the history of mathematics [13, p. 67]. Although it was criticized as unbalanced in content, unduly cautious in tone, and somewhat lacking judgment, the dictionary has served as a valuable source for historians of mathematics. [10, vol. 5, p. 577]

Hutton is also famous as editor of *The Ladys Diary*, a journal that appeared from 1704 to

1841 [15]. In August 1798, the 'Notices of works in hand' section of the *Monthly Magazine* lauded Hutton's *A Course in Mathematics* before it appeared:

> From Dr Hs talents and long experience in his profession, there is every reason to expect that this will not only be a most useful and valuable work, but will completely supersede every other of the same description. [13, p. 67].

It did prove to be popular, appearing in numerous editions over fifty years. There were several editions that were published in North America and there was even an Arabic edition. Thus, it is not surprising that this text was used in the US and at USMA, for the British influence on American education was extremely strong at this time.

Ozanam based his *Récréations* on the Greek Anthology and on works by Bachet, Mydorge, Leurechon, and Schwenter. According to Hutton, the most important influence on Ozanam's *Récréations* was Claude Gaspar Bachet de Méziriac's *Problémes plaisans et délectable sur les Nombres* (1626). This was a solid piece of scholarship that seems to have been used by all writers on the topic for a century or so. The first work containing the words 'Mathematical Recreations' in its title was *Recreation Mathematiques, composé de plusieurs Problémes plaisans et facetieux, par H. van Etten* (1627; English translation 1633). Hutton seems unaware that 'van Etten' was a pseudonym for the Jesuit Jean Leurechon (1591–1670). The work was popular for it went through thirty editions/printings before 1700. But Hutton was not impressed with the book, referring to it, in his Preface, as "a mere wretched rhapsody" which prompted Claude Mydorge, in his *Examen du livre des Récréations mathématiques* (1630), to correct "with some asperity the wretched things it contained." Both Montucla and D. E. Smith agree with this appraisal. But, in the mind of Hutton, Mydorge fares no better:

> This work exhibits a confused collection of questions, the greater part of which are silly and childish, and expressed in barbarous language, sufficient to disgust any person of only common taste. [5, p. v]

The final source for Ozanam was *Deliciae physico-mathematicae* (1636) by Daniel Schwenter (1585–1613). It is this sad state of affairs, according to Hutton, that prompted Ozanam to prepare his own book on recreations. The work had gone through several editions by the time Hutton was preparing his translation, yet Hutton finds Montucla's "book was both very faulty, and incomplete." Consequently, "To render this work more worthy of the enlightened age in which we live, it was necessary to make numerous corrections and considerations." [5, p. vii]. It seems that Hutton felt a need to justify his own work by denegrating his predecessors.

The Contents of Hutton's Recreations

Volume 1.	I	Arithmetic	p. 1
	II	Geometry	pp. 263–447
Volume 2.	III	Mechanics	p. 1
	IV	Optics	p. 169
	V	Acoustics and Music	pp. 375–464
Volume 3.	VI	Astronomy and Geography	p. 1
	VII	Dialling	p. 259

	VIII	Navigation	p. 354
	IX	Architecture	p. 392
	X	Pyrotechny	pp. 438–501
Volume 4.	XI	Elements	p. 1
	XII	Magnets	p. 269
	XIII	Electricity	p. 313
	XIV	Chemistry	pp. 391–516

Although the three last volumes contain much of interest, we shall concentrate our attention on the first, for it is most closely connected with mathematics. It is divided into two sections, one on arithmetic the other on geometry. Now arithmetic is not as boring a section as one might imagine, for this is not a textbook, but a book for those who already know the basics of computation. Immediately after introducing the concept of a prime number (not surprising for the time, it is not clear if 1 is a prime) Hutton states

> One curious property of prime numbers is, that every prime number, 2 and 3 excepted, if increased or diminished by unity, is divisible by 6. This may be readily seen in any numbers taken at pleasure, as 5, 7, 11, 13, 17, 19, 23, 29, 31, &c; but I do not know, that any one has ever yet demonstrated this property *a priori*. [5, p. 31].

At the bottom of the page of the copy that I am using, someone has inked in a proof:

> The property is easily demonstrated. Universally, if $a - 1$, a, & $a + 1$ represent any three consecutive numbers, some one of them must be divisible by 3. For if in the case of $a - 1$, the remainder is 1, in that of a it must be 2, & in that of $a + 1$ there must be no remainder. If a be prime, as a is not divisible, either $a - 1$ or $a + 1$ must be divisible by 3, and since a

That is all that I can read of the proof, for the volume has been rudely trimmed, and the conclusion of the proof is missing, but the proof seems to be right on track. It is curious how Ozanam, or was it Hutton, could have made such an elementary error as to think this result was hard to prove.

There is only one other annotation in the book but before giving it — to give you a feel for the style of writing in the book — I quote the problem it concerns in full:

> A gentleman taking a fancy to a horse, which a horsedealer wished to dispose of at as high a price as he could, the latter, to induce the gentleman to become a purchaser, offered to let him have the horse for the value of a twenty-fourth nail in his shoes, reckoning one farthing for the first nail, two for the second, four for the third, and so on to the twenty-fourth. The gentleman, thinking he should have a good bargain, accepted the offer; what was the price of the horse?
>
> By calculating as before, the 24th term of the progression 1, 2, 4, 8, &c, will be found to be 8388608, equal to the number of farthings the purchaser ought to give for the horse. The price therefore amounted to 8738£ 2*sh* 8*d*, which is more than any Arabian horse, even of the noblest breed, was ever sold for.

The problem is of a familiar type and the solution involves nothing more than a computation, but the setting prompted this marginal annotation:

> Blair Atholl a horse which won the Derby in 1864 was recently sold by another for £12000. [5, p. 81]

That got my interest for the Kentucky Derby is the only sporting event I pay attention to and the race did not begin until 1875. Google set me straight: Blair Atholl won the Derby at Epson Downs.

There are more interesting things in the section on arithmetic. After discussing the convergence of the geometric series, the author considers the harmonic series, noting that the sum "is not a finite number." But instead of giving a proof he gives a vague citation to a paper in the *Journal de Trevoux* which uses "mere paralogisms" to argue that the sum is finite [5, p. 87]. A nice little research project would be to find this paper and to see what these paralogisms are.

How many anagrams of the Latin word 'amor' are there? One might quickly answer 24, but only 7 of them are Latin words (p. 98). The Genoese lottery is discussed (without mentioning Euler), p. 120. There are river crossing problems: the wolf-goat-cabbage and the three jealous husbands and their wives, p. 171. The Knight's tour is here (p. 178) as well as the age of Diophantus, complete with the problem posed in a Latin verse (p. 190). This should be enough to give you the flavor. The book is interesting reading — recreational mathematics is supposed to be interesting — but there is nothing very deep here.

Chapter XII of Hutton's *Recreations* is a long presentation (pp. 211–240) on magic squares. After defining what a magic square is he comments on their origin:

> These squares have been called *magic squares*, because the ancients ascribed to them great virtues; and because this disposition of numbers formed the bases and principle of many of their talismans.
>
> According to this idea, a square of one cell, filled up with unity, was the symbol of the deity, on account of the unity and immutability of God; for they remarked that this square was by its nature unique and immutable; the product of unity by itself being always unity.
>
> The square of the root of 2 was the symbol of imperfect matter, both on account of the four elements, and of the impossibility of arranging this square magically, as will be shown hereafter.
>
> A square of 9 cells was assigned or consecrated to Saturn; that of 16 to Jupiter; that of 25 to Mars; that of 36 to the Sun; that of 49 to Venus; that of 64 to Mercury, and that of 81, or nine on each side, to the Moon.
>
> Those who can find any relation between the planets and such an arrangement of numbers, must no doubt have minds strongly tinctured with superstition; [5, p. 212]

The connections with astrology were standard lore of the time. One thing we find missing is a citation of the Lo Shu as the source of the 3 by 3 magic square. Another is the promised proof for the non-existence of a 2 by 2 magic square.

Hutton notes several properties of magic squares, e.g., that numbers symmetrically placed with respect to the center add to twice that central value. Methods attributed to de la Loubere, Moscopulus, Bachet, Poignard, and de la Hire are given in the texts as rules to follow for constructing magic squares using the first n^2 integers, n odd.

Then he turns to the even case. He provides several rules, but in this case no attributions are given (except vaguely to de la Hire). What is interesting is that he notes the uniqueness of the 3 by 3 magic square: Whatever method is employed "the same square will always arise, except that it will be inverted, or turned from left to right, which is not a variation." [5, p. 235]. That is a trivial result, but then the result of Bernard Frenicle de Bessey is quoted that there are "at most" 880 magic squares of order 4 (the exactness of this result was verified several times, but an arithmetic proof was not presented until 1973). The estimate of the number of 5 by 5 magic squares, using the construction method of de la Hire, namely 57600, is grotesquely small; the correct answer is 275305224. The situation is even worse for 6 by 6 where 4055040 is given and 7 by 7 where 405425600 is given although it is recognized as too small; indeed it is, for the number of 7 by 7 squares is around 10^{34}. This is sequence A006052 in Neal Sloane's On-Line Encyclopedia of Integer Sequences.

The Chapter ends with a discussion of the magic squares of "the ingenious Dr. Franklin" who has "carried this curious speculation further than any of his predecessors". His 16 by 16 square is given on a plate as is his magic circle of circles. The properties of the former are discussed in detail.

One thing missing from this chapter is the first magic square in Europe, but this provides a segue to the most famous magic square in art.

Dürer's Magic Square

Albrecht Dürer (1471–1528) was a superb artist. He created art works and wrote books that are of considerable interest to mathematicians, but I don't believe that his work is as appreciated as it should be. Dürer was born in Nuremberg, learned goldsmithing from his father and studied painting and engraving, on both wood and copper, with Michael Wolgemut. As was the custom of the times he took his *Wanderjahre*, which actually lasted four years. He became fascinated with the work of Italian artists and "became convinced that the new art must be based on science — in particular, upon mathematics, as the most exact, logical, and graphically constructive of the sciences." [10, vol. 4, p. 258]. He then traveled to Venice where he began the study of mathematics and the theory of art, including linear perspective. He met Luca Pacioli whose *Divina proportione* (1509) contained Leonardo da Vinci's drawings of the skeletons of the regular polyhedra. Incidentally Pacioli wrote on double-entry bookkeeping as so is known as the father of accounting. He is also the first mathematician that we have an authentic portrait of [28].

Dürer published the second book on mathematics that was published in the German language, *Underweysung der Messung mit Zirkel und Richtscheyt in Linien, Ebnen, und gantzen Corporen (Treatise on Mensuration with the Compass and Ruler in Lines, Planes, and Whole Bodies)* (1525). This is a work in four parts. The first treats the construction of curves including the conics, spiral of Archimedes and the conchoid, using techniques that anticipated descriptive geometry [21]. The second deals with the regular polygons; the fourth with the Platonic and Archimedean solids. The third designs the letters of the alphabet using Euclidean tools. Dürer's second book was a technical book on fortifications, while his third was the posthumous *Vier Bücher von menschlicher Proportion (Four Books on the Proportions of Men)* (1528), a treatise on proportion.

Art critics are agreed that Albrecht Dürer created three 'master engravings,' all of which are copperplates. The earliest was 'The Knight, Death and the Devil' (1513). These were followed the next year by 'St. Jerome in his Study' and 'Melencolia I,' which is the object of our interest. This is certainly Dürer's most mysterious work and many interpretations were attempted before Erwin Panofski and Fritz Saxl in *Dürer's 'Melencolia I', eine quellen- und typengeschichtliche Untersuchung (A Source- and Type-History of Dürer's 'Melenchoia I')* (1923) argued that it was a symbolic self-portrait.

Hutton's discussion quoted earlier about the connection between magic square and the planets is just one thing involved in the interpretation of this magnificent etching. To discuss this will take us too far afield, but you can consult Panofsky [29] or Finkelstein [23] to get you started. You will also find lots of junk on the web on this topic.

Our interest in this work is the 4 by 4 magic square that appears in the upper right corner:

16	3	2	13
5	10	11	8
9	6	7	12
4	15	14	1

There are two things to note about this square. The middle two numbers in the bottom row are the date that this work was done 15 14. The outer two, 4 and 1 can be thought of as the fourth and first letter of the Latin alphabet, representing the letters D A, Dürer's monogram. The most important source that Dürer drew upon in composing *Melencolia* was Cornelius Agrippa of Nettesheim's *De Occulta Philosophia* (1509-1510). He was also most likely Dürer's source for the magic square [20, 25]. The magic square in 'Melencolia I' has the additional property that any pair of entries symmetrical to the center add to 17. It is just one of four magic squares with this property and with the numerals 15 and 14 centered on the bottom of the square [18].

The second object in the engraving of obvious mathematical interest is the strange shaped block of stone. Its precise mathematical shape has been the subject of conjecture for more than a century, but of all the papers I have read about it, I regret to say that I am unconvinced about what its true shape is. There are many papers dealing with this solid and I encourage you to consult some of them, e.g., [22, 24, 26, 27, 31, 33, 34, 35].

Cardano's Rings

Perhaps the oldest mathematical recreation is the Chinese Rings puzzle or Cardano's Rings as I shall call them. The puzzle consists of seven or ten or whatever number of rings, each threaded through the eye of a short post that is able to slide up and down in a solid base. Connecting the rings is a long oval or slotted bar that goes through them and over the posts. The goal is to manipulate the rings and free the bar from the rest of the device. The rings, posts, and base are permanently hooked together but there is considerable room for movement as the posts can slide up and down through the base and the rings can slide around the eyes in the posts. It is sure to provide you with several hours of mathematical recreation.

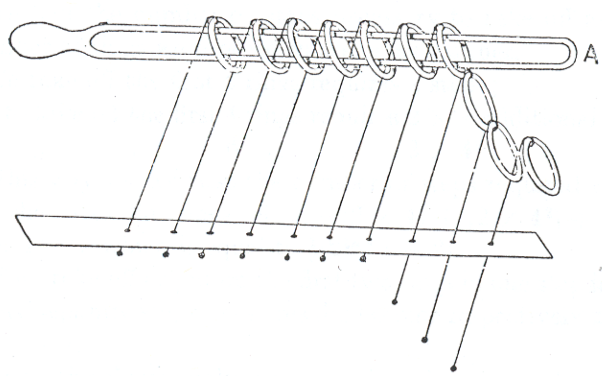

Figure 1. Cardano's Rings

According to the ethnographer Stewart Culin (1858–1929) this puzzle was known in the Chinese Sung Dynasty (960–1279). His *Games of the Orient: Korea, China, Japan* (1895 with another title; reprinted 1959 and 1991 under this title) attributes the puzzle to Hung Ming (181–234), a famous Chinese hero. I am skeptical of this early origin and have been unable to track down a solid modern reference to such an early date. Another puzzle of supposedly ancient Chinese roots is the Tower of Hanoi puzzle. It was published by Édouard Lucas (1842–1891) in his four volume work, *Récréations mathématiques* (1883) under the name 'M. Claus,' an anagram of Lucas. The next year the scientist Henri de Parville (1838–1909) made up the story about the monks of Siam moving 64 gold disks between three diamond needles. The world was to end when the task was completed [36, p. 304]. While that has not yet happened, the world of Lucas ended in an unusual way. At a banquet he was attending, a waiter dropped some dinnerware and Lucas was cut on the cheek with a shard. A few days later he died of septicemia.

The earliest mention of this puzzle in print, where it is called a 'meleda,' is in Girolamo Cardano's *De subtilitate libri XXI* (A Treatise on Subtelty, 1550; book XV, paragraph 2), an encyclopedia of wide range which contains sound subjects ranging from natural philosophy, cosmology, mechanics and cryptography, and the construction of machines to such unsound subjects as alchemy, the occult, and the evil influence of demons. This work was widely read, there being five editions in the remainder of the sixteenth century [39]. Cardano is best known to mathematicians as author of the *Ars Magna* (1545)[38, 41], a book that explains how to solve third and fourth degree polynomial equations.

Recently, however, a manuscript of Luca Pacioli (1445–1517) has turned up that contains a description of the Chinese Rings [42]. This places their European origin around 1500.

After Cardano, the next mention of the puzzle is not in the 1685 English edition of John Wallis's *Algebra* [46], as is often incorrectly said, but in the Latin edition of 1693 [49], where it is described in Caput CXI (pp. 472–478), which is entitled 'De complicatis annulis,' the complicated rings. Wallis gives a description of how to solve the puzzle and it is for this reason that I refer to it as 'Cardano's Rings.'

Ball and Coxeter have a footnote [36, p. 305] indicating that Cardano's Rings are pictured without explanation in the *Récreations Mathématiques* in the 1723 edition (volume 4, p. 439), but I have not seen this edition. There is no mention of the rings in Hutton's English translation of this work. Nonetheless, we include them here because they are part of the Ozanam-Hutton tradition.

One sign of the popularity of this puzzle is that it has been patented at least 34 times in half-a-dozen countries in the twentieth-century, including twenty-one times in the United States [42] and is still sold under a number of names. When traveling to the midwest, my wife and I occasionally stop at Yoder's Shopping Center in Shipshewana, Indiana, an Amish area of the state. While my wife looks at fabric, I look first for material for neckties, and then at the wide variety of tools and household goods that this old-fashioned shop has (if anyone needs a lamp wick, a hand-cranked five-quart ice cream maker, or replacement blades for a meat grinder, this is the place to look: http://www.yodershardware.com). On a trip earlier this summer, I was pleased to be able to purchase a set of Cardano's Rings, or a 'patience puzzle,' as it was called. This is one of the twenty puzzles in the 'Tavern Puzzle Collection' manufactured by the blacksmith Dennis Sucilsky (http://www.tavernpuzzle.com).

I encourage you to find a copy of Cardano's Rings and spend a few dozen hours trying to solve the puzzle. If you become frustrated you can buy one from blacksmith Sucilsky (they come with solutions), consult [37], or read on. But if you want to try to master it yourself, skip the remainder of this section.

W. W. Rouse Ball, in his delightful book on mathematical recreations gives a nice presentation of the solution to this puzzle. First he describes the puzzle and gives the key to its solution:

> It consists of a number of rings hung upon a bar in such a manner that the ring at one end (say A) can be taken off or put on the bar at pleasure; but any other ring can be taken off or put on only when the one next to it towards A is on, and all the rest towards A are off the bar. The order of the rings cannot be changed. [36, p. 305]

Using straightforward counting techniques, Ball develops a recursive relationship for the solution, and shows that if the number n of rings is odd then $(2^{n+1}-1)/3$ steps are needed whereas $(2^{n+1}-2)/3$ are needed if n is even. A 'step' is when a ring is put on or taken off the bar.

Then he gives "another solution, more elegant, though rather artificial." Ball attributes this to L. Gros, *Théorie du Baguenodier (Theory of the Time-Waster)* (1872). What is artificial about this solution is the way binary numerals are assigned to the different arrangements of the rings:

> Denote the rings which are on the bar by the digits 1 or 0 alternately, reckoning from left to right, and denote a ring which is on the bar by the digit assigned to that ring on the bar which is nearest to it on the left of it, or by 0 if there is no ring to the left of it. [36, p. 308].

When the 10 rings of the exemplar in Figure 1 are all on the bar, then the Gros code is 1010101010. When they are all off the bar, the code is 0000000000. In Figure 1, the arrangement is 1010101111. With a little practice it is easy to convert back and forth between the rings and the Gros coding. Now solution of the puzzle is particularly easy with this notation. Start with 1010101010 as the initial position and subtract 1 (using binary subtraction). This is the next position. Now repeatedly subtract 1 until you get to a string of 0s; these Gros codes give the intermediate positions. To see how many moves this will take, subtract the ending position from the starting position and convert to decimal; in this case it takes 682 moves. To put the rings back on the bar, repeatedly add 1. Of course this method works if someone has left your puzzle partway worked and you want to put it back in the original shape; you can even compute how many moves it will take in advance by doing the binary subtraction. Ball was right, the coding is artificial, but the solution is elegant.

There is a second way of coding that also solves the ring puzzle. If a ring is on the bar denote it by 1, if off by 0. This assigns a string of 0s and 1s to every position of the puzzle. Now in solving the puzzle only one ring can move on or off the bar at a time. The codes for those two positions differ in only one digit. This prompts us to define a *Gray code* as an ordering of the 2^n sequences of length n consisting of 0s and 1s in such a way that adjacent sequences differ by just one digit. Such sequences were patented by Frank Gray in 1953 (U.S. patent 2 632 058); hence the name. These Gray codes are never unique and it is still an open question as to how many of them there are for a given n [40].

Now not any Gray sequence will provide a solution to Cardano's Ring puzzle, because some adjacent codes do not represent posssible moves on the puzzle. The only Gray code that will work is the reflected Gray code. Here is how to construct it. Begin with $G_1 = 0, 1$, i.e., the sequence of 0 followed by 1. Then let

$$G_{n+1} = 0G_n \cup 1G'_n$$

i.e., to go from G_n to G_{n+1} concatnenate 1 with each code in G_n and follow that sequence with 0 concatenated with each code in G_n but with the order reversed. Here are the first few codes:

$$
\begin{aligned}
G_1 &= 0, 1 \\
G_2 &= 00, 01, 11, 10 \\
G_3 &= 000, 001, 011, 010, 110, 111, 101, 100
\end{aligned}
$$

Let us now give an example comparing the two codes and how they are used to solve the 4-ring puzzle. Note that although G_4 contains 16 elements, we only use those between 0000 and 1111:

Gray	Gros
1111	1010
1101	1001
1100	1000
0100	0111
0101	0110
0111	0101
0110	0100
0010	0011
0011	0010
0001	0001
0000	0000

Note that with the reflecting Gray code one must write out the code before one can use it to solve the puzzle. This time the coding is elegant, but the solution is artificial.

The line between recreational mathematics and the mathematics of the professional mathematician is never entirely clear. In a conversation about this paper earlier this summer, a former colleague Charles Holland remarked "Isn't all mathematics recreational mathematics?" While I am inclined to agree with him, this puzzle crossed the line between recreational and research by motivating a paper by Przytycki and Sikora that is published in the *Proceedings of the AMS* [43]. The paper uses low-dimensional topology and group theory to prove a conjecture of L. Kauffman that the minimum number of moves to solve Cardano's Rings with n-rings is 2^{n-1}, but they use a different method of counting moves.

Geometrical Problems

The "Part Second" of volume I of Hutton's *Mathematical and Philosophical Recreations* begins with 72 geometrical problems (pp. 262–441). Many of these will be familiar to any student of high school geometry (even today), but others are rather esoteric. Let us illustrate with some examples. We begin with the alpha and omega of these problems and then sample a few more.

> Problem I. For the extremity of a given right line to raise a perpendicular, without continuing the line, and even without changing the opening of the compass if necessary.
>
> Problem LXXII. If each of the sides of any irregular polygon whatever, as $A\ B\ C\ D\ E\ A$, (fig. 122 pl. 14) be divided into two equal parts, as a, b, c, d, e; and if the points

> of division in the contiguous sides be joined; the result will be a new polygon *a b c d e a*: if the same operation be performed on this polygon; then on the one resulting from it; and so on ad infinitum; it is required to find the point where these divisions will terminate.

Hutton comments that Problem LXXII is “impossible to be resolved perhaps by considerations purely geometrical” but that it is “susceptible of a very simple solution, deduced from another consideration.” He leaves this to the second volume [5, vol. 2, pp. 8–9] so that “our readers may exercise their ingenuity upon it.”

One of the nice things that Hutton does do on many problems is to give some indication of their origins. This time he adds that “it was proposed in 1750 by M. D —, who said he had it from M. Buffon.” I have no idea why Hutton hides Monsieur D’s surname, but it seems likely that the Compte de Buffon (1707–1788), who is best known to us via his needle problem, presented it orally in a letter to M. D —.

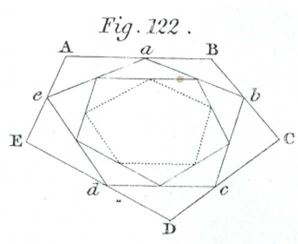

Figure 2. Diagram for Problem LXXII

Problem I is not so well known today and certainly does not show up in common geometry texts, but is has long been of interest to artisans who want a erect a right angle at the end of a line which cannot be extended because of the limited physical circumstances. Hutton provides two solutions. If you can vary the opening of the compass, then he constructs a 3-4-5-triangle at the end of the line segment. If the compass opening is fixed (shades of Abu’l Wafa (940–998)) then he constructs an equilateral triangle at the end of the line segment and then extends one side by a length equal to one of the equal legs of the equilateral triangle. He indicates that “The demonstration of this is so easy, that it requires no illustration.” But then he provides one nonetheless.

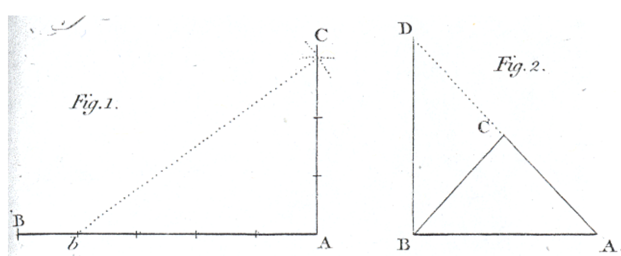

Figure 3. Two solutions to Problem I

Problem XXVII. Of inscribing regular polygons in a circle.

One would hardly call this a problem in recreational mathematics, but it is something that has always interested the learned layman. Hutton describes how to construct the regular triangle, quadrilateral, pentagon, and pentadecagon and is aware that the number of sides can be continually doubled with “rule and compass.” Thus he knows just what Euclid knew two millennia ago.

> The rest, such as the heptagon, ennaegon, endecagon, &c, cannot be described by means of the rule and compasses alone, without trial; and all those who have attempted this method, have failed, or have produced ridiculous paralogisms. [5, p. 304]

This sounds like an empirical statement, not one informed by the work of Gauss in his *Disquisitiones arithmeticae* (1801) which was just published two years before. Although Gauss's construction of the regular 17-gon spread quickly, what is written here indicates that Hutton did not find out immediately, for this was something that he surely would have mentioned. Sadly, the text in the 1851 edition is word for word the same [6].

> Problem XXXV. On the form in which the bees construct their combs.

> Problem XLVI. To make the same body pass through a square hole, a round hole, and an elliptical hole.

The penultimate problem is a classic, although it seems to have disappeared from the current literature:

> Problem LXXI. In the island of Delos, a temple consecrated to Geometry was erected, on a circular basis, (fig. 118 pl. 14), and covered by a hemispherical dome, having four windows in its circumference, with a circular aperture at the top, so combined, that the remainder of the hemispherical surface of the dome was equal to a rectilineal figure; and in the cylindric part of the temple was a door, absolutely squarable or equal to a rectilineal space. What geometrical means did the architect employ in the construction of this monument?

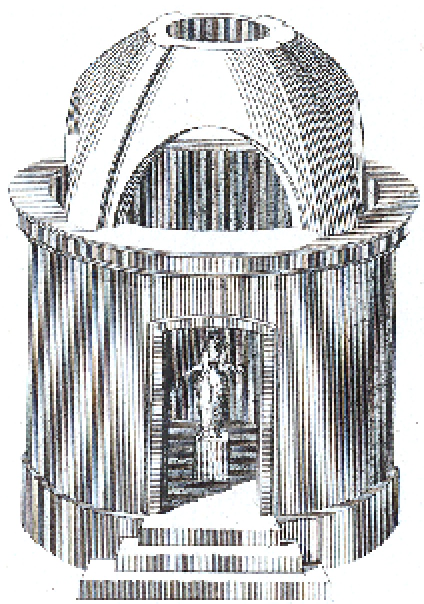

Figure 4. Viviani's Temple

This was an important challenge to Leibniz and the other devotees of the new calculus, for its poser, Vincenzo Viviani (1622–1703), believed that the calculus was "nothing but a kind of game that could only solve its own problems." Of course Leibniz solved the problem promptly [17]. This problem became more important later for it was one of the things that motivated Euler to develop surface integrals.

Prince Rupert's Cube

We end our discussion of the first volume of Hutton's *Recreations* by discussing the final problem that is mentioned in the title of this paper:

> Problem XXX. To cut a hole in a cube, through which another cube of the same size shall be able to pass.

When one first hears this problem, one is incredulous. This cannot be done. But it can and that is what gives the problem its appeal. The Poles have an even more amazing way of stating this problem:

> We have two cubes, a smaller one with an edge of 30 in., and a bigger one with an edge of 31 in. Is it possible to bore a hole through the smaller cube so that the bigger one can slide through it? [55, p. 77]

The poser of this puzzle was Prince Rupert of Palantine (1619–1682), but to understand who he was we need to review our (limited) knowledge of the kings of seventeenth-century England. King Charles I of the House of Stewart was born in 1600 and became King in 1625. He married Henrietta Marie, the daughter of King Henry IV of France, and a staunch Catholic. They produced nine children, four boys and five girls. Charles I inherited financial problems from his father, James I, and fought with Parliament over money. His wife was a meddler and economic and religious issues, partly of her making, led to Civil War between the backers of the king, the Cavalier's, and the supporters of Parliament, the Roundheads. The Roundheads were destined to win due to their superior numbers and finances, and indeed they did win. This led to the execution of Charles I in 1649. After the fall of the Parliamentarians, the second son became King Charles II, reigning from 1660 to 1685 (the first son lived less than a day). Charles II is important in the history of mathematics as a supporter of science and founder of the Royal Society of London. The third son became King James II who reigned from 1685 to 1688 was the last Catholic monarch of England. One daughter, Elizabeth Stuart, married the Winter King Frederick V of the Palatinate and a prime mover in the start of the Thirty Years' War (which lasted from 1618 to 1648).

Elizabeth and Frederick had three sons and one daughter that survived to adulthood. The daughter, Sophia of Hanover married Ernst August, Duke of Brunswich-Lüneburg and an employer of Leibniz. Their son became George I of Great Britain, reigning 1714–1727. The first son became Elector Palatine after the Peace of Westphalia in 1648 at the conclusion of the Thirty Years' War. The other two sons were Rupert (1619–1652) and Maurice (1620-1652) both of whom served in the English Civil War.

The year after Rupert was born the family was forced into exile and so he grew up in Holland. His father died when he was 12 and the next year he became a soldier. Two years later he traveled to London, where he became a favorite of his uncle King Charles I. Later, during an invasion of Westphalia he was captured and held prisoner for three years. During this time he was allowed to keep his pet poodle Boy; he taught him many tricks, including jumping in the air whenever he heard the words "King Charles I." But he also spent his time as a prisoner profitably studying military manuals. When the English Civil War broke out, he joined the cause and was appointed General of Horse. He practically invented the cavalry charge, but was unable to control his troops after they routed the enemy, so his victories were not as successful as they should have been. Nonetheless, he became the most brilliant, the most dashing, and the most successful of the King's generals. Eventually though, he realized the war was lost and so advised his uncle. This led to his being relieved of command [61].

He settled in France and Germany studying warfare, chemistry, and even art. Mezzotint is a method of engraving which dates from the mid-seventeenth century. A plate is first laboriously worked over completely with a tool with a serrated edge, called a rocker. This brings up a uniform burr on the plate and the artist scrapes away some of the roughened surface to create lighter areas. The advantage of this method, over that of a woodcut, is that subtle shades of gray can be produced. The technique was invented by Ludwig von Seigen who produced his first plate, just one of seven, in 1642. Prince Rupert met Seigen in 1654 and learned his methods but did not take up mezzotint until 1657 with his *Head of Titan*. It was Rupert who produced the first masterpiece of the genre, *The Great Executioner* (1658). He also introduced the technique into England in 1660 where it was used primarily to reproduce paintings. Collecting these was popular in from about 1870 to 1929 when the stock market crash killed the trade. Rupert is also credited with the invention of the hand rocker. The work of preparing the plate is time consuming and boring, so artists hired out the task. Doing this days on end caused some of these folks to go mad, hence the phrase 'off his rocker.'

It was immediately after his return to England after the Restoration that Prince Rupert enters the history of mathematics. We have already posed the Prince Rupert's question about the cube, now we look at the solution as told by Ozanam/Hutton:

> If we conceive a cube raised on one of its angles, in such a manner, that the diagonal passing through that angle shall be perpendicular to the plain which it touches; and if we suppose a perpendicular let fall on that plane from each of the elevated angles, the projection thence resulting will be a regular hexagon, each side and each radius of which may be found by the following manner.
>
> On the vertical line AB (fig. 53 pl. 7), equal to the diagonal of the cube, or the sphere of which is triple to that of the cube, describe a semicircle, and make AC equal to the side of the cube, and AD equal to the diagonal of one of its faces; if from the point C there be let fall, on the horizontal tangent of the circle in B, the perpendicular CE, passing through the point D, BE will be the side and the radius of the required hexagon a b c d fig. 54.

If one takes a cube in ones hand, holding opposite vertices between ones thumb A and index finger B (using the notation in Hutton's diagram), then one can turn the cube around this axis between thumb and forefinger. If one turns the cube so that one of the upper vertices is at C, then you can 'see' the line CE. If the cube has edge length 1, it's diagonal AB has length $\sqrt{3}$; then since the triangle ACB is inscribed in a triangle, we have $CB = \sqrt{2}$. Now if one rotates the cube by $\pi/6$ one of the lower vertices will be at D, a point directly below where the vertex at C was. Incidentally, Hugo Steinhaus points out in his delightful *Mathematical Snapshots* that a cube rotating this way generates two cones connected by a hyperboloid of revolution.

> When this operation is finished, describe on this hexagonal projection, and around the same centre, the square which forms the projection of the given cube placed on one of its bases, so that one of its sides shall be parallel, and the other perpendicular to the diameter ac: it may be demonstrated, that this square can be contained within the hexagon, in such a manner, as not to touch with its angles and of the sides: a square hole therefore, equal to one of the bases of the cube, may be made in it, in a direction parallel to one of its diagonals, without destroying the continuity of any side; and consequently another cube of equal size may pass through it, provided it be made to move in the direction of the diagonal of the former.

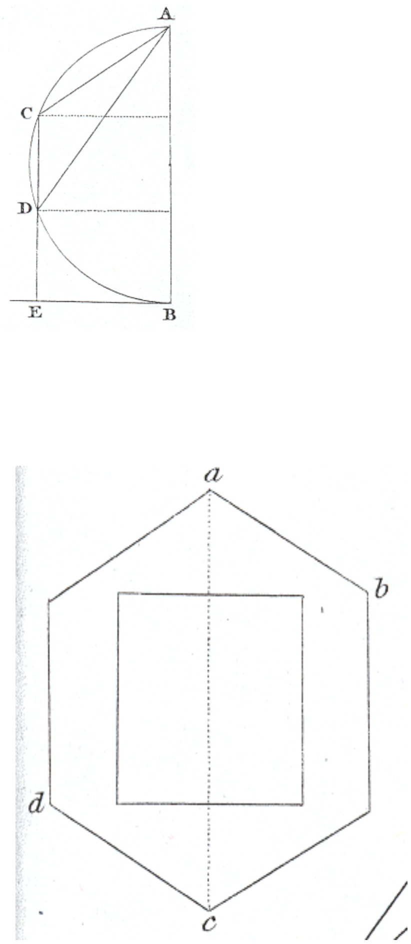

Figure 5. Hutton's figures for Rupert's Cube

Note that while the claim is made that "it may be demonstrated" he does not do it. The first to do so, to my knowledge, was John Wallis in chapter 109 of his Latin *Algebra*, "Perforatio cubi, alterum ipsi aequalem recipiens" [49, II, pp. 470–471]. He made a mistake in his computations and even tried to correct himself in a paper in the *Philosopical Transactions* [50]. But he still did not have it right, as was pointed out by the historian Christoph Scriba in 1968 [60]. Scriba is puzzled by the whole thing, for the computation is not that hard. You should be able to prove by elementary geometry that the largest square that can be inscribed in the hexagon of side length $\sqrt{2}/\sqrt{3}$ has side length $\sqrt{6} - \sqrt{2} = 1.03527$ (where the original cube has side length 1). Thus a cube can be passed through a cube of the same size.

As always seems to be the case in mathematics, there is more to the story. The solution given above makes an assumption about the problem, an unwarranted assumption. Perhaps you would like to pause a moment and think about what it is.

As posed the question just asks about passing one cube through another of the same size. No assumption is made about how that is to be done. But Wallis makes one. He assumes that the hole be cut parallel to a main diagonal of one cube. What if we cut at an angle slightly oblique to this diagonal? This problem was attacked by Pieter Nieuwland (1764–1794) who was appointed professor at Leiden University in 1793, but died a year later. His teacher, J. H. van Swinden, found Nieuwland's solution among his papers and published it in the second edition of his geometry book, *Grondbeginsels der Meetskunde* (1816). This work was highly valued in Germany and translated and enlarged by C. F. A. Jacobi, *Elemente der Geometrie*; this is not the famous Carl Gustav Jakob Jacobi (1804–1851). Nieuwland's solution shows that a cube of edge $3\sqrt{2}/4 = 1.06066172$ can be passed through a cube with edge length 1. For details see Schrek [59]. This problem also occurs in the Montucla edition of Ozanam [4], but I have not verified that. A generalization of this problem is found in [57].

We turn finally to a very interesting object that bears Prince Rupert's name, even if it is not connected with mathematics.

> Problem XXXVIII. Of Prince Rupert's Drops, or Batavian Tears.
>
> This appellation is given to a sort of glass drops terminating in a long tail, which posess a very singular property; for if you give one of them a pretty smart blow on the belly, it opposes a considerable resistance; but if the smallest bit be broken off from the tail, it immediately bursts into a thousand pieces, and it is reduced to dust.
>
> These drops are made by letting glass, in a state of fusion, fall drop by drop into a vessel filled with water. They are then found at the bottom completely formed. A great number of them however generally burst in the water, or immediately after they have been take from it. As these drops were first made in Holland, they are called by the French *Larmes Bataviques*. [5, Vol. 4, pp. 144–145]

When Prince Rupert returned to England, he showed some of these tear-drop shaped pieces of glass to his cousin King Charles II and on March 4, 1660–61, the King sent "five little glass bubbles, two with liquor in them, and the other three solid" to the Royal Society as Thomas Birch wrote in his *History* (1756–57). The philosophers, as scientists were then called, did several experiments on them but were puzzled by the phenomenon. Robert Hooke pictured one in his great *Macrographia* (1665) and gave a good explanation of their behavior: When the drop is quickly cooled the outer layers harden first, forming a rigid exterior. The still hot interior cannot change its volume and so usually produces the 'liquor' mentioned above, which are really air bubbles. The compression of the outer layers is balanced against the high tensile stress in the interior. Thus the head is very strong, but once the outer layer is penetrated the tear bursts into tiny needles [53]. For a picture of several Prince Rupert's Drops, and especially one shattering, see [56]. After describing several experiments with the drops Hutton admits that

> Philosophers have always been much embarrassed respecting the cause of this extraordinary phenomena; and it must indeed be confessed that it is still very obscure. . . . [This problem is left] to the sagacity and researches of our readers. [5, Vol. 4, pp. 145-146]

Indeed this problem remained unsettled until the twentieth century when the concept of tempered glass was understood. These strange drops led eventually to safe eyeglasses, basketball backboards, car windows that shatter instead of breaking into lethal shards, and windows for the space shuttle that can withstand high velocity impacts. To learn more, I encourage you to pay a visit to the Corning Museum of Glass in Corning, New York, where you can pound on the head of a Prince Rupert drop yourself. On a recent visit there I met glass artist B. Brian (http://greentreeglass.com) and he agreed to make some Prince Rupert's drops for me. I can assure you that the phenomenon is truly spectacular!

References

Editions of Ozanam's Récréations

[1] **Ozanam 1694:** *Récréations mathématiques et physiques, qui contiennent plusiers problémes utiles & agreables, d'arithmetique, de geometrie, d'optique, de gnomonique, de cosmographie, de mecanique, de pyrotechnie, & de physique. Avec un traité nouveau des horloges elementaires.* Paris: Jean Jombert. 3 parts in 2 volumes, octavo.

The section on horology is a translation of *Horologie elementari* by Domenico Martinelli (1650–1718); it has separate title-page, pagination, and signatures. In the 1696 edition this is vol. 2, pp. [469]–583.

Reissued Paris 1696, 1697, 1698. Reissued Amsterdam 1696, 1698.

[2] **Ozanam 1708:** *Recreations Mathematical and Physical; laying down, and solving many profitable and delightful problems of arithmetick, geometry, opticks, gnomonicks, consmography, mechanicks, physicks, and pyrotechny*, London: R. Bonwick [etc.], 1708. [38], 530 p. illus., plates, diagrs. 20 cm.

Reissued 1756, 1759 (with a new title), 1790. Cornell University Library has the 1759 and 1790 in electronic form but a password is needed.

[3] **Ozanam-Grandin 1723** *Récréations mathématiques et physiques, qui contiennent plusieurs problêmes d'arithmétique, de géométrie, de musique, d'optique, de gnomonique, de cosmographie, de méchanique, de pyrotechnie, & de physique. Avec un traité de horologes elementaires. Par feu M. Ozanam . . . Nouvelle edition, revûe, corrigée & augmentée. A Paris, Chez Claude Jombert, rue S. Jacques, au coin de la rue des Mathurins, à l'image Notre Dame. M.DCC.XXIII. Avec privilege du roy.* Octavo, 4 volumes, 136 plates (partly folding).

This is the first four volume edition. "The editor is said to be Grandin" [NUC].

Reissued 1725, 1735, 1737, 1741, 1750/1749, 1770.

In his "Chronology of Recreational Mathematics" (check the web for the latest edition), *the* master of the history of recreational mathematics, David Singmaster, notes "First appearance of many topological problems: Scissors on String; People joined by Ropes at Wrists; Cherries Puzzle; Solomon's Seal. First mention of the Knight's Tours outside the chess literature. First orthogonal Latin Squares. First Cutting a Card so One can Pass Through It." Alas, Ozanam-Montucla 1778 dropped the topological problems.

[4] **Ozanam-Montucla 1778:** *Récréations mathématiques et physiques, qui contiennent les problémes et les questions les plus remarquables, et les plus propres à piquer la curiosité, tant des mathématiques que de la physique; le tout traité d'une maniere à la portée des lecteurs qui ont seulement quelques connoissances légeres de ces sciences par feu M. Ozanam. Nouvelle édition, totalement refondue et considérablement augmentée par M. de C. g. f.* A Paris, rue Dauphine, Chez Cl. Ant. Jombert, fils aîné, libraire du roi pour le génie & l'artillerie. M.DCC. LXXVIII. Avec approbation, et privlidge du roi. 4 volumes, quarto, 89 folding plates.

Reissued 1790. Ozanam-Hutton is a translation of this edition.

[5] **Ozanam-Hutton 1803:** *Recreations in Mathematics and Natural Philosophy: Containing Amusing Dissertations and Enqueries Concerning a Variety of Subjects the Most Remarkable and Proper to Excite Curiosity and Attention to the Whole Range of the Mathematical and Philosophical Sciences: The Whole treated in a pleasing and easy Manner, and adapted to the

Comprehension of all who are the least initiated in those Sciences: viz. Arithmetic, Geometry, Trigonometry, Mechanics, Optics, Acoustics, Music, Astronomy, Geography, Chronology, Dialling, Navigation, Architecture, Pyrotechny, Pneumatics, Hydrostatics, Hydraulics, Magnetism, Electricity, Chemistory, Palingensy, &c. First composed by M. Ozanam, of the Royal Academy of Sciences, &c. Lately recomposed, and greatly enlarged, in a new Edition, by the celebrated M. Montucla. And now translated into English, and improved with many Additions and Observations by Charles Hutton LL.D. and F.R.S. and Professor of Mathematics in the Royal Military Academy, Woolwich. In Four Volumes; with near one hundred quarto plates. London: Printed for G. Kearsley, Fleet-Street, by T. Davison, White-Friars. 1803.

This is the edition cited in this paper.

Reissued 1814.

[6] **Ozanam-Riddle 1840:** *Recreations in mathematics and natural philosophy translated from Montucla's ed. of Ozanam by Charles Hutton; and illustrated with upwards of four hundred woodcuts by Edward Riddle.* London: Thomas Tegg, 1840. New and revised edition.

Reissued 1844, 1851 (with a new title), 1854, 1856. The 1844 edition is available on line from the Cornell University Library Historical Mathematics Monographs: http://historical.library.cornell.edu/cgi-bin/cul.math/docviewer?did=07160001&seq=7 . It is this edition I quote. The entire collection of 512 volumes can be accessed at http://historical.library.cornell.edu/math/

NOTA BENE: The above bibliography of editions of the Ozanam-Grandin-Montucla-Hutton-Riddle text is very preliminary. Only a few of the editions listed have been examined and it will take considerable work — including the physical examination of the texts — to sort out the printing history of this work, for items listed as reissues contain considerable differences in publishers, pagination, problems, etc.

Charles Hutton

[7] Alexanderson, Gerald L., and Klosinski, Leonard F., "Mathematicians and old books," *The Mathematical Intelligencer*, vol. 27, no. 2 (Spring 2005), pp. 70–79.

Discusses their penchant for collecting first editions of famous mathematics book. Prices are quoted for the Diophantus and Ratdolt Euclid. There are illustrations of many works.

[8] Anderson, R. E. "Charles Hutton," *Dictionary of National Biography*, XXVIII, 351–355.

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V. Frederick Rickey
Department of Mathematical Sciences
United States Military Academy
West Point, NY 10996
fred-rickey@usma.edu (if this fails, try fred.rickey@usma.edu).