

HZ-ALGEBRA SPECTRA ARE DIFFERENTIAL GRADED ALGEBRAS

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Abstract: We show that the homotopy theory of differential graded algebras coincides with the homotopy theory of *HZ*-algebra spectra. Namely, we construct Quillen equivalences between the Quillen model categories of (unbounded) differential graded algebras and *HZ*-algebra spectra. We also construct Quillen equivalences between the differential graded modules and module spectra over these algebras. We use these equivalences in turn to produce algebraic models for rational stable model categories. We show that basically any rational stable model category is Quillen equivalent to modules over a differential graded \mathbb{Q} -algebra (with many objects).

1. INTRODUCTION

In this paper we show that the algebra of spectra arising in stable homotopy theory encompasses homological algebra. Or, from another perspective, this shows that the algebra of spectra is a generalization of homological algebra. In the algebra of spectra, spectra take the place of abelian groups and the analogue of the tensor product is a *symmetric monoidal smash product*. Using this smash product, the definitions of module spectra, associative and commutative ring spectra, and algebra spectra are easy categorical exercises. The main difference here is that in the algebra of spectra the sphere spectrum, \mathbb{S} , which encodes all of the information about the stable homotopy groups of spheres, takes the place of the integers, the free abelian group on one generator.

Originally, ring spectra (or A_∞ -ring spectra) were built from topological spaces with ring structures which are associative up to coherent homotopies. Since a discrete ring is strictly associative, rings are examples of ring spectra; associated to any ring R is the Eilenberg-Mac Lane ring spectrum HR . One good expository introduction to spectra and ring spectra appears in [Gr06]. As in classical algebra, to understand ring spectra we study their modules. Since we are considering rings up to homotopy though, we focus on the type of homological invariants and derived functors best captured by the derived category rather than the category of modules itself. Robinson, in [Rob87], defined a notion of A_∞ -modules over an Eilenberg-Mac Lane spectrum HR and a notion of homotopy. He then showed that up to homotopy this category of A_∞ -modules is equivalent to $\mathcal{D}(R)$, the unbounded derived category of R . The complicated notions of algebraic structures up to coherent homotopies prevented anyone from generalizing this result to algebras or even defining a notion of A_∞ - HR -algebra spectra.

In the modern settings of spectra referred to in the first paragraph, defining algebra spectra is simple. Here the sphere spectrum is the initial ring and spectra are \mathbb{S} -modules, the modules over the sphere spectrum \mathbb{S} . Ring spectra are then the (strictly associative) \mathbb{S} -algebras. For a commutative \mathbb{S} -algebra A , A -algebra spectra are the monoids in the category of A -modules. Given a discrete ring R , one can construct HR as an \mathbb{S} -algebra which is commutative if R is [HSS, 1.2.5]. Since Robinson showed that HR -module spectra agree with differential graded R -modules “up to homotopy,” one might guess that HR -algebra spectra should capture the same

Date: June 21, 2006; 2000 AMS Math. Subj. Class.: 55P43, 18G35, 55P62, 18D10.

Research partially supported by NSF Grants No. 0417206 and No. 0134938 and a Sloan Research Fellowship.

“up to homotopy” information as differential graded R -algebras. Verifying this conjecture is the subject of this paper.

To formulate this statement rigorously we use Quillen model category structures. One may define the derived category of a ring R as the category of dg R -modules with quasi-isomorphisms formally inverted. The problem with this formal process of inversion is that one loses control of the morphisms; the morphisms between two objects may not form a set. Quillen model structures give a way around this problem. A *Quillen model structure* on a category \mathcal{C} is a choice of three subcategories called weak equivalences, cofibrations and fibrations which satisfy certain axioms [Qui67]. [DS95] is a good introduction to model categories; our standard reference though is [Hov99]. Given a Quillen model structure, inverting the weak equivalences, W , produces a well-defined homotopy category $\mathrm{Ho}(\mathcal{C}) = \mathcal{C}[W^{-1}]$. The category of dg R -modules, $R\text{-Mod}$, forms a model category and one can show that $\mathrm{Ho}(R\text{-Mod})$ is equivalent to $\mathcal{D}(R)$. A *Quillen adjoint pair* (or Quillen map) between two Quillen model categories \mathcal{C} and \mathcal{D} is an adjoint pair of functors $F: \mathcal{C} \rightleftarrows \mathcal{D}: G$ which preserves the model structures and thus induces adjoint derived functors on the homotopy categories $LF: \mathrm{Ho}(\mathcal{C}) \rightleftarrows \mathrm{Ho}(\mathcal{D}): RG$. A *Quillen equivalence* is a Quillen map such that the derived functors induce an equivalence on the homotopy categories. Two model categories are *Quillen equivalent* if they can be connected by a zig-zag of Quillen equivalences. Quillen equivalent model categories represent the same ‘homotopy theory.’ A subtle but important point is that a Quillen equivalence is stronger than just having an equivalence of the associated homotopy categories; see examples at [SS03a, 3.2.1] or [Shi, 4.5].

In [SS03a, 5.1.6], we strengthened Robinson’s result mentioned above by showing that the model category of HR -module spectra is Quillen equivalent to $R\text{-Mod}$. It follows that $\mathrm{Ho}(HR\text{-Mod})$, $\mathrm{Ho}(R\text{-Mod})$ and $\mathcal{D}(R)$ are all equivalent. Connective versions of these results for modules also appeared in [Sch00, 4.4, 4.5]; see also [SS03b, 1.1]. If R is a commutative ring, $R\text{-Mod}$ and $HR\text{-Mod}$ both have symmetric monoidal products, but the Quillen maps used in [SS03a] do not respect these products and hence do not induce functors on the associated categories of algebras. Another approach is thus needed if one wants to compare the homotopy theory of HR -algebra spectra, $HR\text{-Alg}$, and dg R -algebras, DGA_R . The main result of this paper is then the construction of a new zig-zag of Quillen equivalences between $R\text{-Mod}$ and $HR\text{-Mod}$ which preserve the relevant product structures and induce Quillen equivalences between $HR\text{-Alg}$ and DGA_R .

Theorem 1.1. *For any discrete commutative ring R , the model categories of (unbounded) differential graded R -algebras and HR -algebra spectra are Quillen equivalent. The associated composite derived functors are denoted $\mathbb{H}: DGA_R \rightarrow HR\text{-Alg}$ and $\Theta: HR\text{-Alg} \rightarrow DGA_R$.*

The proof of this theorem appears in Section 2. We also construct Quillen equivalences between the categories of modules over these algebras; see Corollary 2.15. We then use these Quillen equivalences to construct algebraic models for *rational stable model categories*; see Corollary 2.16.

In [Man03], an analogue of Theorem 1.1 is proved for E_∞ -algebras, the commutative analogue of A_∞ -algebras which are associative and commutative up to coherent homotopies. Mandell shows that the homotopy category of E_∞ - HR algebras is equivalent to the homotopy category of dg E_∞ - R -algebras [Man03, 7.11]. He also claims these equivalences can be lifted to Quillen equivalences using the techniques of [SS03b]. The model categories of E_∞ - HR algebras and commutative HR -algebras are shown to be Quillen equivalent in [Man03, 7.2]. If R is not an algebra over the rationals though, the category of commutative dg R -algebras does not even have a well behaved homotopy theory; that is, there is no model category on commutative dg R -algebras with weak equivalences and fibrations determined on the underlying dg R -modules (the quasi-isomorphisms and surjections).

In general then, the Quillen equivalence in Theorem 1.1 cannot be extended to categories of commutative algebras. Rationally though, it should hold. We do not consider this extension here though because it would require different techniques. The following related statement, however, is simple to prove here and is used in the construction of an algebraic model of rational T^n -equivariant spectra in [GS]. Its proof appears near the end of Section 4. Recall Θ from Theorem 1.1.

Theorem 1.2. *For \mathcal{C} any commutative $H\mathbb{Q}$ -algebra, $\Theta\mathcal{C}$ is naturally weakly equivalent to a commutative differential graded \mathbb{Q} -algebra.*

Acknowledgments: I would like to thank Stefan Schwede for many fruitful conversations on this material. In fact, this project grew out of a sequence of projects with him. In particular, Corollary 2.16 is really unfinished business from [SS03a] and even appeared in various preprint versions. I would also like to thank Dan Dugger and Mike Mandell for many helpful conversations. I would like to thank John Greenlees, David Gepner and the referee for their careful reading of an earlier version of this paper.

2. \mathbb{Z} -GRADED CHAIN COMPLEXES AND EILENBERG-MAC LANE SPECTRA

Throughout this paper let R be a discrete commutative ring. In this section we define the functors involved in comparing differential graded R -algebras (DGAs) and HR -algebra spectra. As mentioned in the introduction, the functors used in previous comparisons of dg R -modules and HR -module spectra do not induce functors on the associated categories of algebras because they do not respect the product structures on these module categories. The main input for showing the categories of algebras are Quillen equivalent is then a new comparison of the module categories via functors which do respect the product structures. Namely, we construct a zig-zag of Quillen maps between $R\text{-Mod}$ and $HR\text{-Mod}$ in which the right adjoints are lax monoidal functors. More precisely, we show that each of the adjoint pairs in this zig-zag are *weak monoidal Quillen equivalences* in the sense of [SS03b]. The results of [SS03b] then show that they induce the required Quillen equivalences of categories of algebras and modules.

Here a *monoid* (or algebra) is an object R together with a “multiplication” map $R \otimes R \xrightarrow{\mu} R$ and a “unit” $\mathbb{1} \xrightarrow{\eta} R$ which satisfy certain associativity and unit conditions (see [MacL, VII.3]). If R is a monoid, a *left R -module* (“object with left R -action” in [MacL, VII.4]) is an object N together with an action map $R \otimes N \xrightarrow{\alpha} N$ satisfying associativity and unit conditions (see again [MacL, VII.4]).

We begin this section by briefly recalling the necessary definitions and results from [SS03b]. After this we define the categories and functors involved in the zig-zag of Quillen maps between $R\text{-Mod}$ and $HR\text{-Mod}$. We defer the verification that these functors are weak monoidal Quillen pairs to Section 4. We then prove our main results about the Quillen equivalence of categories of algebras and modules arising in spectral algebra and homological algebra. We end this section by using these equivalences to construct algebraic models for rational stable model categories.

2.1. Weak monoidal Quillen pairs. Although our main focus here is describing conditions under which a Quillen map induces a Quillen equivalence on the associated categories of algebras and modules, we first need to consider conditions on the categories themselves. Namely, the categories of algebras and modules must have model structures created from the model structure on the underlying category. In [SS00, 4.1] it is shown that if every object in \mathcal{C} is small and \mathcal{C} is a cofibrantly generated, monoidal model category which satisfies the monoid axiom, then the categories of monoids, modules and algebras over \mathcal{C} have model structures created by the forgetful functor to \mathcal{C} , that is, with the weak equivalences and fibrations determined on \mathcal{C} . We now recall

the definitions of a monoidal model category and the monoid axiom. We refer to [SS00, 2.2] or [Hov99, 2.1.17] for the definition of cofibrantly generated model categories. In Section 3, we verify these conditions for the categories considered in this paper.

The requirement that \mathcal{C} is a *monoidal model category* ensures that \mathcal{C} is a monoidal category where the monoidal product is compatible with the model structure. In general, this would also require a compatibility condition for the unit [Hov99, 4.2.6 (2)], but this condition is not necessary when the unit is cofibrant which is the case for all of the categories considered in this paper.

Definition 2.1. [SS00, 3.1] A model category \mathcal{C} is a *monoidal model category* if it has a closed symmetric monoidal structure with product \otimes and cofibrant unit object $\mathbb{I}_{\mathcal{C}}$ and satisfies the following axiom.

Pushout product axiom. Let $f: A \rightarrow B$ and $g: K \rightarrow L$ be cofibrations in \mathcal{C} . Then the map

$$f \square g: A \otimes L \coprod_{A \otimes K} B \otimes K \rightarrow B \otimes L$$

is also a cofibration. If in addition one of the former maps is a weak equivalence, so is the latter map.

The *monoid axiom* is the extra condition used in [SS00] to extend a model structure on a monoidal model category to model structures on the associated categories of algebras and modules.

Definition 2.2. [SS00, 3.3] A monoidal model category \mathcal{C} satisfies the *monoid axiom* if any map obtained by pushouts and (possibly transfinite) compositions from maps of the form $f \wedge \text{Id}_Z: A \wedge Z \rightarrow B \wedge Z$ for f a trivial cofibration and Z any object in \mathcal{C} is a weak equivalence.

Now we return to the conditions on a Quillen map necessary for inducing a Quillen equivalence on the associated categories of algebras and modules. The main required property is that the right adjoint is lax monoidal and induces a Quillen equivalence on the underlying categories. Additional homotopy properties on the units and the left adjoint are also required though.

Definition 2.3. [SS03b] A Quillen adjoint pair (λ, R) between monoidal model categories (\mathcal{C}, \otimes) and (\mathcal{D}, \wedge) with cofibrant unit objects, $\mathbb{I}_{\mathcal{C}}$ and $\mathbb{I}_{\mathcal{D}}$, is a *weak monoidal Quillen pair* if the right adjoint R is lax monoidal [MMSS, 20.1], $\tilde{\nu}: \lambda(\mathbb{I}_{\mathcal{D}}) \rightarrow \mathbb{I}_{\mathcal{C}}$ is a weak equivalence and $\tilde{\varphi}: \lambda(A \wedge B) \rightarrow \lambda A \otimes \lambda B$ is a weak equivalence for all cofibrant objects A and B in \mathcal{D} . See [SS03b, 3.6] for the definition of $\tilde{\nu}$ and $\tilde{\varphi}$, the co-monoidal structure on λ adjoint to the monoidal structure on R . A *strong monoidal Quillen pair* is a weak pair for which $\tilde{\nu}$ and $\tilde{\varphi}$ are isomorphisms; for example, this holds when λ is strong monoidal [MMSS, 20.1]. When (λ, R) is a Quillen equivalence, we change these names accordingly.

In [SS03b, 3.3] it is shown that if $\lambda: \mathcal{D} \rightleftarrows \mathcal{C}: R$ is a weak monoidal Quillen pair then the monoid valued lift $R: \mathcal{C}\text{-Monoids} \rightarrow \mathcal{D}\text{-Monoids}$ has a left adjoint $L^{\text{mon}}: \mathcal{D}\text{-Monoids} \rightarrow \mathcal{C}\text{-Monoids}$. L^{mon} does not agree with λ on underlying objects in general unless (λ, R) is a strong monoidal Quillen pair. In the strong monoidal case we usually abuse notation and denote the induced functors on monoids as λ and R again.

Standing assumptions 2.4. In the following theorem, assume that in \mathcal{C} and \mathcal{D} every object is small and \mathcal{C} and \mathcal{D} are cofibrantly generated, monoidal model categories which satisfy the monoid axiom. Assume further that the unit objects in \mathcal{C} and \mathcal{D} are cofibrant. We will refer to these as our *standing assumptions*. It follows by [SS00, 4.1] that model structures on monoids in \mathcal{C} and \mathcal{D} are created by the forgetful functors.

Theorem 2.5. [SS03b, 3.12] *Let $\lambda: \mathcal{D} \longrightarrow \mathcal{C}: R$ be a weak monoidal Quillen equivalence. If \mathcal{C} and \mathcal{D} satisfy the standing assumptions listed above, then the adjoint functor pair*

$$L^{mon}: \mathcal{D}\text{-Monoids} \rightleftarrows \mathcal{C}\text{-Monoids}: R$$

is a Quillen equivalence between the respective model categories of monoids. If R is the right adjoint of a strong monoidal Quillen equivalence, then L^{mon} agrees with λ on underlying objects.

Remark 2.6. Unfortunately, we have not found a single weak monoidal Quillen equivalence between HR -module spectra and differential graded R -modules. Instead, we construct a three step zig-zag of weak monoidal Quillen equivalences; see the subsection below. For strong monoidal Quillen pairs, the functors induce adjoint pairs on the associated categories of modules and algebras and it is relatively easy to show these are Quillen equivalences, following [MMSS, Section 16] for example. Two of our three steps will be via strong monoidal Quillen equivalences. For the middle step though, the argument is a bit more involved. Thus, it turns out to be simplest for all three steps to use the general framework of weak monoidal Quillen equivalences [SS03b, 3.12] for extending Quillen equivalences to categories of algebras and modules.

2.2. The zig-zag of functors. Here we describe the categories and functors involved in the zig-zag of Quillen maps between $HR\text{-Mod}$ and $R\text{-Mod}$. In Section 3 we show that the Standing assumptions 2.4 apply to these categories and in Section 4 we show that each of the adjoint functor pairs is a weak monoidal Quillen equivalence. For simplicity we concentrate on $R = \mathbb{Z}$ but in every case \mathbb{Z} could be replaced by any discrete commutative ring. We denote the category $\mathbb{Z}\text{-Mod}$ by $\mathcal{C}h$. First we describe the two intermediate categories which are both analogues of symmetric spectra as considered in [Hov01, Section 7]. We recall the basic definitions here. Let \mathcal{C} be a closed symmetric monoidal category with monoidal product \otimes and unit $\mathbb{I}_{\mathcal{C}}$.

Definition 2.7. Let Σ be the category with objects the sets $\bar{n} = \{1, 2, \dots, n\}$ for $n \geq 0$, (note that $\bar{0}$ is the empty set) and morphisms the isomorphisms. The category of *symmetric sequences in \mathcal{C}* is the functor category \mathcal{C}^{Σ} . \mathcal{C}^{Σ} is a symmetric monoidal category; the monoidal product is defined by

$$(X \otimes Y)_n = \bigvee_{p+q=n} (\Sigma_n)_+ \wedge_{\Sigma_p \times \Sigma_q} (X_p \otimes Y_q).$$

Given an object K in \mathcal{C} , define $\text{Sym}(K)$ as the symmetric sequence $(\mathbb{I}_{\mathcal{C}}, K, K \otimes K, \dots, K^{\otimes n}, \dots)$ where the symmetric group Σ_n acts on $K^{\otimes n}$ by permutation. $\text{Sym}(K)$ is a commutative monoid in \mathcal{C}^{Σ} ; we consider the category of $\text{Sym}(K)$ -modules.

Definition 2.8. The category of *symmetric spectra over \mathcal{C}* , $Sp^{\Sigma}(\mathcal{C}, K)$, is the category of modules over $\text{Sym}(K)$ in \mathcal{C}^{Σ} . $Sp^{\Sigma}(\mathcal{C}, K)$ is a symmetric monoidal category; the monoidal product $X \wedge Y$ is defined as the coequalizer of the two maps

$$X \otimes \text{Sym}(K) \otimes Y \rightrightarrows X \otimes Y.$$

Let \mathcal{S}_* , sAb and ch^+ denote the categories of simplicial sets, simplicial abelian groups and non-negatively graded chain complexes. Let $\tilde{\mathbb{Z}}: \mathcal{S}_* \longrightarrow sAb$ be the functor such that $\tilde{\mathbb{Z}}(K)_n$ is the free abelian group on the non-basepoint simplices in K_n . Through this paper, let $S^1 = \Delta[1]/\partial\Delta[1]$ denote the simplicial circle, $S^n = (S^1)^{\wedge n}$ and $\mathbb{Z}[n]$ be the chain complex which contains a single copy of the group \mathbb{Z} in dimension n . We consider $Sp^{\Sigma}(sAb, \tilde{\mathbb{Z}}S^1)$ and $Sp^{\Sigma}(ch^+, \mathbb{Z}[1])$. To ease notation, we refer to these categories as $Sp^{\Sigma}(sAb)$ and $Sp^{\Sigma}(ch^+)$ from now on.

$Sp^{\Sigma}(sAb)$ and $Sp^{\Sigma}(ch^+)$ are the two intermediate categories in our comparison between $HR\text{-Mod}$ and $\mathcal{C}h$. Next we define the adjoint functors involved in the three step comparison between $HR\text{-Mod}$ and $\mathcal{C}h$ as displayed below with left adjoints on top.

$$HZ\text{-Mod} \begin{array}{c} \xrightarrow{Z} \\ \xleftarrow{U} \end{array} Sp^\Sigma(sAb) \begin{array}{c} \xleftarrow{L} \\ \xrightarrow{\phi^*N} \end{array} Sp^\Sigma(ch^+) \begin{array}{c} \xleftarrow{D} \\ \xrightarrow{R} \end{array} Ch$$

These functors induce functors on the categories of monoids. The composite derived functors (compare with [Hov99, 1.3.2]) mentioned in the introduction in Theorem 1.1 are then

$$\Theta = Dc\phi^*N\tilde{Z}c \text{ and } \mathbb{H} = UL^{\text{mon}}cR$$

where c denotes the cofibrant replacement functors in each of the model categories of monoids. Fibrant replacement functors are not needed because each of the right adjoints U , ϕ^*N , and R preserve all weak equivalences. In Section 4 we show that (Z, U) and (D, R) are both strong monoidal Quillen equivalences. It follows that the left adjoints induced on the categories of monoids are just the restrictions of the underlying functors. Thus we still denote the induced left adjoints by Z and D . Since (L, ϕ^*N) is only a weak monoidal Quillen equivalence, the induced left adjoint on monoids here is L^{mon} as discussed above Theorem 2.5.

First consider the functors between $HZ\text{-Mod}$ and $Sp^\Sigma(sAb)$. Since $\tilde{\mathbb{Z}}$ is strong monoidal and we defined $S^n = (S^1)^{\wedge n}$, then $(\tilde{\mathbb{Z}}S^1)^{\otimes n} \cong \tilde{\mathbb{Z}}S^n$. Therefore the forgetful functor from simplicial abelian groups to pointed simplicial sets takes $\text{Sym}(\tilde{\mathbb{Z}}S^1) = \tilde{\mathbb{Z}}\mathbb{S}$ to the symmetric spectrum

$$HZ = (\mathbb{Z}, \tilde{\mathbb{Z}}S^1, \tilde{\mathbb{Z}}S^2, \dots, \tilde{\mathbb{Z}}S^n, \dots)$$

as defined in [HSS, 1.2.5]. This forgetful functor induces the functor $U: Sp^\Sigma(sAb) \rightarrow HZ\text{-Mod}$; we define its left adjoint Z in the proof of Proposition 4.3.

The next step is to compare $Sp^\Sigma(sAb)$ and $Sp^\Sigma(ch^+)$. Let $N: sAb \rightarrow ch^+$ denote the normalization functor and Γ denote its inverse as defined in [SS03b, 2.1], for example. Applying the normalization functor from sAb to ch^+ to each level induces a functor from $Sp^\Sigma(sAb)$ to the category of modules in $(ch^+)^\Sigma$ over $\mathcal{N} = N(\text{Sym}(\tilde{\mathbb{Z}}S^1)) = N\tilde{\mathbb{Z}}\mathbb{S} = (N\tilde{\mathbb{Z}}S^0, N\tilde{\mathbb{Z}}S^1, N\tilde{\mathbb{Z}}S^2, \dots)$. \mathcal{N} is a commutative monoid since N is a lax symmetric monoidal functor by [SS03b, 2.6]. By identifying $N\tilde{\mathbb{Z}}S^1$ as $\mathbb{Z}[1]$, we see there is a ring map $\phi: \text{Sym}(\mathbb{Z}[1]) \rightarrow \mathcal{N}$ which in degree n is induced by the monoidal structure on N , $(\mathbb{Z}[1])^{\otimes n} \cong (N\tilde{\mathbb{Z}}S^1)^{\otimes n} \rightarrow N(\tilde{\mathbb{Z}}S^{1 \otimes n})$; see [May67, 29.7]. The composition of N and forgetting along ϕ gives a functor $\phi^*N: Sp^\Sigma(sAb) \rightarrow Sp^\Sigma(ch^+)$. A left adjoint to ϕ^*N exists by [SS03b, Section 3.3]; denote it by L . Note, L is not isomorphic to the composite of underlying left adjoints, $\Gamma\phi_*$, because the adjoint of the identity on \mathcal{N} , namely $\text{Sym}(\tilde{\mathbb{Z}}S^1) \rightarrow \Gamma N \text{Sym}(\tilde{\mathbb{Z}}S^1)$, is not a ring map; see [SS03b, 2.14].

Define a functor $R: Ch \rightarrow Sp^\Sigma(ch^+)$ by setting $(RY)_m = C_0(Y \otimes \mathbb{Z}[m])$. Here $\mathbb{Z}[m]$ is the chain complex with a single copy of \mathbb{Z} concentrated in degree m . C_0 is the connective cover; it is the right adjoint to the inclusion of ch^+ in Ch . One can check that there are isomorphisms $(RY)_m \xrightarrow{\cong} C_0(\mathbb{Z}[-1] \otimes (RY)_{m+1})$ with adjoints $\mathbb{Z}[1] \otimes (RY)_m \rightarrow (RY)_{m+1}$ which provide the module structure over $\text{Sym}(\mathbb{Z}[1])$. We define D , the left adjoint of R , in the paragraph above Proposition 4.5.

2.3. Statements of results. Here we summarize the properties of the categories and functors which are verified in Sections 3 and 4. We then prove Theorem 1.1.

The following statement is proved as Corollary 3.4.

Proposition 2.9. *$HZ\text{-Mod}$, $Sp^\Sigma(sAb)$, $Sp^\Sigma(ch^+)$ and Ch satisfy the Standing assumptions 2.4. It follows by [SS00, 4.1] that there are model structures on the categories of modules and algebras over these categories with fibrations and weak equivalences defined on the underlying model category.*

Proposition 2.10. *The following statements are proved as Propositions 4.3, 4.4 and 4.7.*

- (1) $(Sp^\Sigma(sAb), H\mathbb{Z}\text{-Mod}, \mathbb{Z}, U)$ is a strong monoidal Quillen equivalence.
- (2) $(Sp^\Sigma(sAb), Sp^\Sigma(ch^+), L, \phi^*N)$ is a weak monoidal Quillen equivalence.
- (3) $(Ch, Sp^\Sigma(ch^+), D, R)$ is a strong monoidal Quillen equivalence.

Moreover, the right adjoint (U, ϕ^*N, R) in each of these pairs preserves all weak equivalences.

Theorem 1.1 then follows from Theorem 2.5.

Proof of Theorem 1.1. Proposition 2.10 and Proposition 2.9 verify all of the hypotheses required to apply Theorem 2.5 to the three step zig-zag between $H\mathbb{Z}\text{-Mod}$ and Ch . These three adjoint pairs thus induce Quillen equivalences on the categories of monoids.

$$H\mathbb{Z}\text{-Alg} \begin{array}{c} \xrightarrow{Z} \\ \xleftarrow{U} \end{array} Sp^\Sigma(sAb)\text{-Monoids} \begin{array}{c} \xleftarrow{L^{\text{mon}}} \\ \xrightarrow{\phi^*N} \end{array} Sp^\Sigma(ch^+)\text{-Monoids} \begin{array}{c} \xleftarrow{D} \\ \xrightarrow{R} \end{array} DGA_{\mathbb{Z}}$$

Note, monoids in $H\mathbb{Z}\text{-Mod}$ are the $H\mathbb{Z}$ -algebra spectra, denoted $H\mathbb{Z}\text{-Alg}$ above. Also, the monoids in Ch are differential graded algebras, $DGA_{\mathbb{Z}}$. For an arbitrary discrete commutative ring R one can replace \mathbb{Z} and abelian groups by R and R -modules in all of the statements in Propositions 2.9 and 2.10. The composite derived functors mentioned in the introduction are then $\mathbb{H} = UL^{\text{mon}}cR$ and $\Theta = Dc\phi^*NZc$ where c denotes the cofibrant replacement functor in each category of monoids. Fibrant replacements are not necessary here because the right adjoints preserve all weak equivalences. \square

Remark 2.11. An earlier version of this paper had a four step zig-zag of Quillen equivalences instead of the above three step zig-zag. The earlier comparison of $Sp^\Sigma(ch^+)$ and Ch used two steps involving somewhat simpler and more naturally defined functors than D and R .

$$Ch \begin{array}{c} \xleftarrow{F_0} \\ \xrightarrow{\text{Ev}_0} \end{array} Sp^\Sigma(Ch) \begin{array}{c} \xleftarrow{i} \\ \xrightarrow{C_0} \end{array} Sp^\Sigma(ch^+)$$

The inclusion of non-negatively graded chain complexes into \mathbb{Z} -graded chain complexes induces a functor $i: Sp^\Sigma(ch^+) \rightarrow Sp^\Sigma(Ch)$ with right adjoint C_0 , the prolongation of the connective cover. Then evaluation at the zeroth level gives a functor $\text{Ev}_0: Sp^\Sigma(Ch) \rightarrow Ch$ with left adjoint F_0 [Hov01, 7.3]. Here $Sp^\Sigma(Ch)$ is $Sp^\Sigma(Ch, \mathbb{Z}[1])$, the category of modules over $\text{Sym}(\mathbb{Z}[1])$ in Ch^Σ .

We show in Proposition 4.9 that $(Sp^\Sigma(Ch), Sp^\Sigma(ch^+), i, C_0)$ and $(Sp^\Sigma(Ch), Ch, F_0, \text{Ev}_0)$ are strong monoidal Quillen equivalences. The arguments for Proposition 2.9 also extend to $Sp^\Sigma(Ch)$. Thus, the arguments used in Theorem 1.1 would also apply to give a four step zig-zag of equivalences involving $\mathbb{H} = ULcC_0fF_0c$ and $\Theta = \text{Ev}_0fi\phi^*NZc$ where c and f are the appropriate cofibrant and fibrant replacement functors. Here U , ϕ^*N and i preserve all weak equivalences so we have deleted the respective fibrant replacement functors. We should mention that with this longer zig-zag of functors we could only prove a non-natural version of Theorem 1.2.

2.4. Extension to modules. In this section we construct Quillen equivalences between the categories of modules over differential graded algebras and HR -algebra spectra. Again, we use the criteria developed in [SS03b] to show that weak monoidal Quillen equivalences induce equivalences on these categories of modules. The relevant statement, [SS03b, 3.12], requires one condition in addition to the Standing assumptions 2.4 on the source and target model categories.

Definition 2.12. *Quillen invariance of modules* is said to hold in a monoidal model category if any weak equivalence of monoids $A \rightarrow B$ induces a Quillen equivalence between the associated model categories of modules $A\text{-Mod}$ and $B\text{-Mod}$; see [SS03b, 3.11] and [SS00, 4.3].

As in the case of monoids, if $\lambda: \mathcal{D} \rightleftarrows \mathcal{C}: R$ is a weak monoidal Quillen pair, then for any monoid A in \mathcal{C} the lax monoidal right adjoint R lifts to a functor $R: A\text{-Mod} \rightarrow RA\text{-Mod}$. In [SS03b, 3.3], this module valued functor is shown to have a left adjoint $L^A: RA\text{-Mod} \rightarrow$

$A\text{-Mod}$. Similarly, for B a monoid in \mathcal{D} , the right adjoint R induces a functor $(L^{\text{mon}}B)\text{-Mod} \rightarrow B\text{-Mod}$ (which factors through $R(L^{\text{mon}}B)\text{-Mod}$). Again, [SS03b, 3.3] shows that there is a left adjoint denoted by $L_B: B\text{-Mod} \rightarrow (L^{\text{mon}}B)\text{-Mod}$. As with monoids, in general L^A and L_B do not agree with λ on underlying objects unless (λ, R) is a strong monoidal Quillen pair.

Theorem 2.13. [SS03b, 3.12] *Let $\lambda: \mathcal{D} \rightleftarrows \mathcal{C}: R$ be a weak monoidal Quillen equivalence such that R preserves all weak equivalences. Assume the Standing assumptions 2.4 hold for \mathcal{C} and \mathcal{D} .*

- (1) *For any cofibrant monoid B in \mathcal{D} the adjoint functor pair $L_B: B\text{-Mod} \rightleftarrows (L^{\text{mon}}B)\text{-Mod}: R$ is a Quillen equivalence.*
- (2) *Suppose as well that Quillen invariance holds for \mathcal{C} and \mathcal{D} . Then, for any monoid A in \mathcal{C} the adjoint functor pair $L^A: RA\text{-Mod} \rightleftarrows A\text{-Mod}: R$ is a Quillen equivalence.*

The analogues of these statements for modules over rings with many objects also hold; see [SS03b, 6.5].

The statement in [SS03b, 3.12] is more general since it does not require that the right adjoint R preserves all weak equivalences. Instead, one would require that A is fibrant in (2) above. In each of the monoidal Quillen equivalences considered in this paper the right adjoint does preserve all weak equivalences though. We should remark that here (2) follows from (1) for $B = c(RA)$ by using Quillen invariance since $A \rightarrow L^{\text{mon}}_c(RA)$ is a weak equivalence by Theorem 2.5.

The following proposition is proved as Corollary 3.4.

Proposition 2.14. *Quillen invariance for modules holds in $H\mathbb{Z}\text{-Mod}$, $Sp^\Sigma(sAb)$, $Sp^\Sigma(ch^+)$ and Ch .*

Equivalences of the associated categories of modules then follow from Theorem 2.13.

Corollary 2.15. *Using the composite functors $\mathbb{H}: DGA_{\mathbb{Z}} \rightarrow H\mathbb{Z}\text{-Alg}$ and $\Theta: H\mathbb{Z}\text{-Alg} \rightarrow DGA_{\mathbb{Z}}$ defined above we have the following equivalences of module categories.*

- (1) *For an (associative) differential graded algebra A , there is a Quillen equivalence between differential graded A -modules and $\mathbb{H}A$ -module spectra.*

$$A\text{-Mod} \simeq_Q \mathbb{H}A\text{-Mod}$$

- (2) *For B an (associative) $H\mathbb{Z}$ -algebra spectrum, there is a Quillen equivalence between differential graded ΘB -modules and B -module spectra.*

$$\Theta B\text{-Mod} \simeq_Q B\text{-Mod}$$

The analogues of these statements for modules over rings with many objects also hold.

Proof. Both parts follow from repeated applications of Theorem 2.13 and Quillen invariance statements. The statements for modules over enriched categories (or rings with many objects) follow from Theorem 6.5, Parts 1 and 2 from [SS03b]. \square

We use the Quillen equivalences of modules over $H\mathbb{Z}$ -algebras and differential graded algebras from Corollary 2.15 to construct algebraic models for *rational stable model categories*, the pointed model categories \mathcal{C} where suspension is an invertible functor on the homotopy category and for any two objects X, Y the homotopy classes of maps $[X, Y]^{\text{Ho } \mathcal{C}}$ form a rational vector space. We show that basically any rational stable model category with a small generator is Quillen equivalent to (right) modules over a rational differential graded algebra. This is the rational version of [SS03a, 3.1.1] which shows that basically any stable model category \mathcal{C} with a small generator \mathcal{G} is Quillen equivalent to (right) modules over a ring spectrum $\mathcal{E}(\mathcal{G})$, the endomorphism ring of \mathcal{G} . We must say “basically” here because in fact we need \mathcal{C} to be Quillen equivalent to a spectral model category. A *spectral model category* is the analogue of a simplicial model category

where simplicial sets have been replaced by symmetric spectra (or one of the other monoidal model categories of spectra); see [SS03a, 3.5.1] and Proposition 2.17 below. We now state the more general case where we allow a set of compact generators, this is based on [SS03a, 3.3.3, 3.9.3].

Corollary 2.16. *Let \mathcal{C} be a rational stable model category with a set \mathcal{G} of small generators which is Quillen equivalent to a spectral model category. Then there exists a differential graded algebra with many objects \mathcal{A} and a chain of Quillen equivalences between \mathcal{C} and the category of right \mathcal{A} -modules. The objects of \mathcal{A} correspond to the objects in \mathcal{G} and there is an isomorphism of graded \mathbb{Q} -categories between the homology category $H_*\mathcal{A}$ and the full graded subcategory of $\mathrm{Ho}(\mathcal{C})$ with objects \mathcal{G} .*

$$\mathcal{C} \simeq_{\mathcal{Q}} \mathrm{Mod}\text{-}\mathcal{E}(\mathcal{G}) \simeq_{\mathcal{Q}} \mathrm{Mod}\text{-}\mathcal{A}$$

To specify \mathcal{A} , recall the functor Θ from Theorem 1.1. This functor can be extended to rings with many objects; see [SS03b, 6] and [DS, 6]. Then $\mathcal{A} \cong \Theta(H\mathbb{Q} \wedge c\mathcal{E}(\mathcal{G}))$ where c is a cofibrant replacement functor for ring spectra (with many objects) from [SS03b, 6.3].

In general, $\mathrm{Mod}\text{-}\mathcal{A}$ is not a practical algebraic model; it is difficult to make explicit and \mathcal{A} is quite large. This model can be used as a stepping stone to a practical model though. For example, this corollary applies to the category of rational G -equivariant spectra for any compact Lie group G . In [Shi02] and [GS] this large model is used to show that there is an explicit algebraic model Quillen equivalent to the category of rational T^n -equivariant spectra for T^n the n -dimensional torus. For G finite this also extends the results of [GM95, Appendix A] to incomplete universes; see [SS03a, 5.1.2]. On the other hand, although this corollary applies to the rational motivic stable homotopy theory of schemes from [Jar98, Voe98] no such explicit algebraic model is known.

Several different sets of hypotheses can be used to ensure that \mathcal{C} is Quillen equivalent to a spectral model category. A partial converse from [SS03a, 3.5.2] shows that any spectral model category is stable and simplicial.

Proposition 2.17. *If one of the following lists of hypotheses holds, then \mathcal{C} is Quillen equivalent to a spectral model category.*

- (1) [Du06, 1.8, 4.7a, 4.9a] \mathcal{C} is stable and combinatorial.
- (2) [Hov01, 8.11, 9.1] \mathcal{C} is stable, simplicial, left proper, cellular and the domains of the generating cofibrations are cofibrant.
- (3) [SS03a, 3.8.2] \mathcal{C} is stable, simplicial, proper and cofibrantly generated.

Proof of Corollary 2.16. If \mathcal{C} is a rational stable model category with a set of small generators which is Quillen equivalent to a spectral model category \mathcal{D} , then it follows that \mathcal{D} is also rational and has a set \mathcal{G} of small generators. In [SS03a, 3.9.3] we show that any spectral model category \mathcal{D} with a set \mathcal{G} of small generators is equivalent to (right) modules over $\mathcal{E}(\mathcal{G})$, the endomorphism ring spectrum of \mathcal{G} (a symmetric ring spectrum with many objects) defined in [SS03a, 3.7.5]. Since there is a natural isomorphism of the homotopy groups of $\mathcal{E}(\mathcal{G})$ and the graded homotopy classes of maps between elements in \mathcal{G} by [SS03a, 3.5.2], the homotopy groups of $\mathcal{E}(\mathcal{G})$ are rational when \mathcal{D} is rational.

If R is a ring spectrum with rational homotopy groups which is cofibrant as an underlying module, then the map $R \rightarrow H\mathbb{Q} \wedge R$ is a stable equivalence of ring spectra. Since \mathbb{Q} is flat over π_*^s , this follows from the spectral sequence for computing the homotopy of $H\mathbb{Q} \wedge R$ [EKMM, IV.4.1] and the Quillen equivalences of ring spectra established in [Sch01]. Similarly, if \mathcal{R} is a ring spectrum with many objects with rational homotopy groups such that each $\mathcal{R}(i, j)$ is cofibrant as a module, then $\mathcal{R} \rightarrow H\mathbb{Q} \wedge \mathcal{R}$ is a stable equivalence of ring spectra with many objects.

Quillen invariance then shows that $\text{Mod-}\mathcal{E}(\mathcal{G})$, $\text{Mod-}c\mathcal{E}(\mathcal{G})$ and $\text{Mod-}(H\mathbb{Q}\wedge c\mathcal{E}(\mathcal{G}))$ are Quillen equivalent, where c denotes the cofibrant replacement of ring spectra (with many objects) from [SS03b, 6.3]. Here we are using the fact that, since the sphere spectrum is cofibrant, the cofibrant replacement is pointwise cofibrant as a module. Then $\text{Mod-}(H\mathbb{Q}\wedge c\mathcal{E}(\mathcal{G}))$ is Quillen equivalent to the model category of differential graded $\mathcal{A} = \Theta(H\mathbb{Q}\wedge c\mathcal{E}(\mathcal{G}))$ -modules, $\text{Mod-}\mathcal{A}$, by the many objects analogue of Corollary 2.15, Part 2. This analogue holds by Theorem 6.5, Parts 1 and 2 from [SS03b] applied to each of the three adjoint pairs in Proposition 2.10.

Since Θ and \mathbb{H} induce an equivalence between the homotopy categories of differential graded algebras and $H\mathbb{Z}$ -algebras, the calculation of $H_*\mathcal{A} = H_*\Theta(H\mathbb{Q}\wedge c\mathcal{E}(\mathcal{G}))$ follows from [SS03a, 3.5.2], the stable equivalence of $\mathcal{E}(\mathcal{G})$ and $H\mathbb{Q}\wedge c\mathcal{E}(\mathcal{G})$ and the stable equivalence of $\mathbb{H}\mathbb{Z}[0]$ and $H\mathbb{Z}$. This last fact follows since $\mathbb{Z}[0]$ is the unit of the monoidal structure on \mathcal{Ch} and each of the functors U , L^{mon} and R preserve these units. This is simple to verify for U and R and is verified for L^{mon} in the proof of Proposition 4.4; see also [SS03b, 3.7]. \square

3. MONOIDAL MODEL CATEGORIES AND THE MONOID AXIOM

In this section we prove Proposition 2.9 which states that the Standing assumptions 2.4 hold for each of the four categories $H\mathbb{Z}\text{-Mod}$, $Sp^{\Sigma}(sAb)$, $Sp^{\Sigma}(ch^+)$ and \mathcal{Ch} . Recall, the standing assumptions require that these categories are cofibrantly generated, monoidal model categories (see Definition 2.1) which satisfy the monoid axiom (see Definition 2.2), have cofibrant unit objects and have all objects small. We also establish Proposition 2.14 by showing that Quillen invariance (see Definition 2.12) holds in each of these four categories.

The proof that all objects are small in each of these four categories is delayed to the end of this section in Propositions 3.7 and 3.8. All of the other statements for $H\mathbb{Z}$ -module spectra follow from [HSS, 3.4.9, 5.3.8, 5.4.2, 5.4.4] and the fact that $H\mathbb{Z}$ is cofibrant in $H\mathbb{Z}\text{-Mod}$. We are left with establishing the standing assumptions and Quillen invariance for the other three categories.

We first verify that $Sp^{\Sigma}(sAb)$, $Sp^{\Sigma}(ch^+)$ and \mathcal{Ch} are cofibrantly generated, monoidal model categories. \mathcal{Ch} is shown to be a cofibrantly generated, monoidal model category in [Hov99, 2.3.11, 4.2.13] and the proof obviously applies to ch^+ as well. These properties follow for sAb from [Qui67, SM7] by using the strong symmetric monoidal functor $\tilde{\mathbb{Z}}$ from pointed simplicial sets to sAb ; see also [SS03b]. Since sAb and ch^+ are cofibrantly generated, monoidal model categories, the same properties for $Sp^{\Sigma}(sAb)$ and $Sp^{\Sigma}(ch^+)$ then follow from [Hov01, 8.11]. Here we are also using the fact that the generating cofibrations in both sAb and ch^+ are maps between cofibrant objects.

[Hov01] was not able to verify in general though that $Sp^{\Sigma}(\mathcal{C})$ satisfies the monoid axiom if \mathcal{C} does. Instead we follow an approach for verifying the monoid axiom for $Sp^{\Sigma}(sAb)$, $Sp^{\Sigma}(ch^+)$ and \mathcal{Ch} modeled on the approach for $Sp^{\Sigma}(\mathcal{S}_*)$ in [HSS, 5.4]. To establish the general pattern we consider \mathcal{Ch} first.

We prove the monoid axiom for \mathcal{Ch} by using both the projective and injective model structures. The *projective model structure* is the structure we have referred to so far in this paper; see [Hov99, 2.3.11]. The weak equivalences are the quasi-isomorphisms, the (projective) fibrations are the surjections and the (projective) cofibrations are the maps with the left lifting property with respect to the trivial fibrations. We now introduce the *injective model structure* from [Hov99, 2.3.13]. The weak equivalences are again the quasi-isomorphisms, the injective cofibrations are the monomorphisms and the injective fibrations are the maps with the right lifting property with respect to the trivial cofibrations.

Proposition 3.1. *The monoid axiom holds for \mathcal{Ch} .*

Proof. We first consider the monoid axiom for generating trivial (projective) cofibrations. Namely, we show that any map obtained by pushouts and compositions of maps of the form $j \otimes \text{Id}_Z: A \otimes Z \rightarrow B \otimes Z$ is a weak equivalence where $j: A \rightarrow B$ is any generating trivial (projective) cofibration and Z is any object. It then follows by [SS00, 3.5(2)] that the full monoid axiom holds. The generating trivial (projective) cofibrations are all of the form $j: 0 \rightarrow D^n$ [Hov99, 2.3.3]. Here D^n is the acyclic chain complex with all degrees zero except $(D^n)_n = \mathbb{Z}$ and $(D^n)_{n-1} = \mathbb{Z}$. One can show that $D^n \otimes Z$ is acyclic for any Z and thus $j \otimes \text{Id}_Z: 0 \rightarrow D^n \otimes Z$ is a monomorphism and a quasi-isomorphism. The monoid axiom then follows since maps which are monomorphisms and quasi-isomorphisms are preserved under pushouts and compositions. This last statement follows, for example, from the fact that the trivial cofibrations in the injective model structure on Ch are the maps which are monomorphisms and quasi-isomorphisms. \square

As with Ch , we establish the monoid axiom on $Sp^\Sigma(sAb)$ and $Sp^\Sigma(ch^+)$ by forming *injective stable model structures*. We then refer to the stable model structures introduced above from [Hov01] as the projective stable model structure. Throughout this paper we use ‘*injective*’ to refer to this new model structure; so ‘*injective*’ implies certain lifting properties, and not necessarily ‘*monomorphism*’.

Proposition 3.2. *There is an injective stable model structure on both of the categories $Sp^\Sigma(sAb)$ and $Sp^\Sigma(ch^+)$ with injective cofibrations the monomorphisms, weak equivalences the (projective) stable equivalences from [Hov01] and fibrations the maps with the right lifting property with respect to the trivial injective cofibrations. Note, here all objects are injective cofibrant.*

The reason these injective stable model categories are so useful is that there is an action of the respective projective stable model categories on these injective model categories which satisfies an analogue of the pushout product axiom. This action appeared implicitly in the proof of Proposition 3.1 to show that the tensor of a trivial projective cofibration and an injective cofibrant object is a trivial injective cofibration. This action is analogous to the structure of simplicial model categories which have an action by simplicial sets. As with the definition of monoidal model categories, the unital condition is automatic and thus suppressed because the units here are all cofibrant.

Proposition 3.3. *If $i: A \rightarrow B$ is an injective cofibration in $Sp^\Sigma(sAb)$, $Sp^\Sigma(ch^+)$ or Ch and $i': K \rightarrow L$ is a projective cofibration on the same underlying category then*

$$i \square i': A \otimes L \coprod_{A \otimes K} B \otimes K \rightarrow B \otimes L$$

is an injective cofibration which is a weak equivalence if either one of i or i' is. That is, the injective stable model structure in each case is a Quillen module over the projective stable model structure; see also [Hov99, 4.2.18].

We prove this proposition for Ch first and delay the more involved proofs of Propositions 3.2 and 3.3 for $Sp^\Sigma(sAb)$ and $Sp^\Sigma(ch^+)$ until after Corollary 3.4 below.

Proof of Proposition 3.3 for Ch . Recall the injective and projective model structures on Ch defined above Proposition 3.1. We use the criterion in [SS00, 3.5] to reduce to checking the pushout product axiom for $K \rightarrow L$ a generating (trivial) projective cofibration and $A \rightarrow B$ any (trivial) monomorphism (that is, a (trivial) injective cofibration).

The generating set of projective cofibrations is given by $I = \{i_n: \mathbb{Z}[n-1] \rightarrow D^n\}$. Here $\mathbb{Z}[n]$ is the complex concentrated in degree n and D^n is the acyclic complex modeling the n -disk. As graded abelian groups (ignoring differentials) the pushout product of $i: A \rightarrow B$ and i_n is the map $i \square i_n: (B \otimes \mathbb{Z}[n-1]) \oplus (A \otimes \mathbb{Z}[n]) \rightarrow (B \otimes \mathbb{Z}[n-1]) \oplus (B \otimes \mathbb{Z}[n])$. This is a (trivial)

injective cofibration if $i: A \rightarrow B$ is. Also, for $j_n: 0 \rightarrow D^n$, $i \square j_n: A \otimes D^n \rightarrow B \otimes D^n$ is a map between acyclic complexes and hence is a quasi-isomorphism. \square

Next we deduce the monoid axiom and the Quillen invariance properties for $Sp^\Sigma(sAb)$, $Sp^\Sigma(ch^+)$ and Ch assuming the Quillen module structure from Proposition 3.3. Propositions 2.9 and 2.14 then follow for these three categories since the unit objects are all cofibrant and all objects are shown to be small in Propositions 3.7 and 3.8. As mentioned above, the statements about $H\mathbb{Z}$ -modules follow from [HSS, 5.4.2, 5.4.4].

Corollary 3.4. *The stable model structures on $Sp^\Sigma(sAb)$, $Sp^\Sigma(ch^+)$ and Ch are cofibrantly generated, monoidal model categories which satisfy the monoid axiom. Quillen invariance for modules also holds in each of these categories.*

Proof. We verified that these model structures are cofibrantly generated, monoidal model categories at the beginning of this section. We use Proposition 3.3 to establish the monoid axiom. Since the injective cofibrations are monomorphisms, any object A is injective cofibrant. Hence, $A \wedge -$ takes stably trivial projective cofibrations to stably trivial injective cofibrations by Proposition 3.3. Pushouts and directed colimits of stably trivial injective cofibrations are again stably trivial injective cofibrations. Since the injective stable equivalences agree with the projective stable equivalences this establishes the monoid axiom.

In [DS, 6.2(c)], Quillen invariance is shown to follow from the following two properties (see also [SS00, 4.3]):

- (QI1) For any projective cofibrant object A and any weak equivalence $X \rightarrow Y$, the map $A \otimes X \rightarrow A \otimes Y$ is also a weak equivalence.
- (QI2) Suppose $A \rightarrow B$ is a projective cofibration, and X is any object. Then for any map $A \otimes X \rightarrow Z$, the map from the homotopy pushout of $B \otimes X \leftarrow A \otimes X \rightarrow Z$ to the pushout is a weak equivalence.

These two properties in turn follow from Proposition 3.3. For A a projective cofibrant object, $A \wedge -$ preserves stably trivial injective cofibrations by Proposition 3.3. Hence, by Ken Brown's lemma [Hov99, 1.1.12], $A \wedge -$ preserves stable equivalences between injective cofibrant objects. Since all objects are injective cofibrant, this implies (QI1).

By Proposition 3.3, $A \otimes X \rightarrow B \otimes X$ is an injective cofibration whenever $A \rightarrow B$ is a projective cofibration since every object X is injective cofibrant. (QI2) then follows from the fact that the injective stable model structure is left proper since all objects are cofibrant; see also [Hov99, 5.2.6]. \square

We now turn to the proofs of Propositions 3.2 and 3.3. The proofs of these propositions are very similar for $Sp^\Sigma(sAb)$ and $Sp^\Sigma(ch^+)$. We consider in detail the case of $Sp^\Sigma(ch^+)$. For $Sp^\Sigma(sAb)$, one can follow the same outline at every stage; see Proposition 4.1 for an identification of the generating trivial cofibrations.

We will construct the injective stable model categories as localizations of injective level model categories. This follows the outline in [Hov01] which first constructs projective level model categories on $Sp^\Sigma(sAb)$ and $Sp^\Sigma(ch^+)$ with weak equivalences the level equivalences, fibrations the level fibrations and cofibrations the maps with the left lifting property with respect to the level trivial fibrations. These are cofibrantly generated, monoidal model categories by [Hov01, 8.3]. The next lemma establishes the analogous injective level model categories.

Lemma 3.5. *There is an injective level model category on both of the categories $Sp^\Sigma(sAb)$ and $Sp^\Sigma(ch^+)$ with cofibrations the monomorphisms, weak equivalences the level equivalences and fibrations the maps with the right lifting property with respect to the level trivial cofibrations.*

Proof. First consider the injective model structure on ch^+ with cofibrations the monomorphisms and weak equivalences the quasi-isomorphisms; [Hov99, 2.3.13] discusses the analogue on Ch . This model category is cofibrantly generated; a set of generating cofibrations I' contains a map from each isomorphism class of monomorphisms between countable chain complexes and J' is the set of weak equivalences in I' . Here the cardinality of a chain complex is given by the cardinality of the union of all of the levels.

The proof that the injective level model structure on $Sp^\Sigma(ch^+)$ forms a cofibrantly generated model category follows as in [HSS, 5.1.2]. The only changes needed are that one uses a chain homotopy instead of a simplicial homotopy in [HSS, 5.1.4(5)] and for [HSS, 5.1.6, 5.1.7] the arguments for simplicial sets are replaced by the analogues of [Hov99, 2.3.15, 2.3.21] for ch^+ instead of Ch as above. (For $Sp^\Sigma(sAb)$ the proof follows [HSS, 5.1.2] almost verbatim.) Note that every spectrum is cofibrant here. Again the generating cofibrations are the monomorphisms between countable objects and the set of generating trivial cofibrations is the subset of level weak equivalences. Call the fibrant objects here the *injective fibrant objects*.

One could also establish the injective level model structures by using Jeff Smith's work on combinatorial model categories. We sketch the verification of the criteria listed in [Be00, 1.7]. Criteria (c0) and (c3) follow from [Be00, 1.15, 1.18, 1.19] since the functors $ev_n: Sp^\Sigma(ch^+) \rightarrow ch^+$ are accessible and ch^+ is locally presentable by Proposition 3.7; see also [Du01b, 7.3] and [AR94, 2.23, 2.37]. Maps with the right lifting property with respect to monomorphisms must be levelwise weak equivalences (and levelwise fibrations) since the generating projective cofibrations are monomorphisms [Hov01, 8.2]; criterion (c1) thus holds. Finally, directed colimits and pushouts preserve levelwise weak equivalences which are monomorphisms since colimits are created on each level; this verifies criterion (c2). \square

We next show that the analogue of Proposition 3.3 holds for the associated level model categories.

Lemma 3.6. *The injective and projective level model categories on $Sp^\Sigma(sAb)$ and $Sp^\Sigma(ch^+)$ satisfy the property stated in Proposition 3.3. That is, the injective level model structure in each case is a Quillen module over the projective level model structure [Hov99, 4.2.18].*

Proof. Again, use [SS00, 3.5] to reduce to considering only the generating (trivial) projective cofibrations. Each part of the proof that the injective level model structure is a Quillen module over the projective level model structure then follows from checking that the injective model structure on ch^+ is a Quillen module over the projective model structure on ch^+ [Hov99, 4.2.18]. This follows for ch^+ as outlined in the proof of Proposition 3.3 for Ch above. \square

Proof of Proposition 3.2. The injective level model structure is left proper and cellular. Thus, we may localize with respect to the set of maps $\mathcal{F} = \{F_{n+1}(A \otimes \mathbb{Z}[1]) \rightarrow F_n A\}_{n \geq 0}$ for one object A from each isomorphism class of countable objects in $Sp^\Sigma(ch^+)$. See [Hov01, Section 2] for a brief summary of Bousfield localization; the definitive reference is [Hir00]. One could also use the machinery developed by Jeff Smith [Sm] to establish the existence of these local model categories because these categories are also left proper and combinatorial (cofibrantly generated and locally presentable, see Proposition 3.7); see also [Du01a, Section 2].

We are left with showing that the injective stable (local) equivalences agree with the (projective) stable equivalences defined in [Hov01, 8.7] for the projective stable model structure on ch^+ . Following [Hov01, 8.8] one can show that the \mathcal{F} -local objects, or injective stably fibrant objects are the injective fibrant objects which are also Ω -spectra [Hov01, 8.6]. Since injective fibrations are the maps with the right lifting property with respect to all level trivial monomorphisms, they are in particular level fibrations. Any injective fibrant Ω -spectrum is thus a levelwise fibrant Ω -spectrum, that is, a projective fibrant Ω -spectrum.

By definition, a map f is an injective stable (local) equivalence if and only if $\text{map}_{\text{inj}}(f, E)$ is a weak equivalence for any injective fibrant Ω -spectrum E . Here $\text{map}_{\text{inj}}(f, E)$ is the homotopy function complex in the injective level model structure as constructed in [Hov01, Section 2], [Hov99, 5.4] or [Hir00, 17.4.1]. Similarly, a map f is a (projective) stable equivalence if and only if $\text{map}_{\text{proj}}(Qf, F)$ is a weak equivalence for any projective fibrant Ω -spectrum F . Here Q is the projective cofibrant replacement functor.

The identity functors induce a Quillen equivalence between the injective level model structure and the projective level model structure since weak equivalences in both cases are level equivalences. Thus, by [Hir00, 17.4.16], if X is projective cofibrant and Y is injective fibrant then $\text{map}_{\text{inj}}(X, Y)$ is weakly equivalent to $\text{map}_{\text{proj}}(X, Y)$. As above, denote by Q the projective cofibrant replacement functor. Since $QX \rightarrow X$ is a level equivalence, f is an injective stable equivalence if and only if $\text{map}_{\text{inj}}(Qf, E)$, or equivalently $\text{map}_{\text{proj}}(Qf, E)$, is a weak equivalence for any injective fibrant Ω -spectrum E . It follows that if f is a projective stable equivalence, then f is also an injective stable equivalence since injective fibrant Ω -spectra are also projective fibrant Ω -spectra. Similarly, denote by R the injective fibrant replacement functor. Since $Y \rightarrow RY$ is a level equivalence, f is a projective stable equivalence if and only if $\text{map}_{\text{proj}}(Qf, RF)$, or equivalently $\text{map}_{\text{inj}}(Qf, RF)$, is a weak equivalence for any projective fibrant Ω -spectrum F . It follows that if f is an injective stable equivalence, then f is also a projective stable equivalence. \square

Proof of Proposition 3.3. We now show that the injective stable model structure is a Quillen module over the projective stable model structure. The pushout product of an injective cofibration and a projective cofibration is an injective cofibration by Lemma 3.6.

We next consider the case where $j: K \rightarrow L$ is a stably trivial injective cofibration and $i: F_k(B) \rightarrow F_k(C)$ is a generating projective cofibration. The functor $- \otimes_{\text{Sym}(\mathbb{Z}[1])} F_k(B) = - \wedge F_k B$ preserves injective cofibrations by Lemma 3.6. It also preserves stably trivial injective cofibrations for B a countable projective cofibrant complex in ch^+ . This follows by the construction of the injective stable model category since this functor takes maps in \mathcal{F} into \mathcal{F} up to isomorphism; see the last sentence of [Hov01, 2.2]. In fact, for each map in \mathcal{F} it just replaces A by $A \otimes B$ and n by $n + k$. Since B and C are countable complexes, $j \wedge F_k(B)$ and $j \wedge F_k(C)$ are both stably trivial injective cofibrations. Let $P = (L \wedge F_k(B)) \wedge_{K \wedge F_k(B)} K \wedge F_k(C)$; it follows that $j': K \wedge F_k(B) \rightarrow P$ is a stably trivial injective cofibration since it is the pushout of $j \wedge F_k(A)$. The composite of j' and $j \square i: P \rightarrow L \wedge F_k(B)$ is $j \wedge F_k(B)$, so by the two out of three property, $j \square i$ is a stable equivalence as well. By the criterion in [SS00, 3.5], it follows that this holds for i any projective cofibration.

We next consider the case where $j: K \rightarrow L$ is an injective cofibration and $i: A \rightarrow B$ is a stably trivial projective cofibration. In the next paragraph we show that $K \wedge -$ takes stably trivial projective cofibrations to stably trivial injective cofibrations. It then follows by the same argument as in the last paragraph that the pushout product of an injective cofibration and a stably trivial projective cofibration is a stably trivial injective cofibration.

Since K is injective cofibrant for any object, $K \wedge -$ takes (level trivial) projective cofibrations to (level trivial) injective cofibrations by Lemma 3.6. So to show $K \wedge -$ takes stably trivial projective cofibrations to stably trivial injective cofibrations, it is enough to show that $K \wedge -$ takes stably trivial projective cofibrations to stable equivalences. By [SS00, 3.5] it is enough to consider $K \wedge -$ on the generating stably trivial projective cofibrations. Since the generating stably trivial projective cofibrations are maps between projective cofibrant objects, we reduce our problem to showing that $K \wedge Q(-)$ preserves stable equivalences where Q is a projective cofibrant replacement functor.

From Lemma 3.6 we see that for a projective cofibrant object QX , the functor $- \otimes QX$ preserves level trivial injective cofibrations. Thus, by Ken Brown's lemma [Hov99, 1.1.12], $- \otimes QX$

also preserves level equivalences between injective cofibrant objects, i.e. all level equivalences. Since $QK \rightarrow K$ is a level equivalence, it follows that $K \otimes Q(-)$ is level equivalent to $QK \otimes Q(-)$. So we only need to see that $QK \wedge Q(-)$ preserves stable equivalences. Since the projective stable model category is a monoidal model category by [Hov01, 8.11], $QK \wedge -$ preserves projective stably trivial cofibrations. Then, by Ken Brown's lemma again, $QK \wedge -$ preserves stable equivalences between projective cofibrant objects; that is, $QK \wedge Q(-)$ preserves stable equivalences. \square

Finally, we verify the requirement that each object is small by showing that these categories are locally presentable. The small object argument, see [Hov99, 2.1.14] or [DS95, 7.12], then applies to any set of maps and is used in [SS00] to construct the model structures on the categories of algebras and modules.

Proposition 3.7. *Each of the categories $H\mathbb{Z}\text{-Mod}$, $Sp^\Sigma(sAb)$, $Sp^\Sigma(ch^+)$ and Ch is locally presentable. The categories of monoids, modules and algebras over these categories are also locally presentable.*

Proof. Recall D^n is the acyclic chain complex modeling the n -disk. Ch is locally presentable by [AR94, 1.20] because the set $\{D^n\}_{n \in \mathbb{Z}}$ is a strong generator with each object finitely presentable; see [AR94, 0.6, 1.1]. Similar arguments apply to ch^+ . Also, the category of simplicial sets, \mathcal{S} , is locally presentable by [Bor94, 5.2.2b] because it is set-valued diagrams over a small category.

[Bor94, 5.5.9] and [Bor94, 5.3.3, 5.7.5] show that over a locally presentable category any category of algebras over a monad which commutes with directed colimits or any functor category from a small category to a locally presentable category is again a locally presentable category. Since $H\mathbb{Z}\text{-Mod}$, $Sp^\Sigma(sAb)$ and $Sp^\Sigma(ch^+)$ can be built using these two methods from ch^+ or \mathcal{S} , the statement follows. \square

Proposition 3.8. [AR94, 1.13, 1.16, 1.17], [Bor94, 5.2.10] *Each object in a locally presentable category is small relative to the whole category.*

4. PROOF OF PROPOSITION 2.10

Before considering each of the three parts of Proposition 2.10 separately, we identify the generating cofibrations and trivial cofibrations in $Sp^\Sigma(sAb)$. This is useful in the two parts involving this model category. Recall that $\tilde{\mathbb{Z}}: \mathcal{S}_* \rightarrow sAb$ is the reduced free abelian group functor.

Proposition 4.1. *The generating cofibrations and trivial cofibrations in the cofibrantly generated stable model structure on $Sp^\Sigma(sAb)$ defined in [Hov01] are given by applying the prolongation of $\tilde{\mathbb{Z}}$ to the generating trivial cofibrations in $Sp^\Sigma(\mathcal{S}_*)$.*

Proof. Define $\tilde{U}: Sp^\Sigma(sAb) \rightarrow Sp^\Sigma(\mathcal{S}_*)$ by composing $U: Sp^\Sigma(sAb) \rightarrow H\mathbb{Z}\text{-Mod}$ with the forgetful functor from $H\mathbb{Z}$ -modules to underlying symmetric spectra. The left adjoint of \tilde{U} is defined by applying $\tilde{\mathbb{Z}}$ to each level. Denote this prolongation by $\tilde{\mathbb{Z}}$ as well. Define a new model structure on $Sp^\Sigma(sAb)$ with weak equivalences and fibrations the maps which are underlying weak equivalences and fibrations in $Sp^\Sigma(\mathcal{S}_*)$. The fibrant objects are then the Ω -spectra and the trivial fibrations are the level trivial fibrations. It follows that the cofibrations must be the maps with the left lifting property with respect to the level trivial fibrations. To see that these structures satisfy the axioms of a model category we note that this structure agrees with the stable model structure defined in [Hov01]. To see this, note that the trivial fibrations agree. Then the cofibrations and hence the cylinder objects must also agree. Also, the fibrant objects agree and hence the weak equivalences agree; see for example [Hir00, 7.8.6].

Since the weak equivalences and fibrations of $Sp^\Sigma(sAb)$ are determined by \tilde{U} , they are detected by $\tilde{Z}(I)$ and $\tilde{Z}(J)$ where $I = FI_\partial$ and $J = FI_\Lambda \cup K$ are the generating cofibrations and trivial cofibrations of $Sp^\Sigma(\mathcal{S}_*)$ defined in [HSS, 3.3.2, 3.4.9]. It follows that $Sp^\Sigma(sAb)$ is a cofibrantly generated model category with generating cofibrations $\tilde{Z}(I)$ and generating trivial cofibrations $\tilde{Z}(J)$. \square

To show that the three adjoint pairs in Proposition 2.10 are weak (or strong) monoidal Quillen equivalences, we use the following criterion for Quillen pairs between monoidal stable model categories from [SS03b]. Since the unit objects are cofibrant in each case here, the unit condition is simpler than the one appearing in [SS03b]. An object A in a stable model category \mathcal{C} is said to *stably detect weak equivalences* if $f: Y \rightarrow Z$ is a weak equivalence if and only if $[A, Y]_k^{\text{Ho}(\mathcal{C})} \rightarrow [A, Z]_k^{\text{Ho}(\mathcal{C})}$ is an isomorphism for all $k \in \mathbb{Z}$.

Proposition 4.2. [SS03b, 3.17] *Let $\lambda: \mathcal{D} \rightleftarrows \mathcal{C}: R$ be a Quillen adjoint pair between monoidal stable model categories such that R is lax monoidal and the unit objects $\mathbb{I}_{\mathcal{C}}$ and $\mathbb{I}_{\mathcal{D}}$ are both cofibrant. Suppose further that*

- (1) *the adjoint to the monoidal structure map $\nu: \mathbb{I}_{\mathcal{D}} \rightarrow R(\mathbb{I}_{\mathcal{C}})$, namely $\tilde{\nu}: \lambda(\mathbb{I}_{\mathcal{D}}) \rightarrow \mathbb{I}_{\mathcal{C}}$, is a weak equivalence in \mathcal{C} ; and*
- (2) *the unit $\mathbb{I}_{\mathcal{D}}$ stably detects weak equivalences in \mathcal{D} .*

Then (λ, R) is a weak monoidal Quillen pair.

Now we verify the three parts of Proposition 2.10 separately in Propositions 4.3, 4.4 and 4.7.

Proposition 4.3. *$(Sp^\Sigma(sAb), H\mathbb{Z}\text{-Mod}, Z, U)$ is a strong monoidal Quillen equivalence. Moreover, U preserves all weak equivalences.*

Proof. We first define the left adjoint Z . As with \tilde{U} in Proposition 4.1, one expects the left adjoint to involve \tilde{Z} . Applying \tilde{Z} to each level of an $H\mathbb{Z}$ -module in $Sp^\Sigma(\mathcal{S}_*)$ produces a $\tilde{Z}(H\mathbb{Z})$ -module in $Sp^\Sigma(sAb)$. The monad structure on \tilde{Z} then induces a ring homomorphism $\mu: \tilde{Z}(H\mathbb{Z}) \rightarrow H\mathbb{Z} \cong \text{Sym}(\tilde{Z}S^1)$ in $Sp^\Sigma(sAb)$ which induces a push-forward $\mu_*: \tilde{Z}(H\mathbb{Z})\text{-Mod} \rightarrow Sp^\Sigma(sAb)$. One can then check that $Z(X) = \mu_*(\tilde{Z}X) = \tilde{Z}X \wedge_{\tilde{Z}(H\mathbb{Z})} H\mathbb{Z}$ is left adjoint to U . Since both μ_* and \tilde{Z} are strong symmetric monoidal so is Z . It follows that U is lax symmetric monoidal. Note that $\tilde{\nu}: Z(H\mathbb{Z}) \rightarrow \text{Sym}(\tilde{Z}S^1)$ is an isomorphism and both units are cofibrant.

It is clear that $U: Sp^\Sigma(sAb) \rightarrow H\mathbb{Z}\text{-Mod}$ detects and preserves weak equivalences and fibrations since in both cases these are determined by the forgetful functor to $Sp^\Sigma(\mathcal{S}_*)$; see Proposition 4.1. Hence, U and Z form a Quillen adjoint pair. Since $H\mathbb{Z}$ stably detects weak equivalences in $H\mathbb{Z}$ -modules, it follows by Proposition 4.2 that (U, Z) is a weak monoidal Quillen pair. Since Z is strong monoidal, (U, Z) is in fact a strong monoidal Quillen pair.

We are left with showing that this Quillen pair is a Quillen equivalence. U and Z induce adjoint total derived functors \bar{U} and \bar{Z} on the homotopy categories. Since U detects and preserves weak equivalences, by [HSS, 4.1.7] we only need to check that $\psi: \text{Id} \rightarrow \bar{U}\bar{Z}$ is an isomorphism to establish the Quillen equivalence. First, note that ψ is an isomorphism on $H\mathbb{Z}$. These functors are exact and preserve coproducts, so ψ is an isomorphism on any object built from $H\mathbb{Z}$ via suspensions, triangles or coproducts; that is, ψ is an isomorphism on the localizing subcategory generated by $H\mathbb{Z}$. Since $H\mathbb{Z}$ detects the weak equivalences in $H\mathbb{Z}$ -modules, it is a generator by [SS03a, 2.2.1] and this localizing subcategory is the whole category. \square

Proposition 4.4. *$(Sp^\Sigma(sAb), Sp^\Sigma(ch^+), L, \phi^*N)$ is a weak monoidal Quillen equivalence. Moreover, ϕ^*N preserves all weak equivalences.*

Proof. Since normalization from sAb to ch^+ is a lax symmetric monoidal functor, its prolongation $N: Sp^\Sigma(sAb) \rightarrow \mathcal{N}\text{-Mod}$ is also lax symmetric monoidal. Recall, $\mathcal{N} = N(\text{Sym}(\tilde{\mathbb{Z}}S^1)) = N\tilde{\mathbb{Z}}S$. Since $\phi: \text{Sym}(\mathbb{Z}[1]) \rightarrow \mathcal{N}$ is a ring homomorphism between commutative monoids, pulling back modules $\phi^*: \mathcal{N}\text{-Mod} \rightarrow Sp^\Sigma(ch^+)$ is also a lax symmetric monoidal functor.

We show that ϕ^* and N are each the right adjoint of a Quillen equivalence and each preserve all weak equivalences, thus ϕ^*N has these properties as well. We delay the verification that (L, ϕ^*N) is a weak monoidal Quillen pair until the end of this proof.

Since in degree n the map ϕ is induced by the monoidal structure on N , $(\mathbb{Z}[1])^{\otimes n} \cong (N\tilde{\mathbb{Z}}S^1)^{\otimes n} \rightarrow N(\tilde{\mathbb{Z}}S^{1 \otimes n})$, it is a level weak equivalence by [May67, 29.4, 29.7]. By the Quillen invariance property, verified in Corollary 3.4, ϕ^* is then the right adjoint of a Quillen equivalence. The model category on $\mathcal{N}\text{-Mod}$ has underlying weak equivalences and fibrations, as defined by applying [SS00, 4.1]. Hence ϕ^* preserves all weak equivalences by definition.

For the normalization functor, we first show that N preserves all equivalences. Since normalization takes weak equivalences in sAb to weak equivalences in ch^+ , N preserves all levelwise weak equivalences. Given any stable equivalence it can be factored as a stably trivial cofibration followed by a stably trivial fibration. Since stably trivial fibrations are levelwise equivalences, we only need to show that N takes stably trivial cofibrations to stable equivalences. We proceed by showing that N takes the generating stably trivial cofibrations to stable equivalences which are also monomorphisms, that is trivial injective cofibrations. Since N commutes with colimits and trivial injective cofibrations are preserved under pushouts and directed colimits, this then shows that N takes each stably trivial cofibration to a stable equivalence.

Recall from Proposition 4.1 that a set of generating stably trivial cofibrations for $Sp^\Sigma(sAb)$ is given by $\tilde{\mathbb{Z}}(J)$ where $J = FI_\Lambda \cup K$ is defined in [HSS, 3.4.9]. Here the maps in FI_Λ are level equivalences and the maps in K are level equivalent to maps of the form $\lambda_n \wedge X: F_{n+1}^{S^*}(X \wedge S^1) \rightarrow F_n^{S^*}X$. Here we have added a superscript to denote $F_n^C: \mathcal{C} \rightarrow Sp^\Sigma(\mathcal{C})$ the left adjoint to evaluation at level n . Since N preserves level equivalences and monomorphisms, it is enough to show that N of the map $\tilde{\mathbb{Z}}(\lambda_n \wedge X)$ is a stable equivalence. Note that in level k , $(N\tilde{\mathbb{Z}}F_m^{S^*}Y)_k \cong \Sigma_k \wedge_{\Sigma_{k-m}} N\tilde{\mathbb{Z}}(Y \wedge S^{k-m})$. The shuffle map induces maps $N\tilde{\mathbb{Z}}X \otimes N\tilde{\mathbb{Z}}A \rightarrow N\tilde{\mathbb{Z}}(X \wedge A)$ which are weak equivalences; see [SS03b, 2.7]. This in turn induces a levelwise weak equivalence $\mathcal{N} \otimes_{\text{Sym}(\mathbb{Z}[1])} F_{n+1}^{ch^+}(N\tilde{\mathbb{Z}}X \otimes N\tilde{\mathbb{Z}}S^1) \rightarrow N\tilde{\mathbb{Z}}F_{n+1}^{S^*}(X \wedge S^1)$. Similarly, there is a map $\mathcal{N} \otimes_{\text{Sym}(\mathbb{Z}[1])} F_n^{ch^+}(N\tilde{\mathbb{Z}}X) \rightarrow N\tilde{\mathbb{Z}}F_n^{S^*}X$ which is also a levelwise weak equivalence. So the two horizontal maps in the diagram below are level weak equivalences.

$$\begin{array}{ccc} \mathcal{N} \otimes_{\text{Sym}(\mathbb{Z}[1])} F_{n+1}^{ch^+}(N\tilde{\mathbb{Z}}X \otimes N\tilde{\mathbb{Z}}S^1) & \longrightarrow & N\tilde{\mathbb{Z}}F_{n+1}^{S^*}(X \wedge S^1) \\ \downarrow & & \downarrow^{N(\lambda_n \wedge X)} \\ \mathcal{N} \otimes_{\text{Sym}(\mathbb{Z}[1])} F_n^{ch^+}(N\tilde{\mathbb{Z}}X) & \longrightarrow & N\tilde{\mathbb{Z}}F_n^{S^*}(X) \end{array}$$

We next show the left vertical map is a stable equivalence. Since $N\tilde{\mathbb{Z}}X$ is cofibrant in ch^+ and $N\tilde{\mathbb{Z}}S^1 \cong \mathbb{Z}[1]$, [Hov01, 8.8] shows that $F_{n+1}^{ch^+}(N\tilde{\mathbb{Z}}X \otimes \mathbb{Z}[1]) \rightarrow F_n^{ch^+}(N\tilde{\mathbb{Z}}X)$ is a stable equivalence. Since ϕ is a weak equivalence, $\phi_* = \mathcal{N} \otimes_{\text{Sym}(\mathbb{Z}[1])} -$ is a left Quillen functor by Corollary 3.4. Hence, ϕ_* preserves stable equivalences between cofibrant objects and the left vertical map is a stable equivalence. We conclude that $N(\lambda_n \wedge X)$ is level equivalent to a stable equivalence and hence is itself a stable equivalence.

We now show $N: Sp^\Sigma(sAb) \rightarrow \mathcal{N}\text{-Mod}$ is the right adjoint of a Quillen equivalence. Denote its left adjoint by L' . Since $N: sAb \rightarrow ch^+$ preserves weak equivalences and fibrations, the prolongation preserves fibrations and weak equivalences between fibrant objects because they are levelwise fibrations and levelwise weak equivalences [Du01a, A.3]. By [Du01a, A.2], this

shows N is a right Quillen functor. Since weak equivalences between fibrant objects are level equivalences, N also detects such weak equivalences. So, by [HSS, 4.1.7] we just need to show that $\psi: X \rightarrow \overline{NL}X$ is an isomorphism for all \mathcal{N} -modules X . First consider the unit $\mathcal{N} = N\tilde{\mathbb{Z}}\mathbb{S}$. Since N commutes with evaluation at level zero (in sAb or ch^+), the associated left adjoints also commute.

$$\begin{array}{ccc}
Sp^\Sigma(sAb) & \begin{array}{c} \xleftarrow{L'} \\ \xrightarrow{N} \end{array} & \mathcal{N}\text{-Mod} \\
\begin{array}{c} \uparrow F_0^{sAb} \\ \downarrow ev_0 \end{array} & & \begin{array}{c} \uparrow \mathcal{N} \otimes F_0^{ch^+}(-) \\ \downarrow ev_0 \end{array} \\
sAb & \begin{array}{c} \xleftarrow{\Gamma} \\ \xrightarrow{N} \end{array} & ch^+
\end{array}$$

So L' of the free \mathcal{N} -module on $\mathbb{Z}[0]$ is isomorphic to the free $\text{Sym}(\tilde{\mathbb{Z}}S^1)$ -module on $\Gamma\mathbb{Z}[0] = \mathbb{Z}S^0$; that is, $L'\mathcal{N} = L'N\tilde{\mathbb{Z}}\mathbb{S} \cong \text{Sym}(\tilde{\mathbb{Z}}S^1) = \tilde{\mathbb{Z}}\mathbb{S}$. See also [SS03b, 3.7]. Since N preserves all weak equivalences, $\overline{N} \cong N$ and ψ is an isomorphism on $N\tilde{\mathbb{Z}}\mathbb{S}$. Both model categories here are stable, so \overline{N} and $\overline{L'}$ are both exact functors. Thus, ψ is an isomorphism on the localizing subcategory generated by \mathcal{N} . Since $\mathcal{N} = N\tilde{\mathbb{Z}}\mathbb{S}$ is a generator for \mathcal{N} -modules, it follows that ψ is an isomorphism on all \mathcal{N} -modules.

We use Proposition 4.2 to establish $(Sp^\Sigma(sAb), Sp^\Sigma(ch^+), \phi^*N)$ as a weak monoidal Quillen pair. The unit objects $\text{Sym}(\mathbb{Z}[1])$ and $\text{Sym}(\tilde{\mathbb{Z}}S^1)$ are both cofibrant and $\tilde{\nu}: L'\phi_*\text{Sym}(\mathbb{Z}[1]) \rightarrow \text{Sym}(\tilde{\mathbb{Z}}S^1)$ is an isomorphism since $\phi_*\text{Sym}(\mathbb{Z}[1]) = \mathcal{N}$. Finally, $\text{Sym}(\mathbb{Z}[1]) = F_0^{ch^+}\mathbb{Z}[0]$ is a generator for $Sp^\Sigma(ch^+)$ and thus detects weak equivalences. This follows from [SS03a, 2.2.1] since $[F_0\mathbb{Z}[0], X]_k = [F_k\mathbb{Z}[0], X] = [\mathbb{Z}[0], X_k]$ and weak equivalences between fibrant objects are level equivalences. \square

Next we define the left adjoint of the functor $R: Ch \rightarrow Sp^\Sigma(ch^+)$. Let I be the skeleton of the category of finite sets and injections with objects \mathbf{n} . Given X in $Sp^\Sigma(ch^+)$ define a functor $\mathcal{D}_X: I \rightarrow Ch$ by $\mathcal{D}_X(\mathbf{n}) = \mathbb{Z}[-n] \otimes X_n$. For X in $Sp^\Sigma(ch^+)$ there is a structure map $\sigma: \mathbb{Z}[m-n] \otimes X_n \rightarrow X_m$ with adjoint $\tilde{\sigma}: X_n \rightarrow \mathbb{Z}[n-m] \otimes X_m$. For a standard inclusion of a subset $\alpha: \mathbf{n} \rightarrow \mathbf{m}$ the map $\mathcal{D}_X(\alpha)$ is $\mathbb{Z}[-n] \otimes \tilde{\sigma}$. For an isomorphism in I , the action is given by the tensor product of the action on X_n and the sign action on $\mathbb{Z}[-n]$. The functor $D: Sp^\Sigma(ch^+) \rightarrow Ch$ is defined by $DX = \text{colim}_I \mathcal{D}_X$. This functor is similar to the detection functor introduced in [Shi00, 3.1.1]. Now we show that D is the left adjoint of R .

Proposition 4.5. *The functors D and R are adjoint.*

To prove this we first calculate D on free spectra.

Lemma 4.6. $D(F_m K) = \mathbb{Z}[-m] \otimes K$.

Proof. For $n \geq m$, $(F_m K)_n = \Sigma_n^+ \otimes_{\Sigma_{n-m}} \mathbb{Z}[n-m] \otimes K$ and for $n < m$, $(F_m K)_n = 0$. Since $\text{hom}_I(\mathbf{m}, \mathbf{n}) \cong \Sigma_n / \Sigma_{n-m}$ as Σ_n sets, $(F_m K)_n$ can be rewritten as $\text{hom}_I(\mathbf{m}, \mathbf{n}) \otimes \mathbb{Z}[n-m] \otimes K$. Since $\text{hom}_I(\mathbf{m}, -)$ is a free diagram (it is left adjoint to evaluation of a diagram at \mathbf{m}), the proposition follows. \square

Proof of Proposition 4.5. Recall that D^n is the acyclic chain complex modeling the n -disk. For $n > 0$, $\text{Hom}_{Sp^\Sigma(ch^+)}(F_m D^n, RY)$ is the n th level of $(RY)_m$, and for $n = 0$, $\text{Hom}_{Sp^\Sigma(ch^+)}(F_m \mathbb{Z}[0], RY)$ is the zeroth level of $(RY)_m$. Thus, it is enough to note that $\text{Hom}_{Ch}(D(F_m D^n), Y) = \text{Hom}_{Ch}(D^n, Y) = Y_{n-m}$ and $\text{Hom}_{Ch}(D(F_m \mathbb{Z}[0]), Y) = \text{Hom}_{Ch}(\mathbb{Z}[-m], Y) = Z^{-m}(Y)$ where $Z^{-m}(Y)$ is the kernel of the differential $Y_{-m} \xrightarrow{d} Y_{-m-1}$. \square

Proposition 4.7. *($\mathcal{C}h, Sp^\Sigma(ch^+), D, R$) is a strong monoidal Quillen equivalence. Moreover, R preserves all weak equivalences.*

Proof. First we show that R preserves fibrations. Actually, we show that R takes fibrations to level fibrations between fibrant objects. Since the stable model category on $Sp^\Sigma(ch^+)$ is a localization of the levelwise structure, these are in fact fibrations by [Du01a, A.3]. It is easy to check that R takes fibrations in $\mathcal{C}h$ to level fibrations in $Sp^\Sigma(ch^+)$ because a fibration in $\mathcal{C}h$ is a surjective map and a fibration in ch^+ is a surjection above degree zero. Recall that a fibrant object X in $Sp^\Sigma(ch^+)$ is the analogue of an Ω -spectrum; each X_n is levelwise fibrant and $X_n \rightarrow C_0(\mathbb{Z}[-1] \otimes X_{n+1})$ is a quasi-isomorphism. Since all objects in ch^+ are fibrant, the only condition to verify follows from checking that $(RY)_n \cong C_0(\mathbb{Z}[-1] \otimes (RY)_{n+1})$.

Next we show that R preserves and detects weak equivalences. If $X \rightarrow Y$ is a quasi-isomorphism in $\mathcal{C}h$, then $C_0(X \otimes \mathbb{Z}[n]) \rightarrow C_0(Y \otimes \mathbb{Z}[n])$ is also a quasi-isomorphism for any n . Thus $RX \rightarrow RY$ is a level equivalence and hence also a stable equivalence. Also, if $RX \rightarrow RY$ is a stable equivalence, then it must be a level equivalence since both RX and RY are fibrant. Since $H_n(RX)_m = H_{n-m}X$ for $n \geq 0$, it follows that $X \rightarrow Y$ is a quasi-isomorphism.

Since R preserves and detects weak equivalences, to show that D and R form a Quillen equivalence it is enough to check that the derived adjunction is an isomorphism on a generator as in the proofs of Proposition 4.3 and 4.4. Since $F_0(\mathbb{Z}[0])$ is a generator for $Sp^\Sigma(ch^+)$ and $RD(F_0K) = RK = F_0K$ for K in ch^+ , this follows.

Next we show that D is strong symmetric monoidal; it then follows that R is lax symmetric monoidal. First, D is symmetric monoidal because there is a natural transformation $\gamma_{X,Y} : DX \otimes DY \rightarrow D(X \wedge Y)$. Since colimits commute with tensor, the source can be rewritten as $\text{colim}_{I \times I}(X_n \otimes Y_m \otimes \mathbb{Z}[-n-m])$. The colimit in the target can be pulled back over the functor $p : I \times I \rightarrow I$ given by $p(n,m) = n+m$. Then $\gamma_{X,Y}$ is induced by the inclusion $X_n \otimes Y_m \rightarrow (X \wedge Y)_{n+m}$. To show that γ is always an isomorphism we first verify this for free spectra. By Lemma 4.6, $D(F_mK) \otimes D(F_nL) = \mathbb{Z}[-m-n] \otimes K \otimes L$ which is isomorphic to $D(F_mK \wedge F_nL) = DF_{n+m}(K \otimes L)$. It follows that γ is always an isomorphism since any spectrum Z is the coequalizer of the two maps from FFZ to FZ where $FZ = \bigoplus_n F_n(Z_n)$. Since the unit objects $F_0\mathbb{Z}[0]$ and $\mathbb{Z}[0]$ are both cofibrant, $D(F_0\mathbb{Z}[0]) = \mathbb{Z}[0]$ and $\mathbb{Z}[0]$ stably detects weak equivalences, it follows from Proposition 4.2 that this is a strong monoidal Quillen equivalence. \square

We now turn to the proof of Theorem 1.2 which states a partial extension of the main result of this paper to the commutative case over the rationals. We first need the following fact about D .

Lemma 4.8. *D preserves all weak equivalences over \mathbb{Q} .*

Proof. Any weak equivalence can be factored as a trivial cofibration followed by a trivial fibration. D preserves trivial cofibrations, since R preserves fibrations. Any trivial fibration is in fact a level equivalence, so to show that D takes trivial fibrations to weak equivalences we only need to show that rationally $DX \cong \text{hocolim}_I(X_n \otimes \mathbb{Z}[-n])$.

First, note that $\text{hocolim}_I = \text{colim}_n(\text{hocolim}_{I_n})$ where I_n is the full subcategory with objects \mathbf{i} for $i \leq n$. Similarly $\text{colim}_I = \text{colim}_n(\text{colim}_{I_n})$, so we just need to consider hocolim_{I_n} . Consider the category I/n of objects over \mathbf{n} . There is a Σ_n action on $\text{hocolim}_{I/n}$ coming from the action on \mathbf{n} . For any diagram F the quotient $(\text{hocolim}_{I/n} F)/\Sigma_n$ is isomorphic to $\text{hocolim}_{I_n} F$. Also $\text{hocolim}_{I/n} F \rightarrow F(\mathbf{n})$ is a Σ_n equivariant map and a quasi-isomorphism since $\text{Id} : \mathbf{n} \rightarrow \mathbf{n}$ is the final object in I/n . Since taking the quotient by this Σ_n action is exact rationally, $(\text{hocolim}_{I/n} F)/\Sigma_n \rightarrow F(\mathbf{n})/\Sigma_n$ is a quasi-isomorphism over \mathbb{Q} . Thus, $\text{hocolim}_{I_n} F \rightarrow F(\mathbf{n})/\Sigma_n = \text{colim}_{I_n} F$ is a quasi-isomorphism. \square

Proof of Theorem 1.2. Throughout this proof let C be a commutative $H\mathbb{Q}$ -algebra spectrum. We consider the restriction of $\Theta = Dc\phi^*NZc$ to commutative $H\mathbb{Q}$ -algebra spectra and show that for each such C , ΘC is naturally weakly equivalent to a commutative differential graded \mathbb{Q} -algebra. The three main functors D , ϕ^*N and Z in Θ are lax symmetric monoidal functors and hence strictly preserve commutative rings. The cofibrant replacement functors of monoids involved in Θ are not symmetric monoidal though. This is why ΘC is only weakly equivalent and not isomorphic to a commutative dg \mathbb{Q} algebra.

Since D rationally preserves all weak equivalences by Proposition 4.8, the cofibrant replacement transformation $c \rightarrow \text{Id}$ induces a natural weak equivalence $\Theta \rightarrow \tilde{\Theta} = D\phi^*NZc$ when restricted to $H\mathbb{Q}$ -algebras. To complete this proof we show that the functor $\tilde{\Theta}$ is related by a zig-zag of natural transformations, each of which induces a weak equivalence on any $H\mathbb{Q}$ -algebra, to a lax symmetric monoidal functor Θ' .

We have one remaining cofibrant replacement functor to consider in $\tilde{\Theta}$. We show that there is a zig-zag of natural transformations between Zc and a lax symmetric monoidal functor $\alpha^*\tilde{\mathbb{Q}}$ which induces weak equivalences on $H\mathbb{Q}$ -algebra spectra. Define $\Theta' = D\phi^*N\alpha^*\tilde{\mathbb{Q}}$; since each factor is lax symmetric monoidal so is Θ' . Since D and ϕ^*N rationally preserve all weak equivalences, the zig-zag between Zc and $\alpha^*\tilde{\mathbb{Q}}$ induces a zig-zag of weak equivalences between $\Theta'C$ and $\tilde{\Theta}C$. Hence ΘC is also naturally weakly equivalent to $\Theta'C$, a rational commutative dga, for C any $H\mathbb{Q}$ -algebra spectrum.

As above let $\tilde{\mathbb{Z}}: \mathcal{S}_* \rightarrow sAb$ be the free abelian group functor on the non-basepoint simplices. Define $\tilde{\mathbb{Q}}$ similarly. We must compare $Z = \tilde{\mathbb{Z}}(-) \wedge_{\tilde{\mathbb{Z}}(H\mathbb{Z})} H\mathbb{Z}$, defined in detail in the proof of Proposition 4.3, with its rational analogues. Define $\tilde{Q} = \tilde{\mathbb{Q}}(-) \wedge_{\tilde{\mathbb{Q}}(H\mathbb{Z})} H\mathbb{Q}$ and $Q = \tilde{\mathbb{Q}}(-) \wedge_{\tilde{\mathbb{Q}}(H\mathbb{Q})} H\mathbb{Q}$. Then $\tilde{Q} = Q \circ [(-) \wedge_{H\mathbb{Z}} H\mathbb{Q}]$ because $\tilde{\mathbb{Q}}$ is strong monoidal. The inclusion $\mathbb{Z} \rightarrow \mathbb{Q}$ induces a natural monoidal transformation $Z \rightarrow \tilde{Q}$. One can check that for c a cofibrant replacement functor for $H\mathbb{Z}$ -algebras, $c_{\mathbb{Q}} = c(-) \wedge_{H\mathbb{Z}} H\mathbb{Q}$ is a cofibrant replacement functor of $H\mathbb{Q}$ -algebras. This follows since $c \rightarrow c_{\mathbb{Q}}$ is a weak equivalence on $H\mathbb{Q}$ -algebras by [EKMM, IV.4.1] for example, since \mathbb{Q} is flat over \mathbb{Z} . So there is a natural transformation $Zc \rightarrow \tilde{Q}c \xrightarrow{\cong} Qc_{\mathbb{Q}}$. Proposition 4.3 shows that Z and U form a Quillen equivalence. Thus, since U preserves all weak equivalences, $c \rightarrow UZc$ induces a weak equivalence on any $H\mathbb{Z}$ -algebra. Similarly one can show that $c_{\mathbb{Q}} \rightarrow UQc_{\mathbb{Q}}$ induces a weak equivalence on any $H\mathbb{Q}$ -algebra. Since U detects weak equivalences it follows that $ZcA \rightarrow Qc_{\mathbb{Q}}A$ is a natural weak equivalence of algebras on any $H\mathbb{Q}$ -algebra A .

Now consider the maps $H\mathbb{Q} \xrightarrow{\alpha} \tilde{\mathbb{Q}}(H\mathbb{Q}) \xrightarrow{\beta} H\mathbb{Q}$ given by the unit and multiplication of the monad structure on $\tilde{\mathbb{Q}}$. Both of these maps induce isomorphisms on π_* , so they induce two Quillen equivalences via extension and restriction of scalars, (α_*, α^*) , (β_*, β^*) , between the respective categories of algebras over $Sp^{\Sigma}(sAb)$ [HSS, 5.4.5]. Since $\beta \circ \alpha = \text{Id}$, $\alpha^*\beta^* = \text{Id}$. So the functor Q can be rewritten as $Q \cong \beta_*\tilde{\mathbb{Q}} \cong \alpha^*\beta^*\beta_*\tilde{\mathbb{Q}}$. The functor $\tilde{\mathbb{Q}}: Sp^{\Sigma}(\mathcal{S}_*) \rightarrow Sp^{\Sigma}(s\mathbb{Q}\text{-Mod})$ is a left Quillen functor since its right adjoint, the forgetful functor, preserves weak equivalences and fibrations. So $\tilde{\mathbb{Q}}$ preserves cofibrant objects and the Quillen equivalence (β_*, β^*) induces a weak equivalence $\tilde{\mathbb{Q}}c_{\mathbb{Q}} \rightarrow \beta^*\beta_*\tilde{\mathbb{Q}}c_{\mathbb{Q}}$. Since α^* preserves all weak equivalences, this gives a natural weak equivalence $\alpha^*\tilde{\mathbb{Q}}c_{\mathbb{Q}} \rightarrow \alpha^*\beta^*\beta_*\tilde{\mathbb{Q}}c_{\mathbb{Q}} \cong Qc_{\mathbb{Q}}$. Finally, since $\tilde{\mathbb{Q}}$ also preserves all weak equivalences, there is a natural weak equivalence $\alpha^*\tilde{\mathbb{Q}}c_{\mathbb{Q}} \rightarrow \alpha^*\tilde{\mathbb{Q}}$. This produces the promised zig-zag between Zc and $\alpha^*\tilde{\mathbb{Q}}$. Notice, since α^* is lax symmetric monoidal and $\tilde{\mathbb{Q}}$ is strong symmetric monoidal functor, $\alpha^*\tilde{\mathbb{Q}}$ is lax symmetric monoidal as required. \square

Finally, we consider the two step alternative comparison of $Sp^\Sigma(ch^+)$ and Ch .

Proposition 4.9. *($Sp^\Sigma(Ch), Sp^\Sigma(ch^+), i, C_0$) and ($Sp^\Sigma(Ch), Ch, F_0, Ev_0$) are strong monoidal Quillen equivalences.*

Proof. Since fibrations in ch^+ are the maps which are surjections above degree zero, $C_0: Ch \rightarrow ch^+$ preserves fibrations and weak equivalences. Thus, C_0 preserves fibrations and weak equivalences between stably fibrant spectra because they are levelwise fibrations and levelwise weak equivalences by [Du01a, A.3]. So by the criterion given in [Du01a, A.2], i and C_0 form a Quillen adjunction on the stable model categories. For an Ω -spectrum X in $Sp^\Sigma(Ch)$ each negative homology group at one level is isomorphic to a non-negative homology group at a higher level, $H_{-k}X^n \cong H_0X^{n+k}$ for $k \geq 0$. Thus the functor C_0 also detects weak equivalences between fibrant objects. By [HSS, 4.1.7], to show (i, C_0) is a Quillen equivalence, it is thus enough to check that the derived adjunction is an isomorphism on the generator. Since $\text{Sym}(\mathbb{Z}[1])$ is concentrated in non-negative degrees, this follows.

Since the inclusion $i: ch^+ \rightarrow Ch$ is strong symmetric monoidal and $i(\text{Sym}(\mathbb{Z}[1]) \cong \text{Sym}(\mathbb{Z}[1]))$, the prolongation of i is also strong symmetric monoidal. Similarly, $C_0: Ch \rightarrow ch^+$ is lax symmetric monoidal and so is its prolongation. Since both units are cofibrant, the first pair is a strong monoidal Quillen pair.

Since Ch is a stable model category, the second pair of adjoint functors form a Quillen equivalence by [Hov01, 9.1]. Both F_0 and Ev_0 are strong symmetric monoidal functors. Since both units are cofibrant, the second pair is also a strong monoidal Quillen pair. \square

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