Minkowski Sums of Polytopes: Combinatorics and Computation

THÈSE N° 3883 (2007)

PRÉSENTÉE LE 17 AOÛT 2007

À LA FACULTÉ DES SCIENCES DE BASE

Chaire de recherche opérationnelle SO

PROGRAMME DOCTORAL EN MATHÉMATIQUES

ÉCOLE POLYTECHNIQUE FÉDÉRALE DE LAUSANNE

POUR L'OBTENTION DU GRADE DE DOCTEUR ÈS SCIENCES

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Abstract

Minkowski sums are a very simple geometrical operation, with applications in many different fields. In particular, Minkowski sums of polytopes have shown to be of interest to both industry and the academic world. This thesis presents a study of these sums, both on combinatorial properties and on computational aspects.

In particular, we give an unexpected linear relation between the f-vectors of a Minkowski sum and that of its summands, provided these are relatively in general position. We further establish some bounds on the maximum number of faces of Minkowski sums with relation to the summands, depending on the dimension and the number of summands. We then study a particular family of Minkowski sums, which consists in summing polytopes we call *perfectly centered* with their own duals. We show that the face lattice of the result can be completely deduced from that of the summands.

Finally, we present an algorithm for efficiently computing the vertices of a Minkowski sum of polytopes. We show that the time complexity is *linear* in terms of the output for fixed size of the input, and that the required memory size is *independent* of the size of the output. We also review various algorithms computing different faces of the sum, comparing their strong and weak points.

Keywords: Minkowski sums, polytopes, zonotopes, f-vectors, perfectly centered.

Résumé

Les sommes de Minkowski sont une opération géométrique très simple qui a des applications dans de nombreux domaines. En particulier, les sommes de Minkowski de polytopes se sont révélées intéressantes pour l'industrie comme pour le monde académique. Cette thèse présente une étude de ces sommes, aussi bien du point de vue des propriétés combinatoires que des aspects calculatoires.

En particulier, nous montrons une relation linéaire inattendue entre le f-vecteur d'une somme de Minkowski et ceux des sommants, si ceux-ci sont relativement en position générale. Nous présentons aussi certaines bornes sur le nombre maximal de faces d'une somme de Minkowski par rapport aux sommants, selon la dimension et le nombre de sommants. Nous étudions ensuite une famille particulière de sommes de Minkowski qui consiste à sommer des polytopes que nous appelons *parfaitement centrés* avec leur propre dual. Nous montrons que le treillis de faces du résultat peut être complètement déduit de celui des sommants.

Finalement, nous présentons un algorithme pour calculer de manière efficace les sommets d'une somme de Minkowski de polytopes. La complexité de son temps de calcul est *linéaire* en fonction de la taille du résultat, pour une taille fixe du problème, et la taille mémoire nécessaire est *indépendante* de la taille du résultat. Nous passons aussi en revue divers algorithmes calculant différentes faces de la somme, comparant leurs avantages et leurs points faibles.

Keywords: Sommes de Minkowski, polytopes, zonotopes, f-vecteurs, parfaitement centrés.

Part I

Introduction

Chapter 1

Overview

τὰς δ᾽ ἔτι τούτων ἀρχὰς ἄνωθεν θεὸς οἶδεν
καὶ ἀνδρῶν ὃς ἂν ἐκείνῳ φίλος ᾖ.
But the principles ruling them are only known to God,
and those of men who are his friends.
Plato, The Timaeus.

1.1 Combinatorial geometry

The field of combinatorial geometry can roughly be divided into two families. The first is of theoretical nature, and attempts to understand the combinatorial properties of geometrical objects. Of particular interest are *polytopes* which are a generalization of polygons and three dimensional polyhedra.

A well-known topic of combinatorial geometry, and probably the oldest to be studied, is that of *Platonic solids*. Platonic solids are three-dimensional polyhedra whose faces are identical regular polygons, and whose vertices are contained in the same number of faces. There are five of them, which are represented in Figure 1.1. Plato associated the "four most beautiful bodies" (minus the dodecahedron) to the four elements, considering they were the basis of all matter.

In the 16th century, Johannes Kepler attempted in his *Myserium Cosmographicum* to build a model of the solar system using the Platonic solids. Though the attempt failed, the study led to Kepler's laws.

Another famous result of combinatorial geometry is Euler's formula, which is a linear relation between the number of vertices, edges and faces of a polyhedron:

$$V - E + F = 2$$

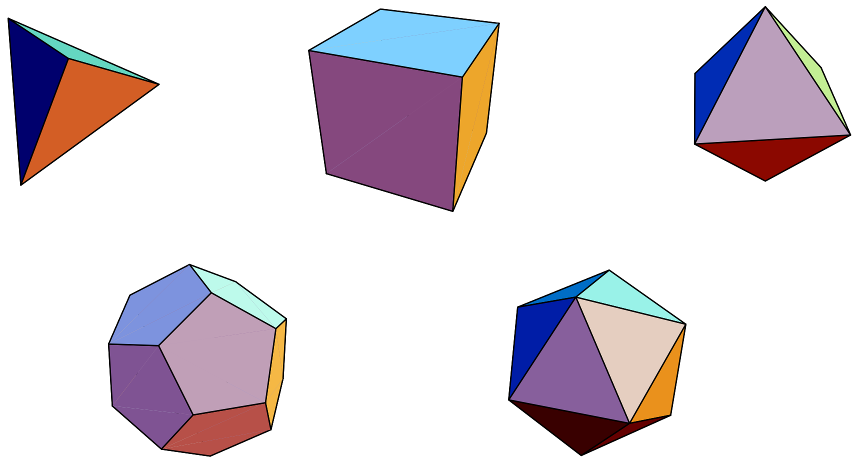


Figure 1.1: The five platonic solids.

That is, the Euler characteristic of a polyhedron is 2. Besides of being important in combinatorial geometry, this result also led to the creation of topology, by looking for solids which had different Euler characteristics, such as the torus, the Moebius strip and the Klein Bottle.

The second part of combinatorial geometry is of applied nature, and deals with algorithms computing geometrical objects and solving geometrical problems. This branch, called computational geometry, only started its real ascent with the development of computers. Since many general problems in computation, visualization, graphics, engineering and simulation can be solved by geometrical models, the field of applications is very large.

1.2 Notions

We will present here the principal notions used in this thesis. More details are given in Chapters 2 and 3. For a complete introduction, we refer to [21] and [43].

Two concepts are fundamental to combinatorial geometry, radically opposed in definition, and yet dual to each other. The first concept is *vectors*, and the second *half-spaces*.

A vector can be considered most simply as a point in a space. Since this thesis mainly deals with Euclidean geometry, the space will usually be $\mathbb{R}^d$.

Vectors are then used to build larger sets. Two vectors $\mathbf{v_1}$ and $\mathbf{v_2}$ define a unique line, whose equation is $\lambda_1\mathbf{v_1} + \lambda_2\mathbf{v_2}$, with $\lambda_1 + \lambda_2 = 1$. If additionally, we ask that λ_1 and λ_2 be non-negative, then the result is the line segment linking $\mathbf{v_1}$ to $\mathbf{v_2}$. This is known as the *convex hull* of $\mathbf{v_1}$ and $\mathbf{v_2}$. Extending the definition, we state that the convex hull of the vectors $\mathbf{v_1}, \ldots, \mathbf{v_r}$ is defined as $\lambda_1\mathbf{v_1} + \cdots + \lambda_r\mathbf{v_r}$, with $\lambda_1 + \cdots + \lambda_r = 1$, and λ_i all non-negative. The convex hull of a finite number of points is called a *polytope*. We can consider this as a *constructive* definition, since each point of the result can be written a weighted sums of vectors.

By contrast, half-spaces are used for *restrictive* definitions of geometrical bodies. A pair $(\mathbf{a}, \beta)$, of one vector in $\mathbb{R}^d$ and one scalar in $\mathbb{R}$, defines the inequality $\langle \mathbf{a}, \mathbf{x} \rangle \leq \beta$ on $\mathbb{R}^d$. A *half-space* is the subset of $\mathbb{R}^d$ consistent with an inequality: $\{\mathbf{x} \in \mathbb{R}^d \mid \langle \mathbf{a}, \mathbf{x} \rangle \leq \beta\}$, with $\mathbf{a} \neq \mathbf{0}$. By combining half-spaces, we define the subset of $\mathbb{R}^d$ consistent with all inequalities, that is the intersection of the half-spaces. The intersection of a finite number of half-spaces is called a *polyhedron*.

An important theorem of combinatorial geometry, the *Minkowski-Weyl theorem*, states that bounded polyhedra and polytopes are the same objects.

Let P be a polytope. We say $(\mathbf{a}, \beta)$ is a *valid inequality* for P if the inequality $\langle \mathbf{a}, \mathbf{x} \rangle \leq \beta$ holds for any point $\mathbf{x} \in P$. The set of points $\mathbf{x}$ of P so that $\langle \mathbf{a}, \mathbf{x} \rangle = \beta$ is then called a *face* of P. The empty set $\emptyset$ and the polytope P itself are faces, for the inequalities $(\mathbf{0}, 1)$ and $(\mathbf{0}, 0)$ which are always valid. For this reason, they are called *trivial faces*. All faces of a polytope are polytopes themselves.

Faces of polytopes can be *partially ordered* by inclusion, that is, some faces are contained in the others. The partially ordered set of faces of a polytope is called its *face lattice*. A *chain* is a subset of a face lattice which is *totally ordered*, that is, for any two distinct faces F and G in the set, either $F \subset G$ or $G \subset F$. The *length* of a chain is its cardinality minus one.

For any face F of a polytope, we define its *rank* as the length of the largest chain made of faces contained in F. For instance, the rank of the empty set, which only contains itself, is zero. The *dimension* of a face is equal to its rank minus one. This is consistent with the usual meaning of dimension. The faces of dimension 0 of a polytope, i.e. faces containing only one vector, are called its *vertices*, and the faces of dimension 1 are called *edges*. Again, these definitions are consistent with the usual meanings for geometrical bodies in dimension 2 and 3. Additionally, the faces which are only contained in themselves and the polytope are called *facets*.

Any polytope is the convex hull of its vertices. Conversely, any polytope is the intersection of the half-spaces defined by the valid inequalities corresponding to its facets.

The faces of polytopes are a fundamental subject of combinatorial geometry. Vectors in a same face share the same properties. Therefore, algorithms of computational geometry usually do not need to distinguish them. Thus, polytopes have a natural discrete decomposition into a finite number of faces. This allows us to model continuous geometric objects with discrete objects which are easier to use.

Polytopes also have a combinatorial structure. For any polytope P in $\mathbb{R}^d$, let $f_k(P)$ denote the number of k-dimensional faces of P. The series $(f_{-1}(P), (f_0(P), (f_1(P), \ldots)$ is called the *f-vector* of P. Then by Euler's formula,

$$\sum_{i=-1}^{d} (-1)^k f_k(P) = 0.$$

As a combinatorial structure, the face lattice of polytopes is also a subject of considerable interest. Many tools of both topology and combinatorics, such as CW-complexes and oriented matroids, can be said to have evolved from it.

The *Minkowski sum* of two sets of vectors S_1 and S_2 is the set of vectors which can be written as the sum of one vector of S_1 and one of S_2: $S_1 + S_2 := \{\mathbf{x_1} + \mathbf{x_2} \mid \mathbf{x_1} \in S_1,\ \mathbf{x_2} \in S_2\}$. This definition can easily be generalized to more than two summands. It is easy to see that the Minkowski sum is commutative and associative.

The Minkowski sum of polytopes is also a polytope, since it is the convex hull of the Minkowski sum of the vertices of the summands.

For each face F of a Minkowski sum of polytopes $P_1 + \cdots + P_r$, there is a unique decomposition of F into faces of the summands, that is, a list $F_1 \in P_1, \ldots, F_r \in P_r$ so that $F_1 + \cdots + F_r = F$. If $dim(F)$ is the dimension of the face F, then we have that $dim(F) \leq dim(F_1) + \cdots + dim(F_r)$, and $dim(F) \geq dim(F_i)$ for all i. Consequently vertices of a Minkowski sum are decomposed into sum of vertices of the summands.

Also, if F and G are both faces of a Minkowski sum of polytopes $P_1 + \cdots + P_r$, and their decompositions are $F_1, \ldots, F_r$ and $G_1, \ldots, G_r$, then $F \subseteq G$ if and only if $F_i \subseteq G_i$ for all i.

Thus, Minkowski sums of polytopes are of interest not only as a geometric operation, but also as a combinatorial operation on the face lattices of the polytopes.

1.3 Goals

The first goal of this thesis is to better understand the combinatorial properties of Minkowski sums of polytopes.

The number of faces of the result is of particular interest. The only exact formulas which have been found up to now are for *zonotopes*, which are the Minkowski sums of line segments ([42]). Our goal is to find such formulas for new families of Minkowski sums.

If exact equations cannot be found, it is interesting to have *bounds* on the number of faces of Minkowski sums in terms of that of the summands. The combinatorial interest aside, this helps us find complexity results for algorithms. The only bounds currently known have been published in [18]. They prove that for any k, the number of k-faces of a Minkowski sum is smaller than that of the zonotope which is the sum of all the non-parallel edges of the summands. Our goal is to find other bounds on the f-vector of Minkowski sums in terms of that of the summands, depending on the dimension and the number of summands.

The second goal of this thesis is to investigate and develop algorithms to compute Minkowski sums of polytopes.

As most applications of Minkowski sums up to now have been in low dimension, little is known about the general complexity of algorithms enumerating their faces. A problem of such algorithms is that the output can easily be exponential. For instance, the sum of d orthogonal line segments in $\mathbb{R}^d$ is the hypercube, which has 2^d vertices. Therefore, their efficiency should be evaluated not only in terms of the input, but also of the output.

A further problem is that the complexity of finding the vertices of a polytope from its facets and inversely is an *open problem*. All currently known algorithms can take exponential time and memory size, even if both the input and the output is small.

Consequently, barring a significant advance in computational geometry, any algorithm computing the facets of a Minkowski sum from the vertices of the summands, or the vertices of the sum from the facets of the summands, will have such an exponential complexity.

An algorithm has been proposed by Fukuda for computing the vertices of the sum for those of the summands. It is our goal to implement this algorithm and test its efficiency.

1.4 Contents and results

Part I of the thesis is an introduction to the subject. Part II contains a theoretical study on faces of Minkowski sums. Part III part contains all studies about algorithms computing Minkowski sums of polytopes in various forms. Part IV is the conclusion of the thesis.

1.4.1 Part I – Introduction

Chapter 1 introduces and resumes the thesis. Section 1.1 is an introduction to combinatorial geometry. In Section 1.2, we present the necessary terms. Section 1.3 introduces the goals of this thesis. This section, Section 1.4 resumes each part of the thesis. Section 1.5 presents the history of Minkowski sums.

In Chapter 2, we give an introduction to polytopes, and all relevant related notions. Section 2.1 introduces sets generated by sums of vectors, and hulls. Section 2.2 presents sets defined restrictively by half-spaces, and explains their identities with generated sets. Section 2.3 defines in detail faces and their properties. Section 2.4 introduces the important notion of duality. Section 2.5 presents particular families of polytopes, such as hypercubes and simplices.

Chapter 3 presents Minkowski sums. Section 3.1 shows their general properties. Section 3.2 introduces particular geometric constructions in which Minkowski sums occur. Section 3.3 presents the simplest of Minkowski sums, which are zonotopes, and the formulas to compute the number of their faces.

1.4.2 Part II – Face study

In Chapter 4, we present different bounds on the number of faces of Minkowski sums of polytopes. In Section 4.1, we show a trivial bound on the number of vertices of the sum, which is the product of the number of vertices of the summands. We give the exact conditions on the dimension and the summands for this trivial bound to be attainable. Namely, for the sum of r polytopes in dimension d, the trivial bound can be attained if and only if $r < d$, or $r = d$ and all polytopes are line segments.

In Section 4.2, we give a construction showing how the trivial bound presented in preceding section can be attained. Furthermore, we show two specific constructions for polytopes in dimension three, maximizing respectively minimizing the number of facets for a fixed number of vertices.

In Section 4.3, we give a general bound on the number of k-dimensional faces of a Minkowski sum of r polytopes in dimension d, by extending the trivial bound presented in Section 4.2. Furthermore, we prove that this bound is tight if $2(r+k) \leq d$, by showing that it is attained by polytopes which have all their vertices on the *moment curve*: $\lambda \mapsto (\lambda, \lambda^2, \ldots, \lambda^d)$.

In Chapter 5, we present a special family of Minkowski sums of polytopes. For any face F of a sum of polytopes and its decomposition $F_1, \ldots, F_r$, we say that F has an *exact decomposition* if $dim(F) = dim(F_1) + \cdots + dim(F_r)$. If all facets of a Minkowski sum have exact decompositions, we say that the summands are *relatively in general position.*

In Section 5.1, we present two important results about Minkowski sums of polytopes which are relatively in general position. First, we show that the maximum number of faces in a Minkowski sum is always attained by summands relatively in general position. Then, we show that the f-vector of the sum and that of summands are linked by a linear relation. Namely, if $P = P_1 + \cdots + P_r$ is a Minkowski sum of d-dimensional polytopes relatively in general position, then

$$\sum_{k=0}^{d-1} (-1)^k k \left(f_k(P) - (f_k(P_1) + \cdots + f_k(P_r))\right) = 0.$$

The relation breaks down if the summands are of lower dimension.

In Section 5.2, we use this linear relation with the constructions of Section 4.2 to show tight maximal bounds on the number of vertices, edges and facets of the Minkowski sum of two polytopes in dimension three, for fixed number of vertices or facets of the summands.

In Chapter 6, we present another special family of Minkowski sums. Let P be a d-dimensional polytope containing the vector $\mathbf{0}$ in its interior. Its *dual polytope* P^* is the set of vectors $\mathbf{a}$ so that $(\mathbf{a}, 1)$ is a valid inequality for P. The dual polytope P^* has a face lattice which is anti-isomorphic to that of P. That is, for any nonempty face F of P, and any vector $\mathbf{x}$ in the relative interior of F, the inequality $(\mathbf{x}, 1)$ is valid for P^* and defines the *associated dual face* F^D of P^*, so that $dim(F^D) = d - 1 - dim(F)$.

In Section 6.1, we present a result of Nesterov proving that the Minkowski sum of a polytope P with its dual polytope is more spherical than P. For this reason, we call this operation *Nesterov rounding.*

In Section 6.2, we define the following notion. Let P be a polytope. We say that P is *perfectly centered* if for any nonempty face F of P, there is a vector $\mathbf{m_F}$ in the relative interior of F so that $(\mathbf{m_F}, \langle \mathbf{m_F}, \mathbf{m_F} \rangle)$ is a valid inequality defining the face F.

We show that the Nesterov rounding of a perfectly centered polytope P can be completely deduced from the face lattice of P. That is, the faces of $P + P^*$ can be characterized as the sum $G + F^D$, for any nonempty faces G

and F of P so that $G \subseteq F$. Also, for any faces $G_1 + F_1^D$ and $G_2 + F_2^D$ of the Nesterov rounding, we have that $G_1 + F_1^D \subseteq G_2 + F_2^D$ if and only if $G_1 \subseteq G_2$ and $F_2 \subseteq F_1$. Finally, we prove that a perfectly centered polytope and its dual are always relatively in general position, and that the Nesterov rounding of a perfectly centered polytope is also a perfectly centered polytope.

In Section 6.3, we show that successive Nesterov roundings of a perfectly centered polytope in dimension three have a facet to vertices ratio which tends towards one.

The *fatness* of a polytope in dimension four is the number of faces of dimension 1 and 2 divided by the number of faces of dimension 0 and 3. In Section 6.4, we show that successive Nesterov roundings of a perfectly centered polytope in dimension four have a *fatness* which tends towards three.

In Section 6.5, we compute the f-vectors of the Nesterov roundings of perfectly centered hypercubes and simplices.

1.4.3 Part III – Algorithms

In Chapter 7, we present an algorithm of Fukuda to compute the vertices of a Minkowski sum from the vertices of its summands. Section 7.1 states the properties of edges and vertices of Minkowski sums which are used by the algorithm.

Section 7.2 presents the reverse search method, on which the algorithm is based. The reverse search method consists in enumerating the vertices of a graph by covering them with an arborescence, which is explored by a depth-first search. It requires two oracles, the first giving the list of adjacent vertices in the graph, the second giving the parent of a vertex in the arborescence. If properly implemented, a reverse search method has a number of steps which is *linear* in the size of the output, and the required memory size is *independent* of the size of the output.

Section 7.3 then shows how we can apply the reverse search method to Minkowski sums of polytopes, and defines the two oracles with the help of linear programming.

In Section 7.4, we present an implementation we did of the algorithm, and we study its efficiency when used to solve different problems.

In Chapter 8, we study algorithms to compute the facets of Minkowski sums. In particular, we explain why the problem is much harder than computing vertices. In Section 8.1, we present the difficulty of computing the facets of a polytope from its vertices, taking as example the *double description method* and the *beneath and beyond method*.

In Section 8.2, we present an algorithm proposed in [22], which is efficient for computing the facets of a Minkowski sum in dimension three, but difficult to generalize to more dimensions.

In Section 8.3, we present a very fast algorithm developed and implemented by Peter Huggins ([23]) for a special case of Minkowski sums, computing both facets and vertices. It is actually an implementation of the beneath and beyond method, which uses a black box to find the vertices as it needs them.

1.4.4 Part IV – Conclusion

In Chapter 9, we present possible future developments of the thesis, on the subject of bounds and combinatorial aspects as well as algorithms. Though some progress was made during the course of this PhD, it is with some pleasure that we can assert the combinatorial study of Minkowski sums has barely begun.

1.5 Historical review

Minkowski sums are a very simple and intuitive operation. Though the German mathematician Hermann Minkowski was not the first one to study them, they were named after him due to the extensive research he did on them. A first result about Minkowski sums was the Brunn-Minkowski Theorem, which was presented by Brunn for his inaugural dissertation in Münich in 1887 ([7]), and later completed and refined by Minkowski ([32]).

A formal definition of the sums can be found in *Volumen und Oberfläche* [31], albeit in a rather different manner from today's definitions. The article examines in particular the so-called *mixed volumes* of the sums. Mixed volumes have a multitude of theoretical applications in conjunction with Minkowski inequalities.

This approach of Minkowski sums seems to have been the only one used for almost a century. No studies appear to have been made on their combinatorial properties. A tentative explanation may be found in the preface of Grünbaum's "Convex Polytopes" ([20]):

> *About the turn of the century, however, a steep decline in the interest in convex polytopes was produced by two causes working in the same direction. Efforts at enumerating the different combinatorial types of polytopes, started by Euler and pursued with much patience in ingenuity during the second half of the XIXth*

> *century, failed to produce any significant results even in the three-dimensional case; this lead to a widespread feeling that the interesting problems concerning polytopes are hopelessly hard. Simultaneously, the ascendance of Klein's "Erlanger Program" and the spread of its normative influence tended to cast the preoccupation with the combinatorial theory of convex polytopes into a rather disreputable rôle-and that at a time when such "legitimate" fields as algebraic geometry and in particular topology started their spectacular development.*

It is understandable that the scientific community lost interest in a field which brought comparatively little theoretical results. It is because of *applications* that interest in combinatorial geometry was rekindled, in particular via linear programming.

Linear programming consists in optimizing a linear function on a polyhedron. Though the initial studies of the geometrical problem go back as far as Fourier, it is only during Second World War that linear programming was developed and applied, as a means to enhance warfare and logistics. Before computers, little point was seen in studying such problems, since the computational work seemed likely to outweigh the benefits of the solution.

However, in 1947, Dantzig proposed independently the *simplex method* which solves linear programs by following a path to the optimal vertex along the edges of the polyhedron, as had been suggested by Fourier. Though the method can be inefficient in worst cases, it turned out to work well for most instances encountered, and linear programming was soon to be applied to a wide range of problems in management and engineering. If only for reasons of efficiency, the combinatorial properties of polyhedra were then the object of study again.

With the rise of computers, many problems of operations research, control theory and computer graphics came to be formulated and solved as problems of combinatorial geometry. The resulting field became known as *computational geometry*.

It is interesting to note that even then, little seems to have been done on the particular subject of Minkowski sums of polytopes.

It is in 1979 that a seminal paper of Lozano-Pérez and Wesley ([29]) brought Minkowski sums to attention in a completely different field, showing their usefulness in the planning of collision-free paths. These sums have since then been studied extensively in related subjects, such as computer-aided design ([10]) and robot motion planning ([26]).

These works, motivated by industrial applications, usually concentrated on the geometric property of Minkowski sums, i.e. the *growing* of shapes by spherical balls first, then by polytopes. They did not address the combinatorial properties, especially since the applications were mostly in low dimensions. However, it was the start of the search for algorithms computing Minkowski sums of polytopes. These algorithms were for the largest part limited to dimension two or three ([9],[17],[22], [24]).

The first (and almost only) study of the complexity of Minkowski sums of polytopes was done in [18]. The paper also introduces an application of Minkowski sums in algebra, namely the computation of Gröbner bases.

In 2004, Fukuda published an algorithm for summing $\mathcal{V}$-polytopes of any dimension ([13]). The motivation came from an industrial application [37]. It was the starting point of this PhD.

Hermann Minkowski

Hermann Minkowski is born in Lithuania in 1864, to a German family. He is the younger brother of Oskar Minkowski, who will later become a renowned pathologist. When Hermann is eight, his family returns to Germany and settles in Königsberg.

He does most of his studies there, occasionally spending a semester at the University of Berlin. He becomes friends with Hilbert, who is also a student in Königsberg. In 1883, he is awarded a prize from the French Academy of Sciences for solving the problem of the number of representations of an integer as the sum of five squares. In 1885, he submits his PhD about quadratic forms.

Minkowski starts teaching in 1887 at the University of Bonn, where he is named professor in 1892. He then teaches at Königsberg for two years, before receiving in 1894 a position at the Eidgenössisches Polytechnikum in Zürich, where he has Albert Einstein as a student in his lectures.

Minkowski marries in 1897, and accepts in 1902 a chair at the University of Göttingen, which is created for him at the instigation of Hilbert. There, he supervises the PhD of Carathéodory.

In 1907, Minkowski invents a way of coupling time and space in a non-Euclidean manner which explains in an elegant way the work of Einstein and Lorentz. This “space-time continuum” will to be the foundation of all mathematical works about relativity.

Minkowski dies suddenly from appendicitis at age 44.

Chapter 2

An introduction to polytopes

The time has come, the Walrus said,
To talk of many things:
Of shoes–and ships–and sealing-wax–
Of cabbages–and kings–
And why the sea is boiling hot–
And whether pigs have wings.
Lewis Caroll, Through the Looking-Glass.

In this chapter, we define the basic terms used throughout this thesis. Though we define as many terms as possible for the sake of completeness, we assume the reader has some understanding of vector spaces on real numbers.

Notations

The field of real numbers is denoted by $\mathbb{R}$. Accordingly, vector spaces defined on $\mathbb{R}$ are denoted by $\mathbb{R}^1$, $\mathbb{R}^2$, $\mathbb{R}^d$, where the number in exponent is an integer representing the dimension of the space.
Vectors are denoted in boldface type, such as $\mathbf{x}$, $\mathbf{y}$, $\mathbf{z}$, $\mathbf{x}_0$, $\mathbf{x}_1$. The zero vector is denoted by $\mathbf{0}$.
Matrices are denoted by capital letters, such as A, B, M_0, M_1.
Scalars are represented in italics. Scalars taking noninteger values are denoted by Greek letters, such as α, β, λ, λ_0, λ_1, λ_2. Integers are represented by Latin letters, such as a, b, c_0. To simplify reading, some of these keep the same signification for most of the thesis. The letter d always represents the full dimension of the vector space, k represents the dimension of objects in the vector space, and the letter r represents the number of objects in a finite set.

For the reader's convenience, we introduce the main concepts without proof, except when the proof itself offers insight into the subject. We advise those wishing to study the topics in more details to refer to [21] or [43].

2.1 Generated sets

We introduce here different continuous combination of vectors which are essential in the geometry of polytopes.

Definition 2.1.1 (Linear, Affine combination) Let $\mathbf{x}_1, \ldots, \mathbf{x}_n$ be vectors in a real vector space, and let $\lambda_1, \ldots, \lambda_n$ be scalars in $\mathbb{R}$. Then $\lambda_1\mathbf{x_1} + \cdots + \lambda_n\mathbf{x_n}$ is called a *linear combination* of the vectors $\mathbf{x}_1, \ldots, \mathbf{x}_n$. If, in addition, $\lambda_1 + \cdots + \lambda_n = 1$, then it is an *affine combination* of $\mathbf{x}_1, \ldots, \mathbf{x}_n$.

Definition 2.1.2 (Conic, Convex combination) Let $\mathbf{x}_1, \ldots, \mathbf{x}_n$ be vectors in a real vector space, and let $\lambda_1, \ldots, \lambda_n$ be non-negative scalars in $\mathbb{R}$. Then $\lambda_1\mathbf{x_1}+\cdots+\lambda_n\mathbf{x_n}$ is called a *conic combination* of the vectors $\mathbf{x}_1, \ldots, \mathbf{x}_n$. If, in addition, $\lambda_1+\cdots+\lambda_n = 1$, then it is a *convex combination* of $\mathbf{x}_1, \ldots, \mathbf{x}_n$.

Definition 2.1.3 (Independence) Vectors $\mathbf{x}_1, \ldots, \mathbf{x}_n$ are called *independent* or *linearly independent* if the representation of any of their linear combinations is unique.

$$\lambda_1\mathbf{x_1} + \cdots + \lambda_n\mathbf{x_n} = \mu_1\mathbf{x_1} + \cdots + \mu_n\mathbf{x_n} \ \Leftrightarrow\ \lambda_i = \mu_i,\ \forall i.$$

Similarly, they are called *affinely independent* if the representation of any of their affine combinations is unique.

Definition 2.1.4 (Dimension) The *dimension* of a set S of vectors, denoted by $dim(S)$ is equal to the cardinality of the largest affinely independent subset of S minus one.

Definition 2.1.5 (Hulls) The *linear hull*, respectively *affine hull*, *conic hull*, and *convex hull* of a set S of vectors is the set of vectors which can be written as linear, respectively affine, conic, and convex combinations of elements of S.

Hulls of a set are necessarily larger than the set itself. Here is the opposite concept:

Definition 2.1.6 (Generated) We say a vector set S is *generated* by the vector set T if S is the hull of T.

There is an essential distinction to make between these different types of hulls. Inside a space of finite dimension (such as $\mathbb{R}^d$, for instance), linear and affine hulls can always be generated by a finite number of vectors. This is not always the case for conic and convex hulls. Therefore, for the latter, we sometimes say a set is *finitely generated*, which means it can be generated by a finite number of vectors.

We now define for the first time the main concept of our work, the Minkowski sum. Though the concept is needed in following paragraphs, it is defined and studied in detail later in chapter 3.

Definition 2.1.7 (Minkowski sum) Let S_1 and S_2 be two sets of vectors. Their *Minkowski sum* is defined as the set of vectors which can be written as the sum of a vector in S_1 and a vector in S_2. Namely:

$$S_1 + S_2 := \{\mathbf{x_1} + \mathbf{x_2} \mid \mathbf{x_1} \in S_1,\ \mathbf{x_2} \in S_2\}$$

Definition 2.1.8 (Linear subspace) A *linear subspace* is a nonempty set which is equal to its linear hull.

The smallest possible linear subspace only contains $\mathbf{0}$.

Definition 2.1.9 (Basis) An independent set is called a *basis* of the linear subspace it generates.

The usual representation of a vector in $\mathbb{R}^d$ is the list of coefficients used for writing the vector as the linear combination of a chosen basis of $\mathbb{R}^d$, which is called the *canonical basis*.

If the canonical basis of $\mathbb{R}^d$ is denoted by the vectors $\mathbf{e_1}, \ldots, \mathbf{e_d}$, then a vector $\mathbf{x} \in \mathbb{R}^d$ such that $\mathbf{x} = \lambda_1\mathbf{e_1} + \cdots + \lambda_d\mathbf{e_d}$ is represented as

$$\begin{pmatrix} \lambda_1 \\ \vdots \\ \lambda_d \end{pmatrix} := \lambda_1\mathbf{e_1} + \cdots + \lambda_d\mathbf{e_d}$$

Definition 2.1.10 (Affine space) An *affine space* is a possibly empty set which is equal to its affine hull.

Linear spaces are equivalent to affine spaces containing $\mathbf{0}$.

Definition 2.1.11 (Closed (convex) cone) A *closed cone* is a nonempty set which is equal to its conic hull.

Again, the smallest possible closed cone only contains the zero vector ($\mathbf{0}$). The literature often defines these objects simply as cones or convex cones. However, we later make a extensive usage of cones which are not closed, and which should not be confused with those we define here.

Definition 2.1.12 (Convex set) A *convex* set is a possibly empty set which is equal to its convex hull.

All closed cones are convex.

It is easy to see from these definitions that the linear hull of a set S is the smallest linear subspace containing S, and equivalently for other hulls.

2.2 Polyhedral sets

While the definitions in the preceding section can be said to be *constructive*, this section presents definitions which are on the contrary *restrictive*. The *duality principle* ruling the relations between these two types of definitions is considered by many as one of the most fascinating concepts of geometry, if not mathematics. The subject is presented in details in Section 2.4.

The main element of restrictive definitions is the following:

Definition 2.2.1 (Half-space) Let $\mathbf{a} \neq \mathbf{0}$ be a vector of $\mathbb{R}^d$ and β a scalar. The pair $(\mathbf{a}; \beta)$ defines a linear inequality in $\mathbb{R}^d$:

$$\langle \mathbf{a}, \mathbf{x} \rangle \leq \beta$$

The *half-space* of $\mathbb{R}^d$ defined by a linear inequality is the set of vectors in $\mathbb{R}^d$ for which the inequality holds.

By combining a number of such restrictions, we get:

Definition 2.2.2 (Polyhedron) A *polyhedron* (plur: *polyhedra*), or *polyhedral set* in $\mathbb{R}^d$ is the intersection of a finite number of half-spaces.

Rather than enumerating a list of n pairs $(\mathbf{a}_i; \beta_i)$ defining single inequalities, it is usual to define a $n \times d$ matrix A which has the different $\mathbf{a}_i$ as its n line vectors, and a vector $\mathbf{b} \in \mathbb{R}^n$ with the β_i, so that it is possible to write the polyhedron as:

$$\{\mathbf{x} \mid A\mathbf{x} \leq \mathbf{b}\}$$

Though the meaning of polyhedra in everyday speech is usually three-dimensional bounded bodies, the definition we have of polyhedra allows for bodies of any dimension, which may well be unbounded.

For instance, points, straight lines, the empty set and $\mathbb{R}^d$ are all polyhedra of $\mathbb{R}^d$.

Theorem 2.2.3 (Minkowski-Weyl) *Any polyhedron is the Minkowski sum of a finitely generated closed cone and a finitely generated convex set, and conversely. That is, P is a polyhedron if and only if there are vectors $\mathbf{v_1}, \ldots, \mathbf{v_n}, \mathbf{r_1}, \ldots, \mathbf{r_m}$ of $\mathbb{R}^d$ such that:*

$$P = \{\lambda_1 \mathbf{v_1} + \ldots + \lambda_n \mathbf{v_n} + \mu_1 \mathbf{r_1} + \ldots + \mu_m \mathbf{r_m} \mid \lambda_1 + \ldots + \lambda_n = 1,\ \lambda_i, \mu_i \geq 0\}$$

As we can see, polyhedra can be described either as a set of inequalities or as a set of generators. These two representations are commonly called $\mathcal{H}$-representation (for half-space) and $\mathcal{V}$-representation (for vertex). If the representation is minimal, the vectors $\mathbf{v_i}$ and $\mathbf{r_i}$ in the $\mathcal{V}$-representation presented here are called *vertices* and *rays*.

There are two natural restrictions of this theorem, by excluding either of the conical and convex combinations.

Theorem 2.2.4 (Minkowski-Weyl for cones) *Any polyhedral cone is a finitely generated closed cone, and conversely. That is, P is a polyhedral cone if and only if there are vectors $\mathbf{r_1}, \ldots, \mathbf{r_m}$ of $\mathbb{R}^d$ such that:*

$$P = \{\mu_1 \mathbf{r_1} + \ldots + \mu_m \mathbf{r_m} \mid \mu_i \geq 0\}$$

Let us now define our main object of study:

Definition 2.2.5 (Polytope) A *polytope* is a bounded polyhedron.

We should note that the word polytope is sometimes also used in the literature to describe non-convex objects, in which case the words *convex polytope* are used instead for this definition. Nevertheless, this thesis uses consistently the word polytope for convex objects.

We say a polytope P in $\mathbb{R}^d$ is *full-dimensional* if its dimension is d. Unless specifically stated otherwise, we always assume the polytopes we are dealing with are full-dimensional.

Theorem 2.2.6 (Minkowski-Weyl for polytopes) *A polytope is a finitely generated convex set, and conversely. That is, P is a polytope if and only if there are vectors $\mathbf{v_1}, \ldots, \mathbf{v_n}$ of $\mathbb{R}^d$ such that:*

$$P = \{\lambda_1 \mathbf{v_1} + \ldots + \lambda_n \mathbf{v_n} \mid \lambda_1 + \ldots + \lambda_n = 1,\ \lambda_i \geq 0\}$$

Depending on the situation, it is often more efficient to consider a polytope either as a convex combination or as an intersection of half-spaces. When working on theoretical properties of polytopes, the previous theorem tells us that we can jump from one representation to the other as it fits our needs.

However, the actual conversion between the $\mathcal{H}$-representation and the $\mathcal{V}$-representation of polytopes, and more generally polyhedra, is a nontrivial problem. The known methods can take an exponential number of steps compared to the complexity of the representation. These methods are described later in Chapter 8.1.

2.3 Faces

Now has come the time to define a list of terms which are commonly used to describe polytopes.

Definition 2.3.1 (Valid inequality) Let S be a set in $\mathbb{R}^d$. A *valid inequality* for S is an inequality which holds for all vectors in S. That is, the pair $(\mathbf{a};\beta)$ is a valid inequality of S if and only if

$$\langle \mathbf{a}, \mathbf{x} \rangle \leq \beta, \quad \forall \mathbf{x} \in S.$$

Definition 2.3.2 (Face) For any valid inequality of a polytope, the subset of the polytope of vectors which are tight for the inequality is called a *face* of the polytope. That is, the set F is a face of the polytope P if and only if

$$F = \{\mathbf{x} \in P \mid \langle \mathbf{a}, \mathbf{x} \rangle = \beta\},$$

for some valid inequality $(\mathbf{a};\beta)$ of P.

The set of faces of a polytope P is denoted by $\mathcal{F}(P)$

Two important faces are included in this definition, the empty set $\emptyset$ and the polytope itself, which are defined by the always valid inequalities $(\mathbf{0};1)$ and $(\mathbf{0};0)$. For this reason, these two faces are called *trivial faces*.

The faces of dimension 0, 1, $d-2$ and $d-1$ are respectively called *vertices*, *edges*, *ridges* and *facets*. The set of vertices of a polytope P is denoted by $\mathcal{V}(P)$.

Theorem 2.3.3 *A polytope is the convex hull of its vertices.*

Proof. Let P be a polytope in $\mathbb{R}^d$. We know P can be written as the convex hull of a finite set of points S. Let us suppose that S is minimal.

Let $\mathbf{v} \in S \setminus \mathcal{V}(P)$. Since S is minimal, $\mathbf{v} \notin conv(\mathcal{V}(P))$. So there is a valid inequality $(\mathbf{a}, \beta)$ for $conv(\mathcal{V}(P))$ so that $\langle \mathbf{a}, \mathbf{v} \rangle > \beta$. This means that $\mathbf{a}$ defines a face F of P with $F \subset P \setminus conv(\mathcal{V}(P))$, with $\mathcal{V}(F) \subseteq \mathcal{V}(P)$, a contradiction. ∎

Theorem 2.3.4 *Any face of a polytope is also a polytope.*

Proof. Let P be a polytope in $\mathcal{H}$-representation, and F a face of P defined by the valid inequality $(\mathbf{a}, \beta)$. It is enough to add the inequalities $(\mathbf{a}, \beta)$ and $(-\mathbf{a}, -\beta)$ to P to obtain an $\mathcal{H}$-representation of F. ∎

Definition 2.3.5 (Face lattice) Let P be a polytope. The *face lattice* of the polytope $\mathcal{L}(P)$ is the set of faces $\mathcal{F}(P)$ partially ordered by inclusion. That is, for F and G in $\mathcal{L}(P)$, we have $F \leq G$ if and only if $F \subset G$.

When the term face lattice is used, it generally implies we are considering the faces as abstract elements ordered by inclusion, leaving aside geometrical notions.

Definition 2.3.6 (Chain) If $\mathcal{L}(P)$ is the face lattice of a polytope, a *chain* S is a subset of $\mathcal{L}(P)$ which is totally ordered, that is, for any distinct $F, G \in S$, we have either $F \subset G$ or $G \subset F$. The *length* of a chain is its cardinality minus one.

If $\{F_1, \ldots, F_n\}$ is a chain, there is an ordering $i_1, \ldots, i_n$ such that $F_{i_1} \subset \cdots \subset F_{i_n}$.

Theorem 2.3.7 (Face lattices) *Let P be a polytope, and $\mathcal{L}(P)$ its face lattice. Then we have the following:*

1. *The face lattice $\mathcal{L}(P)$ has a unique minimal element, which is the empty set $\emptyset$, and a unique maximal element, which is P.*

2. *The face lattice $\mathcal{L}(P)$ is* graded, *which means that all maximal chains of $\mathcal{L}(P)$ have the same length.*

3. *Let F and G be two faces in $\mathcal{L}(P)$. Then there is a unique maximal face $F \wedge G$ they both contain, and a unique minimal face $F \vee G$ containing them.*

Definition 2.3.8 (Rank) Let P be a polytope, and F a face of P. The *rank* of F in the face lattice $\mathcal{L}(P)$ is the length of the longest chain of $\mathcal{L}(P)$ which has $\emptyset$ and F as minimal and maximal elements respectively.

The following chart resumes some of these definitions.

Dimension	Rank	Name
d	$d+1$	polytope
$d-1$	d	facets
$d-2$	$d-1$	ridges
$\dots$	$\dots$	$\dots$
1	2	edges
0	1	vertices
-1	0	empty set

While defining the dimension of the empty set $\emptyset$ as minus one might seem strange, it not only follows the logic of Definition 2.1.4, it will also be very useful for stating some relations later.

A recurrent question in computational geometry is the study of the complexity of polytopes. How many vertices do they have? How many facets? What about other faces? Although the question seems rather straightforward, it is often difficult to find bounds on these numbers for particular families of polytopes.

Definition 2.3.9 (F-vector) Let P be a polytope. we denote by $f_k(P)$ the number of faces of dimension k of P. The series $(f_{-1}(P), f_0(P), f_1(P), \dots)$ is called the *f-vector* of P

By definition, we have $f_{-1}(P) = f_{dim(P)}(P) = 1$. We also consider sometimes that $f_k(P) = 0$ for $k < -1$ and $k > dim(P)$.

The most basic fact about f-vectors of polytopes is the following:

Theorem 2.3.10 (Euler) *Let P be a d-dimensional convex polytope. Then:*

$$\sum_{k=-1}^{d} (-1)^k f_k(P) = 0.$$

Equivalently, we can write this formula the following ways:

$$\sum_{k=0}^{d} (-1)^k f_k(P) = 1,$$

$$\sum_{k=0}^{d-1} (-1)^k f_k(P) = 1 - (-1)^d.$$

In low dimensions, this means that 1-dimensional polytopes have two vertices, and a 2-dimensional polytopes have as many vertices as edges. For

3-dimensional polytopes, it means that if V is the number of vertices of a polytope, E its number of edges and F its number of facets, then

$$V - E + F = 2,$$

which is the form it is usually taught in high school.

We show now that the Euler formula also applies to certain subsets of face lattices.

Definition 2.3.11 (Interval) Let P be a polytope in $\mathbb{R}^d$, and F and G two faces of P so that $F \subseteq G$. The set of faces of P containing F and contained in G is called an *interval*:

$$[F, G]_P \;:=\; \{H \in \mathcal{F}(P) \mid F \subseteq H \subseteq G\}.$$

Intervals inherit a partial order from the lattice they are part of.

Let us now state a theorem about intervals which is proved in the next section, after the necessary definitions.

Theorem 2.3.12 (Intervals) *Let P be a polytope in $\mathbb{R}^d$, and F and G two faces of P so that $F \subset G$. Then the Euler formula applies to the interval $[F, G]_P$. That is,*

$$\sum_{k=dim(F)}^{dim(G)} (-1)^k f_k([F, G]_P) = 0.$$

As we can see, the face lattices of polytopes are very structured. Let us now present some more of their properties.

Definition 2.3.13 (Flag f-vector) Let P be a polytope in $\mathbb{R}^d$. We define the *flag f-vector* of P as follows: For any $S = \{i_1, \ldots, i_n\}$ subset of $\{-1, 0, 1, \ldots, d\}$, $f_S(P)$ is defined as the number of distinct chains $\{F_1, \ldots, F_n\}$ of faces of P, with $dim(F_k) = i_k$ for all k.

For instance, the value $f_{\{0,1\}}(P)$ is the number of pairs of a vertex and an edge of the polytope P, with the edge containing the vertex. It is easy to see that the f-vector of a polytope is contained in its flag f-vector: $f_i(P) = f_{\{i\}}(P)$. To simplify, brackets and commas are often omitted from the notation: $f_{023} := f_{\{0,2,3\}}$.

Flag f-vectors partially encode the information of the face lattice of a polytope. They also contain an equivalent of Euler's formula, called *Bayer-Billera relations* or sometimes *extended Dehn-Sommerville relations*:

Theorem 2.3.14 ([2],[27]) *Let P be polytope of dimension d. Let S be a subset of $\{-1, \ldots, d\}$, and i and k be in S so that $i < k$, and S contains no j so that $i < j < k$. Then*

$$\sum_{j=i}^{k} (-1)^j f_{S \cup j}(P) = 0$$

Proof. These relations are direct consequences of the fact that Euler's formula applies to all intervals of a polytope. Thus, the proof consists in showing that they are essentially a sum of multiples of Euler's Formula.

Let P be a polytope of dimension d. Let S be a subset of $\{-1, \ldots, d\}$, and i and k be in S so that $i < k$, and S contains no j so that $i < j < k$. Let j be so that $i \leq j \leq k$. The value $f_{S \cup j}(P)$ represents a number of choices of chains which we now decompose into three choices:

1. Choice of two faces F_i and F_k so that $F_i \subset F_k$, $dim(F_i) = i$ and $dim(F_k) = k$,

2. Choice of faces of dimension lower than i and higher than k,

3. Choice of a face F_j so that $F_i \subseteq F_j \subseteq F_k$, and $dim(F_j) = j$.

Let us now rewrite the sum to emphasize these three choices.

$$\sum_{j=i}^{k} (-1)^j f_{S \cup j}(P) =$$

$$\sum_{F_i \subset F_k} \left(f_{S \cap [-1,i]}([\emptyset, F_i]_P) \cdot f_{S \cap [k,d]}([F_k, P]_P) \left(\sum_{j=i}^{k} (-1)^j f_j([F_i, F_k]_P) \right) \right)$$

The sum between parentheses corresponds to Euler's formula applied to an interval, so it is equal to zero, and the whole equation is equal to zero. ∎

It has been proved by Bayer and Billera that every linear relation holding for the flag f-vector of all polytopes can be derived from these equalities ([3]).

2.4 Duality

The basis of duality is the study of real linear functions on a vector space. If V is a vector space of finite dimension, it is not difficult to prove that linear functions form another vector space, denoted by V^*, which has the same dimension as V. It is called *dual space*, or sometimes *adjoint space*.

The easiest way to describe a linear function on a vector space V is by a vector of V. The linear function described by a vector $\mathbf{a}$ is then:

$$f_{\mathbf{a}}(x) = \langle \mathbf{a}, \mathbf{x} \rangle$$

Another convention is to represent vectors of V as *column vectors*, and linear functions on V as *row vectors*, or *transposed vectors*. The scalar product in the previous description is then replaced by the standard matrix multiplication:

$$f_{\mathbf{a}}(x) = \mathbf{a}^t\mathbf{x} = (\alpha_1 \ \ldots \ \alpha_d) \begin{pmatrix} \xi_1 \\ \vdots \\ \xi_d \end{pmatrix}$$

Usually, the vector $\mathbf{a}$ is considered as belonging to a different space as the vector $\mathbf{x}$. However, it is sometimes desirable to consider both spaces as the same (See Chapter 6).

The $\mathcal{V}$-representation of a polytope, which we called constructive, contains a set of vectors. The polytope represented is the smallest polytope containing them. If the representation is minimal, then the vectors correspond to the vertices of the polytope.

By contrast, the $\mathcal{H}$-representation, or restrictive, can be said to describe upper bounds for the values attained by different linear functions on the polytope. Namely, each inequality $(\mathbf{a}, \beta)$ defines a linear function $\mathbf{a}$ and its maximal value β. The polytope represented is the largest polytope within these limits. If the representation is minimal, then the linear functions and their maximal values correspond to the facets of the polytope.

In this sense, the $\mathcal{H}$-representation can be said to be dual of the $\mathcal{V}$-representation.

Let us briefly introduce a property on valid inequalities:

Lemma 2.4.1 *Let S be a nonempty bounded set in $\mathbb{R}^d$. Let $\mathbf{a}$ be a vector. Then there is a unique $\beta_{\mathbf{a}}$ so that an inequality $(\mathbf{a}; \beta)$ is valid if and only if $\beta \geq \beta_{\mathbf{a}}$.*

Proof. Let $\beta_{\mathbf{a}} = \sup_{\mathbf{x} \in S} \langle \mathbf{a}, \mathbf{x} \rangle$. Since S is not empty and bounded, $\beta_{\mathbf{a}}$ is well defined. ∎

This leads us to the following definition:

Definition 2.4.2 (Supporting function) Let S be a nonempty bounded set in $\mathbb{R}^d$. We call *supporting function* of S the function H_S defined as:

$$\begin{aligned} \mathbb{R}^d &\rightarrow \mathbb{R} \\ \mathbf{a} &\mapsto H_S(\mathbf{a}) = \sup_{\mathbf{x} \in S} \langle \mathbf{a}, \mathbf{x} \rangle \end{aligned}$$

We shall see in a few paragraphs that supporting functions of polytopes are *piecewise linear*. That is, for any polytope P, we can subdivide $\mathbb{R}^d$ into a finite number of parts on which its supporting function $H_P(\mathbf{a})$ is linear. Let us just state for now a quite trivial property:

Lemma 2.4.3 *Let S be a nonempty bounded set in $\mathbb{R}^d$. Let $\mathbf{a} \in \mathbb{R}^d$ be a vector. Then for any $\lambda \geq 0$, $\lambda H_S(\mathbf{a}) = H_S(\lambda \mathbf{a})$.*

This means that in any direction, the supporting function increases linearly with distance from the origin.

Let us now define the principal notion of this chapter:

Definition 2.4.4 (Dual) Let S be a set in $\mathbb{R}^d$, the *dual* or *polar* of S, denoted by S^*, is the set defined by:

$$S^* = \{\mathbf{a} \mid H_S(\mathbf{a}) \leq 1\}$$

The dual should be considered as a subset of the dual space, though it is not uncommon to represent a set and its dual as being part of the same space.

The dual can be considered as an attempt to build an $\mathcal{H}$-representation of the set, allowing only half-spaces which contain the zero vector $\mathbf{0}$. The reason of this restriction is that if the polytope doesn't contain the zero vector $\mathbf{0}$, then the supporting function has negative values in certain directions, and so, never reaches 1.

Theorem 2.4.5 (Duals) *Duals have the following properties:*

1. *The dual of a set S is the same as that of the closure of the convex hull of $S \cup \{\mathbf{0}\}$.*

2. *The dual of a polyhedron is a polyhedron.*

3. *The dual of a convex set is bounded if and only if the interior of the set contains the zero vector $\mathbf{0}$.*

4. *If S is a closed convex set containing the zero vector $\mathbf{0}$, then $(S^*)^* = S$.*

In particular, if P is a polytope containing the zero vector $\mathbf{0}$ in its interior, then P^* is a polytope, and $(P^*)^* = P$.

To simplify, let us use the following definition:

Definition 2.4.6 (Centered) A convex set is called *centered* if it contains the zero vector $\mathbf{0}$ in its interior.

A centered polytope is therefore full-dimensional. Let us now study the combinatorial properties of the dual polytope.

Definition 2.4.7 (Associated dual face) Let P be a centered polytope. For every face F of P, we define an *associated dual face* F^D in the dual polytope P^* as

$$F^D = \{\mathbf{a} \in P^* \mid \langle \mathbf{a}, \mathbf{x} \rangle = 1, \ \forall \mathbf{x} \in F\}$$

As their name indicates, associated dual faces are faces of the dual polytope. This can be deduced from the fact all equalities in the definition are valid inequalities for P^*. Here are the properties of associated dual faces:

Theorem 2.4.8 (Associated dual faces) *Let P be a centered polytope in $\mathbb{R}^d$, and F, G faces of P. Then:*

1. $F \subseteq G \quad \Leftrightarrow G^D \subseteq F^D$
2. $dim(F^D) = d - 1 - dim(F)$

From this, we can see that the face lattice of a centered polytope P is symmetrical to that of P^*. We get the following associations between types of faces in P and P^*:

F		F^D
$\emptyset$	$\Leftrightarrow$	P^*
vertices	$\Leftrightarrow$	facets
edges	$\Leftrightarrow$	ridges
$\cdots$	$\Leftrightarrow$	$\cdots$
ridges	$\Leftrightarrow$	edges
facets	$\Leftrightarrow$	vertices
P	$\Leftrightarrow$	$\emptyset$

Since we have now introduced the properties of the face lattices of dual polytopes, we now make a parenthesis intended to prove Theorem 2.3.12.

Theorem 2.4.9 *Let P be a polytope in $\mathbb{R}^d$, and F and G two faces of P so that $F \subseteq G$. Then there is a polytope P' of dimension $dim(G) - 1 - dim(F)$ such that its face lattice is identical to that of the interval $[F, G]_P$.*

Proof. Let P be a polytope in $\mathbb{R}^d$, and F and G two faces of P so that $F \subseteq G$. We know that any face of a polytope is a polytope. Therefore, G is a polytope of dimension $dim(G)$ containing the face F. If we project G in $\mathbb{R}^{dim(G)}$ so that G is centered, we can take the dual G^* of G. The polytope G^* has a face lattice inversed from that of G, with G^D as minimal element, and contains the face F^D of dimension $dim(G) - 1 - dim(F)$. Again, F^D is a face of the polytope G^*, so it is a polytope. If we project F^D in $\mathbb{R}^{dim(G)-1-dim(F)}$ so that it is centered, then its dual $(F^D)^*$ is a polytope which has the same face lattice as the interval $[F, G]_P$. ∎

The Theorem 2.3.12 is now trivially proved.

Let us now introduce a notion which is essential for the study of Minkowski sums of polytopes.

Let P be a polytope. Since P is closed, for any $\mathbf{a}$, $H_S(\mathbf{a})$ is attained by some $\mathbf{x} \in S$. That is, $H_S(\mathbf{a}) = \max_{\mathbf{x}\in S}\langle \mathbf{a}, \mathbf{x}\rangle$. And so we can define:

Definition 2.4.10 (Maximizers) Let S be a set in $\mathbb{R}^d$, and $\mathbf{a}$ a vector of $\mathbb{R}^d$. The *set of maximizers* of $\mathbf{a}$ over S is defined as

$$\mathcal{S}(S; \mathbf{a}) = \{\mathbf{x} \in S \mid \langle \mathbf{a}, \mathbf{x}\rangle = H_S(\mathbf{a})\}$$

It is not difficult to see that the faces of a polytope are equivalent to its sets of maximizers, except for the empty set. It should be noted that different vectors can have the same set of maximizers. In fact, if $\{\mathbf{a_1}, \ldots, \mathbf{a_r}\}$ is a list of vectors which have the same set of maximizers, then the vectors in

$$\{\lambda_1 \mathbf{a_1} + \cdots + \lambda_r \mathbf{a_r} \mid \lambda_1, \ldots, \lambda_r > 0\}$$

also have the same set of maximizers (note that the λ_i factors should not be zero). This brings us to define the equivalence classes of vectors which have the same maximizer sets:

Definition 2.4.11 ((Outer) Normal cones) Let P be a polytope in $\mathbb{R}^d$. For any face F of P, we define its *normal cone* $\mathcal{N}(F; P)$ as the set of vectors for which F is the maximizer set over P. That is,

$$\mathcal{N}(F; P) = \{\mathbf{a} \mid F = \mathcal{S}(P; \mathbf{a})\}$$

It is important to understand that normal cones are generally *not* closed cones. For instance, closed cones always contain the $\mathbf{0}$ vector, but only the normal cone of the polytope itself $\mathcal{N}(P; P) = \{\mathbf{0}\}$ contains it. Rather, the cones are all relatively open, that is, they are open sets in their affine hulls for the usual topology. However, their closure are polyhedral closed cones:

Theorem 2.4.12 (Closure of normal cones) *Let P be a polytope in $\mathbb{R}^d$, and let F and G be faces of P. Then $cl(\mathcal{N}(F; P))$ and $cl(\mathcal{N}(G; P))$ are polyhedral closed cones. Also, $F \subseteq G$ if and only if $cl(\mathcal{N}(G; P))$ is a face of $cl(\mathcal{N}(F; P))$.*

Proof. By definition, we have that

$$\mathbf{a} \in \mathcal{N}(F;P) \Leftrightarrow \langle \mathbf{a};\mathbf{x}\rangle = \langle \mathbf{a};\mathbf{y}\rangle > \langle \mathbf{a};\mathbf{z}\rangle,\ \forall \mathbf{x},\mathbf{y} \in F, \mathbf{z} \in P \setminus F.$$

We deduce from this that

$$\mathbf{a} \in cl(\mathcal{N}(F;P)) \Leftrightarrow \langle \mathbf{a};\mathbf{x}\rangle = \langle \mathbf{a};\mathbf{y}\rangle \geq \langle \mathbf{a};\mathbf{z}\rangle,\ \forall \mathbf{x},\mathbf{y} \in F, \mathbf{z} \in P.$$

Since all vectors in F and P are convex combinations of vertices of F and P, we can also write

$$\mathbf{a} \in cl(\mathcal{N}(F;P)) \Leftrightarrow \langle \mathbf{a};\mathbf{x}\rangle = \langle \mathbf{a};\mathbf{y}\rangle \geq \langle \mathbf{a};\mathbf{z}\rangle,\ \forall \mathbf{x},\mathbf{y} \in \mathcal{V}(F), \mathbf{z} \in \mathcal{V}(P).$$

This clearly defines a polyhedral closed cone.

Since $F \subseteq G$ if and only if $\mathcal{V}(F) \subseteq \mathcal{V}(G) \subseteq \mathcal{V}(P)$, we can write

$$cl(\mathcal{N}(G;P)) = cl(\mathcal{N}(F;P)) \cap \{\mathbf{a} \mid \langle \mathbf{a};\mathbf{y}\rangle = \langle \mathbf{a};\mathbf{z}\rangle,\ \forall \mathbf{y},\mathbf{z} \in \mathcal{V}(G)\},$$

Since all equalities of $cl(\mathcal{N}(G;P))$ are equalities or inequalities of $cl(\mathcal{N}(F;P))$, the former is a face of the latter. ∎

We can now prove the piecewise linearity of supporting functions of polytopes:

Theorem 2.4.13 *The supporting function $H_P(\mathbf{a})$ of a polytope P is linear on the closure of any normal cone of P.*

Proof. Let P be a polytope in $\mathbb{R}^d$, and let F be a face of P. Let $\mathbf{a}$ and $\mathbf{b}$ be two vectors in $cl(\mathcal{N}(F;P))$. By definition, $H_S(\mathbf{a}) = \max_{\mathbf{x}\in P}\langle \mathbf{a},\mathbf{x}\rangle$. Since $\mathbf{a} \in cl(\mathcal{N}(F;P))$, $H_S(\mathbf{a}) = \langle \mathbf{a},\mathbf{x}\rangle$, for any $\mathbf{x} \in F$. Let $\lambda \geq 0$. Since $cl(\mathcal{N}(F;P))$ is a polyhedral closed cone, $\mathbf{a} + \lambda\mathbf{b}$ is also in $cl(\mathcal{N}(F;P))$ and $H_S(\mathbf{a}+\lambda\mathbf{b}) = \langle \mathbf{a}+\lambda\mathbf{b},\mathbf{x}\rangle = \langle \mathbf{a},\mathbf{x}\rangle + \lambda\langle \mathbf{b},\mathbf{x}\rangle = H_S(\mathbf{a}) + \lambda H_S(\mathbf{b})$, for any $\mathbf{x} \in F$. ∎

Theorem 2.4.14 *Let P be a polytope in $\mathbb{R}^d$, and let F be a face of P. Then $dim(\mathcal{N}(F;P)) = d - dim(F)$.*

Proof. By definition, the linear hull of the normal cone of a face is the space of vectors orthogonal to the affine hull of the face. ∎

Theorem 2.4.15 *Let P be a centered polytope. Let F be a nontrivial face of P. Then $\mathcal{N}(F;P)$ is the cone generated by the points in the relative interior of the associated dual face F^D. Namely,*

$$\mathcal{N}(F;P) = \{\lambda x \mid \lambda > 0,\ x \in relint(F^D)\}.$$

Consequently,

$$cl(\mathcal{N}(F;P)) = \{\lambda x \mid \lambda \geq 0,\ x \in F^D\}.$$

Proof. By definition, we have that

$$F^D = \{\mathbf{a} \mid \langle \mathbf{a}, \mathbf{x} \rangle = 1 \geq \langle \mathbf{a}, \mathbf{y} \rangle,\ \forall \mathbf{x} \in F, \mathbf{y} \in P\}.$$

Therefore, the relative interior is defined by

$$relint(F^D) = \{\mathbf{a} \mid \langle \mathbf{a}, \mathbf{x} \rangle = 1 > \langle \mathbf{a}, \mathbf{y} \rangle,\ \forall \mathbf{x} \in F, \mathbf{y} \in P\}.$$

This means that

$$relint(F^D) = \{\mathbf{a} \mid F = \mathcal{S}(P;\mathbf{a}),\ \langle \mathbf{a}, \mathbf{x} \rangle = 1\ \forall \mathbf{x} \in F\},$$

and so,

$$\{\lambda x \mid \lambda > 0,\ x \in relint(F^D)\} = \{\mathbf{a} \mid F = \mathcal{S}(P;\mathbf{a})\},$$

Which proves the theorem. ∎

As we can see, the essential combinatorial properties of the polytope can be found in the normal cones of its faces.

Definition 2.4.16 (Normal fan) Let P be a polytope in $\mathbb{R}^d$. The subdivision of $\mathbb{R}^d$ into normal cones of the faces of P is called the *normal fan* of P.

If P is a polytope in a vector space V, the normal fan should then usually be considered as a subdivision of V^*.

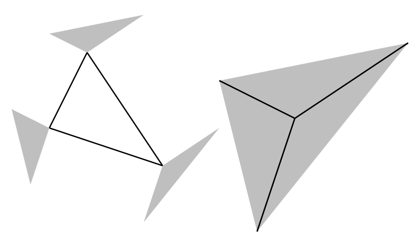


Figure 2.1: A polytope and its normal cones, and the resulting normal fan.

As we can see, the normal fan of a polytope, its supporting function and its dual are three different structures coding information in V^* about a polytope in V (See Figure 2.2).

In certain cases, it is useful to think of the normal fan as being a subdivision of the unit sphere rather than the whole space:

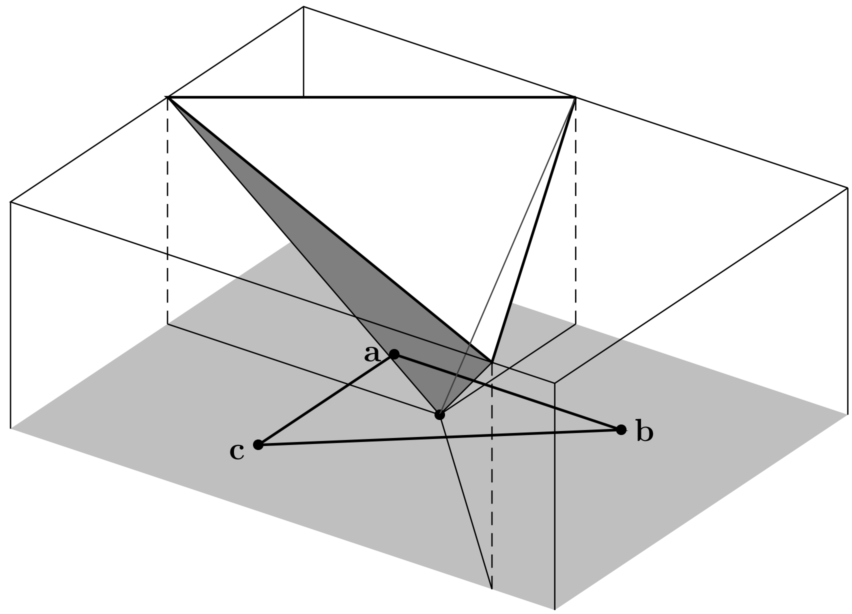


Figure 2.2: A polytope $(\mathbf{a}, \mathbf{b}, \mathbf{c})$, its normal fan, supporting function (in dark gray) and dual (in white).

Definition 2.4.17 (Normalized normal cones) For any polytope P, we call *normalized normal cones* of P the intersections of its normal cones with the unit sphere:

$$\hat{\mathcal{N}}(F;P) = \mathcal{N}(F;P) \cap \{\mathbf{a} \mid \langle \mathbf{a}; \mathbf{a} \rangle = 1\}.$$

In dimension three, for instance, it becomes difficult to represent a normal fan. An easy way to avoid this problem is to represent the normalized normal cones on the unit sphere. The normal cone of facets of the polytope corresponds to a vertex on the sphere, and the normal cones of edges correspond to great circle arcs. This way, we can represent the normal fan of a polytope as a graph on the unit sphere.

For complicated normal fans, it is also possible to project the graph from the sphere on a plane using a stereographic projection, as shown in Figure 2.3.

The disadvantage of this is that the normal cones of edges appear as arcs of circles, which may look unintuitive. Also, the pole used for the stereographic projection is projected on the "infinity" point of the plane. The advantage is that it gives us a way to represent the normal fan of a three-dimensional object on a plane. The stereographic projection also preserves angles, which helps to understand the resulting figure.

Definition 2.4.18 (Polyhedral complex) A polyhedral complex $\mathcal{C}$ is a

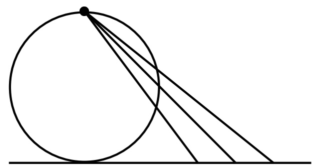


Figure 2.3: The stereographic projection

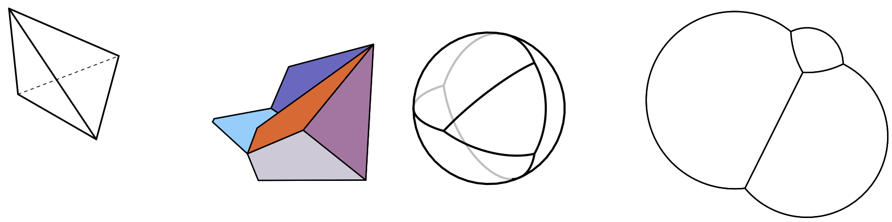


Figure 2.4: A tetrahedron, its normal fan, the intersection with the sphere, and the stereographic projection

finite set of polyhedra such that:

1. Any face of a polyhedron in $\mathcal{C}$ is also in $\mathcal{C}$,
2. The intersection of two polyhedra in $\mathcal{C}$ is also in $\mathcal{C}$,
3. The empty set $\emptyset$ is in $\mathcal{C}$.

Furthermore, the polyhedral complex is called *homogeneous* if all its maximal elements by inclusion have the same dimension.

Theorem 2.4.19 (Polyhedral complex) *Let P be a polytope in $\mathbb{R}^d$. Then the closure of the normal cones of the faces of P form an homogeneous polyhedral complex.*

2.5 Interesting polytopes

Any introduction to polytopes should at least contain a presentation of some of the most well-known families of polytopes.

Definition 2.5.1 (Simple, Simplicial) Let P be a polytope in $\mathbb{R}^d$. Then P is called *simple* if each of its vertices is contained in exactly $dim(P)$ edges. Similarly, P is called *simplicial* if each of its facets contain exactly $dim(P)$ ridges. The dual of a centered simple polytope is a centered simplicial polytope, and conversely.

Definition 2.5.2 (Simplex) A *simplex* (Plural: *simplices*) is a polytope P which has exactly $dim(P) + 1$ vertices.

Since any full-dimensional polytope has at least that many vertices, Simplices can be said to be the simplest polytopes in a given dimension, hence their name. In fact, all simplices in $\mathbb{R}^d$ are combinatorially equivalent. Here are some of their interesting properties:

Theorem 2.5.3 (Simplex) *Let Δ_d be a d-dimensional simplex. Then:*

1. *Every k-dimensional face of Δ_d is a k-dimensional simplex.*
2. *The convex hull of any set of faces of Δ_d is also a face of Δ_d.*
3. *The number of k-dimensional faces of Δ_d is equal to the number of possible choices of $k+1$ vertices among the $d+1$ vertices of Δ_d:*

$$f_k(\Delta_d) = \binom{d+1}{k+1}.$$

4. *The total number of faces of Δ_d is 2^{d+1}.*
5. *The dual of Δ_d is also a d-dimensional simplex.*
6. *Δ_d is simple and simplicial. Additionally, the only polytopes to be both simple and simplicial are simplices and two-dimensional polytopes.*

Definition 2.5.4 (Hypercube) Formally, a *hypercube* in $\mathbb{R}^d$, or *d-cube*, is the Minkowski sum of d orthogonal line segments.

If the line segments are the convex hulls of $\{-\mathbf{e_i}, \mathbf{e_i}\}$ for all i in $1, \ldots, d$, then the restrictive description of the hypercube is

$$\{\mathbf{x} \mid -1 \leq \langle \mathbf{e_i}, \mathbf{x} \rangle \leq 1,\ \forall i\}$$

Here are some properties of the hypercube

Theorem 2.5.5 (Hypercube) *Let $\square_d$ be a d-dimensional hypercube. Then:*

1. *$\square_d$ is simple.*

2. *Every k-dimensional face of $\square_d$ is a k-dimensional hypercube.*

3. *The Minkowski sum of any set of faces of $\square_d$ is also a face of $\square_d$.*

4. *The number of k-dimensional faces of $\square_d$ is equal to:*

$$f_k(\square_d) = \binom{d}{k} 2^{d-k}.$$

5. *The total number of faces of $\square_d$ is $3^d + 1$.*

Definition 2.5.6 (Cross-polytope) A d-dimensional *cross-polytope* is the dual of a hypercube.

Definition 2.5.7 (Cyclic polytopes) We call *moment curve* in dimension d the image of the following function:

$$\begin{array}{ccc} \mathbb{R} & \rightarrow & \mathbb{R}^d \\ \lambda & \mapsto & \begin{pmatrix} \lambda \\ \lambda^2 \\ \vdots \\ \lambda^d \end{pmatrix} \end{array}$$

A *cyclic polytope* is a polytope which has all its vertices on the moment curve.

Cyclic polytopes have very interesting properties:

Theorem 2.5.8 (Cyclic polytopes) *Let C_d^n be a cyclic polytope, formed by n distinct vertices on the moment curve in d-dimension.*

1. *([30]) C_d^n is $\lfloor \frac{d}{2} \rfloor$-*neighbourly*. That is, the convex hull of any k vertices of C_d^n with $2k \leq d$ is a face of C_d^n of dimension $k-1$,*

2. *Every face of C_d^n except C_d^n itself is a simplex,*

3. *(Gale's evenness condition) Let V be a set of d distinct vertices of C_d^n. Then conv(V) is a facet of C_d^n if and only if every two vertices of C_d^n not in V are separated on the moment curve by an even number of vertices in V.*

4. *For any polytope P in $\mathbb{R}^d$ with n vertices, $f_k(P) \leq f_k(C_d^n)$.*

Cyclic polytopes have the maximal number of faces of all polytopes for fixed number of vertices and dimension.

Chapter 3

Minkowski sums

AH? WELL, MATHS. GENERALLY I NEVER GET MUCH FURTHER THAN SUBTRACTION.
Terry Pratchett, THIEF OF TIME.

In this chapter, we define Minkowski sum again, in a slightly more general manner than before. We also study the result of the Minkowski sum on the various dual structures of the polytopes. We then present various constructions.

3.1 Properties

Definition 3.1.1 (Minkowski sum) Let $S_1, \ldots, S_r$ be sets of vectors. We define their *Minkowski sum*, or *vector sum*, as the set of vectors which can be written as the sum of a vector of each set. Namely:

$$S_1 + \cdots + S_r := \{\mathbf{x_1} + \cdots + \mathbf{x_r} \mid \mathbf{x_i} \in S_i,\ \forall i\}$$

It is easy to see this definition inherits some of the attributes of the usual sum. It is commutative and associative, and it has a neutral element, which is the set $\{\mathbf{0}\}$.

In the case of polytopes, the Minkowski sum is equivalent to the convex hull of the Minkowski sum of vertices of the summands. This should be made clear by the following theorem:

Theorem 3.1.2 (Decomposition) *Let $P_1, \ldots, P_r$ be polytopes in $\mathbb{R}^d$, and let F be a face of the Minkowski sum $P = P_1 + \cdots + P_r$. Then there are faces $F_1, \ldots, F_r$ of $P_1, \ldots, P_r$ respectively such that $F = F_1 + \cdots + F_r$. What's more, this decomposition is unique.*

Proof. Let $\mathbf{a}$ be a vector in $\mathcal{N}(F;P)$. Then $F = \mathcal{S}(P;\mathbf{a})$. By the definition of the set of maximizers, it is obvious that $\mathcal{S}(P;\mathbf{a}) = \mathcal{S}(P_1;\mathbf{a})+\cdots+\mathcal{S}(P_r;\mathbf{a})$. We choose $F_i = \mathcal{S}(P;\mathbf{a})$ for each i. ∎

Using the same property of sets of maximizers, we can immediately deduce the following corollaries:

Corollary 3.1.3 *Let $P = P_1 + \cdots + P_r$ be a Minkowski sum of polytopes in $\mathbb{R}^d$, let F be a nonempty face of P, and let $F_1, \ldots, F_r$ be its decomposition. Then $\mathcal{N}(F;P) = \mathcal{N}(F_1;P_1) \cap \cdots \cap \mathcal{N}(F_r;P_r)$.*

Corollary 3.1.4 *Similarly, let $F_1, \ldots, F_r$ be nonempty faces of the polytopes $P_1, \ldots, P_r$ respectively, then $F_1 + \cdots + F_r$ is a face of $P_1 + \cdots + P_r$ if and only if the intersection of their normal cones $\mathcal{N}(F_1;P_1) \cap \cdots \cap \mathcal{N}(F_r;P_r)$ is not empty.*

As we can see, the normal cones of a Minkowski sum are identical to the set of nonempty intersections of normal cones of the summands. The resulting normal fan is called the *common refinement* of the normal fans of the summands. An illustration is shown in Figure 3.1, using the stereographic representation of normal fans presented in Section 2.4.

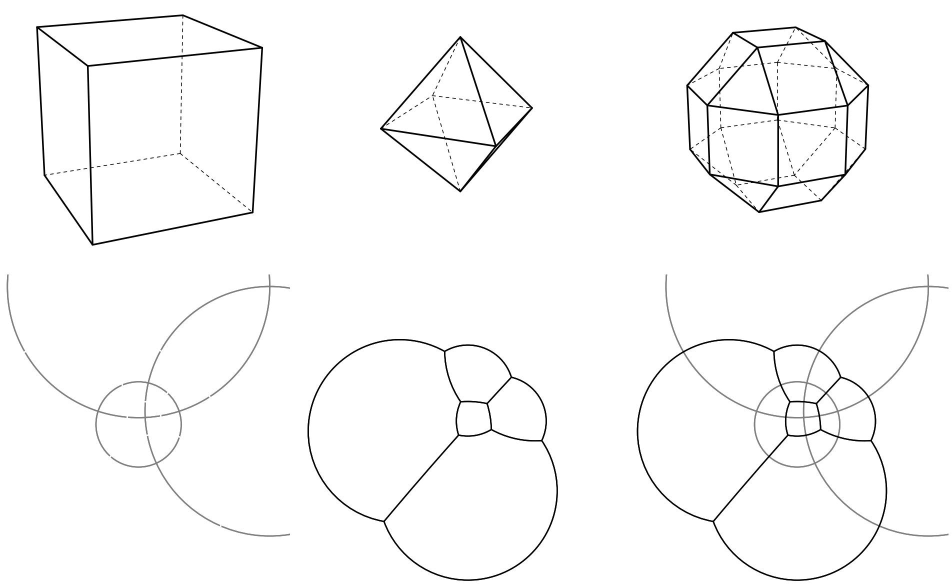


Figure 3.1: A cube, an octahedron and their sum, with their respective normal fans.

As we can see, the normal fan of a Minkowski sum is really easy to find from the normal fans of the summands. The next corollary defines the relations between faces of the Minkowski sum:

Corollary 3.1.5 *Let $P = P_1 + \cdots + P_r$ be a Minkowski sum of polytopes in $\mathbb{R}^d$, let $F \subseteq G$ be faces of P, and let $F_1, \ldots, F_r$ and $G_1, \ldots, G_r$ be their decomposition. Then $F_i \subseteq G_i$, for all i.*

As for supporting functions, the result of a Minkowski sum is also very simple:

Theorem 3.1.6 *The supporting function of a Minkowski sum is the sum of the supporting functions of its summands.*

Proof. Let $P = P_1, \ldots, P_r$ be a Minkowski sum of polytopes in $\mathbb{R}^d$. Let $\mathbf{a}$ be a vector in $\mathbb{R}^d$. Let $\mathbf{x} = \mathbf{x_1} + \cdots + \mathbf{x_r}$, with $\mathbf{x} \in \mathcal{S}(P; \mathbf{a})$ and $\mathbf{x_i} \in \mathcal{S}(P_i; \mathbf{a})$ for all i. Then

$$H_P(\mathbf{a}) = \langle \mathbf{a}, \mathbf{x} \rangle = \langle \mathbf{a}, \mathbf{x_1} \rangle + \cdots + \langle \mathbf{a}, \mathbf{x_r} \rangle = H_{P_1}(\mathbf{a}) + \cdots + H_{P_r}(\mathbf{a}).$$

∎

3.2 Constructions of Minkowski sums

There are many constructions which give raise to Minkowski sums. Some of them illustrate the closeness between Minkowski sums and the convex hull problem.

3.2.1 The Cayley embedding

Let $P_1, \ldots, P_r$ be polytopes in $\mathbb{R}^d$. Let P be their *weighted* Minkowski sum, that is:

$$P(\lambda_1, \ldots, \lambda_r) = \lambda_1 P_1 + \cdots + \lambda_r P_r$$

With $\lambda_i \geq 0$ for all i. (We consider here $\lambda_i P_i$ to be the scaling of P_i by a factor λ_i.)

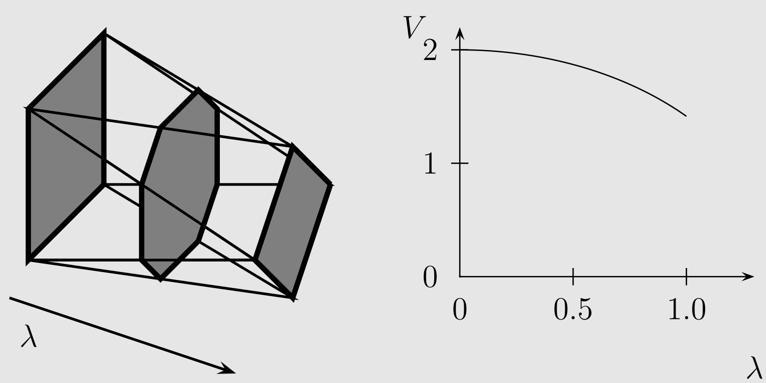
It is easy to see that the normal fan of $\lambda_i P_i$ does not change as long as λ_i is greater than zero. Since the normal fan of a Minkowski sum can be deduced from that of its summands, we can deduce from this that the combinatorial properties of $P(\lambda_1, \ldots, \lambda_r)$ stay the same as long as all λ_i are greater than zero.

The Brunn-Minkowski theorem

If $P(\lambda_1, \ldots, \lambda_r)$ is the weighted sum of $P_1, \ldots, P_r$ polytopes in $\mathbb{R}^d$, with $\lambda_i \geq 0$ for all i. It is possible to write the volume of the Minkowski sum as a polynomial over the factors λ_i. The factors of the polynomial are called *mixed volumes*. They are at the center of the Brunn-Minkowski theory, and the reason Minkowski's name was attached to the sums. Here is the best-known result of this theory, useful in many fields ([14]):

Theorem 3.2.1 (Brunn-Minkowski) *Let S_1 and S_2 be convex sets in $\mathbb{R}^d$. Let S_λ be their weighted Minkowski sum $S_\lambda = \lambda S_1 + (1-\lambda)S_2$, and let $V(\lambda)$ be the dth root of the volume of S_λ. Then $V(S_\lambda)$ is a concave function. That is, for any λ_1, λ_2, and ρ in $[0,1]$, we have that*

$$V(\rho\lambda_1 + (1-\rho)\lambda_2) \geq \rho V(\lambda_1) + (1-\rho)V(\lambda_2)$$



Example of weighted sum: $V(\lambda) = \sqrt{4 - 2\lambda^2}$

Let us define the simplex $\Delta = \{(\lambda_1, \ldots, \lambda_r) \mid \lambda_1 + \cdots + \lambda_r = 1,\ \lambda_i \geq 0\ \forall i\}$ We now define the *Cayley embedding* of the Minkowski sum of the polytopes $P_1, \ldots, P_r$ as follows:

$$\mathcal{C}(P_1, \ldots, P_r) \subseteq \mathbb{R}^d \times \Delta \subseteq \mathbb{R}^d \times \mathbb{R}^r$$

$$\mathcal{C}(P_1, \ldots, P_r) = conv((P_1 \times \mathbf{e_1}) \cup \cdots \cup (P_r \times \mathbf{e_r}))$$

Where $\mathbf{e_i}$ are the vectors in Δ defined by $\lambda_i = 1,\ \lambda_j = 0\ \forall j \neq i$.

The intersection of the Cayley embedding with a plane fixing the value of the λ_i, for instance $\lambda_1 = \cdots = \lambda_r = 1/r$, is equivalent to $P(\lambda_1, \ldots, \lambda_r)$. On the other hand, the projection of the Cayley embedding from $\mathbb{R}^d \times \mathbb{R}^r$ on $\mathbb{R}^d$ by removing the last coordinates clearly gives us the convex hull of $P_1 \cup \cdots \cup P_r$.

In other words, the polytopes $P(\lambda_1, \ldots, \lambda_r)$ are combinatorially equivalent to the Minkowski sum $P_1 + \cdots + P_r$ for all choices of strictly positive λ_i, but the closure of their union is the convex hull of $P_1 \cup \cdots \cup P_r$.

Let us now examine particularly the Cayley embedding of two polytopes:

$$\mathcal{C}(P_1, P_2) = conv((P_1 \times \mathbf{e_1}) \cup (P_2 \times \mathbf{e_2})) \subseteq \mathbb{R}^d \times conv(\mathbf{e_1}, \mathbf{e_2})$$

By construction, the last two coordinates of the vertices correspond either to $\mathbf{e_1}$ or $\mathbf{e_2}$. We can deduce from this there are two kinds of faces in the Cayley embedding. The first kind has all its vertices either on $\mathbf{e_1}$ or on $\mathbf{e_2}$, the second kind has some vertices on $\mathbf{e_1}$, some on $\mathbf{e_2}$. Since for the computation of the Minkowski sum, we intersect the Cayley embedding with a plane separating the points on $\mathbf{e_1}$ from these on $\mathbf{e_2}$, only faces of the second kind intersect with that plane and induce a face of the Minkowski sum. Therefore, the face lattice of the Minkowski sum is equivalent to a subset of the face lattice of the Cayley embedding. The relation by inclusion of these faces is the same in the Minkowski sum as in the Cayley embedding, though the faces in the Minkowski sum are smaller by one dimension.

The faces of the first type are characterized by the fact their last two coordinates correspond to $\mathbf{e_1}$ or $\mathbf{e_2}$, or equivalently by the fact that they are part of $F_1 = \mathcal{S}(\mathcal{C}(P_1, P_2); \mathbf{e_1} - \mathbf{e_2})$ or $F_2 = \mathcal{S}(\mathcal{C}(P_1, P_2); \mathbf{e_2} - \mathbf{e_1})$. So we have that $\mathcal{L}(P_1 + P_2)$ is equivalent to $\mathcal{L}(\mathcal{C}(P_1, P_2)) \setminus \mathcal{L}(F_1) \setminus \mathcal{L}(F_2)$.

Since the convex hull of P_1 and P_2 is the result of a projection from $\mathbb{R}^d \times conv(\mathbf{e_1}, \mathbf{e_2})$ on $\mathbb{R}^d$, the normal fan of the convex hull is the intersection of the normal fan of the Cayley embedding with the plane $\mathbb{R}^d \times (\mathbf{0}, \mathbf{0})$. As we can see, the face lattice of the convex hull is also equivalent to a subset of the face lattice of the Cayley embedding. As with the Minkowski sum, the relation by inclusion of these faces is the same as in the Cayley embedding, though this time their dimensions stay the same.

3.2.2 Pyramids

In the previous section, we have seen that it is possible to obtain the Minkowski sum of polytopes by doing their convex hull in a certain way, then intersecting the result with a plane.

Let us now explain how to obtain the convex hull of polytopes by doing their Minkowski sum in a certain way, and intersecting the result with a plane.

A standard operation for constructing polytopes is the *pyramid*. It allows us to build a polytope in $\mathbb{R}^{d+1}$ from a polytope in $\mathbb{R}^d$ by placing the polytope in $\mathbb{R}^d \times \{0\}$, adding a point "above" it in $\mathbf{e_{d+1}} = (0, \ldots, 0, 1)$, and taking the convex hull. For example, the pyramid of a square would create an Egyptian pyramid. Simplices in dimension d can be defined as d successive pyramids of a 0-dimensional point.

The position of the point we add is not actually important from the combinatorial point of view.

Let $P_1, \ldots, P_r$ be polytopes in $\mathbb{R}^d$. We examine the Minkowski sum P of their pyramids in $\mathbb{R}^{d+1}$.

For any hyperplane H_λ of type $(\mathbf{e_{d+1}}, \lambda)$, the intersection of H_λ with P can be written in $\mathbb{R}^d$ as the union of all

$$\lambda_1 P_1 + \cdots + \lambda_r P_r$$

with $0 \leq \lambda_i \leq 1$ so that $\lambda_1 + \cdots + \lambda_r = r - \lambda$.

For instance, $H_0 \cap P$ is the Minkowski sum of $P_1, \ldots, P_r$, and $H_{r-1} \cap P$ is their convex hull. In fact, H_n is the convex hull of all Minkowski sums of $r - n$ polytopes chosen in $P_1, \ldots, P_r$.

3.2.3 Cartesian product

Let $P_1, \ldots, P_r$ be polytopes in $\mathbb{R}^d$. Let $P = P_1 \times \cdots \times P_r$ be their Cartesian product in $\mathbb{R}^d \times \cdots \times \mathbb{R}^d$. Then the Minkowski sum $P_1 + \cdots + P_r$ is equal to the image of P in $\mathbb{R}^d$ by the following projection π:

$$\pi : \mathbb{R}^d \times \cdots \times \mathbb{R}^d \rightarrow \mathbb{R}^d$$

$$(\mathbf{x_1}, \ldots, \mathbf{x_r}) \mapsto \mathbf{x_1} + \cdots + \mathbf{x_r}$$

Since the Minkowski sum is a projection of P, the normal fan of the Minkowski sum is equivalent to the intersection of the normal fan of P with the linear space of vectors orthogonal to the kernel of the projection. (Or, to formulate it in the dual space, the space of linear functions $f_\mathbf{a}$ so that $f_\mathbf{a}(\mathbf{x}) = 0$, for any $\mathbf{x}$ in the kernel of π.)

This construction of the Minkowski sum as a projection gives us very powerful tools to analyze Minkowski sums, both computationally and combinatorially (See e.g. [39] and [40]).

3.3 Zonotopes

We present here the simplest of Minkowski sums, which are zonotopes. Zonotopes are Minkowski sums of line segments, that is, polytopes with two vertices. This problem has been studied thoroughly, and quite early. In fact, the dual problem, the common refinement of the normal fan, amounts to computing a central hyperplane arrangement, which is a fundamental problem of combinatorics.

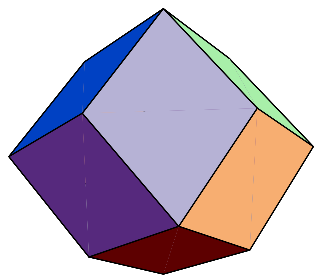


Figure 3.2: A 3-dimensional zonotope, sum of 4 line segments

Hyperplane arrangements consist in subdividing the space with hyperplanes and counting the number of cells it creates. In the words of R. C. Buck ([8]): *how many pieces can be obtained from a round flat cheese by exactly n straight cuts?* To get a more complete description, it suffices to write $\mathbb{R}^d$ instead of "round flat cheese" (though a flat cheese is admittedly a nice topological description of $\mathbb{R}^2$). Buck gives a solution to this problem, and goes on to prove that the number of k-dimensional cells in a subdivision of $\mathbb{R}^d$ by r hyperplanes in general position is:

$$\binom{r}{d-k} \sum_{i=0}^{k} \binom{r+k-d}{i}$$

What is closer to our subject are *central hyperplane arrangements*, that is, arrangements of hyperplanes all containing the vector $\mathbf{0}$. In this case, we can consider each hyperplane to be the normal fan of a segment which is orthogonal to it. This means a central hyperplane arrangement is nothing more than the normal fan of the Minkowski sum of the line segments, that is a zonotope.

The number of k-dimensional faces in a d-dimensional zonotope, sum of r line segments in general position, is therefore equal to the number of $(d-k)$-

dimensional cells in a d-dimensional central arrangement of r hyperplanes in general position. This number was introduced in [42] and is equal to:

$$2\binom{r}{k}\sum_{i=0}^{d-1-k}\binom{r-1-k}{i}.$$

It is proved in [18] that this is the maximal number of faces for sums of polytopes having at most r non-parallel edges in total. This was the first known general bound on the number of faces of a Minkowski sum.

Zonotopes in $\mathbb{R}^d$ which are sums of r line segments can alternately be defined as the projection in $\mathbb{R}^d$ of an n-dimensional hypercube. This is consistent with the construction of Minkowski sums described in Section 3.2.3. We can deduce from this that all faces of a zonotope are zonotopes themselves. If the line segments generating a d-dimensional zonotope are in general position, then its faces are $(d-1)$-dimensional hypercubes.

Part II

Face study

Chapter 4

Bounds on the number of faces

Oh quanto parve a me gran maraviglia
quand'io vidi tre facce a la sua testa!
O, what a marvel it appeared to me,
When I beheld three faces on his head!
Dante Alighieri, THE DIVINE COMEDY.

We discuss in this chapter the question of the maximum complexity of a Minkowski sum in terms of its summands. This problem can itself be divided into a multitude of questions, depending on the way we define either the input or the output.

Let us introduce the different bounds on the number of vertices of the sum, then faces, then present what little we know about facets.

4.1 Bound on vertices

Let us start by defining the problem.

Bound on vertices in terms of vertices: *Let P be the d-dimensional Minkowski sum of n polytopes. What is the maximal number of vertices of P, in terms of n, d, and the summands?*

From the decomposition properties of Minkowski sums, we can deduce the following *trivial bound*:

Theorem 4.1.1 *Let $P_1, \ldots, P_r$ be polytopes and P their Minkowski sum. Then:*

$$f_0(P) \leq \prod_{i=1}^{r} f_0(P_i).$$

The trivial bound can be attained for $n < d$, as will be seen in the construction shown in Section 4.2. However, it can be shown *not* to be tight otherwise. Indeed, the following has been proved:

Theorem 4.1.2 *[40] Let $P_1, \ldots, P_r$ be polytopes with at least $d+1$ vertices and P their Minkowski sum. Then if $r \geq d$,*

$$f_0(P) \leq \left(1 - \frac{1}{(d+1)^r}\right) \prod_{i=1}^{r} f_0(P_i).$$

The complete proof is somewhat intricate, so we show here a weaker theorem proving the unreachability for $r > d$, suggested by Rade Živaljević and Imre Bárány, which uses the *colored Helly theorem* due to Lovász:

Theorem 4.1.3 *[28][1] Let $\mathcal{C}_1, \ldots, \mathcal{C}_r$ be collections of convex sets in $\mathbb{R}^d$ with $r \geq d+1$. If $\cap_{i=1}^{r} C_i \neq \emptyset$ for every choice of $C_i \in \mathcal{C}_i$, then there is a $j \in \{1, \ldots, r\}$ such that $\cap_{C \in \mathcal{C}_j} C \neq \emptyset$, that is a family whose members have a nonempty intersection.*

In our case, the normal cones of vertices of the polytopes $P_1, \ldots, P_r$ form the r collections. If all choices of vertices in the summands add to a vertex of P, it means that their cones intersect, which means that $\cap_{i=1}^{r} C_i \neq \emptyset$ for every choice of $C_i \in \mathcal{C}_i$. However, since the normal cones of a polytope are all disjoint, the intersection of the members in a family is always empty, so we have a contradiction.

These results encouraged us to search for a general result, which we present now:

Theorem 4.1.4 (Attainability) *Let $P_1, \ldots, P_r$ be polytopes with more than one vertex in $\mathbb{R}^d$ and P their Minkowski sum. Then it is possible to reach the trivial bound for the number of vertices in the sum if and only if $r < d$, or $r = d$ and every polytope has exactly two vertices.*

Proof. Let us consider separately all cases:

1. If $r > d$, then the trivial bound is unreachable by the previous theorem.

2. If $r < d$, then we can reach the trivial bound using the construction shown in Section 4.2.

3. If $r = d$ and every polytope has two vertices, then the d-dimensional hypercube meets the criteria.

[1] Unique of all references in this thesis, this article was written in Hungarian. For this reason, the author freely admits he didn't read it, and is merely trusting his sources.

4. If $r = d$ and one polytope has three vertices or more, we now prove that the trivial bound is unreachable.

Let $P = P_1 + \cdots + P_d$ be a Minkowski sum of polytopes in $\mathbb{R}^d$, with P_i having only two vertices for $i > 1$, and P_1 having three vertices. Let us suppose that all vertex decompositions exist. Then the normal fan of P contains $2^{d-1}3$ d-dimensional cells. Since all decompositions exist, the hyperplane $\mathcal{N}(P_d; P_d)$, which is the normal fan of P_d, separates in two each of the $2^{d-2}3$ cells of the normal fan of $P_1+\cdots+P_{d-1}$. So the restriction of the normal fan of P to the $(d-1)$-dimensional hyperplane $\mathcal{N}(P_d; P_d)$ contains $2^{d-2}3$ cells. It is equivalent to the normal fan of the orthogonal projection of $P_1 + \cdots + P_{d-1}$ on a $(d-1)$-dimensional hyperplane orthogonal to P_d.

By induction, the restriction of the normal fan of P to the i-dimensional intersection of hyperplanes $\mathcal{N}(P_i+1; P_i+1) \cap \cdots \cap \mathcal{N}(P_d; P_d)$ contains $2^{i-1}3$ i-dimensional cells, for all $i \geq 1$. In particular, $\mathcal{N}(P_2; P_2) \cap \cdots \cap \mathcal{N}(P_d; P_d)$ is 1-dimensional and contains 3 cells, which is impossible for a normal fan, since a 1-dimensional polytope is a segment and has only 2 vertices. ∎

4.2 Minkowski sums with all vertex decomposition

Let P be the Minkowski sum of the polytopes $P_1, \ldots, P_r$ in $\mathbb{R}^d$. In the previous section, we stated that the number of vertices in the sum is bounded by the number of possible decompositions into vertices of the summands. Therefore, $f_0(P) \leq f_0(P_1) \cdots f_0(P_r)$. We now present a construction reaching this bound for any $r < d$.

Let P_i, $i = 1, \ldots, d-1$, be d-dimensional polytopes, and $\mathbf{v_{i,j}}$ their vertices, $j = 1, \ldots, n_i$ where $n_i \geq 1$ is the number of vertices of the polytope P_i. We set the coordinates of the vertices to be:

$$\mathbf{v_{i,j}} = \cos(\alpha_{i,j}\pi) \cdot \mathbf{e_i} + \sin(\alpha_{i,j}\pi) \cdot \mathbf{e_d},$$

with $0 = \alpha_{i,1} < \cdots < \alpha_{i,n_i} = 1$ for all j, where $\mathbf{e_i}$'s are the unit vectors of an orthonormal basis of the d-dimensional space. So the vertices of P_i are distinct and placed on the unit half-circle in the space generated by $\mathbf{e_i}$ and $\mathbf{e_d}$. Observe that the polytopes are two-dimensional for now. By the construction, one can easily verify that

$$\mathbf{v_{i,j}} \in \mathcal{N}(\{\mathbf{v_{i,j}}\}; P_i).$$

This stays true if we add anything to those vectors in the spaces orthogonal to that of the half-circle:

$$\mathbf{v_{i,j}} + \sum_{k \neq i,d} \beta_k \mathbf{e_k} \in \mathcal{N}(\{v_{i,j}\}; P_i), \forall \beta_k \in \mathbb{R}.$$

So for any choice of $S = \{j_i\}_{i=1}^{d-1}$, $j_i = 1, \ldots, n_i$, we can build this vector:

$$\mathbf{v_S} = \sum_{i=1}^{d-1} \cot(\alpha_{i,j_i}\pi) \cdot \mathbf{e_i} + \mathbf{e_d}.$$

This vector $\mathbf{v_S}$, projected to the space generated by $\mathbf{e_d}$ and any $\mathbf{e_i}$, is equal to $\cot(\alpha_{i,j_i}\pi) \cdot \mathbf{e_i} + \mathbf{e_d}$ which is collinear with $\cos(\alpha_{i,j_i}\pi) \cdot \mathbf{e_i} + \sin(\alpha_{i,j_i}\pi) \cdot \mathbf{e_d}$, and thus belongs to $\mathcal{N}(\{\mathbf{v_{i,j_i}}\}; P_i)$. So we have that:

$$\mathbf{v_S} \in \bigcap_{i=1}^{d-1} \mathcal{N}(\{\mathbf{v_{i,j_i}}\}; P_i),$$

and since this intersection is not empty, it means that $\mathbf{v_{j_1}}, \ldots, \mathbf{v_{j_{d-1}}}$ is a vertex of the Minkowski sum $P_1 + \cdots + P_{d-1}$. This stays true for any choice of $S = \{j_i\}_{i=1}^{d-1}$, so the Minkowski sum has $\prod_{i=1}^{d-1} n_i$ vertices. The polytopes P_i thus defined are 2-dimensional. The property still stands if we add small perturbations to the vertices to make the polytopes full-dimensional.

The resulting Minkowski sum is very similar to a “lifted pile of cube”, as defined in Section 5.1 of [43].

4.2.1 Dimension three

Let us now show that when constructing a polytope in $\mathbb{R}^d$ as in the preceding section, it is possible to perturb the vertices so as to make it simplicial or, if it has an even number of vertices, simple.

Let P be a polytope in $\mathbb{R}^d$ such that its vertices $\mathbf{v_1}, \ldots, \mathbf{v_n}$ can be written as:

$$\mathbf{v_j} = \begin{pmatrix} \cos(\alpha_j \pi) \\ \sin(\alpha_j \pi) \\ \beta_j \end{pmatrix},$$

with $0 = \alpha_1 < \cdots < \alpha_n = 1$ all distinct. Independently of the combinatorial properties of P, it is possible to divide the β_j by a large number to make it flat enough for a sum such as is described in the preceding section.

We suppose we would like P to be simplicial. Let us pose $\beta_1 = 1$, $\beta_n = -1$ and $\beta_j = 0$ otherwise. An example is in Figure 4.1. It is easy to see that

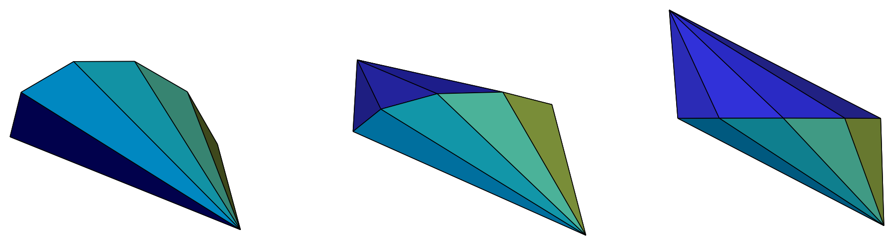


Figure 4.1: Simplicial polytope

$(\mathbf{v_1}, \mathbf{v_j})$ and $(\mathbf{v_j}, \mathbf{v_n})$ are edges, for any $2 \leq j \leq n-1$. Since $(\mathbf{v_1}, \mathbf{v_n})$ and $(\mathbf{v_j}, \mathbf{v_j+1})$ for $2 \leq j \leq n-2$ are edges, the polytope has $3n-6$ edges and is simplicial.

We now suppose that n is even and we would like P to be simple. We now build a polytope combinatorially similar to that on figure 4.2. To begin with, we pose $\beta_j = \pm \sin \alpha_j$, for all j so that vertices are disposed on either of the two facets at the back of the figure. Both of these facets contain $\mathbf{v_1}$ and $\mathbf{v_n}$.

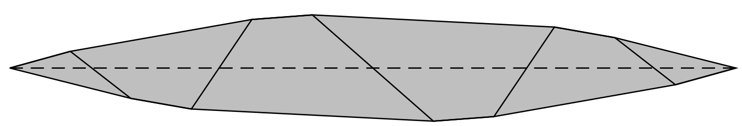


Figure 4.2: Simple polytope

Following the rules, we fix $\mathbf{v_2}$ and $\mathbf{v_3}$ anywhere so that $\alpha_2 < \alpha_3$ and $\beta_2\beta_3 < 0$. We show that for any $\mathbf{v_j}$, $\mathbf{v_{j+1}}$ so that $\alpha_j < \alpha_{j+1}$ and $\beta_j\beta_{j+1} < 0$, it is possible to fix $\mathbf{v_{j+2}}$ and $\mathbf{v_{j+3}}$ so that $\mathbf{v_j}$, $\mathbf{v_{j+1}}$, $\mathbf{v_{j+2}}$ and $\mathbf{v_{j+3}}$ are coplanar, $\alpha_{j+1} < \alpha_{j+2} < \alpha_{j+3}$, and $\beta_{j+2}\beta_{j+3} < 0$.

The principle is explained in Figure 4.3. We fix $\mathbf{v_{j+2}}$ anywhere so that $\alpha_{j+1} < \alpha_{j+2}$ and $\beta_{j+1}\beta_{j+2} > 0$. Since the line $(\mathbf{v_{j+1}}, \mathbf{v_{j+2}})$ is on a plane containing the line $(\mathbf{v_1}, \mathbf{v_n})$, the two intersect in a point $\mathbf{w}$. The line $(\mathbf{v_j}, \mathbf{w})$ in turn intersects the arc of a circle containing $\mathbf{v_j}$ in a different point, where we put $\mathbf{v_{j+3}}$. Since $\mathbf{v_j}$, $\mathbf{v_{j+1}}$, $\mathbf{v_{j+2}}$ and $\mathbf{v_{j+3}}$ are disposed on two lines which intersect, they are coplanar.

To resume, for any n even, we can create a polytope which has n vertices and $n/2+2$ facets, and is therefore simple.

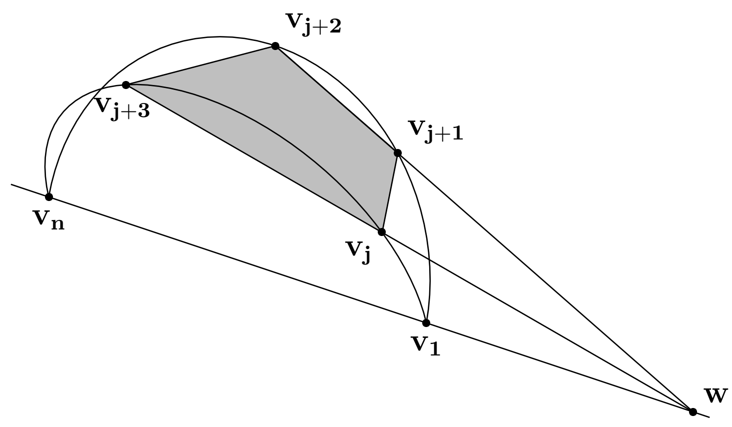


Figure 4.3: Construction step of $\mathbf{v_{j+2}}$ and $\mathbf{v_{j+3}}$

4.3 Bound on faces

We now introduce a general bound on faces. Again, we first give a trivial bound. We then prove that this bound can be reached for lower dimensions by sums of cyclic polytopes (See Section 2.5).

Let $P_1, \dots, P_r$ be d-dimensional polytopes. For each $k = 0, \dots, d-1$ and $r \geq 1$, the number of k-faces of $P_1 + \cdots + P_r$ is bounded by:

$$f_k(P_1 + \cdots + P_r) \leq \sum_{\substack{1 \leq s_i \leq f_0(P_i) \\ s_1 + \cdots + s_r = k + r}} \prod_{i=1}^{r} \binom{f_0(P_i)}{s_i},$$

where s_i's are integral.

To prove this, we show that the decomposition of a face of a Minkowski sum contains a minimal number of vertices of the summands.

Let $P_1, \dots, P_r$ be d-dimensional polytopes, and F a k-dimensional face of $P_1 + \cdots + P_r$. Let $F_i \subseteq P_i$, $i = 1, \dots, r$ be the decomposition of F. Let $k_1, \dots, k_r$ be the dimensions of respectively $F_1, \dots, F_r$. Then $k_1 + \cdots + k_r \geq k$. The minimal number of vertices for a face of dimension k_i is $k_i + 1$. So the total number of vertices contained in faces of the decomposition of F is at least $k + r$. For any fixed $k_1, \dots, k_r$, the number of possible choices of s_i vertices for each P_i is:

$$\prod_{i=1}^{r} \binom{f_0(P_i)}{s_i}.$$

This proves the bound. We now show that the bound is tight for faces of lower dimensions.

Cyclic polytopes are known to have the maximal number of faces for any fixed number of vertices. This property is somewhat carried over to their Minkowski sum.

We recall that cyclic polytopes on the d-dimensional curve of moments are $\lfloor \frac{d}{2} \rfloor$-*neighbourly*, which means that the convex hull of any set of $\lfloor \frac{d}{2} \rfloor$ vertices of a cyclic polytope P is a face of P [21, 4.7].

Note that if we choose a set S of points on the moment curve, with $|S| \leq \lfloor \frac{d}{2} \rfloor$, $conv(S)$ forms a face of any polytope P having S as a subset of its vertices, no matter how the other vertices are chosen. That is, there is always a vector $\mathbf{a_S}$ so that $\mathcal{S}(P; \mathbf{a_S}) = conv(S)$.

Theorem 4.3.1 *In dimension $d \geq 4$, it is possible to choose $r \leq \lfloor \frac{d}{2} \rfloor$ polytopes P_1, P_2, ..., P_r so that the trivial upper bound for the number of k-faces of $P_1 + \cdots + P_r$ is attained for all $0 \leq k \leq \lfloor \frac{d}{2} \rfloor - r$.*

Proof. Let P be the Minkowski sum of polytopes $P_1, \ldots, P_r$ whose vertices are all distinct on the moment curve, with $k = \lfloor \frac{d}{2} \rfloor - r$.

Let $S_1 \subseteq \mathcal{V}(P_1), \ldots, S_r \subseteq \mathcal{V}(P_r)$ be subsets of the vertices of the polytopes such that $S_i \neq \emptyset$, $\forall i$ and $|S_1| + \cdots + |S_r| = k + r$. Since $k + r \leq \lfloor \frac{d}{2} \rfloor$, there is a linear function maximized at $S_1, \ldots, S_r$ on the moment curve. Therefore, $conv(S_i)$ is an $(|S_i| - 1)$-dimensional face of P_i, $\forall i = 1, \ldots, r$.

Since the same linear function is maximized over each P_i on these faces, they sum up to a face of P. Since the set of vertices $S_1 \cup \ldots \cup S_r$ is affinely independent, $dim(conv(S_1) + \cdots + conv(S_r)) = dim(conv(S_1)) + \cdots + dim(conv(S_r)) = |S_1| + \cdots + |S_r| - r = r + k - r = k$. ∎

We should note that this construction is somewhat overkill. For instance, to reach the trivial bound for vertices of the sum of r polytopes, it requires that $d \geq 2r$, while we know from preceding sections that $d \geq r + 1$ is enough.

Chapter 5

Polytopes relatively in general position

Of course it is happening inside your head, Harry,
but why on earth should that mean that it is not real?
Joan K. Rowling, HARRY POTTER AND THE DEATHLY HALLOWS.

We introduce in this chapter a new relation concerning a special family of Minkowski sums of polytopes. We show that, when summands are positioned "relatively" in general position, the f-vector of the sum is linked by a linear relation to the f-vectors of the summands ([15]).

5.1 Introduction

Let $P = P_1 + \cdots + P_r$ be a Minkowski sum of polytopes in $\mathbb{R}^d$, and F a face of P. As stated in Theorem 3.1.2, F can be decomposed into faces of the summands: $F = F_1 + \cdots + F_r$, with F_i face of P_i. The dimension of F is then at least as large as the highest among the dimensions of the summands, and can at most be equal to the sum of the dimensions of the summands:

$$\max_i(dim(F_i)) \leq dim(F) \leq dim(F_1) + \cdots + dim(F_r)$$

If this maximum is attained, we say that F has an *exact decomposition.* The reason for this term is that if $F = F_1 + \cdots + F_r$ has an exact decomposition, then for any $\mathbf{x} \in F$, there is a unique decomposition $\mathbf{x} = \mathbf{x_1} + \cdots + \mathbf{x_r}$ with $\mathbf{x_i} \in F_i$ for all i.

For instance, the sum in $\mathbb{R}^d$ of r line segments in general position is a zonotope whose nontrivial faces are all cubical and have exact decompositions.

If every facet of a polytope has an exact decomposition, then by extension every nontrivial face has an exact decomposition. We say that the summands are *relatively in general position.*

An important point about polytopes relatively in general position is that they can attain the maximal number of faces in a Minkowski sum:

Theorem 5.1.1 *Let $P = P_1 + \cdots + P_r$ be a Minkowski sum of polytopes in $\mathbb{R}^d$. There is a Minkowski sum $P' = P'_1 + \cdots + P'_r$ of polytopes relatively in general position so that $f_k(P'_i) = f_k(P_i)$ for all i and k, and so that $f_k(P') \geq f_k(P)$ for all k.*

Proof. Let $P = P_1 + P_2$ be a Minkowski sum of polytopes not relatively in general position. We show that if we rotate F_1 by a small angle about an axis in general position, then the number of faces does not diminish.

Let $F = F_1 + F_2$ be a face whose decomposition is not exact, that is $dim(F) < dim(F_1) + dim(F_2)$. In terms of normalized cones, it means that $\hat{\mathcal{N}}(F; P) = \hat{\mathcal{N}}(F_1; P_1) \cap \hat{\mathcal{N}}(F_2; P_2)$, and $dim(\hat{\mathcal{N}}(F; P)) > d - 1 - dim(\hat{\mathcal{N}}(F_1; P_1)) - dim(\hat{\mathcal{N}}(F_2; P_2))$. Let us perturb P_1 by a small enough rotation about an axe in general position. There is now a superface $\hat{\mathcal{N}}(G_2; P_2)$ of $\hat{\mathcal{N}}(F_2; P_2)$ so that $\hat{\mathcal{N}}(F_1; P_1) \cap \hat{\mathcal{N}}(G_2; P_2) \neq \emptyset$, and so that $dim(\hat{\mathcal{N}}(F_1; P_1) \cap \hat{\mathcal{N}}(G_2; P_2)) = dim\,\hat{\mathcal{N}}(F; P)$. This means F_1 and G_2 sum to a face F' with $dim(F') = dim(F)$. So for every face with an inexact decomposition, there is now a new face of the same dimension with an exact decomposition. If the angle is small enough, every face with an exact decomposition should still exist. Therefore the number of faces won't diminish, and the new sum is relatively in general position.

By induction, we can slightly rotate the summands $P_1, \ldots, P_r$ so that they are relatively in general position, without diminishing the number of faces in their sum. ∎

Minkowski sums of polytopes in general position can therefore be used when computing the maximum complexity of Minkowski sums. We now present our most important result.

Theorem 5.1.2 *Let $P_1, \ldots, P_r$ be d-dimensional polytopes relatively in general position, and $P = P_1 + \cdots + P_r$ their Minkowski sum. Then*

$$\sum_{k=0}^{d-1} (-1)^k k(f_k(P) - (f_k(P_1) + \cdots + f_k(P_r))) = 0.$$

Note that the form is rather similar to Euler's formula:

$$\sum_{k=0}^{d-1} (-1)^k f_k(P)) = 1 - (-1)^d.$$

By using Euler's formula, we can rewrite the theorem slightly differently.

Corollary 5.1.3 *Let $P_1, \ldots, P_r$ be d-dimensional polytopes, and $P = P_1 + \cdots + P_r$ their Minkowski sum. Then for all a,*

$$\sum_{k=0}^{d-1}(-1)^k(k+a)(f_k(P) - (f_k(P_1) + \cdots + f_k(P_r))) = a(1-r)(1-(-1)^d).$$

To prove the theorem, we first introduce a few lemmas.

Lemma 5.1.4 *Let $P = P_1 + \cdots + P_r$ be a Minkowski sum of d-dimensional polytopes. Let F_i be a face of P_i, and $\Omega(F_i)$ the set of faces of P which have F_i as a subface in their decomposition, P excepted:*

$$\Omega(F_i) = \{G_1 + \cdots + G_r \in \mathcal{F}(P) \setminus \{P\} \mid F_i \subseteq G_i\}.$$

Let $f(\Omega(F_i))$ be the f-vector of $\Omega(F_i)$. Then:

$$\sum_{k=dim(F_i)}^{d-1} (-1)^{d-1-k} f_k(\Omega(F_i)) = 1.$$

Proof. In the normal fan of P, the set $\Omega(F_i)$ corresponds to the set of normalized normal cones contained in the closure of $\hat{\mathcal{N}}(F_i; P_i)$. Each face G in $\Omega(F_i)$ corresponds to a normalized cone of dimension $d - 1 - dim(G)$. Let's call M the resulting cell complex. We have that

$$\sum_{k=dim(F_i)}^{d-1} (-1)^{d-1-k} f_k(\Omega(F_i)) = \sum_{k=0}^{d-1-dim(F_i)} (-1)^k f_k(M).$$

The support of M is the closure of $\hat{\mathcal{N}}(F_i; P_i)$, which is contractible. So we can use Euler's formula to prove the following:

$$\sum_{k=0}^{d-dim(F_i)} (-1)^k f_k(M) = 1.$$

∎

Lemma 5.1.5 *Let $F = F_1 + \cdots + F_r$ be a face of the Minkowski sum $P = P_1 + \cdots + P_r$ with an exact decomposition. Then:*

$$\sum_{k=0}^{d-1}(-1)^k k(f_k(F) - (f_k(F_1) + \cdots + f_k(F_r))) = 0.$$

Proof. Since F has an exact decomposition, all of its subfaces also have one. What's more, for any set $(G_1, \ldots, G_r)$ so that $G_i \subseteq F_i$ for all i, the sum $G = G_1 + \cdots + G_r$ is a subface of F. In other words, the face lattice of F is the same as that of the Cartesian product of its summands $F_1 \times \cdots \times F_r$.

Let d_i be the dimension of F_i and f^i its f-vector. It can be written as $(f^i_0, \ldots, f^i_{d_i})$, with $f^i_{d_i} = 1$. The f-vector f_i verifies Euler's formula, which means

$$\sum_{k=0}^{d_i} (-1)^k f^i_k = 1.$$

Let's define the generating function $p_i(x)$ of the f-vector f^i as follows:

$$p_i(x) = f^i_0 t^0 + \cdots + f^i_{d_i} t^{d_i}.$$

Euler's formula can now be written as $p_i(-1) = 1$.

Let f be the f-vector of F. Since any n-tuple of faces G_i of F_i sums to a face of F, we can write:

$$f_i = \sum_{a_1+\cdots+a_r=i} (f^1_{a_1} \cdots f^n_{a_r}).$$

Therefore, if $p(x)$ is the generating function based on the f-vector of F, we have $p(x) = \prod_{i=1}^n p_i(x)$.

The difference of the f-vectors can be represented by $q(x) = p(x) - (p_1(x) + \cdots + p_r(x)) = q_0 t^0 + \cdots + q_{dim(F)} t^{dim(F)}$. It is easy to see that

$$\sum_{k=0}^{dim(F)} (-1)^k k q_k = -q'(-1).$$

Since $p'(x) = \sum_{i=1}^n \left(p'_i(x) \prod_{j \neq i} p_j(x) \right)$, and we have $p_j(-1) = 1$ for all j, $q'(-1) = (p'_1(-1) + \cdots + p'_r(-1)) - (p'_1(-1) + \cdots + p'_r(-1)) = 0$. ∎

Lemma 5.1.6 *Let $P_1, \ldots, P_r$ be d-dimensional polytopes relatively in general position, and $P = P_1 + \cdots + P_r$ their Minkowski sum. If F is a face of P, we denote by $t_i(F)$ the face of P_i in its decomposition. Then for all $0 \leq i \leq d-1$, we have:*

$$f_i(P) - (f_i(P_1) + \cdots + f_i(P_r)) =$$
$$\sum_{F \in \mathcal{F}(P), F \neq P} (-1)^{d-1-dimF} (f_i(F) - (f_i(t_1(F)) + \cdots + f_i(t_r(F)))).$$

Proof. We are going to show that the sum of coefficients for each separate face is equal to 1.

Let G be a nontrivial face of P. The sum of its coefficients in the right hand term contributed by G can be written as

$$\sum_{F \supseteq G, F \neq P} (-1)^{d-1-dim(F)}.$$

Since polytope lattices are Eulerian posets,

$$\sum_{F \supseteq G} (-1)^{d-1-dim(F)} = 0.$$

So we have that

$$\sum_{F \supseteq G, F \neq P} (-1)^{d-1-dim(F)} = 1.$$

Now, let G_i be a nontrivial face of P_i. the sum of coefficients in the right hand side contributed by G_i can be written as

$$\sum_{F \in \Omega(G_i)} (-1)^{d-1-dim(F)}.$$

By Lemma 5.1.4, this is equal to 1. ■

We can now prove the main theorem:

Proof. In Lemma 5.1.6, we proved that $f(P) - (f(P_1) + \cdots + f(P_r))$ is sum of $f(F) - (f(t_1(F)) + \cdots + f(t_r(F)))$, where $t_i(F)$ is the face of P_i in the decomposition of F. In Lemma 5.1.5, we proved the theorem is true for these vectors. ■

5.2 Applications

Now, let us examine what Theorem 5.1.2 tells us. Let $P = P_1 + \cdots + P_r$ be a Minkowski sum of polytopes in $\mathbb{R}^d$ relatively in general position. If $d = 2$, the result is that

$$f_1(P) = f_1(P_1) + \cdots + f_1(P_r),$$
$$f_0(P) = f_0(P_1) + \cdots + f_0(P_r).$$

In other words, there are as many facets (or vertices) in the sum as in all summands. Up to now, nothing very interesting!

If $d = 3$, we have:

$$2(f_2(P) - (f_2(P_1) + \cdots + f_2(P_r))) = f_1(P) - (f_1(P_1) + \cdots + f_1(P_r)),$$

$$f_2(P) - (f_2(P_1) + \cdots + f_2(P_r)) = f_0(P) - (f_0(P_1) + \cdots + f_0(P_r)) + 2,$$
$$f_1(P) - (f_1(P_1) + \cdots + f_1(P_r)) = 2(f_0(P) - (f_0(P_1) + \cdots + f_0(P_r))) + 4.$$

This is much better! We are now able to link the augmentation of the number of facets and edges during the Minkowski sum to the augmentation of the number of vertices. Using the results of Section 4.1, we can now write bounds for the maximum number of vertices, edges and facets when summing two polytopes in $\mathbb{R}^3$.

Theorem 5.2.1 *Let P_1 and P_2 be polytopes in $\mathbb{R}^3$. Then we have:*

$$f_0(P_1 + P_2) \leq f_0(P_1)f_0(P_2),$$

$$f_1(P_1 + P_2) \leq f_0(P_1) + f_0(P_2) + 2f_0(P_1)f_0(P_2) - 8,$$
$$f_2(P_1 + P_2) \leq f_0(P_1) + f_0(P_2) - 6 + f_0(P_1)f_0(P_2),$$

$$f_0(P_1 + P_2) \leq 4f_2(P_1)f_2(P_2) - 8f_2(P_1) - 8f_2(P_2) + 16,$$
$$f_1(P_1 + P_2) \leq 8f_2(P_1)f_2(P_2) - 17f_2(P_1) - 17f_2(P_2) + 40,$$
$$f_2(P_1 + P_2) \leq 4f_2(P_1)f_2(P_2) - 9f_2(P_1) - 9f_2(P_2) + 26,$$

All these bounds are tight.

Proof. We already now that $f_0(P_1 + P_2) \leq f_0(P_1)f_0(P_2)$ is a tight bound. Therefore, if $f_0(P_1)$ and $f_0(P_2)$ are fixed, we need to maximize $f_1(P_1)$ and $f_1(P_2)$ for the second bound, and $f_2(P_1)$ and $f_2(P_2)$ for the third. This can be done by summing simplicial polytopes as indicated in Section 4.2, so that $f_1(P_i) = 3f_0(P_i) - 6$ and $f_2(P_i) = 2f_0(P_i) - 4$.

if $f_2(P_1)$ and $f_2(P_2)$ are fixed, the maximum augmentation of the number of faces is reached by having $f_0(P_1)$ and $f_0(P_2)$ as large as possible. This can be done by summing simple polytopes as indicated in Section 4.2, so that $f_1(P_i) = 3f_2(P_i) - 6$ and $f_0(P_i) = 2f_2(P_i) - 4$. ∎

Chapter 6

Nesterov rounding

Géomètre : "Nul n'entre ici s'il n'est géomètre."
Gustave Flaubert, Le dictionnaire des idées reçues.

We study in this chapter a special family of Minkowski sums, those of polytopes summed with their own dual. After presenting a geometrical result by Yurii Nesterov, we show that in certain cases, the combinatorial properties of the Minkowski sum can be deduced directly and completely ([16]).

6.1 Asphericity

We recall that a polytope is called centered if its interior contains the origin, and that its dual is then also a centered polytope.

Let us denote the unit ball in $\mathbb{R}^d$ by $\mathcal{B}_d$. That is,

$$\mathcal{B}_d = \{\mathbf{x} \in \mathbb{R}^d \mid \langle \mathbf{x}, \mathbf{x} \rangle \leq 1\}.$$

Let S be a bounded closed centered convex set in $\mathbb{R}^d$. we define its *asphericity* $\gamma(S)$ as follows:

$$r(S) = \max\{\lambda \mid \lambda\mathcal{B}_d \subseteq S\},$$

$$R(S) = \min\{\lambda \mid \lambda\mathcal{B}_d \supseteq S\},$$

$$\gamma(S) = \frac{R(S)}{r(S)}.$$

That is, the asphericity of S is the ratio between the radii of the smallest ball containing and the largest ball contained in S. Obviously, $\gamma(S) \geq 1$ for any S, with equality only if S is a sphere.

It was proved by Nesterov ([35]) that for well chosen factors α and β, we have that

$$\gamma(\alpha S + \beta S^*) \leq \sqrt{\frac{1+\gamma(S)}{2}} < \sqrt{\gamma(S)}.$$

That is, summing a centered polytope (and more generally a centered full-dimensional bounded closed convex set) with its dual has a *rounding effect*. For this reason, we call this operation *Nesterov Rounding*.

Note that for this operation, we consider S and S^* to belong to the same vector space. In fact, for the rest of this chapter, we consider $\mathbb{R}^d$ and its dual space as a unique space.

6.2 Combinatorial properties

As usual, multiplying either term of a Minkowski sum by a factor doesn't affect the combinatorial properties of the sum. Since these properties are our main interest, we write "the" Nesterov rounding instead of "a" Nesterov rounding, to mean the class of all Nesterov roundings with the unique combinatorial type. While the scaling factor is irrelevant for our study, the position of the origin in P does affect the combinatorial structure of the Nesterov rounding. In other words, the combinatorial structure of the Nesterov rounding of P is not uniquely determined by that of the polytope. However, it is the case when the polytope has a certain property, which we introduce now.

Definition 6.2.1 (perfectly centered) Let P be a centered polytope in $\mathbb{R}^d$. We say that P is *perfectly centered* if for any nonempty face F of P, the intersection $relint(F) \cap \mathcal{N}(F; P)$ is nonempty.

It is easy to see that if the intersection is nonempty, then it consists of a single point, since a face is orthogonal to its normal cone.

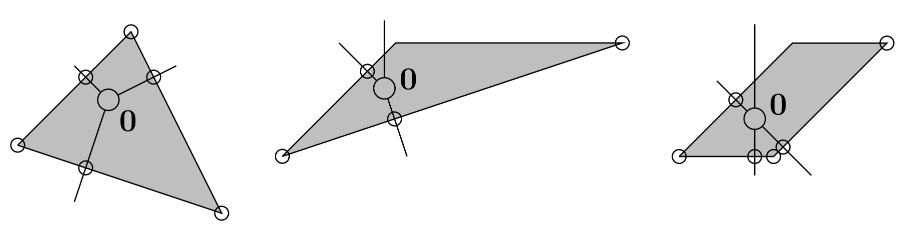


Figure 6.1: A perfectly centered and two non-perfectly-centered polytopes

For instance, the polytope on the left in Figure 6.1 is perfectly centered, and the two others are not. The one in the center can be made perfectly centered by moving the origin, but the one on the right cannot be. The perfectly centered property was previously studied in an article due to Broadie ([6]) where it was called the *projection condition*. We prefer to use "perfectly centered" because it gives a better insight into the core characteristic of these sums.

Using the Theorem 2.4.15, we immediately get the following:

Lemma 6.2.2 *A polytope P is perfectly centered if and only if P is centered and the intersection $\mathcal{N}(F;P) \cap \mathcal{N}(F^D;P^*)$ is nonempty, for every nontrivial face F of P.*

Proof. By Theorem 2.4.15, for any nontrivial face F,

$$\mathcal{N}(F^D;P^*) = \{\lambda\mathbf{x} \mid \lambda > 0,\ \mathbf{x} \in \mathit{relint}(F)\}.$$

Thus, for every nontrivial face F of a polytope P, the relations $\mathit{relint}(F) \cap \mathcal{N}(F;P) \neq \emptyset$ and $\mathcal{N}(F^D;P^*) \cap \mathcal{N}(F;P) \neq \emptyset$ are equivalent. Since $\mathcal{N}(P;P) = \{\mathbf{0}\}$, two statements $\mathit{relint}(P) \cap \mathcal{N}(P;P) \neq \emptyset$ and $\mathbf{0} \in \mathit{relint}(P)$ are also equivalent. ∎

This gives us a simple proof of the following duality

Corollary 6.2.3 ([6], Lemma 4.4) *The dual of a perfectly centered polytope is perfectly centered.*

Here is the main result of Broadie:

Lemma 6.2.4 ([6], Theorem 2.1) *If P is a perfectly centered polytope, then G is a facet of the Minkowski sum $P + P^*$ if and only if G is the sum of a face F of P with its associated dual face F^D in P^*.*

Interestingly, Broadie's article was about a quite different subject, and this theorem appeared about a variant of the Cayley embedding, in a way completely unrelated to Minkowski sums.

Corollary 6.2.5 *A perfectly centered polytope and its dual are always relatively in general position.*

Proof. Let P be a perfectly centered polytope in $\mathbb{R}^d$. For any facet G of $P + P^*$, with $G = F + F^D$, F face of P, we have

$$dim(F) + dim(F^D) = dim(F) + (d - 1 - dim(F)) = d - 1 = dim(G).$$

∎

We now extend the characterization of facets to all faces and to determine the face lattice of the Nesterov rounding $P + P^*$ of a perfectly centered polytope.

Lemma 6.2.6 *Let P be a perfectly centered polytope. If a facet of $P+P^*$ is decomposed into two faces $F \subseteq P$ and $F^D \subseteq P^*$, then any nonempty subface G of F generates with F^D a subface of $F + F^D$ of dimension $dim(G) + dim(F^D)$.*

Proof. This is the case because the faces G and F^D span affine spaces which are orthogonal to each other. ∎

In other words, for any two faces F and G of P with $G \subseteq F$, G and F^D sum to a face of $P + P^*$. We show that there are no other faces in $P + P^*$.

Lemma 6.2.7 *Let P be a polytope. Let two nonempty faces of its Nesterov rounding $P + P^*$ be decomposed into $G_1 + F_1^D$ and $G_2 + F_2^D$. Then*

$$G_1 + F_1^D \subseteq G_2 + F_2^D \quad \Leftrightarrow \quad G_1 \subseteq G_2,\ F_1 \supseteq F_2.$$

Proof. Let two nonempty faces of its Nesterov rounding $P + P^*$ be decomposed into $G_1 + F_1^D$ and $G_2 + F_2^D$.

If $G_1 \subseteq G_2$ and $F_1 \supseteq F_2$, we have $F_1^D \subseteq F_2^D$, and thus $G_1 + F_1^D \subseteq G_2 + F_2^D$.

For the converse direction, observe that for two faces A and B of a polytope P, $A \nsubseteq B$ if and only if $cl(\mathcal{N}(A; P)) \cap \mathcal{N}(B; P) = \emptyset$. Assume $G_1 \nsubseteq G_2$, that is, $cl(\mathcal{N}(G_1; P)) \cap \mathcal{N}(G_2; P) = \emptyset$. This implies

$$\begin{aligned} & cl(\mathcal{N}(G_1 + F_1^D; P + P^*)) \cap \mathcal{N}(G_2 + F_2^D; P + P^*) \\ & = cl(\mathcal{N}(G_1; P) \cap \mathcal{N}(F_1^D; P^*)) \cap \mathcal{N}(G_2; P) \cap \mathcal{N}(F_2^D; P^*) \\ & \subseteq cl(\mathcal{N}(G_1; P)) \cap \mathcal{N}(G_2; P) \cap cl(\mathcal{N}(F_1^D; P^*)) \cap \mathcal{N}(F_2^D; P^*) = \emptyset. \end{aligned}$$

Consequently, $G_1 + F_1^D \nsubseteq G_2 + F_2^D$, and by symmetry, $F_1 \nsupseteq F_2$. ∎

Now we are ready to prove:

Theorem 6.2.8 *Let P be a perfectly centered polytope. A subset H of $P+P^*$ is a nontrivial face of $P + P^*$ if and only if $H = G + F^D$ for some ordered nontrivial faces $G \subseteq F$ of P.*

Proof. By Lemma 6.2.4, the facets of $P + P^*$ are of form $F + F^D$ for some nontrivial face F of P. Lemma 6.2.6 says that if F and G are nontrivial faces of P with $G \subseteq F$, then $G + F^D$ is a face of the sum polytope. Finally, Lemma 6.2.7 shows that all the faces are of that kind, since it proves that there are no other subfaces to the facets. ∎

Corollary 6.2.9 *The face lattice of the Nesterov rounding $P + P^*$ of a perfectly centered polytope is determined by that of P.*

Theorem 6.2.10 *The Nesterov rounding of a perfectly centered polytope is also perfectly centered.*

Proof. Le P be a perfectly centered polytope. Let F and G be nontrivial faces of P with $G \subseteq F$. We'll denote by $\mathbf{m_F}$ and $\mathbf{m_G}$ the unique points in their intersections with their respective normal cones. By Theorem 6.2.8, it suffices to show that $\mathbf{m_G} + \mathbf{m_{F^D}} \in \mathcal{N}(G; P) \cap \mathcal{N}(F^D; P^*)$. By Lemma 6.2.2, $\mathbf{m_G} \in \mathcal{N}(G; P) \cap \mathcal{N}(G^D; P^*)$. Also, $\mathbf{m_{F^D}} \in \mathcal{N}(F; P) \cap \mathcal{N}(F^D; P^*)$. Since $G \subseteq F$, $\mathcal{N}(F; P) \subseteq cl(\mathcal{N}(G; P))$. Since $\mathbf{m_G} \in \mathcal{N}(G; P)$ and $\mathbf{m_{F^D}} \in cl(\mathcal{N}(G; P))$, $\mathbf{m_G} + \mathbf{m_{F^D}} \in \mathcal{N}(G; P)$. By symmetry, $\mathbf{m_G} + \mathbf{m_{F^D}} \in \mathcal{N}(F^D; P^*)$, completing the proof. ∎

Note that the sum of two perfectly centered polytopes is not always perfectly centered. For example, in Figure 6.2, both rectangles are perfectly centered, but their sum is not, since the sum of the two marked vertices is not in its normal cone.

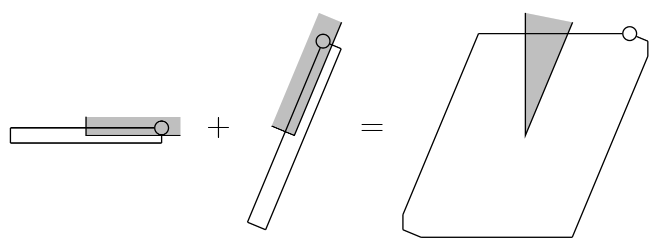


Figure 6.2: A non-perfectly-centered sum of perfectly centered polytopes

6.3 Repeated Nesterov rounding in dimension 3

We examine in this section the result of executing many times the Nesterov rounding on a 3-dimensional polytope. We prove that the ratio of the number of facets of that of vertices approaches 1, hinting that the repeated operation tends towards a self-dual polytope not only geometrically, but also combinatorially.

To simplify, let us use the following notation: $f_k^{(n)}$ denotes the number of k-dimensional faces in a centered polytope P after executing the Nesterov rounding n times.

Theorem 6.3.1 *Let P be a perfectly centered 3-dimensional polytope. Then the following relations hold:*

$$f_0^{(n)} = 4^{n-1} f_0^{(1)},$$

$$f_1^{(n)} = 2 \cdot 4^{n-1} f_0^{(1)}, \text{ and}$$

$$f_2^{(n)} = f_2^{(1)} + (4^{n-1} - 1) f_0^{(1)}.$$

Proof. Let P be a perfectly centered three-dimensional polytope. By Corollary 6.2.9,

$$f_2^{(n)} = f_0^{(n-1)} + f_1^{(n-1)} + f_2^{(n-1)}.$$

It is a general property of face lattices that for two faces $G \subseteq F$ so that $dim(G) + 2 = dim(F)$ there are exactly two faces H_1 and H_2 of dimension $dim(G) + 1$ so that $G \subseteq H_1 \subseteq F$ and $G \subseteq H_2 \subseteq F$. In a Nesterov rounding, it means that all $(d-3)$-dimensional faces, which are sums of a face G and F^D, $G \subseteq F$ so that $dim(G) + 2 = dim(F)$ are contained in four $(d-2)$-dimensional faces, which are $G + H_1^D$, $G + H_2^D$, $H_1 + F^D$ and $H_2 + F^D$, and four $(d-1)$-dimensional faces, which are $G + G^D$, $H_1 + H_1^D$, $H_2 + H_2^D$ and $F + F^D$. In the 3-dimensional case, it means that all vertices are contained in four incident edges and four facets. Since each edge contains exactly two vertices, we have:

$$f_1^{(n)} = 2 f_0^{(n)}, \ \forall n \geq 1.$$

Since the number of vertices in the next Nesterov rounding is equal to the number of pairs of a vertex and its containing facets, it also means that:

$$f_0^{(n+1)} = 4 f_0^{(n)}, \ \forall n \geq 1.$$

Thus we have the following equations:

$$f_0^{(n)} = 4^{n-1} f_0^{(1)},$$

$$f_1^{(n)} = 2 \cdot 4^{n-1} f_0^{(1)}, \text{ and}$$

$$f_2^{(n)} = f_2^{(n-1)} + 3 \cdot 4^{n-2} f_0^{(1)}$$

$$\Rightarrow f_2^{(n)} = f_2^{(1)} + (4^{n-1} - 1) f_0^{(1)}.$$

∎

Note that the ratio of the number of facets to that of vertices tends towards 1.

6.4 Repeated Nesterov rounding in dimension 4

In this section, we examine the repetition of the Nesterov rounding on a 4-dimensional polytope.

As before, we use the following notation: $f_k^{(n)}$ denotes the number of k-dimensional faces in a centered polytope P after executing the Nesterov rounding n times. Additionally, we define the following terms:

$$fat^{(n)} = \frac{f_1^{(n)} + f_2^{(n)}}{f_0^{(n)} + f_3^{(n)}}$$

is called the *fatness* of the polytope. It is a ratio between the number of edges and ridges and that of vertices and facets. Note that the fatness of a polytope is the same as that of its dual.

$$complex^{(n)} = \frac{f_{03}^{(n)}}{f_0^{(n)} + f_3^{(n)}}$$

is called the *complexity* of the polytope. It is a ratio between the number of pairs of a vertex contained in a facet and the total number of vertices and facets. Again, the complexity of a polytope is the same as that of its dual.

These parameters have been introduced in [44], as tools to study the f-vectors and flag f-vectors of polytopes in dimension four. Their study is of some interest in the search for efficient convex hull algorithms. In particular, in the case of algorithms enumerating all faces of a polytope rather than just its facets, the efficiency is lower when computing "fat" polytopes.

Theorem 6.4.1 *Let P be a perfectly centered polytope in $\mathbb{R}^d$. The fatness of the Nesterov roundings of P tends towards* 3*.*

Proof. Let P be a perfectly centered polytope in $\mathbb{R}^d$. Corollary 6.2.5 tells us that P and its dual are relatively in general position, as well as all its Nesterov roundings with their respective dual. Therefore we have:

$$3f_3^{(n+1)} - 2f_2^{(n+1)} + f_1^{(n+1)} = 3f_3^{(n)} - 2f_2^{(n)} + f_1^{(n)} + 3f_0^{(n)} - 2f_1^{(n)} + f_2^{(n)}.$$

Using Euler's formula, we get:

$$\frac{3}{2}f_3^{(n+1)} - \frac{1}{2}f_2^{(n+1)} - \frac{1}{2}f_1^{(n+1)} + \frac{3}{2}f_0^{(n+1)} = 3f_3^{(n)} - f_2^{(n)} - f_1^{(n)} + 3f_0^{(n)}.$$

We can rewrite this as:

$$(f_3^{(n+1)} + f_0^{(n+1)})(3 - fat^{(n+1)}) = 2(f_3^{(n)} + f_0^{(n)})(3 - fat^{(n)}).$$

From this we can conclude:

$$\frac{(3 - fat^{(n+1)})}{(3 - fat^{(n)})} = 2\frac{f_3^{(n)} + f_0^{(n)}}{f_3^{(n+1)} + f_0^{(n+1)}}.$$

Since $f_3^{(n+1)} = f_3^{(n)} + f_2^{(n)} + f_1^{(n)} + f_0^{(n)} = (f_3^{(n)} + f_0^{(n)})(1 + fat^{(n)})$, and that the fatness of a polytope is at least 2, we can conclude that

$$\frac{(3 - fat^{(n+1)})}{(3 - fat^{(n)})} < \frac{2}{3}$$

And that $fat^{(n)}$ tends towards 3. ∎

6.5 Special cases of Nesterov rounding

We examine in this section the Nesterov rounding on hypercubes and simplices. Using Theorem 6.2.8, we show the resulting f-vectors are defined by simple functions.

Theorem 6.5.1 *Let Δ_d be a perfectly centered simplex of dimension d. Then, the f-vector of the Nesterov rounding of Δ_d is given by*

$$f_k(\Delta_d + \Delta_d^*) = \binom{d+1}{k+2}\left(2^{k+2} - 2\right), \qquad \textit{for } 0 \le k \le d-1.$$

Proof. Let Δ_d be a perfectly centered simplex of dimension d. The f-vector of Δ_d is given by

$$f_k(\Delta_d) = \binom{d+1}{k+1}, \qquad \text{for } 0 \le k \le d-1.$$

By Theorem 5.1.2, the faces of $\Delta_d + \Delta_d^*$ can be characterized as the sums $F^D + G$, with $G \subseteq F$ nontrivial faces of Δ_d.

Let S and T be the vertex sets of respectively G and F, with $S \subseteq T$, and denote $U = T \setminus S$. The dimension k of $F^D + G$ is $dim(F^D) + dim(G) = d - 1 + dim(G) - dim(F) = d - 1 + |S| - |T| = d - 1 - |U|$.

So the number of faces of dimension k can be written as pq, where p is the number of possible choices of U with $|U| = d - 1 - k$, and q is the number of choices of S nonempty, so that $S \cap U = \emptyset$ and $|T| = |S \cup U| < d + 1$. Thus we have

$$p = \binom{d+1}{k+2} \qquad \text{and} \qquad q = 2^{k+2} - 2.$$

∎

Theorem 6.5.2 *Let $\Box_d$ be a hypercube of dimension d. Then, the f-vector of the Nesterov rounding of $\Box_d$ is given by*

$$f_k(\Box_d + \Box_d^*) = \binom{d}{k+1} 2^{d-k-1} \left(3^{k+1} - 1\right), \qquad \textit{for } 0 \leq k \leq d-1.$$

Proof. Let $\Box_d$ be a hypercube of dimension d. Then $\Box_d$ has $3^d - 1$ nontrivial faces, which can be decomposed into:

$$f_k(\Box_d) = \binom{d}{k} 2^{d-k}, \qquad \text{for } 0 \leq k \leq d-1.$$

By Theorem 5.1.2, the faces of $\Box_d + \Box_d^*$ can be characterized as the sums $F^D + G$, with $G \subseteq F$ nontrivial faces of $\Box_d$.

Let S and T be the sets of fixed coordinates of respectively G and F, with $T \subseteq S$, and denote $U = S \setminus T$. The dimension k of $F^D + G$ is $dim(F^D) + dim(G) = d-1+dim(G)-dim(F) = d-1+(d-|S|)-(d-|T|) = d-1-|U|$.

So the number of faces of dimension k can be written as pqr, where p is the number of possible choices of U with $|U| = d-1-k$, q is the number of ways to fix the coordinates in U, and r is the number of choices of G, so that $S \cap U = \emptyset$ and $|T| = |S \cup U| < d+1$. We have

$$p = \binom{d}{k+1}, \qquad q = 2^{d-k-1} \qquad \text{and} \qquad r = 3^{k+1} - 1.$$

∎

We would like to thank Günter Ziegler, who has greatly contributed to the simplification of these proofs.

Part III

Algorithms

Chapter 7

Enumerating the vertices of a Minkowski sum

Une pierre deux maisons trois ruines quatres fossoyeurs un jardin des fleurs
un raton laveur
Jacques Prévert, INVENTAIRE.

We introduce in this chapter an algorithm due to Fukuda ([13]) computing the $\mathcal{V}$-representation of the Minkowski sum of polytopes also in $\mathcal{V}$-representation. We also present results of an implementation we did of this algorithm.

7.1 Theory

The algorithm is based on two very simple observations:

1. Vertices of a Minkowski sum are decomposed into vertices of the summands.
2. Edges of a Minkowski sum are decomposed into vertices and parallel edges of the summands.

The algorithm consists in exploring the vertices of the Minkowski sum by moving along edges. For this, we need to identify pairs of vertices which are linked by an edge.

Proposition 7.1.1 *If* $\mathbf{u}$ *and* $\mathbf{v}$ *are adjacent vertices of the sum polytope* $P = P_1 + \cdots + P_r$*, decomposed into* $\mathbf{u} = \mathbf{u_1} + \cdots + \mathbf{u_r}$ *and* $\mathbf{v} = \mathbf{v_1} + \cdots + \mathbf{v_r}$*, then* $\mathbf{u_i}$ *and* $\mathbf{v_i}$ *are either equal or adjacent vertices of* P_i*, and all adjacent pairs are linked by parallel edges.*

Proof. Let E be the edge of P from $\mathbf{u}$ to $\mathbf{v}$, decomposed into a sum of faces $E = E_1 + \cdots + E_r$. E_i are either vertices ($\mathbf{u_i} = \mathbf{v_i}$) or parallel edges ($\mathbf{u_i}$ and $\mathbf{v_i}$ are adjacent). ∎

Therefore, if we have a vertex $\mathbf{v}$ of the sum polytope $P = P_1 + \cdots + P_r$ decomposed into $\mathbf{v} = \mathbf{v_1} + \cdots + \mathbf{v_r}$, the edges incident to the summands $\mathbf{v_i}$ give us possible candidates for edges incident to $\mathbf{v}$ in the sum polytope P. For each candidate, we can test whether it is a real edge by trying to find an hyperplane separating it from the other candidates.

It is easy to find a vertex of the sum and its decomposition, by enumerating the vertices of each summand and choosing the maximal one for any linear function in general position. Using the method we just described, we can find the incident edges, and then the adjacent vertices they link to, with their own decomposition.

From this point, the problem of enumerating all vertices of the Minkowski sum amounts to finding how to explore one by one the vertices from adjacent vertex to adjacent vertex in an way as efficient as possible. The following section describes an ideal solution.

7.2 The reverse search method

The reverse search method was developed by David Avis and Komei Fukuda for the vertex enumeration problem. However, it can be used more generally as a technique for enumerating many kind of discrete objects ([1]).

Let us assume we wish to enumerate a large set of objects, which can be arranged as a graph $G(V, E)$, where the set of nodes V is the set of objects, and the edges E can be deduced using an *adjacency oracle*, which allows us to enumerate the neighbours of a node in the graph.

Let us assume we can define an arborescence on this graph, that is, a tree of oriented edges with a single sink. A property of arborescences is that every node except the sink has a single parent, that is, a single neighbour node in the tree with the edge oriented towards it. We assume we can define a *local search function* which determines the parent of any node except for the sink. We further assume the source node is known.

The reverse search works as follows. The enumeration is initiated at the source node. From then, the tree is explored using depth-first search. On each node N of the tree treated, the neighbours of N are enumerated using the adjacency oracle. The descendants of N in the arborescence are identified by the fact that N is their parent. When a descendant is found, its embranchment is fully explored before looking for the next descendant.

When all descendants are explored, the algorithm goes back one level to the parent node. Thus, the algorithm examines every node of the graph, testing each time every neighbour but continuing the search only on descendants, going down and up every embranchment of the tree until ending on the source node again.

Reverse search algorithms, if properly implemented, have the following properties:

1. The number of calls to the oracles is proportional to the size of the output times a polynomial in the size of the input.

2. Memory size needed is polynomial in the size of the input.

3. Parallel implementation is straightforward.

The first two properties are very important. Many enumeration problems can have an output size which is exponential in terms of the input size (in particular, enumerating the vertices of a Minkowski sum fits in this category). Naturally, at least linear time in terms of the output size is needed for the enumeration (if only to write results). Reverse search can therefore be said to have maximal efficiency in terms of the output size, since the computation time is linear and the memory size *independent* of the size of the output.

7.3 Reverse search and Minkowski sums

We explain here the details of how reverse search can be applied to the enumeration of the vertices of a Minkowski sum.

The set V we wish to enumerate is of course that of the vertices of the Minkowski sum. The graph $G(V, E)$ is the one which is created by the vertices and edges of the Minkowski sum.

Let P be the Minkowski sum of $P_1, \ldots, P_r$ polytopes in $\mathbb{R}^d$. Let $\mathbf{v}$ be a vertex of P, decomposing in $\mathbf{v_1} + \cdots + \mathbf{v_r}$. In Section 7.1, we have seen that it is possible to enumerate the adjacent vertices of $\mathbf{v}$. For this, we need to examine edge candidates which are parallel to the edges incident in P_i to $\mathbf{v_i}$ for all i. A candidate is recognized as an edge if we can separate it from other candidates with an hyperplane, which can be tested by solving a linear program. This method can be used as an adjacency oracle for the reverse search method.

The local search function is somewhat trickier, and needs a trip to the normal fan of the polytope.

For each vertex $\mathbf{v}$ of the Minkowski sum P, we need to define a *maximizer vector* $\mathbf{m_v}$ which is inside its normal cone $\mathcal{N}(P; \mathbf{v})$. The facets of the normal cone of a vertex are the normal cones of incident edges. Therefore, we can list the inequalities defining the normal cone by using once again the adjacency oracle. In fact, it is not even necessary this time to distinguish real edges from the false candidates, because the inequalities defined by the false candidates are merely redundant in the description of the normal cone. Once we have this $\mathcal{H}$-representation of the normal cone, we can find a suitable $\mathbf{m_v}$ in its interior by solving a linear program.

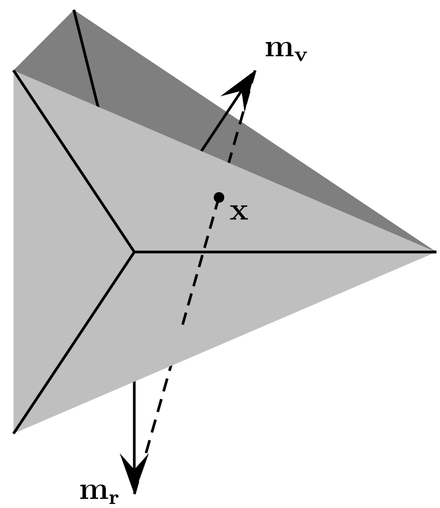


Figure 7.1: Ray shooting to find the parent vertex.

The procedure to find the parent vertex is illustrated on Figure 7.1. Let $\mathbf{r}$ be the vertex of P we chose to be the sink of the arborescence. Let $\mathbf{v}$ be a vertex different from $\mathbf{r}$. We *shoot a ray* which starts from $\mathbf{m_v}$ towards $\mathbf{m_r}$. The ray starts therefore in $\mathcal{N}(P;\mathbf{v})$ and ends in $\mathcal{N}(P;\mathbf{r})$. When leaving $\mathcal{N}(P;\mathbf{v})$, the ray hits a facet of the normal cone (in $\mathbf{x}$ on the figure), which is the normal cone of an edge incident to $\mathbf{v}$. We can determine this facet by a single step of the simplex method. In case of degeneracies, we can choose one of the facets it meets at the same point using symbolic perturbations. In this way, we determine a unique edge E incident to $\mathbf{v}$. We define the parent of $\mathbf{v}$ to be the vertex at the other end of E.

It is important to note that any edge orientation thus created is consistent with the general orientation of all edges of P by the linear function $\mathbf{b_r}$. The linear hull of $\mathcal{N}(P;E)$ divides $(\mathbb{R}^d)^*$ in the two sets of linear functions which orient E one way or the other. Since the ray from $\mathbf{b_v}$ to $\mathbf{b_r}$ crosses $\mathcal{N}(P;E)$, $\mathbf{b_r}$ gives it the same orientation.

Thus, this method defines for each vertex of P except $\mathbf{r}$ a single parent among its adjacent vertices. Since the orientations created are consistent with a linear function, they contain no cycle, and so they form an arborescence with $\mathbf{r}$ as sink.

Let us examine the complexity of the algorithm. Let $P = P_1 + \cdots + P_r$ be a Minkowski sum in $\mathbb{R}^d$ we are computing. Let m_i be the maximal degree of vertices in P_i for all i, and $m = m_1 + \cdots + m_r$ the maximal degree of vertices in P, as implied in Section 7.1. Then the number of times an edge candidate is tested is in $O(mf_0(P))$. Each of these tests implies the resolution of a linear program in dimension d with $O(m)$ constraints. The number of times we need to compute the local search function is in $O(mf_0(P))$, since each edge of P is tested twice. Each of these imply the resolution of a linear program in dimension d with $O(m)$ constraints, and a ray shooting, that is a single pivot of a linear program with $O(m)$ constraints. Therefore, the total complexity of the algorithm is in $O(mf_0(P))$ resolutions of linear programs of $O(m)$ constraints.

We also need to compute the adjacency list of each summand P_i, which is done in $O(f_0(P_i))^2$ resolutions of linear programs with $O(m_i)$ constraints.

7.4 Results

Fukuda's algorithm was implemented in C++, based on a framework developed for `TOPCOM` by Jürg Rambau ([38]). Computations are done in exact precision using the `GMP` library. The computation of the adjacency matrix of each summand, the resolution of linear programs and the ray shooting described in preceding section are executed by calls to Fukuda's `cddlib` library. The complete source code is available on the web ([41]).

Let us illustrate the efficiency of the algorithm by showing the numerical results of some problems we studied in the course of this PhD.

First of all, the program is able to handle very large problems without difficulties. The largest Minkowski sum we computed is that of the Birkhoff polytopes of the Pappus configuration. The sum is in $\mathbb{R}^{28}$. Though each of the nine summands has only 6 vertices, the sum has 2.372.583 of them! Though the computation time was around 4 weeks, the memory size needed was low. Actually, in this particular case, it is faster to actually enumerate all possible 6^9 vertex decompositions, and check for each one whether it is present in the sum. Since it is the case for 23% of them, we test four times too many decompositions, but the test can be made with a single linear program, and so it is very fast. The computation is then done in just over three days. Nevertheless, it was a good test of the stability of the implementation.

7.4.1 Hypercubes

In table 7.1 are reproduced the computation time of summing d orthogonal line segments in $\mathbb{R}^d$, that is, computing the vertices of a d-dimensional hypercube.

dimension	vertices	edges	time	lp	rs
5	32	80	12	5	0
6	64	192	39	25	1
7	128	448	121	71	5
8	256	1'024	362	238	11
9	512	2'304	1'063	733	33
10	1'024	5'120	2'982	2'114	91
11	2'048	11'264	8'250	5'898	219
12	4'096	24'576	21'917	16'065	523
13	8'192	53'248	57'724	42'905	1'310
14	16'384	114'688	149'770	113'289	3'161
15	32'768	245'760	381'683	291'584	7'896

Table 7.1: Computation times (0.01s) for computing d-dimensional hypercubes.

The "lp" and "rs" values indicate the total time used for computing the solution of a linear program or a ray shooting respectively. Though there is one ray shooting executed for each edge of the Minkowski sum, the computation time is very small. The computation time for solving linear programs is quite large as expected. We can see that as dimension grows, it amounts to more than 75% of the total computation time. The rest of the computation time is used for actually building the linear programs before feeding it to the solver.

7.4.2 Hidden Markov Models

In table 7.2 are reproduced the computation time of the different inference functions of an Hidden Markov Model (HMM) of different lengths.

Here, the column "bld" contains the time used just to *build* linear programs testing whether a candidate is an edge or not. We observe that this takes almost as much as solving linear programs! We can deduce from this that the linear programs are quite large, but are solved very fast.

length	vertices	facets	time	lp	bld
2	38	2	26	16	12
3	398	71	1’566	644	419
4	1’570	225	25’502	8’844	7’957
5	5’266	749	413’620	139’139	120’699
6	17’354	2’507	6’176’450	2’109’214	1’526’135
7	55’230	8’516	54’431’755	18’834’517	12’959’941

Table 7.2: Computation times (0.01s) for HMM of different lengths.

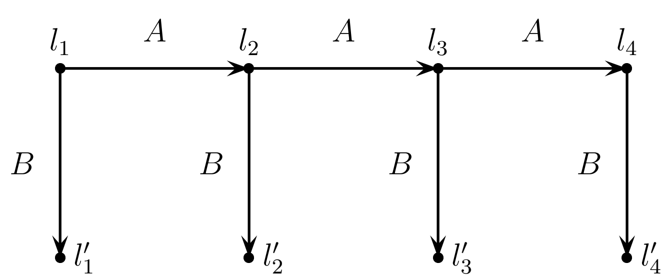


Figure 7.2: A hidden Markov model of length 4.

Hidden Markov Models

Hidden Markov Models are used to study chains of informations which we may observe only indirectly (see Figure 7.2). The chains are modelized by a homogeneous Markov chain $(l_1, \dots, l_n)$, from which we derive observations $(l'_1, \dots, l'_n)$, which are imperfect.
Briefly presented, the goal of the computation is to compute the number of distinct functions inferring the values of an hidden Markov chain of length n on the basis of observed values.
If the values are binary, and the transition probabilities of the chain and the observations are coded in two transition matrices A and B, then the probability of realizations are polynomials in $\mathbb{R}^8 = (\alpha_{00}, \alpha_{01}, \alpha_{10}, \alpha_{11}, \beta_{00}, \beta_{01}, \beta_{10}, \beta_{11})$. For a certain observation $(l'_1, \dots, l'_n)$, the inference function consists in choosing the values $(l_1, \dots, l_n)$ maximizing the likelihood. If we represent the logarithm of the polynomials as vectors in $\mathbb{R}^8 = (\ln(\alpha_{00}), \dots, \ln(\beta_{11}))$, this amounts to solving a linear program on their convex hull.
So for one observation, the different inference functions are represented by the vertices of a polytope. For many observations, the different inference functions are represented by the vertices of the Minkowski sum of the respective polytopes.
Combinatorially, the computation is a sum in $\mathbb{R}^8$ of 2^n polytopes of about 2^n vertices each. The summands are actually 4-dimensional, and the sum 5-dimensional.
For more information on the subject, please refer to [36].

7.4.1 Distributed sums

We study in this section whether it is efficient to decompose sums of many polytopes in distributed sums in order to speed up the computations. We tried for this three different schemes for computing the Hidden Markov Model of length 4, which we compare in Table 7.3. HMM's are ideal for such a study, since the number of summands is a power of two. We name the sixteen summands 1 to 9, then A to G.

The first scheme consists in simply summing all polytopes in a single computation. It is denoted by $1 + \cdots + G$ in Table 7.3.

The second scheme is incremental. It consists in summing first $1+2 = 12$, then $12 + 3 = 123$, $123 + 4 = 1234$, and so on.

The third scheme is pyramidal. We first sum of polytopes by pairs, such as $1 + 2 = 12$, $3 + 4 = 34$, and so on. We then sum the results by pairs, such as $12 + 34 = 1234$ and $56 + 78 = 5678$, and again until all polytopes are summed.

sum	vertices	time	init	diff
$1 + \cdots + G$	1'570	25'502	428	26'074
$1 + 2, 7 + 8, 9 + A, F + G$	36	73	43	30
$3 + 4, 5 + 6, B + C, D + E$	48	127	65	62
$12 + 34, \ldots, DE + FG$	148	845	608	237
$1234 + 5678, \ldots$	494	13'715	12'945	770
$12345678 + 9ABCDEFG$	1'570	381'865	379'784	2'081

Table 7.3: Computation times (0.01s) for different sums.

The "init" column indicates the time necessary for the initial step of the algorithm, which is to compute the adjacency of the summands. The "diff" column indicates the time taken by the rest of the algorithm. Since the complexity of computing the adjacency of a polytope is in the square of its number of vertices, we see immediately the problem of distributed sums: Computing the adjacency of intermediate results takes much more time than solving the whole problem.

It would be possible to modify the algorithm so as to output the adjacency list of the Minkowski sum as well as its vertices, without changing its complexity. In this case, this initial step should become unnecessary and the time gained by distributing the sums would be appreciable. However, this would require larger amounts of memory to store intermediate results.

Chapter 8

Enumerating the facets of a Minkowski sum

Da steh ich nun, ich armer Tor!
Und bin so klug als wie zuvor.
Johann Wolfgang von Goethe, FAUST.

As stated before, some problems of computational geometry are much easier to solve when using polytopes in $\mathcal{V}$-representation than polytopes in $\mathcal{H}$-representation. Unfortunately, Minkowski sums of polytopes is one of them.

Indeed, the vertices of a Minkowski sum are very easy to characterize, since they decompose in a sum of vertices of the summands. By contrast, the facets of a d-dimensional Minkowski sum of r polytopes can be decomposed into a multitude of ways.

For instance, let $P = P_1 + P_2$ be a Minkowski sum in dimension three. Facets of P can decompose into a vertex of P_1 plus a facet of P_2, a facet of P_1 plus a vertex of P_2, or an edge of P_1 plus a edge of P_2. Those are the exact decompositions, where the dimension of the sum is equal to the sum of the dimension of the summands. If we allow the decomposition to be inexact, facets of P can also be the sum of an edge and a facet, or even of two facets.

It is not difficult (and an interesting exercise) to prove that if P is the d-dimensional Minkowski sum of r polytopes, then there are

$$\binom{d+r-2}{d-1}$$

different ways for $(d-1)$-dimensional facets to have an exact decomposition, and if we allow inexact decompositions, the number of possibilities is

$$d^r - \binom{d+r-2}{d-2}.$$

What's more, we have shown in the preceding chapter that it is quite easy to have an adjacency oracle determining the edges incident to the vertex of a Minkowski sum. However, computing the $(d-2)$-dimensional ridges incident to a $(d-1)$-dimensional facet of a Minkowski sum amounts to finding the $(d-2)$-dimensional facets of a $(d-1)$-dimensional Minkowski sum! For this reason, it would generally be very inefficient to use an algorithm based on the same principles as in the preceding chapter. Every step of the algorithm would be in the worst case almost as complex as the initial problem, and we would end up computing recursively all faces of the polytope, down to vertices.

There is however an exception. When the summands are relatively in general position, the facets have an exact decomposition, and so, their face lattices are isomorphic to those of Cartesian products. In this case, an oracle for listing the ridges incident to a facet can be built as follows. Let $P = P_1 + \cdots + P_r$ be a Minkowski sum. Let F be a facet of P with $F_1, \ldots, F_r$ its exact decomposition. Then a set F' is a facet of F if and only if

$$F' = F_1 + \cdots + F_{i-1} + G_i + F_{i+1} + \cdots + F_r$$

For some facet G_i of F_i, with $dim(F_i) \geq 1$. Note that it may be necessary to compute the whole face lattices of the summands to use this oracle, instead of merely their vertex adjacency lists. Nevertheless, it makes it possible to create an algorithm for computing the facets of a Minkowski sum of polytopes, assuming they are relatively in general position.

From there, one might be able to design a general algorithm using some kind of symbolic perturbation which would make input polytopes satisfy this assumption. The complexity of such a technique is uncertain, and should be studied carefully.

Another way to compute the facets of a Minkowski sum is naturally to first compute its vertices, then compute from there its facets. However, this solution is not satisfying, for reasons which are explained in next section.

8.1 Convex hull

A core element in the study of polytopes is their duality, that is, the fact they can be represented as the intersection of a finite number of half-space (the $\mathcal{H}$-representation), or as the convex hull of a finite number of vertices (the $\mathcal{V}$-representation).

The *convex hull problem*, which consists in transforming a $\mathcal{V}$-representation into $\mathcal{H}$-representation, is therefore at the center of the computational geometry on polytopes. The inverse question, called *vertex enumeration*, of transforming a $\mathcal{H}$-representation into a $\mathcal{V}$-representation actually amounts to virtually the same problem: From a polytope in H-representation, it is trivial to compute the V-representation of its dual. If we solve the convex hull problem on the dual to find the H-representation of the dual, we can then easily deduce the V-representation of the starting polytope.

Not only is the convex hull problem one of the most basic questions, it is also the one with the most applications. Many problems about polytopes can be solved naturally and easily by using one representation, but are considerably more difficult when using the other. For instance, the intersection of two polytopes is quite easy to do using $\mathcal{H}$-representation, as it is enough to remove the half-spaces which are redundant. Doing the same using $\mathcal{V}$-representation is considerably more difficult. On the contrary, computing the Minkowski of two polytopes is much easier with $\mathcal{V}$-representations than with $\mathcal{H}$-representations.

This would be unimportant if the convex hull problem were easy to solve. Unfortunately, it is not. In fact, even if we have a list of inequalities, there is currently no known easy way to check whether the list is complete or not. It has even been proved that the more general problem concerning polyhedra is NP-Complete ([25])! We attempt here to explain the difficulties encountered. For simplicity, we take as example the inverse problem of vertex enumeration, which is as we said equivalent by duality.

8.1.1 Double description

Let us outline an algorithm for enumerating the vertices of a d-dimensional polytope in $\mathcal{H}$-representation. It was presented by Motzkin at al. in 1953 ([33]).

Algorithm 8.1.1 (Double description)

```
Input:  h_1,...,h_m half-spaces
Output:  v_1,...,v_n vertices
P = ℝ^d
V = ∅
for i:= 1 to m {
   Add to V vertices created by cutting P with h_i
   Remove from V points which are beyond h_i
   P := P ∩ h_i
}
Output V
```

This corresponds to computing successively the vertices of h_1, $h_1 \cap h_2$, $h_1 \cap h_2 \cap h_3$ and so on until we have run out of half-spaces, removing as we go vertices which are cut off from the polytope by the half-space we are adding.

The problem of this algorithm is that we may need to compute many unnecessary vertices. An example of application to the regular nonagon is shown on Figure 8.1. We can see that we compute six unnecessary vertices for nine in the solution.

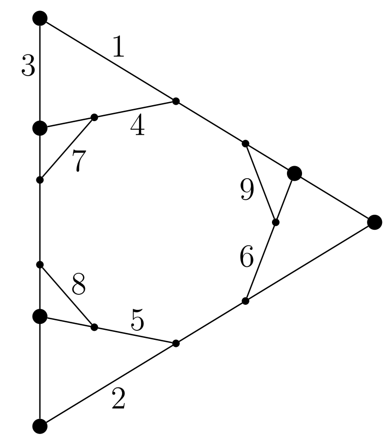


Figure 8.1: Computation of the vertices of a nonagon

The number of unnecessary vertices can sometimes be reduced by using the right insertion order for the half-spaces. However, it has been proved that in some cases, all orderings compute a number of unnecessary vertices which is exponential in terms of the input and the output ([4]).

It is interesting to note that it is possible to transpose the Double Description algorithm in the dual space to compute the convex hull of a set of points. The principles of this dual version, called *beneath and beyond*, were presented by Branko Grünbaum in 1963 ([19]).

In brief, this algorithm works by adding vertices one by one to a polytope, computing for each step the new convex hull by adding new facets formed by the new point with old ones, and removing facets which are visible from the new point. The duality is complete! If both algorithms are used on dual centered polytopes, each computation on one side is mirrored on the dual. The relations between the different steps of both algorithms are shown in this table:

Double description	Beneath and beyond
Input: facets	Input: vertices
Output: vertices	Output: facets
Add facet $\langle \mathbf{a_i}, \mathbf{x} \rangle \leq 1$	Add vertex $\mathbf{a_i}$
Compute vertices of intersection	Compute facets of convex hull
Remove vertices beyond $\langle \mathbf{a_i}, \mathbf{x} \rangle \leq 1$	Remove facets beyond which $\mathbf{a_i}$ is.

8.2 Overlay of normal fans

As we have seen in the preceding section, it is difficult to deduce the facets of a polytope from its vertices. Therefore, we should look for a way to compute the facets of a Minkowski sum directly.

We present here a quite efficient method which was proposed by Guibas and Seidel in [22]. As we said before, the normal fan of polytopes contains all of their combinatorial organization. It is therefore enough to compute the normal fan of a Minkowski sum to have its combinatorial properties. We can then easily deduce the polytope itself by combining these informations with the summand polytopes.

We know that the normal fan of a Minkowski sum is the common refinement of the normal fans of its summands. The particular interest of working with normal fans is that their structure is essentially simpler by one rank compared to the polytopes: We can intersect them with a sphere of radius one without losing any information. For this reason, computing the Minkowski sum of three-dimensional polytopes is equivalent to computing the common refinement of polyhedral complexes on the surface of a three-dimensional sphere, which is a two-dimensional manifold.

This is great news, because many problems of computational geometry are considerably easier in dimension two than three. For instance, computing the common refinement of cell complexes in two dimension can be done in $O(n+k)\log n)$ time, where n and k are the size of the input and the output respectively, by using a sweeping plane algorithm.

So we arrive to a simple and efficient method to compute the facets of a Minkowski sum in dimension three:

1. Compute the normal fan of the polytopes,

2. Compute the intersection of the normal fans with a sphere,

3. Compute the overlay of the resulting polyhedral complexes.

The vertices of the overlay then correspond to facets of the Minkowski sum. The edges of the overlay provide the facet adjacency.

This method was implemented with great success in a slightly modified version (intersecting the normal fans with the surface of a cube instead of a sphere, so as to avoid working in a non-Euclidean space) by Fogel and Halperin ([11]).

Unfortunately, it is impossible to generalize this method to higher dimensions. Even adapting it to the sum of four-dimensional polytopes seems quite difficult. The reason being that while efficient algorithms exist for computing the common refinement of two polyhedral complexes in dimension two, their complexity grows quickly in higher dimensions.

The reason of this is quite simple: The vertex of a common refinement in dimension two is either a vertex of the starting complexes, or the intersection of two edges. But in higher dimensions, there are more possibilities, corresponding to the many decomposition possibilities of a facet in the Minkowski sum. As we hit the same roadblock, we are forced to conclude that an efficient algorithm for computing facets would require a completely different strategy.

8.3 Beneath and beyond

Another approach for computing facets of Minkowski sums has been developed and implemented by Peter Huggins in the special case of sequence alignment problems ([23]). It consists in executing a beneath and beyond algorithm as presented in Chapter 8.1, computing for each step a new vertex of the Minkowski sum.

Let us explain it in more details. We start from the $\mathcal{V}$-representation of the summands. For any linear function $\mathbf{a}$, it is easy to find the vertex of the Minkowski sum which is optimal for $\mathbf{a}$. Indeed, the maximum vertex for the Minkowski sum is the sum of the maximum vertices for each summand. These can be found by a simple vertex enumeration. In the special case of sequence alignments, Huggins uses the Needleman-Wunsch algorithm ([34]).

We start by finding $d+1$ vertices of the Minkowski sums by optimizing in various directions, and compute the facets of their convex hull. We add vertex by vertex to the simplex P thus found, as in a classic beneath and beyond algorithm, until the Minkowski sum is complete.

For each facet $(\mathbf{a}, \beta)$ of the polytope P, we find the vertex $\mathbf{v}$ of the Minkowski sum which is maximal for the linear function $\mathbf{a}$. If $\mathbf{v}$ is on the facet $(\mathbf{a}, \beta)$, then the facet is a facet of the Minkowski sum.

If $\mathbf{v}$ is not on $(\mathbf{a}, \beta)$, then we add it to P like for beneath and beyond: We compute the new facets in $conv(P \cup \mathbf{v})$, and remove the facets of P visible from $\mathbf{v}$.

When all facets have been checked, the Minkowski sum is complete.

The implementation Peter Huggins did of this algorithm is extremely fast: HMM6 and HMM7 are computed in a few *minutes*, when our implementation takes *days*, without even computing facets! The implementation uses floating-point arithmetic, which is faster than exact precision, but its speed is nevertheless impressive. One possible problem of this algorithm is that it shares with beneath and beyond the drawback of its exponential time and memory requirements in terms of the input and the output size. However, this is the case for all incremental algorithms computing the convex hull.

Part IV

Conclusion

Chapter 9

Open Problems

Il lavoro cessa al tramonto. Scende la notte sul cantiere.
È una notte stellata. - Ecco il progetto, - dicono.
Work stops at sunset. Night falls over the construction site.
It's a starry night. Here is the project, they say.
Italo Calvino, The Invisible Cities.

Despite the simplicity of the definition of Minkowski sums, and their large number of applications, we are still relatively ignorant of many of their combinatorial properties, even in quite trivial cases. For instance, we have only conjectures about how many vertices the sum of three polytopes in dimension three can have!

Though the complexity of the problem partly explains the lack of results, another cause is certainly that the combinatorial study of Minkowski sums is a rather recent subject of research. We are convinced that there are many new results just waiting to be discovered.

9.1 Bounds

An obvious direction of research would be to look for more bounds of all kinds on the complexity of the sum.

The results of Section 4.3 state that the trivial bound on vertices can be reached when summing two polytopes in dimension 4. However, we found in Section 4.2 that it was also possible in dimension 3. It is an open problem whether the other bounds from Section 4.3 are optimal for dimension, but it seems unlikely.

We know very little of bounds on the number of vertices when the trivial bound cannot be reached, that is, in Minkowski sums of k polytopes in dimension d, with $k \geq d$. Though a construction has been proposed in [12], its maximality is not established.

It would also be desirable to find bounds on faces of higher dimensions, as the only known bound on facets is for dimension three. For instance, the Hidden Markov Models introduced in Section 7.4 generate a much smaller number of facets than predicted by theory. Though they amount to a sum in dimension five, we are for the moment at a loss to explain the phenomenon.

9.2 Relation

The linear relation presented in Chapter 5 is limited to sums of polytopes relatively in general position. Though the relation fails in other cases, it remains to be seen whether it is possible to find extensions, factoring the faces which have inexact decompositions.

Additionally, it is likely that it is possible to extend the relation to more general families of objects. It can of course be applied to the dual operation of Minkowski sums of polytopes, that is the common refinement of normal fans. With minimum modifications to our proof, it should be possible to show that the relation remains valid for the refinement of all polyhedral complexes in general position, and even for the refinement of non polyhedral complexes, as long as cells before and after the refinement are topologically contractible.

9.3 Nesterov Rounding

Though Nesterov rounding might seem such a specific operation as to have no applications, it has been already extraordinarily helpful for finding new theoretical results. Apart from zonotopes, the Nesterov rounding of perfectly centered polytopes in the only family of Minkowski sums for which we have complete knowledge of the f-vector.

Additionally, the asymptotic properties of a polytope rounded repeatedly are quite remarkable. Though it has already been proved that the rounded object tends to a self-dual one geometrically, there are hints indicating that the f-vector itself tends to become more and more symmetrical. We have proved this for dimension three, in Section 6.3, but the problem remains open for higher dimensions.

9.4 Algorithmic developments

There are many different paths to explore from the algorithmic point of view, be it by extending and improving current ones, or looking for new ones.

While the current implementation of the algorithm described in Chapter 7 is very reliable and versatile, it has shown on some occasions to be much less efficient than other implementations. The most obvious way to improve it would be to write a parallel version. As mentioned, the reverse search method is perfectly suited to parallelization, and the speed-up can be expected to be nearly optimal.

Another way to make the program faster would be to implement a floating-point version. Though it would mean some uncertainty with regards to the combinatorial structure of the result, the geometrical properties would likely be close enough for most applications.

Other algorithms may be extended, for instance, the one presented in Chapter 8.2 is currently limited to dimension three. It should be possible, if not easy, to extend it to dimension four or five .

It would also certainly be interesting to investigate the algorithm we sketched in the introduction of Chapter 8, which computes facets of the Minkowski sum of polytopes relatively in general position. We might then be able to extend it to more general cases.

Generally, it is frustratingly simple to find one particular vertex or facet of a Minkowski sum. The difficulty resides in enumerating all of them efficiently.

A completely new approach would be to exploit the symmetries of the Minkowski sum. Hidden Markov model problems are highly symmetric, and so would be a typical example. This method has recently been thoroughly researched concerning the convex hull problem. (For an excellent survey, see [5].)

Thankfully, there are enough applications of Minkowski sums, from biology to computer graphics, to assume that the research won't stop here.

Thanks

So, good night unto you all.
Give me your hands, if we be friends,
And Robin shall restore amends.
William Shakespeare, A MIDSUMMER NIGHT'S DREAM.

I would like to thank first Prof. Komei Fukuda, who supervised my work with patience and dedication during the course of my PhD. I also thank Prof. Thomas Liebling, my thesis director, equally for his patience as well as his invaluable advices and comments. Special thanks should also go to Prof. Peter Gritzmann and Prof. Günter Ziegler, members of the jury, for having read my report very carefully, and for their stimulating input during my examination. I also thank Prof. Bernd Sturmfels, for providing me with many interesting problems related to Minkowski sums.

Additionally to the help of these people with my research, nothing could have been done without the support and encouragements of many others. I would like to thank the people of the operations research labs in EPFL for the wonderful ambience and their friendship during these years. I also thank the team of Satellite for sweetening the life of thousands of people in EPFL. I thank my parents, for their constant support during the course of my studies. Last but not least, I thank Linqiao for being part of my life since two years, and encouraging me during the last stages of this PhD.

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