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Stability of linear time-periodic delay-differential equations via Chebyshev polynomials

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SUMMARY

This paper presents a new technique for studying the stability properties of dynamic systems modeled by delay-differential equations (DDEs) with time-periodic parameters. By employing a shifted Chebyshev polynomial approximation in each time interval with length equal to the delay and parametric excitation period, the dynamic system can be reduced to a set of linear difference equations for the Chebyshev expansion coefficients of the state vector in the previous and current intervals. This defines a linear map which is the 'infinite-dimensional Floquet transition matrix U'. Two different formulas for the computation of the approximate U, whose size is determined by the number of polynomials employed, are given. The first one uses the direct integral form of the original system in state space form while the second uses a convolution integral (variation of parameters) formulation. Additionally, a variation on the former method for direct application to second-order systems is also shown. An error analysis is presented which allows the number of polynomials employed in the approximation to be selected in advance for a desired tolerance. An extension of the method to the case where the delay and parametric periods are commensurate is also shown. Stability charts are produced for several examples of time-periodic DDEs, including the delayed Mathieu equation and a model for regenerative chatter in impedance-modulated turning. The results indicate that this method is an effective way to study the stability of time-periodic DDEs. Copyright © 2004 John Wiley & Sons, Ltd.

KEY WORDS: delay-differential equations (DDEs); stability properties; Chebyshev polynomials

1. INTRODUCTION

Dynamic systems with time delay are currently found in many problems in engineering as well as the physical and biological sciences. In many applications, the delay term in the

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dynamic system may induce instability or poor performance. In the past decades, engineers and scientists made sustained efforts to study the dynamics of systems with time delay. The response and stability under various excitations and the design of effective controllers are the key issues discussed in the vast amount of literature available on this subject. The stability of delayed systems with constant coefficients has been a focus of several monographs [1, 2] in which a variety of analytical and numerical techniques have been proposed to compute stability boundaries.

However, the situation becomes more complicated when the parameters in the system are periodically varying. The importance of this area is apparent in engineering fields such as machine tool vibrations [3–10], parametric control of robotic systems [11], neural networks [12] and optimal control [13], among others. The mathematical models of such systems are time-periodic delay-differential equations (DDEs), and obtaining their response or stability characteristics and designing control laws for various applications are necessarily more subtle than for either time-delayed systems with constant coefficients or periodically varying systems without delay, for which the well-known Floquet theory allows one to determine stability from the eigenvalues of the Floquet transition matrix (FTM). Recently, however, a few numerical methods have been proposed to study the stability of this kind of system. These include time finite element analysis [3], numerical simulation [10], statistical signal variance [14], harmonic balance and infinite determinants [4, 7, 9, 15], approximation of the delay by weighted integrals [6] and the semi-discretization method [16]. In each of these papers, the traditional stability charts related to different dynamic models are given. Prior to these studies, some authors replaced the periodic coefficients with their time-averaged values prior to employing stability theorems for delayed systems with constant coefficients [8], a technique which may give inaccurate results [17]. It has been shown [18, 19] that an infinite-dimensional version of Floquet theory can be applied to periodic DDEs, allowing one to define a monodromy operator U which can be viewed as an infinite-dimensional FTM. Thus, the stability is determined by infinitely many eigenvalues.

Here, it is proposed that by using a shifted Chebyshev polynomial approximation, the stability of a time-periodic DDE system can be determined by the eigenvalues of a truncated version of the infinite-dimensional FTM expressed in Chebyshev coefficient form. For the special case in which the delay and parametric periods are equal, this matrix defines a linear map which transforms the Chebyshev coefficients of the state in the previous delay interval to the Chebyshev coefficients of the state in the present delay interval. This finite-dimensional approximation of U is found by two different approaches. One utilizes the direct integral form of the original system and the other uses a variation of parameters formula. The former technique can also be applied directly to a set of second-order periodic DDEs as are obtained in applications such as machine tool vibrations. Both methods are efficient and have computational advantages. It is also shown how to extend the method to the case where the delay and parametric periods are commensurate. An error analysis is also included in order to allow the number of polynomials to be selected based on some desired tolerance. For illustrative examples, the stability charts of delayed Mathieu equations are obtained for different cases of proportionality between parametric period and the delay period. Also, stability charts are produced for the dynamic model of impedance modulation in turning and compared with those obtained in a prior study [9]. The results show that the proposed techniques can provide accurate stability boundaries for systems modeled by time-periodic DDEs.

2. MATHEMATICAL TOOLS

2.1. Shifted Chebyshev polynomials

The standard Chebyshev polynomials are defined as [20]:

$$T_r(x) = \cos r\theta, \quad \cos\theta = x, \quad -1 \leqslant x \leqslant 1 \tag{1}$$

Using the change of variable $x = 2t - 1$, $0 \leqslant t \leqslant 1$, the shifted Chebyshev polynomials are defined in the interval $t \in [0, 1]$ as

$$T_r^*(t) = T_r(2t-1) \tag{2}$$

Note $|T_r^*(t)| \leqslant 1$. These polynomials can be generated by noting $T_0^*(t) = 1$, $T_1^*(t) = 2t - 1$ and using the recurrence

$$T_{r+1}^*(t) = 2(2t-1)T_r^*(t) - T_{r-1}^*(t) \tag{3}$$

which yields $T_2^*(t) = 8t^2 - 8t + 1$, $T_3^*(t) = 32t^3 - 48t^2 + 18t - 1$, and so on. Suppose $f(t)$ is a continuous scalar function which can be expanded in shifted Chebyshev polynomials:

$$f(t) = \sum_{j=0}^{\infty} b_j T_j^*(t), \quad 0 \leqslant t \leqslant 1 \tag{4}$$

Using the orthogonality relation

$$\int_0^1 T_j^*(t) T_k^*(t) w(t)\,\mathrm{d}t = 0 \quad \text{if} \;\; j \neq k \tag{5}$$

where $w(t) = (t - t^2)^{-1/2}$, the coefficients b_j are given by

$$\begin{aligned} b_j &= \frac{2}{\pi} \int_0^1 f(t) T_j^*(t) w(t)\,\mathrm{d}t, \quad j = 1, 2, 3, \ldots \\ b_0 &= \frac{1}{2} \int_0^1 f(t) w(t)\,\mathrm{d}t, \quad j = 0 \end{aligned} \tag{6}$$

On the other hand, when using finite expansions $f(t) = \sum_{j=0}^{m-1} a_j T_j^*(t)$, it is also convenient to use shifted Chebyshev polynomial interpolation, which is equivalent to the use of a discrete orthogonality relation [21].

2.2. Shifted Chebyshev operational matrices

Our notation for a finite expansion in the first m shifted Chebyshev polynomials is

$$f(t) = \sum_{j=0}^{m-1} a_j T_j^*(t) = \mathbf{T}(t)^{\mathrm{T}} \mathbf{a} \tag{7}$$

where $\mathbf{T}(t)=\{T_0^*(t)\ T_1^*(t)\ \cdots\ T_{m-1}^*(t)\}^{\mathrm{T}}$ is an $m\times 1$ column vector of the polynomials and $\mathbf{a}$ is an $m\times 1$ column vector of the coefficients.

Linear operations on functions can now be written as matrix operations on vectors of polynomials and coefficients, respectively. In order to build a square monodromy matrix whose eigenvalues will determine the stability of the DDE, we employ square matrix approximations to these operations. For instance, differentiation is a matrix:

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{T}(t)=\mathbf{D}\mathbf{T}(t) \tag{8}$$

where [22]

$$\mathbf{D}=\begin{bmatrix} 0 & 0 & 0 & 0 & 0 & 0 & \ldots & 0 & 0\\ 2 & 0 & 0 & 0 & 0 & 0 & \ldots & 0 & 0\\ 0 & 8 & 0 & 0 & 0 & 0 & \ldots & 0 & 0\\ 6 & 0 & 12 & 0 & 0 & 0 & \ldots & 0 & 0\\ 0 & 16 & 0 & 16 & 0 & 0 & \ldots & 0 & 0\\ 10 & 0 & 20 & 0 & 20 & 0 & \ldots & 0 & 0\\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots & \vdots\\ 0 & 4(m-2) & 0 & 4(m-2) & 0 & 4(m-2) & \ldots & 0 & 0\\ 2(m-1) & 0 & 4(m-1) & 0 & 4(m-1) & 0 & \ldots & 4(m-1) & 0 \end{bmatrix} \tag{9}$$

is an $m\times m$ matrix. Note that the matrix is square exactly because the derivative of a polynomial is a polynomial of lower degree. The matrix has a row of zeros because the derivative of a constant is zero. We write

$$\frac{\mathrm{d}}{\mathrm{d}t}f(t)=\mathbf{T}(t)^{\mathrm{T}}\mathbf{D}^{\mathrm{T}}\mathbf{a} \tag{10}$$

Integration can be represented with a small error by a square matrix [21]:

$$\int_0^t \mathbf{T}(\tau)\,\mathrm{d}\tau=\mathbf{G}\mathbf{T}(t)+O(m^{-1}) \tag{11}$$

where

$$\mathbf{G}=\begin{bmatrix} 1/2 & 1/2 & 0 & 0 & 0 & \dots & 0 \\ -1/8 & 0 & 1/8 & 0 & 0 & \dots & 0 \\ -1/6 & -1/4 & 0 & 1/12 & 0 & \dots & 0 \\ 1/16 & 0 & -1/8 & 0 & 1/16 & \dots & \vdots \\ -1/30 & 0 & 0 & -1/12 & 0 & \ddots & 0 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \ddots & 1/4(m-1) \\ (-1)^m/2m(m-2) & 0 & 0 & \dots & 0 & -1/4(m-2) & 0 \end{bmatrix} \tag{12}$$

is an $m \times m$ matrix. The $O(m^{-1})$ error comes from the fact that integration of an $m-1$ degree polynomial gives an m degree polynomial:

$$\int_0^t T_{m-1}^*(\tau)\,\mathrm{d}\tau = aT_0^*(t) + bT_{m-2}^*(t) + \frac{1}{4m}T_m^*(t) \tag{13}$$

for appropriate a, b. The $(1/4m)T_m^*(t)$ term is dropped when using the square matrix $\mathbf{G}$. We write

$$\int_0^t f(\tau)\,\mathrm{d}\tau = \mathbf{T}(t)^{\mathrm{T}}\mathbf{G}^{\mathrm{T}}\mathbf{a} \tag{14}$$

with the understanding that a small term has been dropped.

The product of two shifted Chebyshev polynomials is

$$T_r^*(t)T_k^*(t) = \tfrac{1}{2}(T_{r+k}^*(t) + T_{|r-k|}^*(t)) \tag{15}$$

which follows from $T_r(x) = \cos(r\theta)$ and $\cos(r\theta)\cos(k\theta) = \frac{1}{2}(\cos((r+k)\theta) + \cos(|r-k|\theta))$. The operation of multiplying a scalar function $f(t)$ by a scalar function $g(t)$ is linear, and

$$f(t)g(t) = \mathbf{T}(t)^{\mathrm{T}}\mathbf{P_a}\mathbf{b} \tag{16}$$

if $f(t)=\mathbf{T}(t)^{\mathrm{T}}\mathbf{a}$ and $g(t)=\mathbf{T}(t)^{\mathrm{T}}\mathbf{b}$, where [23]

$$\mathbf{P_a}=\begin{bmatrix} a_0 & a_1/2 & a_2/2 & a_3/2 & \dots & a_{m-1}/2 \\ a_1 & a_0+a_2/2 & \frac{1}{2}(a_1+a_3) & \frac{1}{2}(a_2+a_4) & \dots & a_{m-2}/2 \\ a_2 & \frac{1}{2}(a_1+a_3) & a_0+a_4/2 & \frac{1}{2}(a_1+a_5) & \cdots & a_{m-3}/2 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m-1} & a_{m-2}/2 & a_{m-3}/2 & a_{m-4}/2 & \dots & a_0 \\ 0 & a_{m-1}/2 & a_{m-2}/2 & a_{m-3}/2 & \dots & a_1/2 \\ 0 & 0 & a_{m-1}/2 & a_{m-2}/2 & \dots & a_2/2 \\ 0 & 0 & 0 & a_{m-1}/2 & \dots & a_3/2 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & \dots & a_{m-1}/2 \end{bmatrix} \tag{17}$$

This matrix has $2m-1$ rows and m columns, which follows because the product of two $m-1$ degree polynomials is a $2m-2$ degree polynomial.

As with the integration operational matrix **G** above, computation will be done with square matrices, and we write

$$f(t)g(t)=\mathbf{T}(t)^{\mathrm{T}}\mathbf{Q_a}\mathbf{b} \tag{18}$$

when using the square $m\times m$ approximation

$$\mathbf{Q_a}=\begin{bmatrix} a_0 & a_1/2 & a_2/2 & a_3/2 & \dots & a_{m-1}/2 \\ a_1 & a_0+a_2/2 & \frac{1}{2}(a_1+a_3) & \frac{1}{2}(a_2+a_4) & \dots & a_{m-2}/2 \\ a_2 & \frac{1}{2}(a_1+a_3) & a_0+a_4/2 & \frac{1}{2}(a_1+a_5) & \cdots & a_{m-3}/2 \\ \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{m-1} & a_{m-2}/2 & a_{m-3}/2 & a_{m-4}/2 & \dots & a_0 \end{bmatrix} \tag{19}$$

which is the first m rows of $\mathbf{P_a}$. This approximation involves dropping the last $m-1$ rows of $\mathbf{P_a}$ or more accurately dropping $m(m-1)/2$ terms from product (16). The truncated terms have sum of absolute values at most $\frac{1}{2}\{|b_1||a_{m-1}|+|b_2|(|a_{m-1}|+|a_{m-2}|)+\cdots+|b_{m-1}|(|a_{m-1}|+|a_{m-2}|+\cdots+|a_1|)\}$, for all $t\in[0,1]$. This is a sum of products $|a_r b_k|$ where $r+k\geqslant m$. Each product is small if the expansions of $f(t)$ and $g(t)$ are good in the sense of having rapidly decaying coefficients. It follows that if f, g are at least twice differentiable then the order of the total error in using $\mathbf{Q_a}$ instead of the exact $\mathbf{P_a}$ goes to zero as m increases [23].

Since we want to manipulate Chebyshev expansions of matrix-valued functions of t we introduce the $nm \times n$ expansion matrix

$$\hat{\mathbf{T}}(t) = \mathbf{I}_n \otimes \mathbf{T}(t) \tag{20}$$

where $\mathbf{I}_n$ is the $n \times n$ identity and $\otimes$ is the Kronecker product. Suppose $\mathbf{A}(t)$ is an $n \times n$ matrix-valued function whose entries $a_{ij}(t)$ can be expanded in m shifted Chebyshev polynomials. Then

$$\mathbf{A}(t) = \hat{\mathbf{T}}(t)^{\mathrm{T}}\mathbf{A} \tag{21}$$

where

$$\mathbf{A} = \begin{bmatrix} \mathbf{a}_{11} & \dots & \mathbf{a}_{1n} \\ \vdots & \ddots & \vdots \\ \mathbf{a}_{n1} & \dots & \mathbf{a}_{nn} \end{bmatrix} \tag{22}$$

and $\mathbf{a}_{ij}$ is an $m \times 1$ matrix of the coefficients of the matrix entry $a_{ij}(t)$.

Therefore, the derivative operation is written $(\mathrm{d}/\mathrm{d}t)\mathbf{A}(t) = \hat{\mathbf{T}}(t)^{\mathrm{T}}\hat{\mathbf{D}}^{\mathrm{T}}\mathbf{A}$ where $\hat{\mathbf{D}} = \mathbf{I}_n \otimes \mathbf{D}$ using Equations (8), (21). The integral operation is written $\int_0^t \mathbf{A}(\tau)\,\mathrm{d}\tau = \hat{\mathbf{T}}(t)^{\mathrm{T}}\hat{\mathbf{G}}^{\mathrm{T}}\mathbf{A}$ where $\hat{\mathbf{G}} = \mathbf{I}_n \otimes \mathbf{G}$ using Equations (11), (21). Note $\hat{\mathbf{D}}, \hat{\mathbf{G}}$ are $nm \times nm$ matrices.

Suppose $\mathbf{B}(t) = \hat{\mathbf{T}}(t)^{\mathrm{T}}\mathbf{B}$ is another matrix function. Then the product is written

$$\mathbf{A}(t)\mathbf{B}(t) = \hat{\mathbf{T}}(t)^{\mathrm{T}}\hat{\mathbf{Q}}_{\mathbf{A}}\mathbf{B} \tag{23}$$

where

$$\hat{\mathbf{Q}}_{\mathbf{A}} = \begin{bmatrix} \mathbf{Q}_{\mathbf{a}_{11}} & \cdots & \mathbf{Q}_{\mathbf{a}_{1n}} \\ \vdots & \ddots & \vdots \\ \mathbf{Q}_{\mathbf{a}_{n1}} & \cdots & \mathbf{Q}_{\mathbf{a}_{nn}} \end{bmatrix} \tag{24}$$

is an $nm \times nm$ matrix and $\mathbf{Q}_{\mathbf{a}}$ has already been defined.

Finally, also note that $\mathbf{A}(t) = \hat{\mathbf{T}}(t)^{\mathrm{T}}\mathbf{A} = \mathbf{A}'\,\hat{\mathbf{T}}(t)$ where

$$\mathbf{A}' = \begin{bmatrix} \mathbf{a}_{11}^{\mathrm{T}} & \cdots & \mathbf{a}_{1n}^{\mathrm{T}} \\ \vdots & \dots & \vdots \\ \mathbf{a}_{n1}^{\mathrm{T}} & \cdots & \mathbf{a}_{nn}^{\mathrm{T}} \end{bmatrix} \tag{25}$$

is not the same as $\mathbf{A}^{\mathrm{T}}$. It follows that $\mathbf{A}(t)^{\mathrm{T}} = \hat{\mathbf{T}}(t)^{\mathrm{T}}(\mathbf{A}')^{\mathrm{T}} = \mathbf{A}^{\mathrm{T}}\hat{\mathbf{T}}(t)$.

2.3. Computation of fundamental solution matrix for ODEs

Consider the linear time-periodic system of ODEs

$$\dot{x}(t) = \mathbf{A}(t)\mathbf{x}(t), \quad \mathbf{x}(t_0) = \mathbf{x}^{\mathbf{0}} \tag{26}$$

where $\mathbf{x}(t)$ is an $n \times 1$ state vector and $\mathbf{A}(t)=\mathbf{A}(t+T)$ is an $n \times n$ periodic matrix with principal period T. If the fundamental solution matrix Φ (t)$andthematrix\Psi(t)$ of the adjoint system are defined as

$$\dot{\Phi}(t)=\mathbf{A}(t)\Phi(t), \quad \Phi(0)=\mathbf{I} \tag{27a}$$

$$\dot{\Psi}(t)=-\mathbf{A}(t)^{\mathrm{T}}\Psi(t), \quad \Psi(0)=\mathbf{I} \tag{27b}$$

then $\Phi^{-1}(t)=\Psi^{\mathrm{T}}(t)$ [24].

Sinha and Wu [25] previously approximated the fundamental solution matrix (27) by normalizing the principal period to $T=1$ and expanding in terms of shifted Chebyshev polynomials. Recall that (26) is equivalent to $\mathbf{x}(t)-\int_{t_0}^{t}\mathbf{A}(s)\mathbf{x}(s)\,\mathrm{d}s=\mathbf{x}(t_0)$. Using the properties of operational matrices introduced in the previous section, the system of differential equations (26) is reduced to a set of linear algebraic equations for the Chebyshev coefficients $\mathbf{m}$ of the state vector $\mathbf{x}(t)=\hat{\mathbf{T}}(t)^{\mathrm{T}}\mathbf{m}$ as

$$[\mathbf{I}-\hat{\mathbf{G}}^{\mathrm{T}}\hat{\mathbf{Q}}_{\mathbf{F}}]\,\mathbf{m}=\hat{\mathbf{x}}(0) \tag{28}$$

where $\hat{\mathbf{Q}}_{\mathbf{F}}$ is a product operational matrix of the Chebyshev coefficients $\mathbf{F}$ in the expansion of $\mathbf{A}(t)=\hat{\mathbf{T}}(t)^{\mathrm{T}}\mathbf{F}$ and $\hat{\mathbf{x}}(0)=\mathbf{x}(0)\otimes(1\;\,0\;\,\cdots\;\,0)^{\mathrm{T}}$ contains the initial condition. Then, from (28), $\Phi(t)$ can be obtained from the $n \times n$ matrix solution over one period which corresponds to the initial state of the identity matrix as

$$\Phi(t)=\hat{\mathbf{T}}(t)^{\mathrm{T}}\begin{bmatrix} \vdots & \vdots & & \vdots \\ \mathbf{m}_1 & \mathbf{m}_2 & \ldots & \mathbf{m}_n \\ \vdots & \vdots & & \vdots \end{bmatrix} \tag{29}$$

where $\mathbf{m}_i$ is the Chebyshev coefficient vector for the ith column of $\Phi(t)$. The matrix $\Psi(t)$ is found similarly. In all, $2n$ sets of nm linear algebraic equations must be solved to compute $\Phi(t)$ and $\Psi(t)$. The Floquet transition matrix is then obtained as $\Phi(1)$ since the normalized period is $T=1$. For details, refer to Reference [25].

3. METHOD OF ANALYSIS

3.1. The infinite-dimensional 'monodromy' or 'delay Floquet transition' matrix

Consider the system of n first-order linear time-periodic delay-differential equations

$$\begin{aligned} \dot{\mathbf{x}}(t)&=\mathbf{A}_1(t)\mathbf{x}(t)+\mathbf{A}_2(t)\mathbf{x}(t-\tau) \\ \mathbf{x}(t)&=\boldsymbol{\phi}(t), \quad t_0-\tau \leqslant t \leqslant t_0 \end{aligned} \tag{30}$$

where $\mathbf{x}(t)$ is the $n \times 1$ state vector, $\mathbf{A}_i(t)=\mathbf{A}_i(t+T)$, $i=1,2$, are $n \times n$ periodic matrices with principal period T, $\boldsymbol{\phi}(t)$ is an $n \times 1$ initial vector function in the interval $[t_0-\tau, t_0]$. Equation (30) represents the mathematical models for a variety of practical problems such as in milling, where the regenerative effect of the cutting process causes the time delay while the varying number of active teeth causes the time periodicity [4–8, 10], and in the remote control of periodic robot motion, where parametric excitation could be used to stabilize long-delay oscillations [11]. In this paper, we only consider a single fixed delay period $\tau > 0$.

Assume $\tau = T$. If $\Phi(t)$ is the fundamental solution matrix to the non-delay part of Equation (30), and $\Psi(\mathbf{t})$ is the fundamental solution matrix for the adjoint system (e.g. Equation (27)) such that $\Phi^{-1}(t) = \Psi(t)^{\mathrm{T}}$, the infinite-dimensional monodromy operator for a periodic DDE system can be defined as [19]

$$(U\mathbf{x})\,(t) = \Phi(t)\left\{\mathbf{x}(T) + \int_0^t \Psi(s)^{\mathrm{T}}\mathbf{A}_2(s)\mathbf{x}(s)\,\mathrm{d}s\right\} \tag{31}$$

which maps continuous functions from the interval $[0, T]$ back to the same interval, i.e. $U : C[0, T] \rightarrow C[0, T]$. We will study the stability of U, that is, the eigenvalues of U. Therefore, it is essential that U act from a vector space of functions back to the same vector space of functions. On the other hand, U is the solution operator of the DDE, Equation (30), in the sense that if $\mathbf{x}_i(t)$ is a solution on the interval $t \in [i-1, i]$, then $\mathbf{x}_{i+1}(t) = (U\tilde{\mathbf{x}}_i)(t-i)$ is the solution on the interval $t \in [i, i+1]$, where $\tilde{\mathbf{x}}_i(s) = \mathbf{x}_i(s-i+1)$ for $s \in [0, 1]$. Note that Equation (31) reduces to a well-known formula of the solution to an ODE system once $\mathbf{A}_2(t) \equiv \mathbf{0}$, in which case, U becomes a Floquet transition matrix $\Phi(T)$. This justifies the name for U—*the delay Floquet transition matrix* (*DFTM*). Note also that the same procedure can be applied to define the U for the neutral DDE system:

$$\dot{\mathbf{x}}(t) = \mathbf{A}_1(t)\mathbf{x}(t) + \mathbf{A}_2(t)\mathbf{x}(t-\tau) + \mathbf{B}(t)\dot{\mathbf{x}}(t-\tau) \tag{32}$$

where the delayed values of the derivatives of the states are also required and $\mathbf{B}(t)$ is a periodic continuous $n \times n$ matrix function of time. On each interval of length $T = \tau$ the delayed parts are known from the previous interval and therefore we can treat them as non-homogeneity of the corresponding ODE.

3.2. Direct method: the case $T = \tau$

We first consider the case where the parametric period is the same as the delay period ($T = \tau$). The direct method proceeds as follows. By integrating Equation (30) once as

$$\mathbf{x}(t) = \mathbf{x}(t_0) + \int_{t_0}^{t} (\mathbf{A}_1(s)\mathbf{x}(s) + \mathbf{A}_2(s)\mathbf{x}(s-\tau))\,\mathrm{d}s \tag{33}$$

where t_0 is an arbitrary initial time. This is similar to the approach used in Reference [25] for time-periodic ODEs. Normalizing the delay period to $T = \tau = 1$ and setting the initial time to $t_0 = 0$, we obtain the solution vector $\mathbf{x}_1(t)$ in the first interval [0, 1] as

$$\mathbf{x}_1(t) = \mathbf{x}_1(0) + \int_0^t (\mathbf{A}_1(s)\mathbf{x}(s) + \mathbf{A}_2(s)\boldsymbol{\phi}(s-1))\,\mathrm{d}s \tag{34}$$

Next, we expand $\mathbf{x}_1(t), \mathbf{A}_1(t)$ and $\mathbf{A}_2(t)$ and the initial function $\boldsymbol{\phi}(t-1)$ in shifted Chebyshev polynomials as

$$\begin{gathered}\mathbf{x}_1(t)=\hat{\mathbf{T}}(t)^{\mathrm{T}}\mathbf{m}_1, \quad \mathbf{A}_1(t)=\hat{\mathbf{T}}(t)^{\mathrm{T}}\mathbf{F}\\ \mathbf{A}_2(t)=\hat{\mathbf{T}}(t)^{\mathrm{T}}\mathbf{D}, \quad \boldsymbol{\phi}(t-1)=\hat{\mathbf{T}}(t)^{\mathrm{T}}\mathbf{m}_0\\ \mathbf{x}(0)=\hat{\mathbf{T}}(t)^{\mathrm{T}}\bar{\mathbf{T}}(1)\mathbf{m}_0, \quad 0\leqslant t\leqslant 1\end{gathered} \tag{35}$$

where $\mathbf{m}_1$ and $\mathbf{m}_0$ are the $nm\times 1$ Chebyshev coefficients vectors of the solution vector $\mathbf{x}_1(t)$ and the initial function $\boldsymbol{\phi}(t-1)$. $\mathbf{F}$ and $\mathbf{D}$ are the $nm\times n$ Chebyshev coefficients matrices of $\mathbf{A}_1(t)$ and $\mathbf{A}_2(t)$ in the first interval [0, 1], respectively. The $nm\times nm$ matrix $\bar{\mathbf{T}}(1)$ is defined as

$$\bar{\mathbf{T}}(1)=\hat{\mathbf{I}}\hat{\mathbf{T}}(1)^{\mathrm{T}}=\mathbf{I}_n\otimes\begin{bmatrix}1 & \dots & \dots & 1\\ 0 & \dots & \dots & 0\\ \vdots & \vdots & \vdots & \vdots\\ 0 & \dots & \dots & 0\end{bmatrix}_{m\times m} \tag{36}$$

where $\mathbf{I}=\hat{\mathbf{T}}(t)^{\mathrm{T}}\hat{\mathbf{I}}$. Using the Chebyshev expansions in Equation (35), Equation (34) takes the form

$$\hat{\mathbf{T}}(t)^{\mathrm{T}}\mathbf{m}_1=\hat{\mathbf{T}}(t)^{\mathrm{T}}\bar{\mathbf{T}}(1)\mathbf{m}_0+\int_0^t(\hat{\mathbf{T}}(s)^{\mathrm{T}}\mathbf{F}\hat{\mathbf{T}}(s)^{\mathrm{T}}\mathbf{m}_1+\hat{\mathbf{T}}(s)^{\mathrm{T}}\mathbf{D}\hat{\mathbf{T}}(s)^{\mathrm{T}}\mathbf{m}_0)\,\mathrm{d}s \tag{37}$$

Applying the operational matrices defined in Section 2, we obtain

$$\hat{\mathbf{T}}(t)^{\mathrm{T}}\mathbf{m}_1=\hat{\mathbf{T}}(t)^{\mathrm{T}}\bar{\mathbf{T}}(1)\mathbf{m}_0+\hat{\mathbf{T}}(t)^{\mathrm{T}}\hat{\mathbf{G}}^{\mathrm{T}}\hat{\mathbf{Q}}_{\mathbf{F}}\mathbf{m}_1+\hat{\mathbf{T}}(t)^{\mathrm{T}}\hat{\mathbf{G}}^{\mathrm{T}}\hat{\mathbf{Q}}_{\mathbf{D}}\mathbf{m}_0 \tag{38}$$

which can be simplified as

$$[\mathbf{I}-\hat{\mathbf{G}}^{\mathrm{T}}\hat{\mathbf{Q}}_{\mathbf{F}}]\mathbf{m}_1=[\bar{\mathbf{T}}(1)+\hat{\mathbf{G}}^{\mathrm{T}}\hat{\mathbf{Q}}_{\mathbf{D}}]\,\mathbf{m}_0 \tag{39}$$

For the ith interval $[i-1, i]$, the linear map $\mathbf{W}$ which relates the Cheybshev coefficient vector $\mathbf{m}_i$ to that in the previous interval is given by inverting the first matrix above as

$$\mathbf{m}_i=[\mathbf{I}-\hat{\mathbf{G}}^{\mathrm{T}}\hat{\mathbf{Q}}_{\mathbf{F}}]^{-1}[\bar{\mathbf{T}}(1)+\hat{\mathbf{G}}^{\mathrm{T}}\hat{\mathbf{Q}}_{\mathbf{D}}]\,\mathbf{m}_{i-1} \tag{40}$$

Hence, the stability matrix which is an approximation to the monodromy operator is given by

$$\mathbf{W}=[\mathbf{I}-\hat{\mathbf{G}}^{\mathrm{T}}\hat{\mathbf{Q}}_{\mathbf{F}}]^{-1}[\bar{\mathbf{T}}(1)+\hat{\mathbf{G}}^{\mathrm{T}}\hat{\mathbf{Q}}_{\mathbf{D}}] \tag{41}$$

for the case $T=\tau$. The stability of Equation (30) is defined by the eigenvalues of $\mathbf{W}$ since it advances the solution forward one period. The matrix $\mathbf{W}$ can be considered as a finite-dimensional approximation to the infinite-dimensional operator which maps continuous functions from the interval [0, 1] back to the same interval. For asymptotic stability, all the eigenvalues of $\mathbf{W}$ must lie within, or, for the neutral stability, on the unit circle.

It is helpful to examine $\mathbf{W}$ for the cases (a) $\mathbf{A}_2(t)=0$, (b) $\mathbf{A}_1(t)=0$ and (c) $\mathbf{A}_1$ and $\mathbf{A}_2$ are constant matrices. In case (a) there is no delay so that Equation (30) is a set of periodic ODEs. Equation (41) then reduces to

$$\begin{aligned}
\mathbf{W} &= [\mathbf{I}-\hat{\mathbf{G}}^{\mathrm{T}}\hat{\mathbf{Q}}_{\mathbf{F}}]^{-1}\bar{\mathbf{T}}(1) = [\mathbf{I}+\hat{\mathbf{G}}^{\mathrm{T}}\hat{\mathbf{Q}}_{\mathbf{F}}+(\hat{\mathbf{G}}^{\mathrm{T}}\hat{\mathbf{Q}}_{\mathbf{F}})^2+\cdots]\hat{\mathbf{I}}\hat{\mathbf{T}}(1)^{\mathrm{T}} \\
&= [\hat{\mathbf{I}}+\hat{\mathbf{G}}^{\mathrm{T}}\mathbf{F}+\hat{\mathbf{G}}^{\mathrm{T}}\hat{\mathbf{Q}}_{\mathbf{F}}\hat{\mathbf{G}}^{\mathrm{T}}\mathbf{F}+\cdots]\hat{\mathbf{T}}(1)^{\mathrm{T}} = \Phi\hat{\mathbf{T}}(1)^{\mathrm{T}} \\
&= \begin{bmatrix} \vdots & \vdots & & \vdots \\ \mathbf{m}_1 & \mathbf{m}_2 & \dots & \mathbf{m}_n \\ \vdots & \vdots & & \vdots \end{bmatrix}\hat{\mathbf{T}}(1)^{\mathrm{T}} \\
&= \begin{bmatrix} \vdots & & \vdots & \vdots & & \vdots & \vdots & \vdots & & \vdots \\ \mathbf{m}_1 & \dots & \mathbf{m}_1 & \mathbf{m}_2 & \dots & \mathbf{m}_2 & \dots & \mathbf{m}_n & \dots & \mathbf{m}_n \\ \vdots & m \text{ times} & \vdots & \vdots & m \text{ times} & \vdots & \vdots & \vdots & m \text{ times} & \vdots \end{bmatrix} \qquad (42)
\end{aligned}$$

where the relation $\hat{\mathbf{Q}}_{\mathbf{F}}\hat{\mathbf{I}}=\mathbf{F}$ (which follows from expanding $\mathbf{A}_1(t)\mathbf{I}=\mathbf{A}_1(t)$) has been used and the $nm\times n$ matrix Φ contains the Chebyshev coefficients of the fundamental solution matrix $\Phi(t)$ as in Section 2.3. Equation (42) is similar to the expansion of the $n\times n$ Floquet transition matrix (FTM)

$$\Phi(1)=\hat{\mathbf{T}}(1)^{\mathrm{T}}\Phi \qquad (43)$$

of Equation (26). In fact, it can be shown that the n eigenvalues of Equation (43) are also eigenvalues of $\mathbf{W}$ in Equation (42) but where each is repeated m times, and hence the stability of the two are the same. Another interpretation is that, because the initial condition (a continuous function) is approximated by expanding in m shifted Chebyshev polynomials, when the delay vanishes only the first polynomial (a constant) is needed to fully describe the initial condition and the remaining $m-1$ terms do not add more information.

The second case ($\mathbf{A}_1(t)=0$) results in the matrix

$$\mathbf{W}=\hat{\mathbf{G}}^{\mathrm{T}}\hat{\mathbf{Q}}_{\mathbf{D}}+\bar{\mathbf{T}}(1) \qquad (44)$$

in which no matrix inversion is needed. Finally, the third case ($\mathbf{A}_1$ and $\mathbf{A}_2$ constants) results in

$$\begin{aligned}
\mathbf{W} &= [\mathbf{I}-\hat{\mathbf{G}}^{\mathrm{T}}\hat{\mathbf{Q}}_{\mathbf{F}}]^{-1}[\hat{\mathbf{G}}^{\mathrm{T}}\hat{\mathbf{Q}}_{\mathbf{D}}+\bar{\mathbf{T}}(1)] \\
&= \Phi\hat{\mathbf{T}}(1)^{\mathrm{T}}+\sum_{n=0}^{\infty}\mathbf{F}^n\mathbf{D}\otimes(\mathbf{G}^{\mathrm{T}})^{n+1} \qquad (45)
\end{aligned}$$

in which the first term in Equation (45) is an $nm\times nm$ expansion of

$$\Phi(1)=\mathrm{e}^{\mathbf{F}}=\hat{\mathbf{T}}(1)^{\mathrm{T}}\Phi \qquad (46)$$

Hence, Equation (45) provides the stability matrix for DDEs with constant coefficients.

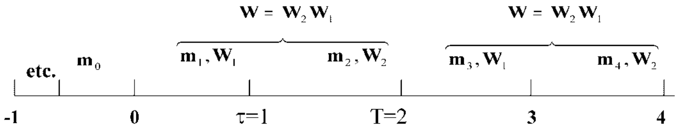

Figure 1. The approach of constructing **W** matrix when $T=2\tau$ with $\tau=1$.

It is clear that a numerical inversion in Equation (41) enables stability boundaries to be obtained numerically, while a power-series expansion of this inverse (which is equivalent to Picard iteration as employed by Sinha and Butcher [26]) would enable symbolic boundaries to be obtained in conjunction with the use of symbolic software [21]. However, only the numerical problem is addressed here.

3.3. *The case $T=q\tau$*

We next consider the case where the forcing period is an integer multiple of the delay period, i.e. $T=q\tau$, $q=2,3,\ldots$. For simplicity, we start by finding the solution when $T=2\tau$. The problem under consideration is still Equations (30). This time, we normalize the time such that $T=2\tau=2$. The main difference between $T=2\tau$ case and τ case is that in the former, the matrices $\mathbf{A}_1(t)$ and $\mathbf{A}_2(t)$ must be re-expanded in the second interval $[1,2]$. Using the relationship defined in Equation (40), in the interval $[1,2]$, we have

$$\mathbf{m}_2=[\mathbf{I}-\hat{\mathbf{G}}^{\mathrm{T}}\hat{\mathbf{Q}}_{\mathbf{F}_2}]^{-1}[\bar{\mathbf{T}}(1)+\hat{\mathbf{G}}^{\mathrm{T}}\hat{\mathbf{Q}}_{\mathbf{D}_2}]\mathbf{m}_1 \tag{47}$$

where $\mathbf{F}_2$ and $\mathbf{D}_2$ are the coefficients of Chebyshev of the expansions of $\mathbf{A}_1(t)$ and $\mathbf{A}_2(t)$ in the second interval $[1,2]$. Note that if we let

$$\mathbf{W}_2=[\mathbf{I}-\hat{\mathbf{G}}^{\mathrm{T}}\hat{\mathbf{Q}}_{\mathbf{F}_2}]^{-1}[\bar{\mathbf{T}}(1)+\hat{\mathbf{G}}^{\mathrm{T}}\hat{\mathbf{Q}}_{\mathbf{D}_2}] \tag{48}$$

and denote the **W** matrix in the first interval [0, 1] as $\mathbf{W}_1$, then substituting (39) and (48) into (47), we obtain

$$\mathbf{m}_2=\mathbf{W}_2\mathbf{m}_1=\mathbf{W}_2\mathbf{W}_1\mathbf{m}_0=\mathbf{W}\mathbf{m}_0 \tag{49}$$

The entire stability matrix is therefore

$$\mathbf{W}=\mathbf{W}_2\mathbf{W}_1=[\mathbf{I}-\hat{\mathbf{G}}^{\mathrm{T}}\hat{\mathbf{Q}}_{\mathbf{F}_2}]^{-1}[\bar{\mathbf{T}}(1)+\hat{\mathbf{G}}^{\mathrm{T}}\hat{\mathbf{Q}}_{\mathbf{D}_2}]\,[\mathbf{I}-\hat{\mathbf{G}}^{\mathrm{T}}\hat{\mathbf{Q}}_{\mathbf{F}_1}]^{-1}[\bar{\mathbf{T}}(1)+\hat{\mathbf{G}}^{\mathrm{T}}\hat{\mathbf{Q}}_{\mathbf{D}_1}] \tag{50}$$

where $\mathbf{F}_1$ and $\mathbf{D}_1$ are the coefficients of the Chebyshev expansions of $\mathbf{A}_1(t)$ and $\mathbf{A}_2(t)$ in the first interval $[0,1]$. We can think of (50) as defining 2 maps $\mathbf{W}_k$, $k=1,2$; $C[0,1]\rightarrow C[0,1]$. The stability of (30) will be determined by the eigenvalues of Equation (50). Figure 1 shows the $\mathbf{W}_k$ matrix for $T=2\tau$ case in the first four intervals.

The general formula of **W** matrix of system (30) for $T=q\tau$ case can be derived in the same way. It is given by

$$\mathbf{W}=\mathbf{W}_q\mathbf{W}_{q-1}\cdots\mathbf{W}_2\mathbf{W}_1 \tag{51}$$

where

$$\mathbf{W}_i=[\mathbf{I}-\hat{\mathbf{G}}^{\mathrm{T}}\hat{\mathbf{Q}}_{\mathbf{F}_i}]^{-1}[\bar{\mathbf{T}}(1)+\hat{\mathbf{G}}^{\mathrm{T}}\hat{\mathbf{Q}}_{\mathbf{D}_i}] \tag{52}$$

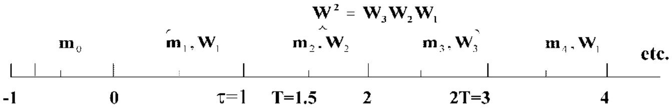

Figure 2. The approach of constructing $\mathbf{W}$ matrix when $pT = q\tau$ with $p = 2$, $q = 3$, $\tau = 1$.

3.4. *The case* $pT = q\tau$

When the excitation period T and the delay period τ of system (30) has the relationship $pT = q\tau$, where p and q are integers, we can first normalize the system with respect to τ, then use the proposed method to obtain the $\mathbf{W}$ matrix as in Equation (51) at the interval $[0, pT]$. Since this matrix is the pth power of the actual Floquet transition matrix, its eigenvalues will be raised to the pth power and, therefore, the question of whether the system is stable/unstable is directly answered (although the type of instability may not be apparent). Figure 2 shows this process on the first four intervals, where $p = 2$, $q = 3$. As was shown in Reference [27], the proposed method above can also be directly applied to a system of second-order equations. This is repeated here in Appendix A.

3.5. *The convolution integral* (*variation of parameters*)

In this section we present an alternate method which is based on the use of a variation of parameters formula (convolution integral) [21]. The solution of (30) can be expressed as [23]

$$\mathbf{x}(t) = \Phi(t)\Psi(t_0)^{\mathrm{T}}\mathbf{x}(t_0) + \Phi(t)\int_{t_0}^{t} \Psi(s)^{\mathrm{T}}\mathbf{A}_2(s)\mathbf{x}(s-\tau)\,\mathrm{d}s \tag{53}$$

Equation (53) takes the form of a term depending on the initial state and an integral involving the delay. It assumes knowledge of the fundamental solution matrix $\Phi(t)$ and its adjoint $\Psi(t)$ which can be calculated by using the method described in Section 2.2.

Again, we first consider the case where the parametric forcing and delay periods are identical ($T = \tau$) and the initial time is $t_0 = 0$. Then the solution $\mathbf{x}_1(t)$ of (30) in the interval $[0, 1]$ is found from (53) as

$$\mathbf{x}_1(t) = \Phi(t)\mathbf{x}_1(0) + \Phi(t)\int_{0}^{t} \Psi(s)^{\mathrm{T}}\mathbf{A}_2(s)\boldsymbol{\phi}(s-1)\,\mathrm{d}s, \quad 0 \leqslant t \leqslant 1 \tag{54}$$

Note that $\Psi^{\mathrm{T}}(0) = \mathbf{I}$. Let the Chebyshev expansion of the matrices $\Phi(t)$ and $\Psi(t)$ be

$$\Phi(t) = \hat{\mathbf{T}}(t)^{\mathrm{T}}\mathbf{S}, \quad \Psi(t) = \hat{\mathbf{T}}(t)^{\mathrm{T}}\mathbf{C}, \quad 0 \leqslant t \leqslant 1 \tag{55}$$

where the matrices $\mathbf{S}$ and $\mathbf{C}$ have dimension $nm \times n$. Substituting Equations (35) and (55) into (54), we have

$$\mathbf{x}_1(t) = \hat{\mathbf{T}}(t)^{\mathrm{T}}\mathbf{S}\hat{\mathbf{T}}(1)^{\mathrm{T}}\mathbf{m}_0 + \hat{\mathbf{T}}(t)^{\mathrm{T}}\mathbf{S}\int_{0}^{t} \mathbf{C}^{\mathrm{T}}\hat{\mathbf{T}}(s)\hat{\mathbf{T}}(s)^{\mathrm{T}}\mathbf{D}\hat{\mathbf{T}}(s)^{\mathrm{T}}\mathbf{m}_0\,\mathrm{d}s \tag{56}$$

which can be simplified as

$$\mathbf{x}_1(t) = \hat{\mathbf{T}}(t)^{\mathrm{T}}[\mathbf{S}\hat{\mathbf{T}}(1)^{\mathrm{T}} + \hat{\mathbf{Q}}_{\mathbf{S}}\hat{\mathbf{G}}^{\mathrm{T}}\hat{\mathbf{Q}}_{\mathbf{C}^{\mathrm{T}}}\hat{\mathbf{Q}}_{\mathbf{D}}]\mathbf{m}_0$$
$$= \hat{\mathbf{T}}(t)^{\mathrm{T}}\mathbf{m}_1, \quad 0 \leqslant t \leqslant 1 \tag{57}$$

To extend the solution past $t=1$, we use the following result from Floquet theory. If we have a time-periodic system (30), the fundamental solution matrix $\Phi(t)$ and its adjoint matrix $\Psi(t)$ defined in (27) satisfy

$$\Phi(t+kT)\Psi(s+kT)^{\mathrm{T}} = \Phi(s)\Psi(s)^{\mathrm{T}} \tag{58}$$

where k is an integer. Using this formula and noticing that $\mathbf{A}_2(t+T) = \mathbf{A}_2(t)$, $\mathbf{B}(t+T) = \mathbf{B}(t)$, and $\Psi(0)^{\mathrm{T}} = \mathbf{I}$, we get the solution $\mathbf{x}_i(t)$ in the ith interval $[i-1, i]$ as

$$\mathbf{x}_i(t) = \hat{\mathbf{T}}(\tilde{t})^{\mathrm{T}}[\mathbf{S}\hat{\mathbf{T}}(1)^{\mathrm{T}} + \hat{\mathbf{Q}}_{\mathbf{S}}\hat{\mathbf{G}}^{\mathrm{T}}\hat{\mathbf{Q}}_{\mathbf{C}^{\mathrm{T}}}\hat{\mathbf{Q}}_{\mathbf{D}}]\mathbf{m}_{i-1}$$
$$= \hat{\mathbf{T}}(\tilde{t})^{\mathrm{T}}\mathbf{m}_i, \quad i-1 \leqslant t \leqslant i \tag{59}$$

where $\tilde{t} = t - i + 1 \in [0, 1]$, $\mathbf{m}_i$ is the Chebyshev coefficient vector in the interval $[i-1, i]$ and $\mathbf{x}_i(i-1) = \mathbf{x}_{i-1}(i-1)$. Note that the linear map between $\mathbf{m}_{i-1}$ and $\mathbf{m}_i$ is

$$\mathbf{m}_i = [\mathbf{S}\hat{\mathbf{T}}(1)^{\mathrm{T}} + \hat{\mathbf{Q}}_{\mathbf{S}}\hat{\mathbf{G}}^{\mathrm{T}}\hat{\mathbf{Q}}_{\mathbf{C}^{\mathrm{T}}}\hat{\mathbf{Q}}_{\mathbf{D}}]\mathbf{m}_{i-1} \tag{60}$$

Hence, the stability matrix is given by

$$\mathbf{W} = \mathbf{S}\hat{\mathbf{T}}(1)^{\mathrm{T}} + \hat{\mathbf{Q}}_s\hat{\mathbf{G}}^{\mathrm{T}}\hat{\mathbf{Q}}_{\mathbf{C}^{\mathrm{T}}}\hat{\mathbf{Q}}_{\mathbf{D}} \tag{61}$$

The stability of the $\mathbf{W}$ matrices in Equation (41) and Equation (61) is identical. For the case $T = q\tau$, as before, we first consider the simpler case $T = 2\tau$ and normalize the time such that $\tau = 1$. Equations (54) and (57) still give the solution in the first interval [0, 1]. By extending the previous technique for $T = \tau$, at the second interval [1, 2], instead of solving Equation (27), we solve the initial value problem

$$\dot{\Phi}_2(t) = \mathbf{A}_1(t)\Phi_2(t), \quad \Phi_2(1) = \Phi_1(1), \quad 1 \leqslant t \leqslant 2 \tag{62a}$$

$$\dot{\Psi}_2(t) = -\mathbf{A}_1(t)^{\mathrm{T}}\Psi_2(t), \quad \Psi_2(1) = \Psi_1(1), \quad 1 \leqslant t \leqslant 2 \tag{62b}$$

where $\Phi_1(t)$ and $\Psi_1(t)$ satisfy Equation (27). (The subscript on Φ or Ψ refers to the first or second interval.) The solution in the second interval $t \in [1, 2]$ is now given as

$$\mathbf{x}_2(t) = \Phi_2(t)\Psi_2(1)^{\mathrm{T}}\mathbf{x}_1(1) + \Phi_2(t)\int_0^t \Psi_2(s+1)^{\mathrm{T}}\mathbf{A}_2(s+1)\mathbf{x}(s)\,\mathrm{d}s \tag{63}$$

where $\mathbf{x}_2(1) = \mathbf{x}_1(1)$. After expanding in Chebyshev polynomials as in Equation (35), Equation (63) takes the form

$$\mathbf{x}_2(t) = \hat{\mathbf{T}}(\tilde{t})^{\mathrm{T}}\mathbf{S}_2\mathbf{C}_1^{\mathrm{T}}\hat{\mathbf{T}}(1)\hat{\mathbf{T}}(1)^{\mathrm{T}}\mathbf{m}_1 + \hat{\mathbf{T}}(\tilde{t})^{\mathrm{T}}\mathbf{S}_2\int_0^{\tilde{t}} \mathbf{C}_2^{\mathrm{T}}\hat{\mathbf{T}}(s)\hat{\mathbf{T}}(s)^{\mathrm{T}}\mathbf{D}_2\hat{\mathbf{T}}(s)^{\mathrm{T}}\mathbf{m}_1\,\mathrm{d}s \tag{64}$$

where $\tilde{t} = t - 1 \in [0, 1]$, the $nm \times n$ matrices $\mathbf{S}_2$, $\mathbf{C}_2$, $\mathbf{D}_2$ are the Chebyshev coefficient matrices of $\Phi_2(t)$, $\Psi_2(t)$ and $\mathbf{A}_2(t)$ in the interval $[1, 2]$. As before, Equation (64) can be rearranged as

$$\begin{aligned}\mathbf{x}_2(t) &= \hat{\mathbf{T}}(\tilde{t})^{\mathrm{T}}[\mathbf{S}_2\hat{\mathbf{T}}(1)^{\mathrm{T}}\hat{\mathbf{Q}}_{\mathbf{C}_1^{\mathrm{T}}} + \hat{\mathbf{Q}}_{\mathbf{S}_2}\hat{\mathbf{G}}^{\mathrm{T}}\hat{\mathbf{Q}}_{\mathbf{C}_2^{\mathrm{T}}}\hat{\mathbf{Q}}_{\mathbf{D}_2}]\mathbf{m}_1 \\ &= \hat{\mathbf{T}}(\tilde{t})^{\mathrm{T}}\mathbf{m}_2 \end{aligned} \tag{65}$$

Note that if we let

$$\mathbf{W}_2 = \mathbf{S}_2\hat{\mathbf{T}}(1)^{\mathrm{T}}\hat{\mathbf{Q}}_{\mathbf{C}_1^{\mathrm{T}}} + \hat{\mathbf{Q}}_{\mathbf{S}_2}\hat{\mathbf{G}}^{\mathrm{T}}\hat{\mathbf{Q}}_{\mathbf{C}_2^{\mathrm{T}}}\hat{\mathbf{Q}}_{\mathbf{D}_2} \tag{66}$$

and denote the $\mathbf{W}$ matrix in the first interval $[0, 1]$ as $\mathbf{W}_1$, then the stability matrices $\mathbf{W}$ for $T = 2\tau$ case and $T = q\tau$ case are the same as the ones given in Equations (50) and (51), where

$$\mathbf{W}_i = \mathbf{S}_i\hat{\mathbf{T}}(1)^{\mathrm{T}}\hat{\mathbf{Q}}_{\mathbf{C}_{i-1}^{\mathrm{T}}} + \hat{\mathbf{Q}}_{\mathbf{S}_i}\hat{\mathbf{G}}^{\mathrm{T}}\hat{\mathbf{Q}}_{\mathbf{C}_i^{\mathrm{T}}}\hat{\mathbf{Q}}_{\mathbf{D}_i} \tag{67}$$

In the case $pT = q\tau$, we can apply the same procedure described in Section 3.3. And the stability of system (30) is again determined by the eigenvalues of $\mathbf{W}$ in Equation (51).

4. ERROR ANALYSIS

4.1. *Exponential rate*

The methods of Section 3 involve repeated use of operational matrices $\mathbf{G}$ and $\mathbf{Q}$ and thus are not exact—see Section 2 for quantitative estimates. Furthermore, the Chebyshev representation of the coefficient functions $\mathbf{A}_1(t)$ and $\mathbf{A}_2(t)$ generally produces a small error. Complete accounting for the error in each step of the methods would require accumulating tedious error estimates.

Consider, instead, that the *goal* is to produce an accurate Chebyshev approximation to the solution $\mathbf{x}(t)$ of the delay equation (30). Now, based on the behaviour of the solution in the previous interval, we will estimate the maximum and minimum (exponential) rates of change of the solution. A requirement of the methods of Section 3 is that sufficiently many polynomials must be used so as to represent these characteristic rates of change. We will use this requirement to determine m, the number of Chebyshev polynomials and thus the dimension of the operational matrices, *in advance* when computing particular examples.

4.2. *Scalar example*

Consider the scalar constant coefficient equation

$$\dot{x} = ax + bx(t-1) \tag{68}$$

for $t \geqslant 0$. By integrating we find

$$\begin{aligned}|x(t)| &\leqslant |x(0)| + \int_0^t |b||x(s-1)|\,\mathrm{d}s + \int_0^t |a||x(s)|\,\mathrm{d}s \\ &= \left(|\boldsymbol{\phi}(1)| + |b|\int_0^t |\boldsymbol{\phi}(s)|\,\mathrm{d}s\right) + |a|\int_0^t |x(s)|\,\mathrm{d}s\end{aligned}$$

$$\leqslant |\boldsymbol{\phi}|(1+|b|t)+|a|\int_0^t |x(s)|\,\mathrm{d}s \tag{69}$$

where $x(s)=\boldsymbol{\phi}(s)$ for $-1\leqslant s\leqslant 0$ and $|\boldsymbol{\phi}|=\max_{-1\leqslant s\leqslant 0}|\boldsymbol{\phi}(s)|$. By Gronwall's inequality [19, Lemma 3.1],

$$|x(t)|\leqslant|\boldsymbol{\phi}|(1+|b|t)\mathrm{e}^{|a|t} \tag{70}$$

That is, from one delay period to the next, the value of $|a|$ determines the maximum exponential rate of change of the solution, though there is an additional sub-exponential factor controlled by $|b|$.

4.3. General case

Much the same applies to the non-scalar systems of this paper if one replaces absolute values by a matrix norm. Suppose $\|\cdot\|$ denotes a vector and matrix norm as appropriate. Integrating the normalized (i.e. $\tau=1$) periodic equation (30) gives

$$\|\mathbf{x}(t)\|\leqslant\|\boldsymbol{\phi}\|(1+\|\mathbf{A}_2\|t)+\|\mathbf{A}_1\|\int_0^t\|\mathbf{x}(s)\|\,\mathrm{d}s \tag{71}$$

where $\mathbf{x}(s)=\boldsymbol{\phi}(s)$ for $-1\leqslant s\leqslant 0$ and $\|\boldsymbol{\phi}\|=\max_{-1\leqslant s\leqslant 0}\|\boldsymbol{\phi}(s)\|$. In the second line

$$\|\mathbf{A}_2\|=\max_{0\leqslant s\leqslant 1}\|\mathbf{A}_2(s)\| \tag{72}$$

is the maximum of the matrix norms on the time interval [0, 1]. By Gronwall's inequality,

$$\|x(t)\|\leqslant\|\boldsymbol{\phi}\|(1+\|\mathbf{A}_2\|t)\mathrm{e}^{\|\mathbf{A}_1\|t} \tag{73}$$

4.4. Estimating *m*

The above examples show that one can determine the exponential growth rate of a system from easily calculated norms of the coefficient matrices. From this information one can choose a sufficient value for the important numerical parameter

$$m=(\text{number of shifted Chebyshev polynomials})$$

to give a desired accuracy in a particular example.

Suppose that $p(t)$ is the best shifted Chebyshev polynomial expansion of $f(t)$ using only the m polynomials $T_0^*(t),\ldots,T_{m-1}^*(t)$. We depend on the following relationship which relates the accuracy of the Chebyshev approximation p to the size of the derivatives of f: In Corollary 8.11 [28] translated to the shifted case, we find that if f has m continuous derivatives on $t\in[0,1]$, then

$$\max_{0\leqslant t\leqslant 1}|f(t)-p(t)|\leqslant\frac{1}{2^{2m-1}m!}\max_{0\leqslant t\leqslant 1}|f^{(m)}(t)| \tag{74}$$

4.5. *Scalar example, cont.*

Suppose we want to apply the methods of this paper to the scalar constant coefficient equation with $a = 3,\ b = -2$:

$$\dot{x} = 3x - 2x(t-1) \tag{75}$$

and with $x(s) = \boldsymbol{\phi}(s),\ -1 \leqslant s \leqslant 0$. Suppose we want to have error at most order $\mathrm{TOL} = 10^{-6}$. From (70) above the solution is a function with exponential growth at most $\mathrm{e}^{|a|t} = \mathrm{e}^{3t}$.

What m is needed to get this accuracy? From Equation (74) we need

$$\frac{1}{2^{2m-1} m!} 3^m \mathrm{e}^3 < 10^{-6} \tag{76}$$

that is, $m \geqslant 10$. On the other hand, if we use the refined estimate in (70), namely $|x(t)| \leqslant |\boldsymbol{\phi}|(1 + |b|t)\mathrm{e}^{|a|t}$, and if we also suppose that $|\boldsymbol{\phi}| < 2.2$ (for example) then we need m so that

$$(2.2)\frac{1}{2^{2m-1} m!} 3^{m-1}(2m+9)\mathrm{e}^3 < 10^{-6} \tag{77}$$

and any $m \geqslant 11$ works. Note that the additional sub-exponential terms in estimate (70) cause only a small increase in the expected m beyond that needed to handle the basic exponential rate.

4.6. *General case, cont:*

The heuristic

$$x(t) \sim \mathrm{e}^{\|\mathbf{A}_1\| t} \tag{78}$$

along with Equation (74) will give a conservative m in the general case, that is, an m which is higher than necessary. Estimate (73) produces an even more conservative m. It is the use of a matrix norm $\|.\|$ which tends to produce unnecessarily conservative values.

We improve this situation by recalling the constant coefficient case wherein the actual maximum exponential rate is the largest eigenvalue of $\mathbf{A}_1$. Recall that

$$\|\mathbf{A}\| \geqslant \rho(\mathbf{A}) \tag{79}$$

for any natural matrix norm of a square matrix $\mathbf{A}$. (By definition, $\rho(\mathbf{A}) = \max\{|\lambda| :\ \lambda$ is an eigenvalue of $\mathbf{A}\}$.) Thus, a more realistic estimate of m comes from substituting

$$\bar{\rho} = \max_{0 \leqslant s \leqslant 1} \rho(\mathbf{A}_1(s)) \tag{80}$$

for the norm $\|\mathbf{A}_1\|$ in the above estimates. The validity of this substitution is not provable in the general case, of course, though it is well known to give the right order of magnitude for the exponential rate in many examples.

4.7. *Example: The delayed Mathieu equation*

Consider the delayed Mathieu equation,

$$\ddot{x}(t) + (\delta + 2\varepsilon \cos 2t)x(t) = bx(t-\pi) \tag{81}$$

Rewritten in state space form and then normalized to τ (= delay) = 1 we get

$$\dot{\mathbf{x}} = \mathbf{A}_1(t)\mathbf{x} + \mathbf{A}_2\mathbf{x}(t-1) \tag{82}$$

with

$$\mathbf{A}_1(t) = \begin{pmatrix} 0 & \pi \\ -\pi(\delta + 2\varepsilon\cos 2\pi t) & 0 \end{pmatrix}$$

We suppose that the solution $\mathbf{x}(t)$ has approximate exponential form $e^{\bar{\rho}t}$ where $\bar{\rho}$ is the maximum of the time-dependent eigenvalues of $\mathbf{A}_1$, as above. In fact

$$\lambda_{\pm} = \pm\,\pi\sqrt{\delta + 2\varepsilon\cos 2\pi t} \tag{83}$$

so $\bar{\rho} = \pi\sqrt{|\delta| + 2|\varepsilon|}$. We are disregarding the parameter b in $\mathbf{A}_2(t)$ because it only contributes a subexponential term in estimate (73).

Suppose, as in Figure 5, that we wish to determine stability for parameter ranges $-5 \leqslant \delta \leqslant 20$, $-4 \leqslant b \leqslant 4$, $0 \leqslant \varepsilon \leqslant 20$. Then $\bar{\rho} = \pi\sqrt{60} \approx 24.3$ is the worst case and we seek m so that

$$\frac{\bar{\rho}^m e^{\bar{\rho}}}{2^{2m-1}m!} \leqslant \mathrm{TOL} \tag{84}$$

Supposing $\mathrm{TOL} = 10^{-6}$ we predict this level of accuracy with $m \geqslant 41$.

5. EXAMPLES

5.1. An example where $\Phi(t)$ is known exactly

Consider the following π-periodic system:

$$\begin{gathered} \frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix}(t) = \begin{pmatrix} -\sin^2 t & 1 - 0.5\sin 2t \\ -1 - 0.5\sin 2t & -\cos^2 t \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}(t) + \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}(t-\pi) \\ \begin{pmatrix} x \\ y \end{pmatrix}(t) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \quad -\pi < t < 0 \end{gathered} \tag{85}$$

Even though $\mathbf{A}_1(t)$ is not commutative, the exact fundamental solution matrix $\Phi(t)$ for the system of 2 ODEs (without the delays) is [29]

$$\Phi(t) = \begin{pmatrix} \cos t & e^{-t}\sin t \\ -\sin t & e^{-t}\cos t \end{pmatrix} \tag{86}$$

Therefore, we can find the exact solution by the method of steps, which we compare with the solution we get by iterative application of the matrix $\mathbf{W}$.

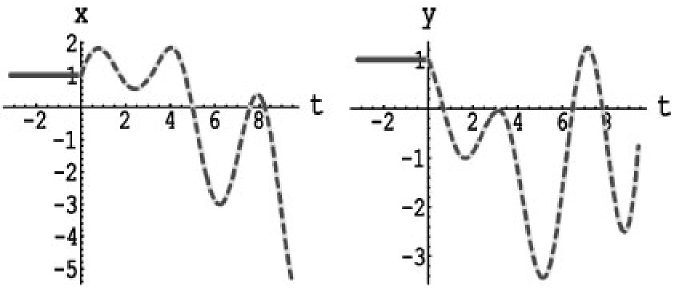

Figure 3. Initial condition (solid black), exact (solid grey) and approximate (dashed—overlapping with 'exact') solutions to system (85) for $m=12$.

Before calculating $\mathbf{W}$ we need to decide on the number of Chebyshev polynomials m needed to achieve the desired accuracy. Recalling that $\mathbf{x}(t)\sim e^{\|\mathbf{A}_1\|t}$, an upper bound for mth derivative of any entry of $\mathbf{A}_1(t)$ is $\pi\cdot(2\pi)^{m-1}=2^{m-1}\pi^m$. So according to Equation (74), we need $\pi^m/2^m m!\leqslant 10^{-6}$, i.e. $m\geqslant 12$. Therefore, we set $m=12$. Now we can calculate $\mathbf{W}$ and obtain the approximate solution to (85) by method of steps [21] (see Figure 3). The maximum error on the interval $[0,3\pi]$ is

$$\approx\begin{pmatrix}1.26\times 10^{-5}\\ 9.33\times 10^{-6}\end{pmatrix}$$

Note that the heuristic in Section 4.6 is not exact. For this reason, and to compensate for the accumulation of rounding and truncation error, it is appropriate in an example like this to use $m=13$ or 14. Figure 4 demonstrates the convergence of the approximate solution as the number of Chebyshev polynomials m increases.

5.2. Delayed Mathieu equation

Consider the delayed Mathieu equation in Equation (81) which is similar to the one studied in Reference [15]. Note that the parametric period is equal to the delay period, i.e. $T=\tau=\pi$. We consider four special cases: (1) $\varepsilon=0$. This special case is well known [2] and the stability diagram was first obtained in Reference [30]. By applying the procedure described in Section 3, we obtain the stability chart on the region $[-4,20]\times[-4,5]$ of (δ,b) plane in Figure 5(a). (2) $b=0$, i.e. no delay term. This special case is just Mathieu's equation for which the stability diagram was first obtained in Reference [31]. The stability chart obtained via the proposed method on the region $[-4,20]\times[0,20]$ is in Figure 5(b). (3) We fix the parameter $\varepsilon=1$ and plot the stability diagram of Equation (81) in (δ,b) plane on the region $[-4,20]\times[-4,4]$ in Figure 5(c). The stability chart for Equation (81) including both delay and parametric excitation was first obtained in Reference [15]. (4) Finally, we select parameter $\varepsilon=2$ and plot the stability chart of Equation (81) in (δ,b) plane on the region $[-4,20]\times[-4,4]$ in Figure 5(d). The CPU times obtained on a Celeron 400 MHz computer with 64 MB RAM are summarized in Table I. The second-order formulation in Appendix A gives the same result in all four cases above.

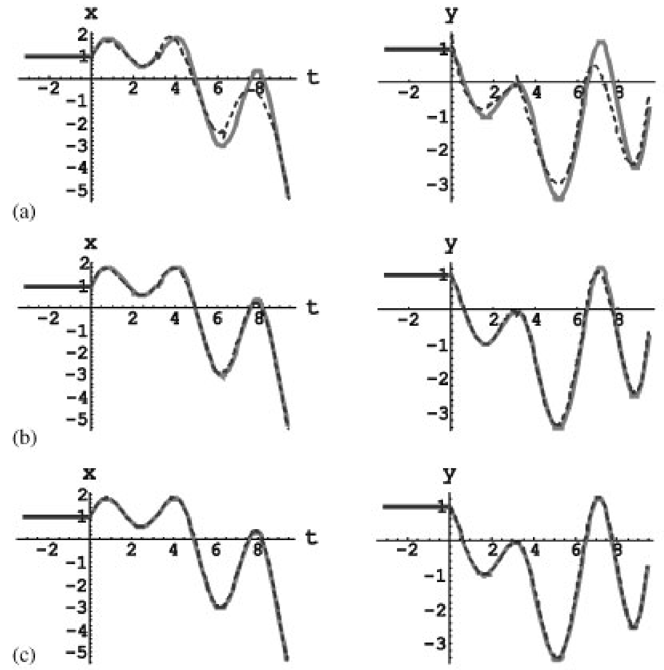

Figure 4. Initial condition (solid black), exact (solid grey) and approximate (dashed) solutions to system (85) for $m =$ (a) 4; (b) 5; and (c) 6.

5.3. Delayed Mathieu equation (2)

Consider the system

$$\ddot{x}(t) + (a + c\cos t)x(t) = (b\cos t)x(t - 2\pi) \tag{87}$$

where the parameters a, b and c are three unknown parameters. Qualitatively, Equation (87) is more complicated than Equation (81) because of the addition of the periodic coefficient in front of the time delay. Again we have $T = \tau$.

Figure 6 shows the stability chart produced by the method described in Section 3.2 on the region $[-4, 20] \times [-15, 15]$ of (a, b) parameter plane with $c = 2$. The number of shifted Chebyshev polynomials used here $m = 50$. The stable region is black while the unstable region is white. The stability chart generated by using the alternate method in Section 3.5 is exactly the same. In order to show how the various formulations compare, Table II gives the eigenvalues and spectral radius of the **W** matrix for the parameter set $a = 0.5$, $b = 0.3$, and $c = 0.2$ using $m = 9$ Chebyshev polynomials.

5.4. Delayed Mathieu equation (3)

Now consider the DDE system

$$\ddot{x}(t) + (a + c\cos t)x(t) = (b\cos t)x(t - \pi) \tag{88}$$

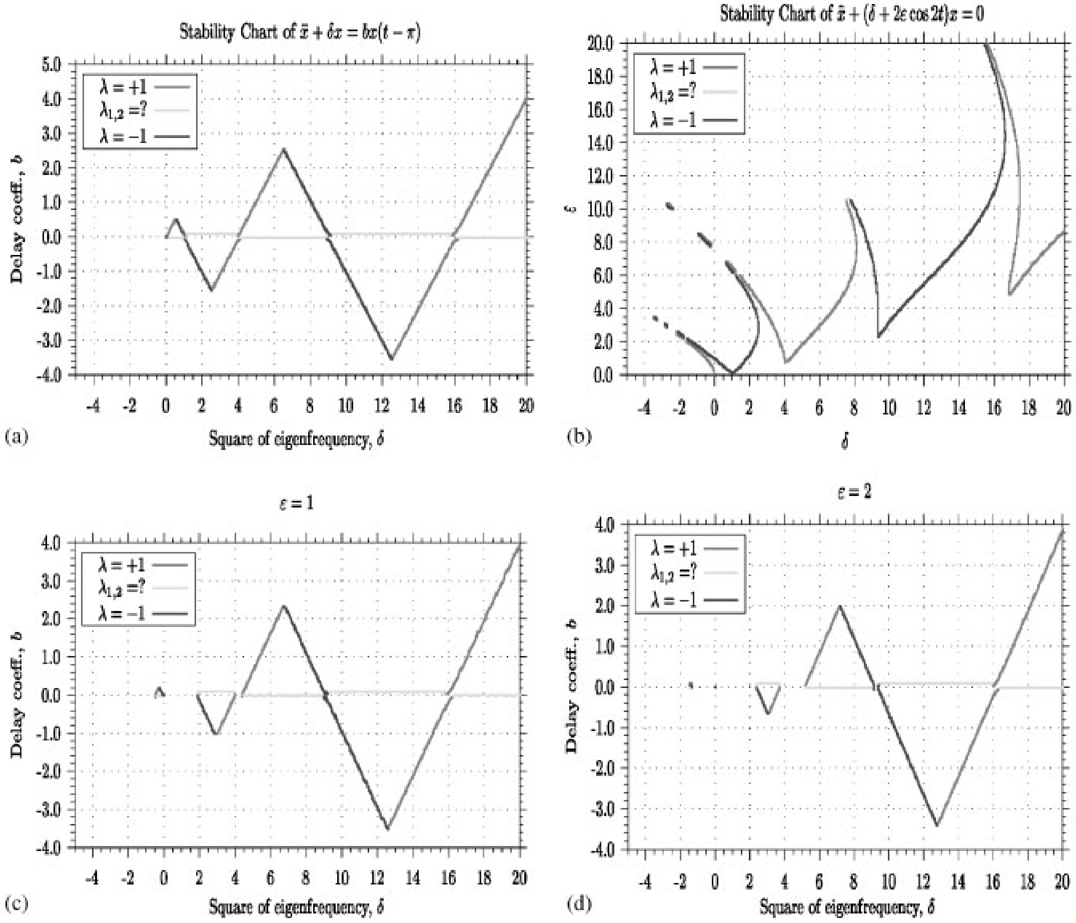

Figure 5. The stability diagrams of system (81).

Table I. Computational statistics for Example 5.2.

	Number of points #	CPU time	CPU/#
1. $\varepsilon = 0$	20271	825 s	0.041
2. $\varepsilon = 1$	12121	461 s	0.038
3. $\varepsilon = 2$	18836	743 s	0.039
4. $b = 0$	23486	966 s	0.041

where the main period T in this case equals 2π and the delay period τ equals π, i.e. $T = 2\tau$. Using $m = 50$ polynomials, we plot the stability chart of $\mathbf{W}$ in Equation (88) on the region $[-4, 20] \times [-15, 15]$ of (a, b) parameter plane with $c = 2$. Again, the result produced by using the alternative method is the same. See Figure 7 for results.

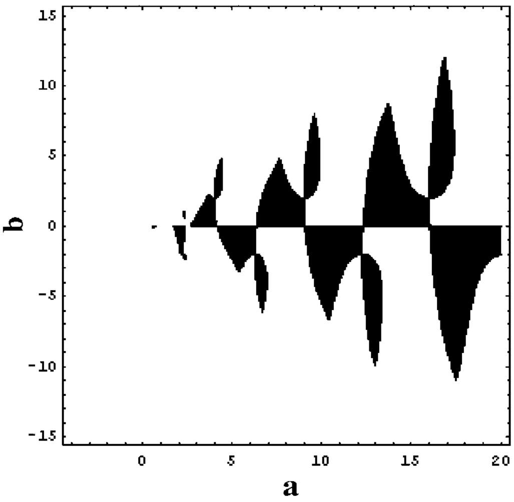

Figure 6. The stability chart of system (87) with $c = 2$, $m = 50$—Stable region (black), unstable region (white).

5.5. *Impedance-modulated turning*

Finally, consider the following DDE system:

$$\ddot{x}(\tau) + 2\xi\omega\dot{x}(\tau) + \omega^2\left(1 + b + \varepsilon\cos\left(\tfrac{1}{2}\tau\right)\right)x(\tau) = b\omega^2 x(\tau - 2\pi) \tag{89}$$

Equation (89) is the dimensionless form of dynamic model of machine tool vibration studied by Segalman and Butcher [9] which models turning with impedance modulation. The independent variable τ is the revolution angle co-ordinate, and the non-dimensional rotation speed is $\Omega = 1/\omega$. It should be noted that the modulation period is twice the period of the delay term, i.e. $T = 2\tau$. Using the method described in Section 3.2, we generate the stability charts of Equation (89) in Figure 8 on the region $[0.1, 0.7] \times [0, 0.05]$ and $[0.1, 0.2] \times [0, 0.03]$ of (Ω, b) parameter plane with $\varepsilon = 0.1$, $\xi = 0.001$, $m = 60$. The plots are the same as the one obtained by Segalman and Butcher [9]. The region of stable motion lies below the curve while the region of unstable motion lies above it. Again, the convolution integral technique gives identical results.

6. CONCLUSION

A new technique was proposed to study the stability of time-periodic delay differential equations. Based on approximation of the state vector in the current and previous delay intervals in terms of shifted Chebyshev polynomials, the original DDE system reduces to a set of linear equations for the Chebyshev coefficients. The relationship between Chebyshev coefficients of the previous and current intervals defines a linear map which approximates the infinite-dimensional Floquet transition matrix U. The stability criteria is then determined by the eigenvalues of this matrix.

Table II. Comparison of eigenvalues of W matrix of different methods for Example 5.3 with $a=0.5$, $c=0.2$, $b=0.3$ by using $m=9$ Chebyshev polynomials.

	Direct method Equation (41)	Variation of parameter method Equation (61)	Second-order form Equation (A8)	Second-order form Equation (A9)
Spectral radius of W matrix	0.709476	0.708464	0.694111	0.684111
Eigenvalues of W matrix	$0.0255273+0.709017i$	$0.0250459+0.708021i$	$0.032571+0.693346i$	$0.032571+0.693346i$
	$0.0255273-0.709017i$	$0.0250459-0.708021i$	$0.032571-0.693346i$	$0.032571-0.693346i$
	-0.686151	-0.68777	-0.683857	-0.683857
	-0.168329	-0.164713	-0.173093	-0.173093
	0.0936568	0.108978	0.0752239	0.0752239
	-0.0326532	$-0.0338029+0.0343959i$	-0.0521132	-0.0521132
	$-0.0174436+0.025378i$	$-0.0338029-0.0343959i$	$0.002125+0.037709i$	$0.00212475+0.0377093i$
	$-0.0174436-0.025378i$	-0.0436358	$0.002125-0.037709i$	$0.00212475-0.0377093i$
	$0.0163755+0.00352852i$	0.0343914	0.011561	0.011561
	$0.0163755-0.00352852i$	0.0180991	0	
	-2.90477×10^{-16}	-1.66386×10^{-16}	0	
	6.78419×10^{-17}	5.75999×10^{-17}	0	
	-4.32603×10^{-17}	4.62716×10^{-17}	0	
	3.38434×10^{-17}	3.27697×10^{-17}	0	
	-1.85474×10^{-17}	2.14507×10^{-17}	0	
	1.68454×10^{-17}	1.08274×10^{-17}	0	
	1.27524×10^{-17}	8.79195×10^{-18}	0	
	-5.30892×10^{-18}	-1.43906×10^{-19}	0	

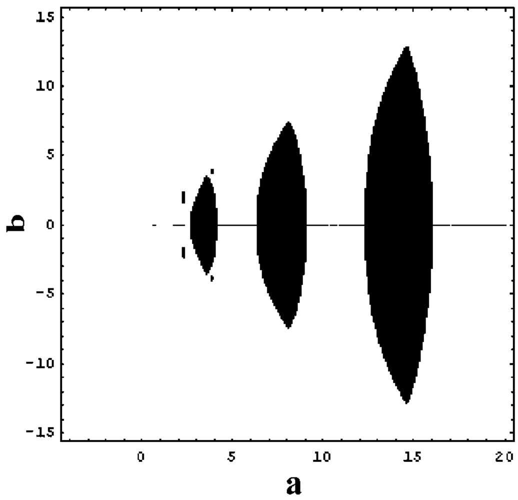

Figure 7. The stability chart of system (88) with $c = 2$ and $m = 50$—Stable region (black), unstable region (white).

This technique is applicable to practical problems such as regenerative chatter vibrations in machining or remote parametric control of robot motion.

Specifically, compared to the traditional numerical simulation method for stability analysis of delayed systems, the current technique is much simpler as all of the computation is implemented by matrix manipulation. And since an approximation to the Floquet transition matrix is found, it is also useful for designing controllers to stabilize the original system. Also, much of the information needed to set up the problem can be stored in the computer in advance. The 'product' and 'integration' matrices associated with the shifted Chebyshev polynomials can be readily constructed from the general expressions. In general, the periodic terms in $\mathbf{A}(t)$ have the forms $\sin(n\pi t/T)$ and/or $\cos(n\pi t/T)$. The expansions of these quantities can be made a part of the subroutine, and one does not have to compute the expansion coefficients each time. The entire computation process can be automated rather easily.

Two different formulas for the computation of the approximate U, whose size is determined by the number of polynomials employed, were given. The first one used the direct integral form of the original system in state space form while the other one used a convolution integral (variation of parameters) formulation. Additionally, a variation on the former method for direct application to second-order systems was also shown. An error analysis was presented which allows the number of polynomials employed in the approximation to be selected in advance for a desired tolerance. An extension of the method to the case where the delay and parametric periods are commensurate was also shown. Examples such as a delayed Mathieu equation and a dynamic model for impedance-modulated turning were studied. The results were found to be very accurate compared with previously known stability boundaries for these systems. It is concluded that the proposed method offers an efficient alternative to other numerical techniques for obtaining the stability characteristics of periodic delayed systems which currently find application in many engineering fields.

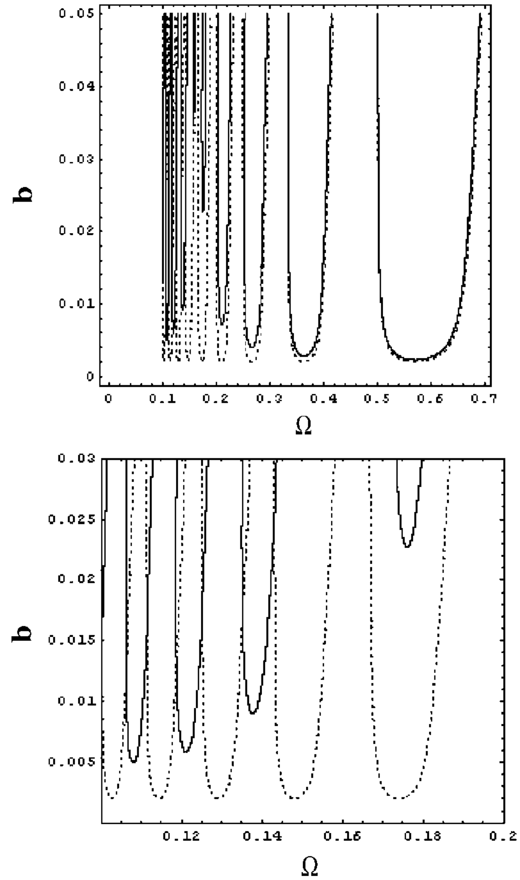

Figure 8. The stability charts of system (89)—impedance modulation for $\varepsilon = 0$ (dashed) and $\varepsilon = 0.2$ (solid) with $\xi = 0.001$ and $m = 60$ with stable region below the curve, unstable region above the curve.

APPENDIX A

Consider a set of periodic second-order DDEs of the form

$$\mathbf{M}(t)\ddot{\mathbf{x}}(t) + \mathbf{C}(t)\dot{\mathbf{x}}(t) + \mathbf{K}(t)\mathbf{x}(t) = \mathbf{B}(t)\mathbf{x}(t - T) \tag{A1}$$

where $\mathbf{M}(t)$, $\mathbf{C}(t)$, $\mathbf{K}(t)$ and $\mathbf{B}(t)$ are periodic matrices with principal period T. Many engineering applications, such as in machine tool vibrations, can be expressed in this form. In order to avoid a transformation to state space form, we can work directly with this second-order form. Normalizing the delay period to $T = 1$ and integrating (A1) from 0 to t, we obtain

$$\mathbf{M}(t)\dot{\mathbf{x}}(t) - \mathbf{M}(0)\dot{\mathbf{x}}(0) - \int_0^t \dot{\mathbf{M}}(\tau)\dot{\mathbf{x}}(\tau)\,\mathrm{d}\tau + \int_0^t \mathbf{C}(\tau)\dot{\mathbf{x}}(\tau)\,\mathrm{d}\tau$$

$$+\int_0^t \mathbf{K}(\tau)\mathbf{x}(\tau)\,\mathrm{d}\tau = \int_0^t \mathbf{B}(\tau)\mathbf{x}(\tau-1)\,\mathrm{d}\tau \tag{A2}$$

Equation (A2) can be integrated again as

$$\begin{aligned}
&\mathbf{M}(t)\mathbf{x}(t) - \mathbf{M}(0)\mathbf{x}(0) - \int_0^t \dot{\mathbf{M}}(\tau)\mathbf{x}(\tau)\,\mathrm{d}\tau + \int_0^t \mathbf{M}(0)\dot{\mathbf{x}}(0)\,\mathrm{d}\tau - \int_0^t\int_0^s \dot{\mathbf{M}}(\tau)\dot{\mathbf{x}}(\tau)\,\mathrm{d}\tau\,\mathrm{d}s\\
&\quad + \int_0^t\int_0^s \mathbf{C}(\tau)\dot{\mathbf{x}}(\tau)\,\mathrm{d}\tau\,\mathrm{d}s + \int_0^t\int_0^s \mathbf{K}(\tau)\mathbf{x}(\tau)\,\mathrm{d}\tau\,\mathrm{d}s = \int_0^t\int_0^s \mathbf{B}(\tau)\mathbf{x}(\tau-1)\,\mathrm{d}\tau\,\mathrm{d}s
\end{aligned} \tag{A3}$$

where τ and s are dummy variables.

Next, we expand $\mathbf{x}(t), \dot{\mathbf{x}}(t), \mathbf{x}(t-1), \mathbf{M}(t), \mathbf{M}(0), \mathbf{C}(t), \mathbf{C}(0), \mathbf{K}(t), \mathbf{B}(t)$ in Chebyshev polynomials as

$$\begin{aligned}
&\mathbf{x}(t) = \hat{\mathbf{T}}(t)^{\mathrm{T}}\mathbf{m}_i, \quad \dot{\mathbf{x}}(t) = \hat{\mathbf{T}}(t)^{\mathrm{T}}\dot{\mathbf{m}}_i, \quad \mathbf{x}(t-1) = \hat{\mathbf{T}}(t)^{\mathrm{T}}\mathbf{m}_{i-1}\\
&\mathbf{M}(t) = \hat{\mathbf{T}}(t)^{\mathrm{T}}\mathbf{M}, \quad \mathbf{C}(t) = \hat{\mathbf{T}}(t)^{\mathrm{T}}\mathbf{C}, \quad \mathbf{M}(0) = \hat{\mathbf{T}}(t)^{\mathrm{T}}\mathbf{M}_0\\
&\mathbf{C}(0) = \hat{\mathbf{T}}(t)^{\mathrm{T}}\mathbf{C}_0, \quad \mathbf{K}(t) = \hat{\mathbf{T}}(t)^{\mathrm{T}}\mathbf{K}, \quad \mathbf{B}(t) = \hat{\mathbf{T}}^{\mathrm{T}}(t)\mathbf{B}\\
&\dot{\mathbf{M}}(t) = \hat{\mathbf{T}}(t)^{\mathrm{T}}\hat{\mathbf{D}}^{\mathrm{T}}\mathbf{M}, \quad \dot{\mathbf{C}}(t) = \hat{\mathbf{T}}(t)^{\mathrm{T}}\hat{\mathbf{D}}^{\mathrm{T}}\mathbf{C}\\
&\mathbf{x}(0) = \hat{\mathbf{T}}(t)^{\mathrm{T}}\bar{\mathbf{T}}(1)\mathbf{m}_{i-1}, \quad \dot{\mathbf{x}}(0) = \hat{\mathbf{T}}(t)^{\mathrm{T}}\bar{\mathbf{T}}(1)\hat{\mathbf{D}}^{\mathrm{T}}\mathbf{m}_{i-1}
\end{aligned} \tag{A4}$$

where $\mathbf{m}_i$, $\mathbf{m}_{i-1}$, $\dot{\mathbf{m}}_i$, $\dot{\mathbf{m}}_{i-1}$ are the Chebyshev coefficients vectors of the displacement and velocity vectors, and $\mathbf{M}$, $\mathbf{C}$, $\mathbf{M}_0$, $\mathbf{C}_0$, $\mathbf{B}$, $\mathbf{K}$ are the Chebyshev coefficients matrices with consistent dimensions. $\bar{\mathbf{T}}(1)$ is defined in (36) and $\hat{\mathbf{D}} = \mathbf{I}_n \otimes \mathbf{D}$, where $\mathbf{D}$ is the *differentiation* operational matrix defined in Equation (9). Substituting (A4) into (A2) and (A3) and after some manipulation, we obtain

$$\hat{\mathbf{Q}}_{\mathbf{M}}\dot{\mathbf{m}}_i + [-\hat{\mathbf{G}}^{\mathrm{T}}\hat{\mathbf{Q}}_{(\hat{\mathbf{D}}^{\mathrm{T}}\mathbf{M})}\hat{\mathbf{D}}^{\mathrm{T}} + \hat{\mathbf{G}}^{\mathrm{T}}\hat{\mathbf{Q}}_{\mathbf{C}}\hat{\mathbf{D}}^{\mathrm{T}} + \hat{\mathbf{G}}^{\mathrm{T}}\hat{\mathbf{Q}}_{\mathbf{K}}]\mathbf{m}_i = [\hat{\mathbf{Q}}_{\mathbf{M}_0}\bar{\mathbf{T}}(1)\hat{\mathbf{D}}^{\mathrm{T}} + \hat{\mathbf{G}}^{\mathrm{T}}\hat{\mathbf{Q}}_{\mathbf{B}}]\mathbf{m}_{i-1} \tag{A5}$$

and

$$\begin{aligned}
&[\hat{\mathbf{Q}}_{\mathbf{M}} - \hat{\mathbf{G}}^{\mathrm{T}}\hat{\mathbf{Q}}_{(\hat{\mathbf{D}}^{\mathrm{T}}\mathbf{M})} - (\hat{\mathbf{G}}^{\mathrm{T}})^2\hat{\mathbf{Q}}_{(\hat{\mathbf{D}}^{\mathrm{T}}\mathbf{M})}\hat{\mathbf{D}}^{\mathrm{T}} + (\hat{\mathbf{G}}^{\mathrm{T}})^2\hat{\mathbf{Q}}_{\mathbf{C}}\hat{\mathbf{D}}^{\mathrm{T}} + (\hat{\mathbf{G}}^{\mathrm{T}})^2\hat{\mathbf{Q}}_{\mathbf{K}}]\mathbf{m}_i\\
&\quad = [\hat{\mathbf{Q}}_{\mathbf{M}_0}\bar{\mathbf{T}}(1) + \hat{\mathbf{G}}^{\mathrm{T}}\hat{\mathbf{Q}}_{\mathbf{M}_0}\bar{\mathbf{T}}(1)\hat{\mathbf{D}}^{\mathrm{T}} + (\hat{\mathbf{G}}^{\mathrm{T}})^2\hat{\mathbf{Q}}_{\mathbf{B}}]\mathbf{m}_{i-1}
\end{aligned} \tag{A6}$$

Equations (A5) and (A6) can be rearranged into matrix form as

$$\begin{aligned}
&\begin{bmatrix} -\hat{\mathbf{G}}^{\mathrm{T}}\hat{\mathbf{Q}}_{(\hat{\mathbf{D}}^{\mathrm{T}}\mathbf{M})}\hat{\mathbf{D}}^{\mathrm{T}} + \hat{\mathbf{G}}^{\mathrm{T}}\hat{\mathbf{Q}}_{\mathbf{C}}\hat{\mathbf{D}}^{\mathrm{T}} + \hat{\mathbf{G}}^{\mathrm{T}}\hat{\mathbf{Q}}_{\mathbf{K}} & \hat{\mathbf{Q}}_{\mathbf{M}}\\ \hat{\mathbf{Q}}_{\mathbf{M}} - \hat{\mathbf{G}}^{\mathrm{T}}\hat{\mathbf{Q}}_{(\hat{\mathbf{D}}^{\mathrm{T}}\mathbf{M})} - (\hat{\mathbf{G}}^{\mathrm{T}})^2\hat{\mathbf{Q}}_{(\hat{\mathbf{D}}^{\mathrm{T}}\mathbf{M})}\hat{\mathbf{D}}^{\mathrm{T}} + (\hat{\mathbf{G}}^{\mathrm{T}})^2\hat{\mathbf{Q}}_{\mathbf{C}}\hat{\mathbf{D}}^{\mathrm{T}} + (\hat{\mathbf{G}}^{\mathrm{T}})^2\hat{\mathbf{Q}}_{\mathbf{K}} & \mathbf{0} \end{bmatrix}\begin{pmatrix}\mathbf{m}_i\\ \dot{\mathbf{m}}_i\end{pmatrix}\\
&\quad = \begin{bmatrix} \hat{\mathbf{Q}}_{\mathbf{M}_0}\bar{\mathbf{T}}(1)\hat{\mathbf{D}}^{\mathrm{T}} + \hat{\mathbf{G}}^{\mathrm{T}}\hat{\mathbf{Q}}_{\mathbf{B}} & \mathbf{0}\\ \hat{\mathbf{Q}}_{\mathbf{M}_0}\bar{\mathbf{T}}(1) + \hat{\mathbf{G}}^{\mathrm{T}}\hat{\mathbf{Q}}_{\mathbf{M}_0}\bar{\mathbf{T}}(1)\hat{\mathbf{D}}^{\mathrm{T}} + (\hat{\mathbf{G}}^{\mathrm{T}})^2\hat{\mathbf{Q}}_{\mathbf{B}} & \mathbf{0} \end{bmatrix}\begin{pmatrix}\mathbf{m}_{i-1}\\ \dot{\mathbf{m}}_{i-1}\end{pmatrix}
\end{aligned} \tag{A7}$$

Equation (A7) represents a linear map of the Chebyshev coefficients of the position and velocity vectors (state variables) from one delay interval of length $\tau = T$ to the next interval. Hence, the $2nm \times 2nm$ stability matrix is given by

$$\mathbf{W} = \mathbf{A}_{\mathrm{L}}^{-1}\mathbf{A}_{\mathrm{R}} \tag{A8}$$

where $\mathbf{A}_{\mathrm{L}}$ and $\mathbf{A}_{\mathrm{R}}$ are the coefficient matrices on the left and right side of Equation (A7), respectively. This stability matrix was the one used in Reference [27] and to produce Figure 5.

Alternatively, it is also possible to get an $nm \times nm$ stability matrix from Equation (A6) which is given by

$$\begin{aligned}\mathbf{W} = {} & [\hat{\mathbf{Q}}_{\mathbf{M}} - \hat{\mathbf{G}}^{\mathrm{T}}\hat{\mathbf{Q}}_{(\hat{\mathbf{D}}^{\mathrm{T}}\mathbf{M})} - (\hat{\mathbf{G}}^{\mathrm{T}})^2\hat{\mathbf{Q}}_{(\hat{\mathbf{D}}^{\mathrm{T}}\mathbf{M})}\hat{\mathbf{D}}^{\mathrm{T}} + (\hat{\mathbf{G}}^{\mathrm{T}})^2\hat{\mathbf{Q}}_{\mathbf{C}}\hat{\mathbf{D}}^{\mathrm{T}} + (\hat{\mathbf{G}}^{\mathrm{T}})^2\hat{\mathbf{Q}}_{\mathbf{K}}]^{-1} \\ & \times[\hat{\mathbf{Q}}_{\mathbf{M}_0}\bar{\mathbf{T}}(1) + \hat{\mathbf{G}}^{\mathrm{T}}\hat{\mathbf{Q}}_{\mathbf{M}_0}\bar{\mathbf{T}}(1)\hat{\mathbf{D}}^{\mathrm{T}} + (\hat{\mathbf{G}}^{\mathrm{T}})^2\hat{\mathbf{Q}}_{\mathbf{B}}]\end{aligned} \tag{A9}$$

which is half the size of Equation (A8). This is possible because the Chebyshev coefficients of the displacement from one interval to the next are sufficient to completely describe the time evolution or the stability problem of the second-order DDE; therefore, the velocity expansion coefficients are not needed. In fact, it can be shown that the stability matrix of (A9) is the top-left $m \times m$ partition of the stability matrix in (A8). The stability matrix in Equation (A9) was verified by duplicating the stability charts in Figures 6 and 7.

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