

Painless Trigonometry: a tool-complementary school mathematics project

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Abstract

Since 2001-02, we have been working in our mathematics classrooms with the materials and digital tools provided by a government-sponsored national programme: the Teaching Mathematics with Technology programme (EMAT). The main computer tools of the EMAT programme are Spreadsheets (Excel), Dynamic Geometry (Cabri-Géomètre), and Logo (MSWLogo). At the beginning we used these tools independently, but in more recent years we have tried to develop long-term projects that incorporate all the tools, and that also serve as means to introduce students to topics of mathematics that are normally considered too advanced for them (such as trigonometry).

In this paper we report the design and results of one of those projects, a trigonometry project, that we first implemented in the academic year 2005-06 with 6 groups of the first two grades of two junior secondary schools in Mexico. Approximately 250 students of 12-14 yrs of age participated, in total, in the project in that year.

We believe in the importance of using, in an integrated and complementary way, a variety of tools for learning since we consider that each tool brings with it a different type of knowledge and constitutes a different epistemological domain (Balacheff & Sutherland, 1994). We also believe in the importance of constructionist (Harel & Papert, 1991) or programming activities for meaningful learning. With those fundamental theoretical premises, we developed the long-term trigonometry project for introducing young students to the Pythagorean theorem, basic trigonometry concepts, and their applications using explorations and constructive activities with Cabri-Géomètre, Excel and Logo (Figure 1). Students thoroughly enjoyed the activities and gained interest in mathematics. They also developed problem-solving and collaborative skills. Furthermore, in written tests after the project, the students showed an understanding of the "advanced" trigonometry concepts, as well as of other algebraic ideas.

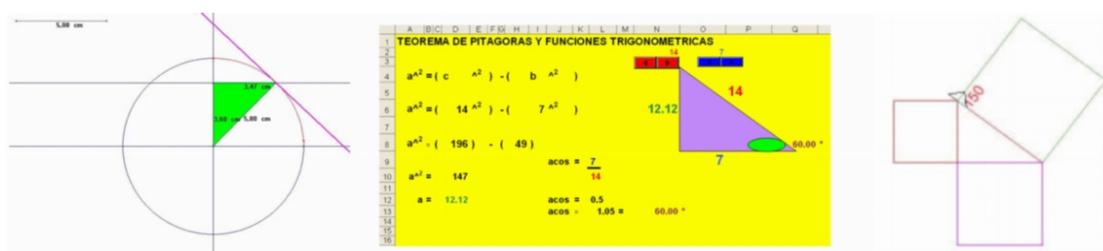


Figure 1. Complementary trigonometry explorations and constructions with Cabri, Excel and Logo

Keywords

Mathematics; trigonometry; Pythagorean theorem; Logo; Excel; Cabri-Géomètre; school project

...children can learn to use computers in a masterful way, and [...] learning to use computers can change the way they learn everything else... (Papert, 1980; p. 8)

Background

In 2001-2002, the Teaching Mathematics with Technology (EMAT) Programme began to be implemented in the junior secondary schools we work at, in Mexico City, Mexico. EMAT is a programme sponsored, since 1997, by the Mexican Ministry of Education to promote the use of new technologies, using a constructivist approach, to enrich and improve the current teaching and learning of junior secondary mathematics in Mexico. A study carried out in Mexico and England (Rojano et al., 1996) involving mathematical practices in science classes, revealed that in Mexico few students were able to close the gap between the formal treatment of the curricular topics and their possible applications. This suggested that it was necessary to replace the formal approach of the then official curriculum, with a “down-up” approach capable of fostering the students’ explorative, manipulative, and communication skills. Thus, a main part of the EMAT programme are pre-designed activities and a pedagogical model that emphasize *exploratory* and *collaborative* learning. And the various EMAT software were chosen on the criteria (Ursini & Rojano, 2000) that they be *open* tools; that is, where the user could be in control and have the power of deciding how to use the software. These open tools had to be flexible enough so that they could be used with different didactical objectives (those of the pre-designed activities, or others as well). The main computer tools of the EMAT programme are Spreadsheets (Excel), Dynamic Geometry (Cabri-Géomètre), and Logo (MSWLogo).

Initially at our schools, we worked with each of these tools separately, covering different themes with each of them. Logo was the last tool that we incorporated into our schools, and one of the immediate things we noticed was how much it enriched, not only children’s motivations for exploring mathematical topics, but also the use of the other tools: because of the programming experience with Logo, students began asking if it was possible to also program the other tools (this led us, for example, to show them how to create macros in Excel and Cabri).

Moreover, we began to look into ways of integrating the three tools. Each tool brings with it a different type of knowledge and constitutes a different epistemological domain (Balacheff & Sutherland, 1994). This, in fact, is one of the reasons why the EMAT designers provided various tools, but their use has generally been independent one from the other. We consider it, however, important to use these tools in an integrated and complementary way.

The trigonometry project

We thus took up the challenge, in the academic year 2005-06, to create an ICT-based project for the learning of a topic which is traditionally difficult to teach and learn, and more so at the junior secondary level (children aged 12 to 15 yrs-old): trigonometry. This is a topic which is in the curriculum only for Grade 3 (the older students) of the junior secondary level; but we decided to work with the younger students of Grades 1 and 2. On the one hand, we concur with the idea that ICT can act as scaffolding for learning more advanced topics; on the other, our strategy was to familiarize students with this topic so that by the time they get to Grade 3 and have to formally learn trigonometry, they will have experiences and useful intuitive ideas (diSessa, 2000) to build upon. For Grade 1 students, we would build upon the curricular themes of *operations with decimal numbers, square root and powers*, and from there create a link to the trigonometry project. In the case of students of Grade 2 we would have less difficulty, as these children already work with algebra; we could thus link it to curricular themes of *polynomials and the use of variables* as well as those of *square root, geometrical figures*.

Our approach was to use not only the EMAT tools –Logo, Cabri and Excel— but also Powerpoint presentations and paper-and-pencil activities, to introduce the theme and the use (for this project) of the various tools.

The stages and implementation of the project

One of the first aims was to combat the apathy that the majority of Grade 1 students have towards mathematics. The use of the technological tools provided a means to motivate our students.

In the first sessions of the school-year we taught students, of both grades, the basic use of the three tools (Logo, Cabri and Excel) through some of the pre-designed EMAT activities (students of neither group had used any of these tools before –although in our school the tools have been available for a few years, the teacher that the Grade 2 students had had the previous year, did not make use of them).

Later, we began to engage students in the trigonometry project explorations: For this project, we began with an integrated theoretical-practical Powerpoint presentation and a paper-and-pencil exercise, followed in later sessions, with technology-based explorations, as described further below.

The explorations were carried over at least a dozen weekly technology-based sessions (i.e. one 50 minute session, once a week). On the one hand, the long-term investigation over the course of the year was motivating and engaging for everyone; it also provided time between sessions to rethink and re-orchestrate the strategies, if necessary. On the other hand, we had the problem, for students, of the long weekly (and sometimes longer) gap between sessions; also, in the case of the group that had their weekly technology sessions, at the end of the week, we noticed weariness that slightly affected the sessions.

The presentation and paper-and-pencil exercise

After covering the curricular theme “Square Roots with Decimals”, we began with a Powerpoint presentation and an exercise on the Pythagorean theorem. In the exercise a square is drawn inside a 7 X 7 grid in such a way that four right-angle triangles are formed (each with legs of sizes 3 and 4), with the sides of the square being the hypotenuses (see Figure 2). In this example, the length of the hypotenuse is the integer 5, but we clarified that it was an exception. The aim of this exercise was to illustrate (see Figure 3) that the sum of the squares of the legs of each right triangle was the same as the square of the length of the hypotenuse (the area of the inside square).

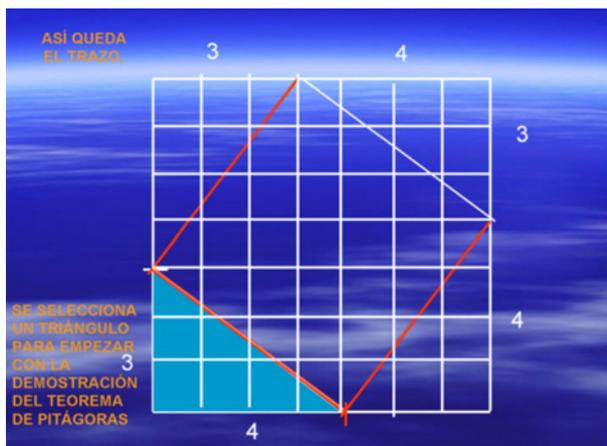


Figure 2. The first step in the Pythagorean Theorem exercise



Figure 3. The last slide in the Pythagorean Theorem presentation

The technology-based explorations

The previous exercise served simply as an introduction to the Pythagorean theorem and as support for the software-based explorations that followed, where students would investigate

further this theorem and its application through construction activities with Cabri, Excel and Logo. In these technology-based activities, the students were the main actors, collaborating and directing the construction process and their peers. As recommended by the EMAT pedagogical model, students worked in teams of two (or sometimes three), with one computer. The structuring of teams was important for discussing and collaborating on the solutions to the problems, and confronting different strategies. The teams then had the opportunity to present to the entire group their work, and engage in whole-classroom discussions (which is also an important recommendation of the EMAT pedagogical model).

(It may be worth mentioning that we did have a difficulty towards the end of the year, when some of our projection equipment was damaged and whole-group presentations were not as easy; but this is part of the challenges of working with technology.)

The Cabri and Excel explorations

The first technology-based activities, were geometric explorations of the Pythagorean theorem using Cabri (see Figure 4), to illustrate that the size of the hypotenuse of a right triangle can always be determined by the sizes of the legs of the triangle; we also introduced the trigonometric circle and basic trigonometric concepts such as sine, cosine and tangent.

With Excel, we introduced exponential formulas for powers and square roots, as well as trigonometric formulas which were related to the explorations with Cabri. In this Excel exploration (see Figure 5), the students programmed the Pythagorean theorem formulas for finding the values of the legs and hypotenuse of a right triangle; as well as formulas, using trigonometric functions, for finding the value of missing angles. They also programmed interactive buttons, something which was motivating for them, and that forced them to understand better the underlying relationships.

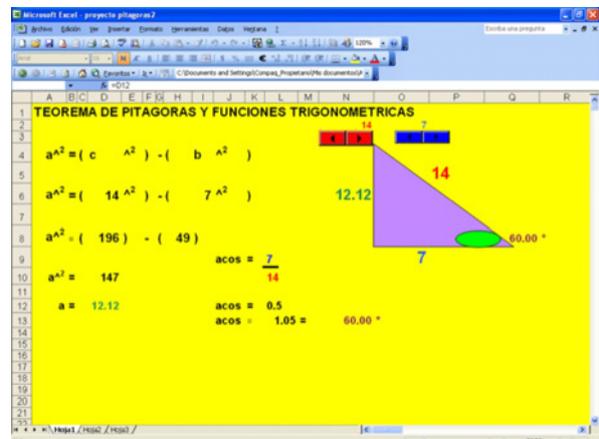
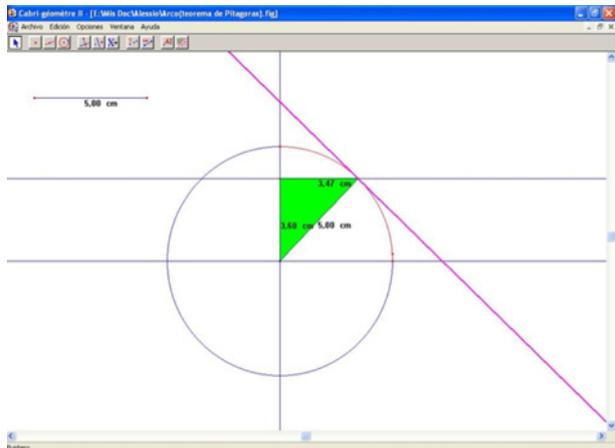


Figure 4. The trigonometry explorations with Cabri.

Figure 5. The Excel explorations of the Pythagorean theorem and of trigonometric functions

The Logo explorations

With Logo, we divided the explorations in several stages:

In the first stage, students had to construct a specific case: a right triangle with legs 30 and 40 (which meant the hypotenuse was a nice, and easy to compute, integer). They soon learned (out of need) to use the primitives *arcsin*, *arctan*, and *arcos*, since without them they would not have been able to compute the rotation angles for drawing the triangle. In this sense, the previous activities with Cabri and Excel were very useful; the first, because it gave them an understanding of the underlying concepts of the trigonometric functions; and the latter, because it familiarized them with the formulas and use of the trigonometric functions. Now, in Logo, in order to use the

formulas for finding the missing angles, they really needed to be able to read them properly, and to understand exactly how to apply them in the new context (a situation where, in many cases – depending on how the procedure was constructed—, the angles needed were the *rotation* angles of the triangle, instead of the inner ones).

In the second stage, the task was to write programs for other right triangles, using any sides (legs) of the triangle they wanted. This time (if they hadn't done it before) they had to use, within their procedure, the formula derived from the Pythagorean theorem and the *sqrt* primitive command for the square root, in order to generate the hypotenuse. At a first attempt, they could compute with paper-and-pencil, or in Logo's direct mode, the rotation angles and simply use the values in their procedure (Procedure 1). But they later had to also program, within the procedure, the formula for the rotation angle (see Procedure 2).

Procedure 1

```
to pitagoras
fd 100
bk 100
rt 90
fd 100
lt 135
fd sqrt(100 * 100 + 100 * 100)
end
```

Procedure 2

```
to pitagoras2
fd 200
bk 200
rt 90
fd 150
lt 180
rt arctan (200 / 150)
fd sqrt (200 * 200 + 150 * 150)
end
```

The latter was the stepping-stone for the final generalization stage, in which students had to write a general program for a right triangle, using variables. Furthermore, in this stage, students not only had to construct the procedure for the right triangle, but at the end, they were also asked to write a procedure that would draw the squares on each side of the triangle and compute their values for labelling the figure. In this way, they built a model of the concept of the Pythagorean theorem (see Procedure 3 and Figure 6).

Procedure 3: Final procedure for the Pythagorean model programmed by one of the teams of students

```
to triangulo :x :y
fd :x bk :x rt 90
setcolor 20
fd :y lt 180
wait 30 rt arctan :x / :y
wait 30 fd sqrt(:x * :x + :y * :y)
wait 30 setcolor 10 rt 90 repeat 3 [fd sqrt(:x * :x + :y * :y) rt 90]
lt 90 + arctan :x / :y
wait 30 setcolor 5 repeat 4 [fd :y rt 90]
rt 90
fd :y
wait 30 setcolor 20 repeat 4 [fd :x rt 90]
rt 90
fd :x rt arctan :x / :y
label sqrt (:x * :x + :y * :y)
end
```

When students had completed their programs, we let them play around freely with their procedures (see Figure 7). This type of activities are important to relax, motivate and reward the students for their hard work.

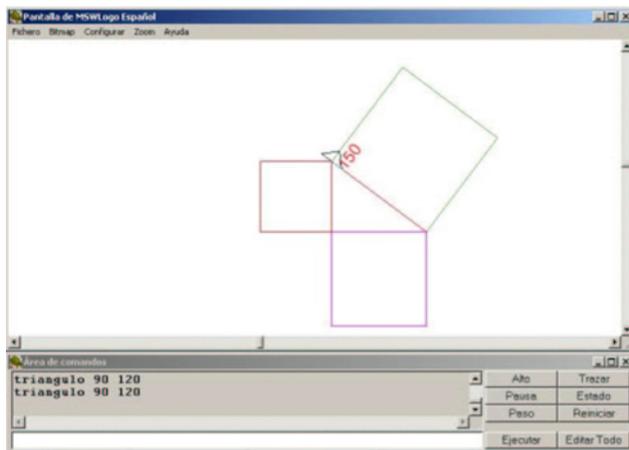


Figure 6. The Pythagorean model produced by the final procedure in Logo.

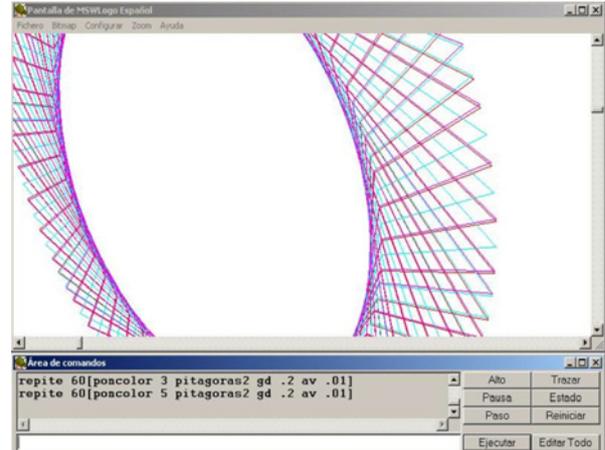


Figure 7. Example of some students playing with their right triangle procedures in Logo.

Some results

In other papers (Sacristán, 2003; Sacristán et al., 2006) some of the benefits (as well as difficulties) of the implementation of the EMAT programme, its tools and pedagogical model, have already been reported, in particular those concerning the development of skills and motivation in students. Here we present the particular observations from our schools.

Observations on the development of problem-solving abilities and collaborative attitudes

Since we began integrating the EMAT ICT tools for our mathematics courses, and particularly Logo and other “programming” or constructionist (Harel & Papert, 1991) activities, we have noticed that our students have developed many valuable learning skills: In particular, there is an increase in their problem-solving abilities, and they tend to reflect more on the problems and even go beyond the tasks required.

In fact, both the students and the teachers involved in the EMAT programme at our schools and in the project presented in this paper, have found new means of solving problems or doing things. Because we do not impose a particular way for solving the tasks, there is more freedom to analyse, conjecture, test, discuss, compare and learn from each other, with a collaboration between the students, the math teacher and the ICT teacher.

We also believe that the team-work during the sessions, and the sharing and discussing of solutions and strategies at the end of each session, are fundamental elements for reaching successful solutions and for stabilizing the learning derived from the technology-based activities and explorations.

Achievements in School Assessments

In the end-of-term mathematics tests, the students involved in our courses had very high scores, with a high percentage of success rate (see Table 1) and it is worth noting that parts of these tests covered trigonometry and algebra, even for first grade students. Samples of solved tests are given in Figure 8.

Group	1° A	1° B	1° C	1° D
Nb. of students with pass grade	35	34	32	42
Nb. of students with fail grade	9	10	15	5

Table 1. Results of the end-of-term mathematics tests for Grade 1

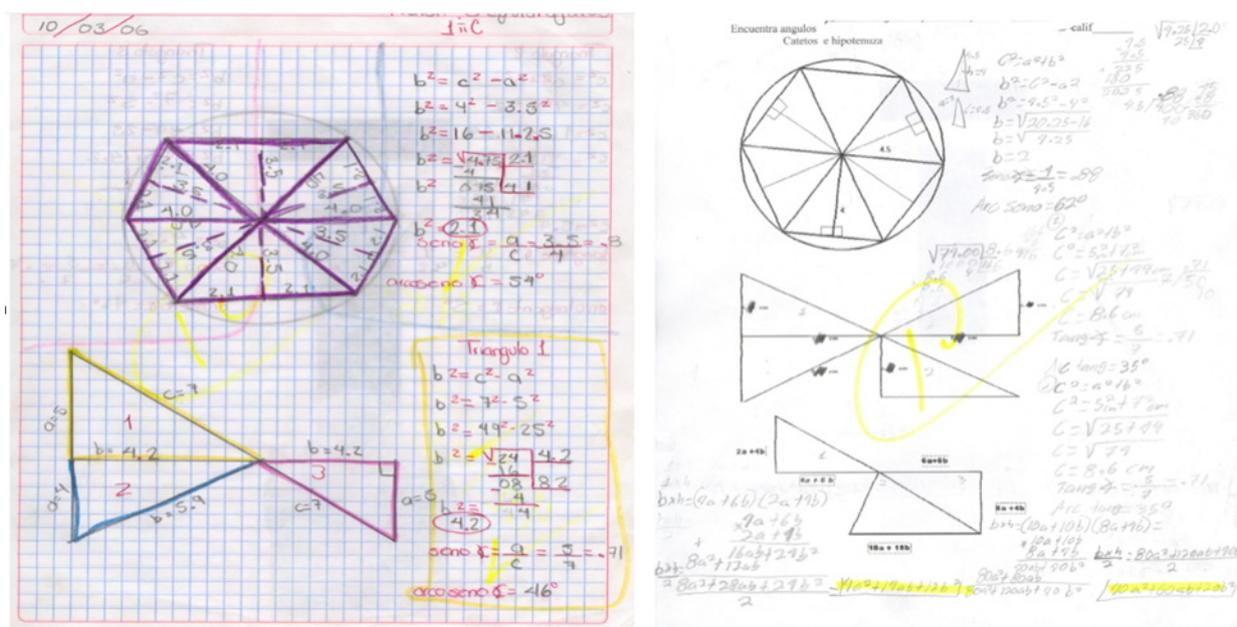


Figure 8: Written tests on Trigonometry by a Grade 1 student and a Grade 2 student, respectively

Observations on affective aspects

With the use of the ICT tools and activities in our mathematics courses, we have noticed a much greater interest on the part of the students for mathematics. This increased when we incorporated Logo, and it became even greater when we began to engage on long-term projects such as the one presented in this paper.

In fact, we face very strong difficulties in making students leave the technology-based sessions, because they are so involved and enjoy their work there so much. When they do leave, they go out discussing amongst themselves their different or possible solutions and procedures.

The atmosphere and dynamics of the classrooms have also changed deeply. We have observed much improved classroom discipline and engagement, with a reduction of behavioural problems. Also students tend to get along better and they have developed values of cooperation and assistance amongst themselves: they appreciate the benefits of collaborative work.

Most importantly, through these technology-based activities, students are given the opportunity to find their own paths for reaching a solution. When they achieve this, they gain confidence in their own knowledge and abilities and this is highly motivating for them.

Remark on students' perceptions of the tools

In terms of the technological tools, the involvement in the project (as well as with other EMAT activities) have shown students the various uses they can give to tools like Logo, Cabri and Excel and how each of these tools complements the other. They have become proficient in the

use of all these tools. Furthermore, students no longer make a memorized “algorithmic” use of the tools, but they actually reflect on what they are doing, why, and how.

Final remarks

What we have achieved with our projects, is to make mathematics explorations, an enjoyable part of students’ life, where working towards finding a solution actually becomes a game for them –like it does for a real mathematician. We observed how they used names for their Logo procedures and objects that perhaps didn’t reflect what they were doing –we let them, because these objects were *their* toys, and the explorations became *their* games. And this is what we believe motivates learning.

Our students are now able to perceive mathematics as something meaningful, something that they can apply for solving a project; instead of something boring, meaningless and forced upon them –as is so often the case. Furthermore, they can use the technological tools as aids for developing their abilities, and learning how to think, instead of simply “clicking buttons”.

From our side, we also achieved to introduce what is normally considered a “too advanced” topic for Grade 1 students, and showed (as is seen in the sample tests in Figure 8) that children did learn and understand some of the ideas they explored: not only the trigonometry ideas, but also the use of algebraic variables and concepts. In its 40 years, Logo has been proved an excellent tool for introducing algebraic concepts and particularly the use of variables. In our project, we have again confirmed that. Whatever may happen, we are now at ease knowing that most of our students are proficient in the curricular topics of Grades 1 and 2, and beyond. Richard Noss told us once, that “knowledge has no age; it rather depends on how the teacher wants to teach it” (personal communication).

We feel that every mathematics programme should have some programming (or at least, constructionist) activities; in our case it was Logo; and its influence made us develop more constructive activities with the other tools. We also appreciate the conceptual perspectives that each of the tools we used, offered. Cabri, with its manipulable dynamic representations allowed us to explore properties of right triangles, such as the relationship between the sides, and between the sides and the angles; it also gave us a means to introduce the trigonometric circle and a basic understanding of some trigonometric functions. With Excel (and Logo), we delved deeper into the structure of the Pythagorean and basic trigonometric formulas and of the algebraic relationships between their elements. With Logo, we gained a further understanding of how and when to apply those concepts for solving a particular project: that of drawing a general right triangle. It was an orchestration of all the tools.

But it was not easy. We faced many challenges –e.g. from difficulties in using the computer labs, to learning how to adapt how we teach. It has taken us several years of experience working with technology in our mathematics classrooms for us to gradually change our practice, the way we teach, and the ways we perceive and use the tools. As is reported in Sacristán et al. (2006), one of the biggest difficulties in the implementation of the EMAT programme, has been teachers’ difficulties in adapting to the proposed pedagogical model and changing their classrooms’ dynamics. We also had those difficulties. A few years ago, we were not used to having children talk and discuss during class. Now, that has changed. But we have to be open to the possibility of change, before that can happen, and we understand how difficult that can be for many of our peers.

In fact, we faced many criticisms (and even obstacles) from some peers and authorities, when we began to develop our technology-based projects, because they didn’t understand what we were doing, despite the fact that the EMAT programme is sponsored by the government.

Again, change is hard. Many of our peers fear the use of technology because they don’t feel proficient enough (as is also reported in Sacristán, 2006); they also fear changing the way they were used to teach. We also faced those fears, but despite our insecurities, we took the risk and have enjoyed discovering we can learn at the same time as the students. Much later we

discovered that, in fact, Papert (1999, p. xv) believes “that there is such a thing as becoming a good learner and therefore that teachers should do a lot of learning in the presence of the children and in collaboration with them”.

Another fear that teachers have, is in engaging in such long-term projects because they feel they lose time that they need to cover the required curricular topics. What we have found out, is that many of the curricular topics naturally arise *from* the explorations. Projects, such as the one described in this paper, not only deal with the topic they are designed for; the need for the use of many other mathematical concepts also emerges¹ and we are reminded of Papert’s vision in his book *Mindstorms* when he described the gears of his childhood (Papert, 1980). This gives us the opportunity to introduce new mathematical topics as children encounter them, and *then* we can develop them. In this way, the students have a means to relate to the mathematics, and it becomes more meaningful for them. Thus, time is far from being lost; it is actually well invested in activities that are much more significant for students than the usual rushing through all the curricular topics.

We believe that in the future, if students learn to use technology in a thoughtful way (moving away from a mechanized use) for understanding, developing projects or solving problems they will become better learners and the teacher will have more of a mediating role between students and mathematics rather than being the traditional presenter of knowledge.

We are a new generation of students and teachers creating experimental environments, computational microworlds and classroom dynamics that are very different from traditional school practices. The catalyst for change are the technological tools, but it is in our power, as teachers, to actually make the change, and use the tools in innovative ways that change mathematical teaching and learning. The tools don’t bring the benefits; it’s the *use* we make of them that does.

In the words of Seymour Papert (1999, p. xv):

Opportunity means more than just “access” to computers. It means an intellectual culture in which individual projects are encouraged and contact with powerful ideas is facilitated.

Doing that means teachers have a harder job. But we believe that it is a far more interesting and creative job and we have confidence that most teachers will prefer “creative” to “easy.”

In that spirit, we continue working, as we have for the past 5 years, in developing interesting mathematical projects for our students with an integral use of technological tools like Logo, Cabri and Excel. And we hope that by sharing our experience, others will also dare change, use technologies in meaningful ways and perhaps also create their own projects, where there is collaboration and partaking of experiences amongst students, between students and teachers, and also amongst teachers, that enriches everyone’s learning.

And if a student of any age comes to us for learning trigonometry, or any other topic, we hope to be able to give him/her the means to learn it “painlessly”.

References

Balacheff, N. and Sutherland, R. (1994) *Epistemological Domain of Validity of Microworlds: The Case of Logo and Cabri-géomètre*. In *Lessons from Learning*, IFIP Conference TC3WG3.3, R. Lewis & P. Mendelsohn (eds), North Holland, pp. 137-150.

¹ In the project described here, as explained before, the students not only explored the basic trigonometric ideas, but it also allowed us to cover many curricular topics and beyond, from number operations to algebraic ideas, such as variables, use of formulas and manipulation of equations.

- diSessa, A. A. (2000). *Changing minds: Computers, learning, and literacy*. Cambridge: MIT Press.
- Harel, I., & Papert, S., (eds.) (1991). *Constructionism*. Norwood: Ablex.
- Papert, S. (1980). *Mindstorms: Children, Computers, and Powerful Ideas*, New York: Basic Books.
- Papert, S. (1999) *What is Logo? Who Needs It?* In Logo Philosophy and Implementation, Logo Computer Systems Inc. Pp. iv-xvi.
- Rojano, T., Sutherland, R., Ursini, S., Molyneux, S., and Jinich, E. (1996). *Ways of solving algebra problems: The influence of school culture*. In Proceedings of the 20th Conference of the International Group for the Psychology of Mathematics Education, Vol. 4. L. Puig & A. Gutierrez (Eds.), Valencia, Spain: Universidad de Valencia. pp. 219-226.
- Sacristán, A. I. (2003) *Mathematical Learning with Logo in Mexican Schools*. In Eurologo'2003 Proceedings: Re-inventing technology on education. Coimbra, Portugal: Cnotinfor, Lda., pp. 230-240.
- Sacristán, A. I., Ursini, S., Trigueros, M. and Gil, N. (2006). *Computational Technologies in Mexican Classrooms: The Challenge of Changing a School Culture*. In Information Technologies at Schools. The 2nd International Conference "Informatics in Secondary Schools: Evolution and perspectives" Selected papers, V. Dagiene and R. Mittermeir (Eds.). Vilnius: Institute of Mathematics and Informatics. Pp. 95-110.
- Ursini, S. & Rojano, T. (2000). *Guía para Integrar los Talleres de Capacitación EMAT*. Mexico: SEP-ILCE.

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