Moduli Problems for Structured Ring Spectra

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In this document we make good on all the assertions we made in the previous paper "Moduli spaces of commutative ring spectra" [20] wherein we laid out a theory a moduli spaces and problems for the existence and uniqueness of E_∞-ring spectra. In that paper, we discussed the the Hopkins-Miller theorem on the Lubin-Tate or Morava spectra E_n; in particular, we showed how to prove that the moduli space of all E_∞ ring spectra X so that $(E_n)_*X \cong (E_n)_*E_n$ as commutative $(E_n)_*$ algebras had the homotopy type of BG, where G was an appropriate variant of the Morava stabilizer group. This is but one point of view on these results, and the reader should also consult [3], [38], and [41], among others.

A point worth reiterating is that the moduli problems here begin with algebra: we have a homology theory E_* and a commutative ring A in E_*E comodules and we wish to discuss the homotopy type of the space $\mathcal{TM}(A)$ of all E_∞-ring spectra so that $E_*X \cong A$. We do not, *a priori*, assume that $\mathcal{TM}(A)$ is non-empty, or even that there is a spectrum X so that $E_*X \cong A$ as comodules.

For a variety of applications we are not simply interested in this absolute problem, but in a relative version as well. We fix an E_∞-ring spectrum Y and write $k = E_*Y$ for the resulting commutative algebra in E_*E comodules. Then we may choose a morphism of commutative algebras $k \to A$ in E_*E-comodules and write $\mathcal{TM}(A/k)$ for the moduli space of Y-algebras X so that $E_*X \cong A$ as a k-algebra. The absolute case can be recovered by setting $Y = S^0$, the zero-sphere. While we are assuming the existence of Y, we are *not* assuming that $\mathcal{TM}(A/k)$ is non-empty or even that there exists a spectrum X with $E_*X \cong A$.

The main results are Theorems 3.3.2, 3.3.3, and 3.3.5 which together give a decomposition of $\mathcal{TM}(A/k)$ as the homotopy inverse limit of a tower of fibrations

$$\cdots \to \mathcal{TM}_n(A/k) \to \mathcal{TM}_{n-1}(A/k) \to \cdots \to \mathcal{TM}_1(A/k)$$

where

1. $\mathcal{TM}_1(A/k)$ is weakly equivalent to $B\,\mathrm{Aut}_k(A)$ where $\mathrm{Aut}_k(A)$ is the group of automorphisms of the k-algebra A in E_*E-comodules; in particular, $\mathcal{TM}_1(A)$ is is non-empty and connected;

2. for all $n > 1$, there is a homotopy pull-back square

$$\begin{tikzcd}
\mathcal{TM}_n(A/k) \arrow[r] \arrow[d] & B\,\mathrm{Aut}_k(A, \Omega^n A) \arrow[d] \\
\mathcal{TM}_{n-1}(A/k) \arrow[r] & \hat{\mathcal{H}}^{n+2}_A(A/k, \Omega^n A).
\end{tikzcd}$$

This last diagram needs a bit of explanation. As a graded abelian group $[\Omega^n A]_k = A_{n+k}$; this is a module over A in the category of E_*E-comodules. The group $\mathrm{Aut}_k(A, \Omega^n)$ is the automorphism group of the pair. If M is an A-module and n a non-negative integer, there is an André-Quillen cohomology *space* so that

$$\pi_i \mathcal{H}^n(A/k, M) = H^{n-i}(A/k, M)$$

where $H^*(-,-)$ denotes an appropriate André-Quillen cohomology functor. The group $\mathrm{Aut}_k(A, M)$ acts on $\mathcal{H}^n(A/k, M)$ and $\hat{\mathcal{H}}^n(A/k, M)$ is the Borel construction of this action. Note that the fiber of $\mathcal{TM}_n(A/k) \to \mathcal{TM}_{n-1}(A/k)$ at any basepoint will either be empty or will be homotopy equivalent to the space $\mathcal{H}^{n+1}(A/k, \Omega^n A)$.

What is notable about this decomposition is that the spaces $B\,\mathrm{Aut}_k(A, \Omega^n)$ and $\hat{\mathcal{H}}^{n+2}_A(A/k, \Omega^n A)$ are determined completely by algebraic data.

By trying to lift the vertex of $\mathcal{TM}_1(A/k)$ up the tower, one gets an obstruction theory for realizing A. The obstructions to both existence and uniqueness lie in André-Quillen cohomology groups. See Remark 3.3.7. This is surely the same obstruction theory as in [41], although we haven't checked this.

This paper is very long – even though we consigned the applications to [20] or to an as-yet-nonexistent paper on elliptic cohomology and topological modular forms. Some of this length is probably gratuitous, as we have repeated a lot of material available elsewhere, notably [7], [10], [17], and [20]. It was tempting to simply point to results in all of these papers, but in the end there were too many small details that needed reworking and, perhaps worse, the result had all the narrative flow of a spreadsheet.

Here are some highlights of what is accomplished here. The main idea, which goes back to Dwyer, Kan, and Stover, is to try to construct a *simplicial* E_∞-algebra whose geometric realization will realize A. Then we use the new simplicial direction and apply Postnikov tower techniques to get the decomposition of the moduli space. Making this work requires an enormous of amount of technical detail. Specifically:

1. The resolution model category structures of [16] and [10] must be reworked to accommodate resolving the E_∞-operad as well. This is necessary, in some cases, to obtain computational control over free objects – for an arbitrary homology theory E_*, the homology of a free E_∞-ring spectrum may be hard to compute. Even more, we are not really interested in the resolution model category itself, but a localization of it at some homology theory E_*. While localization theory is highly developed [23], the hypotheses remain fairly rigid, and this leads us into a discussion of the point-set topology of structured ring spectra. In addition, the standard localization theorems don't apply directly – although the techniques do. All of this is accomplished in the first chapter.

2. The second chapter is a grab-bag of essentially algebraic results. For example, we need to have a description of comodules as diagrams in order to prove the important Corollary 3.1.18 which allows us to identify the module structure on $\Omega^n A$ in our André-Quillen cohomology. We need a theory of Postnikov towers for simplicial algebras in E_*E-comodules, and for that we need a Blakers-Massey excision theorem, and so on. We also have to be a bit careful about what André-Quillen cohomology actually is. And, along the way, we discuss a spectral sequence for computing mapping spaces.

3. If these results ever do get used to discuss topological modular forms, we will need a version suitable for use when E_* is p-completed K-theory. This was not discussed in [20] and takes some pages to set up as well.

4. The third chapter, which is where all the theorems are, is the shortest, and really is a recapitulation of the program set out in [7]. But, again, there are details to be spelled out. Some of these involve the passage to E_*-localization and its effect on the spiral exact sequence; another of these is to spell out exactly what it needed for the relative case; only the absolute case is in the literature.

Throughout this manuscript, we are working with simplicial algebras is spectra over a simplicial operad T. If E_* is a homology theory based on a homotopy commutative ring spectrum E so that E_*E is flat over E_*, then we have a theory of E_*E modules. If X is a simplicial T algebra, then E_*X is an E_*T-algebra in category of simplicial E_*E-comodules. One of the central difficulties we had to confront was to find some condition on T and E_*T so that we could control, at once, the homotopical algebra of T-algebras in simplicial spectra and E_*T algebras in E_*E-comodules. The condition we arrived at – that of *homotopically adapted operad* (See Definition 1.4.16.) – is somewhat cumbersome, but it is satisfied in all the applications we have in mind.

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Contents

Chapter 1

Homotopy Theory and Spectra

1.1 Mapping spaces and moduli spaces

1.1.1 Model category basics

We will assume that the reader is familiar with basics of model categories, cofibrantly generated model categories, and simplicial model categories. These are adequately and thoroughly presented in many references, including [25] and [23]. All our model categories will be, at the very least, cofibrantly generated. This implies, in particular, that given any morphism $f : X \to Y$ in our model category, there are *natural* factorizations

$$\begin{tikzcd} X \arrow[r, "j"] & Z \arrow[r, "q"] & X \end{tikzcd}$$

of f where j is a cofibration and a weak equivalence and q is a fibration; there is also a natural factorization with j a cofibration and q and fibration and a weak equivalence.

Less familiar, perhaps, is the notion of a *cellular* model category, which we now review. The importance of this notion is that cellular model categories are particularly amenable to localization, and this makes for a very clean theory for us. Here are the definitions, all from [23].

1.1.1 Definition. *Fix a category $\mathcal{C}$ with all limits and colimits. If $I = \{A \to B\}$ is some chosen set of maps in $\mathcal{C}$, a* presentation *of a relative I-cell complex $f : X \to Y$ consists of an ordinal number $\lambda = \lambda_f$ and a colimit preserving functor $Y_{(-)} : \lambda \to \mathcal{C}$ so that*

1. *$Y_0 = X$;*

2. *for each β there is a set of maps $T^X_\beta = \{f_i : A \to Y_\beta\}$ with A the source*

of a morphism in I and a push-out diagram

$$\begin{tikzcd}
\sqcup_{T_\beta} A \arrow[r, "\sqcup f_i"] \arrow[d] & Y_\beta \arrow[d] \\
\sqcup_{T_\beta} B \arrow[r] & Y_{\beta+1};
\end{tikzcd}$$

3. *an isomorphism from $X \to \operatorname{colim}_{\beta<\lambda} Y_\beta$ to $f : X \to Y$.*

The *size* of $f : X \to Y$ is the cardinality of the set of cells $\amalg T_\beta$. If X is the initial object of $\mathcal{C}$, then Y is a presented I-cell complex.

We are particularly interested in the case when I generates the cofibrations.

1.1.2 Definition. *A* subcomplex *of a presented relative I-cell complex $X \to Y$, consists of a presented I-cell complex $X \to K$ so that $\lambda_K = \lambda_Y$ and a natural transformation $K_{(-)} \to Y_{(-)} : \lambda \to \mathcal{C}$ so that for all $\beta < \lambda$, the induced map*

$$T^K_\beta \longrightarrow T^Y_\beta$$

is an injection and so that the induced map of push-out squares commutes. If X is the initial object, we may write $K \subseteq Y$.

1.1.3 Definition. *Let $\mathcal{C}$ be a category with all colimits, W an object of $\mathcal{C}$, and I a class of morphisms in $\mathcal{C}$.*

1. *The object W of $\mathcal{C}$ is* small *relative to I if there is a cardinal number κ so that for every regular cardinal $\lambda \geq \kappa$ and every λ sequence*

$$Z_0 \longrightarrow Z_1 \longrightarrow \cdots \longrightarrow Z_\alpha \longrightarrow \cdots$$

   of morphisms in I, the natural map

$$\operatorname{colim}_{\alpha<\kappa} \operatorname{Hom}_{\mathcal{C}}(W, Z_\alpha) \to \operatorname{Hom}_{\mathcal{C}}(W, \operatorname{colim}_{\alpha<\kappa} Z_\alpha)$$

   is an isomorphism.

2. *The object W is* compact *relative to I if there is a cardinal γ so that for every presented relative I-complex $X \to Y$ every map from W to Y factors through a subcomplex of size at most γ.*

Recall that in any category, an *effective monomorphism* is a morphism which can be written as the equalizer of a pair of parallel arrows.

1.1.4 Definition. *A* cellular *model category $\mathcal{C}$ is a cofibrantly generated model category for which there is a set of I of generating cofibrations and set J of generating acyclic cofibrations so that*

1. *the domains and codomains of the elements of I are compact relative to I;*

2. *the domains of the elements of J are small relative to I; and*

3. *the cofibrations are effective monomorphisms.*

1.1.5 Remark. Almost all of our model categories will be, in some way, based on topological spaces – or, more exactly, compactly generated weak Hausdorff spaces. In this case, if a morphism is a cofibration then it will be a Hurewicz cofibration and, hence, a closed inclusion and an effective monomorphism. Furthermore, the domains of the generating sets I and J of cofibrations and acyclic cofibrations will be cofibrant. Finally, if A is the domain of an object in I or J, it will have a stronger compactness property than that required by Definition 1.1.3: the functor $\mathrm{Hom}_{\mathcal{C}}(A, -)$ will commute with all filtered colimits over diagrams of closed inclusions. Thus, many of the conditions of Definition 1.1.4 will be nearly automatic.

We next come to a slight variation on model categories. When considering categories of simplicial algebras in spectra, we will want to stipulate that the weak equivalences be those morphisms $X \to Y$ so that after applying some homology theory E_*, the resulting morphism $E_*X \to E_*Y$ becomes a weak equivalence of simplicial E_*-modules. This won't quite be a model category structure, for reasons which are by now familiar: push-outs along all cofibrations do not necessary preserve these E_*-equivalences – one has to assume that the cofibration has cofibrant source. This situation arose also in [18] and [45]. The latter source supplies an axiomatic framework (there credited to Mark Hovey, see [26]) for coming to terms with this phenomenon. Here is the definition. We highlight where the usual notion of a model category is weakened.

1.1.6 Definition. *Let $\mathcal{C}$ be category with specified classes of weak equivalences, fibrations, and cofibrations. Then $\mathcal{C}$ is a* semi-model category *provided the following axioms hold:*

1. *The category $\mathcal{C}$ has all limits and colimits;*

2. *Weak equivalences, cofibrations, and fibrations are all closed under retracts; fibrations and acyclic fibrations are closed under pull back;*

3. *If f and g are composable morphisms and two of f, g, and gf are weak equivalences, so is the third;*

4. *All cofibrations have the left lifting property with respect to acyclic fibrations, and all acyclic cofibrations with* **cofibrant source** *have the left lifting property with respect to all fibrations.*

5. *Every morphism can be functorially factored as as a cofibration followed by an acyclic fibration and every morphism with* **cofibrant source** *can be functorially factored as an acyclic cofibration followed by a fibration.*

Note that this should really be called a *left* semi-model category, as the definition singles out cofibrations. But this is only kind of semi-model category which will arise in this paper.

The various auxiliary notions of model category also can be similarly modified. For example, we have the following.

1.1.7 Definition. *A semi-model category $\mathcal{C}$ is* cofibrantly generated *if there are sets of morphisms I and J which detect, respectively the acyclic fibrations and the fibrations. Furthermore, the domains of the morphism in I should be small relative to relative I-cell morphisms and the domains of J should be small with respect to relative J-cell morphisms with cofibrant source.*

Here "detect" means, for example, that a morphism is an acyclic fibration if and only if it has the right lifting property with respect to the morphisms in I.

Or again, the following:

1.1.8 Definition. *A semi-model category $\mathcal{C}$ is a* simplicial *semi-model category if it simplicial in the sense of [35]* §*II.2, and if the following corner axiom holds. Let*

$$\mathrm{map}(-,-) : \mathcal{C}^{op} \times \mathcal{C} \longrightarrow s\mathbf{Sets}$$

denote the simplicial mapping space functor. Then if $j : A \to B$ is a cofibration with **cofibrant source** *and $q : X \to Y$ is a fibration, then*

$$\mathrm{map}(B, X) \longrightarrow \mathrm{map}(B, Y) \times_{\mathrm{map}(A,Y)} \mathrm{map}(A, X)$$

is a fibration of simplicial sets which is a weak equivalence if either f is a weak equivalence or j is a weak equivalence.

This gives a working model for mapping spaces in a semi-model category; namely, the simplicial set of maps $\mathrm{map}(X, Y)$ where X is cofibrant and Y is fibrant.

We append here a final definition, mostly because we have no other place to put it. Let I be a small category, $\mathcal{C}$ any category with colimits and $\mathcal{C}^I$ the category of I-diagrams in $\mathcal{C}$. Let I^δ be the category with same objects as I but only identity morphisms; thus, I^δ is I made discrete. An I-diagram $X : I \to \mathcal{C}$ is *I-free* (or simply free) if it is the left Kan extension of some diagram $X_0 : I^\delta \to \mathcal{C}$.

1.1.9 Definition. *Let Δ be the ordinal number category and $\Delta_+ \subset \Delta$ the category with same objects but only surjective morphisms. Let $\mathcal{C}$ be a category and $X : \Delta^{op} \to \mathcal{C}$ a simplicial object. Then X is s-free if the underlying diagram*

$$X : \Delta_+^{op} \longrightarrow \mathcal{C}$$

is free.

The restricted diagram $X : \Delta_+^{op} \to \mathcal{C}$ is the *underlying degeneracy diagram*, and to be s-free is to say that there are objects Z_k so that there are isomorphisms

$$X_n = \coprod_{\phi : n \to k} Z_k$$

where ϕ runs over the surjections in Δ. Furthermore, these isomorphisms should commute with the degeneracies. In many model categories of simplicial objects, the cofibrant objects are retracts of s-free objects. See [35]§II.4.

1.1.2 Moduli spaces

We now recall some of the basic facts about Dwyer-Kan classification spaces, mapping spaces, and moduli spaces. In all cases, these spaces will be the nerve (or classifying space) of some category. The subtlety in this construction will be that often the category $\mathcal{C}$ to which we wish to apply the nerve functor is not small and, therefore, we don't immediately get a simplicial set. However, there are at least three ways to deal with this problem. The first is to notice that the we will obtain *homotopically small* nerves, which determine a well-defined homotopy type. For this, see [14]. The second is to restrict, in each case, to a small subcategory of the category in question which is still large enough to capture enough information to determine the correct homotopy type. In both cases, the constructions are routine, so we employ the third solution: we ignore the problem in order to simplify exposition.

To begin the theory, we need only consider some category $\mathcal{C}$ with a specified class of weak equivalences. Later on, in order to make calculations, we will need a model category or perhaps, only a semi-model category.

If $\mathcal{C}$ is a category with weak equivalences, the Dwyer-Kan hammock localization $L^H\mathcal{C}(X,Y)$ yields a model for the space of morphisms between two objects X and Y of $\mathcal{C}$. See [13]. The following result implies that the hammock localization is a good model for the derived space of maps between two objects.

1.1.10 Proposition. *1.) Suppose $X' \to X$ and $Y \to Y'$ are weak equivalences in $\mathcal{C}$. Then*

$$L^H\mathcal{C}(X,Y) \to L^H\mathcal{C}(X',Y')$$

is a weak equivalence.

2.) Let $\mathcal{C}$ be a simplicial semi-model category, and denote by

$$\mathrm{map}(-,-) : \mathcal{C}^{op} \times \mathcal{C} \longrightarrow s\mathbf{Sets}$$

the mapping space functor. Then if X is cofibrant and Y is fibrant there is a zig-zag of weak equivalences between $\mathrm{map}(X,Y)$ *and $L^H\mathcal{C}(X,Y)$.*

Proof. The first property is Proposition 3.3 of [13]. For the second statement, we note that the argument in §7 of [15] easily adapts to the more general semi-model category. □

For fixed X, the components $\pi_0 L^H\mathcal{C}(X,X)$ of $L^H\mathcal{C}(X,X)$ form a monoid, and we define the derived simplicial monoid of self-equivalences

$$\mathrm{Aut}_{\mathcal{C}}(X) \subseteq L^H\mathcal{C}(X,X) \tag{1.1.1}$$

of X by taking those components which are invertible. We note that if X in some semi-model category is cofibrant and fibrant, then the previous result implies that $\mathrm{Aut}_{\mathcal{C}}(X)$ is weakly equivalent to the components of $\mathrm{map}(X,X)$ which are invertible.

1.1.11 Definition. *Let $\mathcal{C}$ be a semi-model category. A category of weak equivalences in $\mathcal{C}$ is a subcategory of $\mathcal{E}$ of $\mathcal{C}$ which has the twin properties that*

1.) if X is an object in $\mathcal{E}$ and Y is weakly equivalent to X, then $Y \in \mathcal{E}$;

2.) the morphisms in $\mathcal{E}$ are weak equivalences and if $f : X \to Y$ is a weak equivalence in $\mathcal{C}$ between objects of $\mathcal{E}$, then $f \in \mathcal{E}$.

For example, $\mathcal{E}$ might have the same objects as $\mathcal{C}$ and all weak equivalences.

Let $B\mathcal{E}$ denote the nerve of the category $\mathcal{E}$; this is the Dwyer-Kan classification spaces, and we will refer to it as a *moduli space.* In fact, there is a formula for this weak homotopy type: the following is from [14].

1.1.12 Proposition. *Let $\mathcal{E}$ be a category of weak equivalences in some semi-model category $\mathcal{C}$. Then*

$$B\mathcal{E} \simeq \coprod_{[X]} B\mathrm{Aut}_{\mathcal{C}}(\mathrm{X})$$

where $[X]$ runs over the weak homotopy types in $\mathcal{E}$ and $\mathrm{Aut}_{\mathcal{C}}(\mathrm{X})$ is the (derived) monoid of self-weak equivalences of X.

Proof. See §2 of [14]. The proof goes through verbatim in the more general context. Since one of the needed references for this argument can be hard to obtain, we will also offer an outline of the proof below in 1.1.18. □

1.1.13 Example (The moduli space of an object). Fix an object X of some semi-model category $\mathcal{C}$ and let $\mathcal{E}(X)$ be the smallest category of weak equivalences containing X. Then $\mathcal{E}(X)$ has as objects all Y which are weakly equivalent to X and as morphisms all weak equivalences $Y \to Y'$. We will write $\mathcal{M}(X)$ for $B\mathcal{E}(X)$. Then

$$\mathcal{M}(X) \simeq B\,\mathrm{Aut}_{\mathcal{C}}(X).$$

1.1.14 Example (Moduli spaces for diagrams). If $\mathcal{C}$ is a semi-model category and I is some small indexing category, let $\mathcal{C}^I$ be the category of I-diagrams in $\mathcal{C}$. Under many conditions, $\mathcal{C}^I$ has a semi-model category structure with $X \to Y$ a weak equivalence if $X_i \to Y_i$ is a weak equivalence for all i. (See [23], among many references.) But in any case, this always yields a notion of weak equivalence and we can talk about categories $\mathcal{E}$ of weak equivalences as above. For example, let I be the category with two objects and one non-identity arrow; then $\mathcal{C}^I$ is the category of arrows in $\mathcal{C}$. Then we may let $\mathcal{M}(X{\rightsquigarrow}Y)$ denote the classifying space of the category with objects all arrows $U \to V$ with U weakly equivalent to X and Y weakly equivalent to Y. This is not quite the moduli space of arrows $X \to Y$; see the next example, and Proposition 1.1.17.

1.1.15 Example (Mapping spaces as moduli spaces). Let X and Y be two objects in a semi-model category $\mathcal{C}$. We can define a space of morphisms between X and Y as a moduli space. It is the nerve of the category $\mathcal{E}(X, Y)$ whose objects are diagrams

$$\begin{tikzcd} X & U \arrow[l, "\simeq"'] \arrow[r] & V & Y \arrow[l, "\simeq"'] \end{tikzcd}$$

where $U \to X$ and $Y \to V$ are weak equivalences. Morphisms are commutative diagrams of the form

$$\begin{tikzcd}
X \arrow[d, "="'] & U \arrow[l, "\simeq"'] \arrow[r] \arrow[d, "\simeq"'] & V \arrow[d, "\simeq"] & Y \arrow[l, "\simeq"'] \arrow[d, "="] \\
X & U' \arrow[l, "\simeq"'] \arrow[r] & V' & Y \arrow[l, "\simeq"']
\end{tikzcd}$$

in which the indicated maps are weak equivalences. Let $\mathcal{M}_{\mathrm{Hom}}(X,Y)$ denote the moduli space of $\mathcal{E}(X,Y)$. A theorem of Dwyer and Kan [13] implies that if $\mathcal{C}$ is a model category, there is a natural weak equivalence

$$\mathcal{M}_{\mathrm{Hom}}(X,Y) \longrightarrow L^H\mathcal{C}(X,Y).$$

Thus, in a simplicial model category, $\mathcal{M}_{\mathrm{Hom}}(X,Y)$ is weakly equivalent to the derived mapping space.

1.1.16 Example (Mapping spaces in semi-model categories). Now suppose that $\mathcal{C}$ is only a semi-model category. Then the argument that the inclusion $\mathcal{M}_{\mathrm{Hom}}(X,Y) \to L^H\mathcal{C}(X,Y)$ is a weak equivalence will not work for all X and Y, for at some point (see Proposition 8.2 of [13]) one must take the push-out along an acyclic cofibration and claim it is a weak equivalence. This defect can be remedied as follows.

First, let $\mathcal{C}^c \subseteq \mathcal{C}$ be the full subcategory of cofibrant objects, with the inherited class of weak equivalences. Furthermore, if X and Y are cofibrant, let $\mathcal{M}^c_{\mathrm{Hom}}(X,Y)$ be the nerve of the category of diagrams

$$\begin{tikzcd}
X & U \arrow[l, "\simeq"'] \arrow[r] & V & Y \arrow[l, "\simeq"']
\end{tikzcd}$$

where U and V are cofibrant. Then the argument cited above does show that

$$\mathcal{M}^c_{\mathrm{Hom}}(X,Y) \longrightarrow L^H\mathcal{C}^c(X,Y)$$

is a weak equivalence when $\mathcal{C}$ is a semi-model category.

Second, if X and Y are cofibrant, then functorial factorizations make it easy to show that the inclusion

$$\mathcal{M}^c_{\mathrm{Hom}}(X,Y) \to \mathcal{M}_{\mathrm{Hom}}(X,Y)$$

is a weak equivalence. Since $L^H\mathcal{C}^c(X,Y) \to L^H\mathcal{C}(X,Y)$ is a weak equivalence, by the analog of [13] 8.4, we obtain that

$$\mathcal{M}_{\mathrm{Hom}}(X,Y) \longrightarrow L^H\mathcal{C}(X,Y)$$

is a weak equivalence for X and Y *cofibrant* in a semi-model category $\mathcal{C}$.

The relationship between the various mapping objects thus far defined is spelled out in the following result. The proof here is a paradigm for many similar results, and we will often refer to it in later parts of the paper.

1.1.17 Proposition. *Suppose that X and Y are two objects in a model category $\mathcal{C}$. Then there is a homotopy fiber sequence*

$$\mathcal{M}_{\mathrm{Hom}}(X,Y) \to \mathcal{M}(X \rightsquigarrow Y) \to \mathcal{M}(X) \times \mathcal{M}(Y).$$

If $\mathcal{C}$ is only a semi-model category, we must also assume that X and Y are cofibrant.

Proof. This is an application of Quillen's Theorem B (see [21]), which specifies the homotopy fiber of the morphism on nerves $BF : BC \to BD$ induced by a functor $F : C \to D$ between small categories. For $X \in D$, let X/F denote the category with objects the arrows $X \to FY$ in $\mathcal{D}$, with $Y \in C$; the arrows in X/F will be triangles induced by morphisms $Y \to Y'$. If $X' \to X$ is a morphism in D, we get a functor $X/F \to X'/F$ by precomposition, and Theorem B says that

$$B(X/F) \to BC \to BD$$

is a fiber sequence if $B(X/F) \to B(X'/F)$ is a weak equivalence of all $X' \to X$.

The result now follows. The maps are the obvious ones: the morphism $\mathcal{M}_{\mathrm{Hom}}(X,Y) \to \mathcal{M}(X \rightsquigarrow Y)$ is induced by the functor that sends $X \leftarrow U \to V \leftarrow Y$ to $U \to V$; the morphism $\mathcal{M}(X \rightsquigarrow Y) \to \mathcal{M}(X) \times \mathcal{M}(Y)$ sends $U \to V$ to (U, V). One easily checks the conditions of Theorem B, using Example 1.1.15 or Example 1.1.16 as necessary. □

1.1.18 Example (A proof of Proposition 1.1.12). If we let $\mathcal{M}_{\mathrm{Aut}}(X)$ be the moduli space of diagrams

$$\begin{tikzcd} X & U \arrow[l, "\simeq"'] \arrow[r, "\simeq"] & V & X \arrow[l, "\simeq"'] \end{tikzcd}$$

and $\mathcal{M}(X \overset{\mathrm{we}}{\rightsquigarrow} X)$ the moduli space of morphisms

$$\begin{tikzcd} U \arrow[r, "\simeq"] & V \end{tikzcd}$$

where U and V are both weakly equivalent to X, then the kind of argument just given provides a fiber sequence

$$\begin{tikzcd} \mathcal{M}_{\mathrm{Aut}}(X) \arrow[r] & \mathcal{M}(X \overset{\mathrm{we}}{\rightsquigarrow} X) \arrow[r, "q"] & \mathcal{M}(X) \times \mathcal{M}(X). \end{tikzcd}$$

However, there is weak equivalence $\mathcal{M}(X) \to \mathcal{M}(X \overset{\mathrm{we}}{\rightsquigarrow} X)$ sending U to $1 : U \to U$, and the morphism q becomes equivalent to the diagonal. Then Proposition 1.1.12 follows once we identify $\mathcal{M}_{\mathrm{Aut}}(X)$ with $\mathrm{Aut}(X)$. For this see [13] 6.3.

1.1.19 Example (Moduli spaces in the presence of homotopy groups). Suppose that the semi-model category $\mathcal{C}$ has some specified notion of homotopy groups π_i, $i \geq 0$. Then we let $\mathcal{M}(X \looparrowright Y)$ denote the moduli space of arrows $f : U \to V$, where

1. U is weakly equivalent to X and V is weakly equivalent to Y, and

2. the morphism f induces an isomorphism on π_i for all i such that $\pi_i X$ and $\pi_i Y$ are *both* non-trivial.

Note that $\mathcal{M}(X \looparrowright Y)$ is a (possibly empty) disjoint union of components of $\mathcal{M}(X \rightsquigarrow Y)$, as defined in Example 1.1.14.

This kind of moduli space will be mostly used when we have a pair X and Y where $\pi_i Y$ is isomorphic – but not canonically isomorphic – to $\pi_i X$ whenever $\pi_i Y$ is non-zero.

There are many variants on this sort of example. For example, given three spaces, we can form $\mathcal{M}(X \looparrowright Y \looparrowleft Z)$.

In a semi-model category, we will always assume we have cofibrant objects.

1.2 The ground category: basics on spectra

The whole point of this document is to produce a theory of moduli spaces of structured ring spectra; in particular, we wish to discuss E_∞-ring spectra. Thus we need some category of spectra where we can work easily with operads. This works best if the underlying category has a closed symmetric monoidal smash product, so we will choose one of the models of spectra with this property. It turns out that almost any of the categories of this sort built from topological spaces (as opposed to simplicial sets) will do. For example, we could choose the S-modules of [18] or the orthogonal spectra of [33]; however, simply to be concrete, we will select the symmetric spectra in topological spaces, as discussed in [33]. This category owes much to the symmetric spectra in simplicial sets, as developed in [28], but it is not clear that the latter category satisfies Theorem 1.2.3 below.

It turns out that for any of the models of spectra we might consider here, the category of C-algebras in spectra, where C is some operad, depends only on the weak equivalence type of C in the naïvest possible sense, which is in sharp distinction to the usual results about, say, spaces. (The exact result is below, in Theorem 1.2.4.) However, the reasons for this are not very transparent, because they are buried in the construction of the smash product. But it is worth emphasizing this point: the smash product has the property that if X is a cofibrant spectrum, then the evident action of the nth symmetric group on the n-fold iterated smash product of X with itself is free.

The concepts of a monoidal model category and of a module over a monoidal category is discussed in Chapter 4.2 of [25]. Specifically, simplicial sets are a monoidal model category and a simplicial model category is a module category over simplicial sets. For any category of spectra, the action of a simplicial set K on a spectrum X should be, up to weak equivalence, given by the formula

$$X \otimes K = X \wedge |K|_+$$

whenever this makes homotopical sense. Here the functor $| - |$ is geometric realization and $(-)_+$ means adjoin a disjoint basepoint. This is the part 3.) of the next result. Also, whatever category of spectra we have, it should be amenable to localization. This happens most easily when one has a cellular model category, an idea discussed in the previous section; see Definition 1.1.4.

Let $\mathcal{S}$ denote the category of symmetric spectra in topological spaces, as developed in [33]. We fix once and for all the "positive" model category structure on $\mathcal{S}$, as in §14 of that paper.

1.2.1 Theorem. *The category $\mathcal{S}$ of symmetric spectra satisfies the following conditions:*

1.) *The category $\mathcal{S}$ is a cellular simplicial model category Quillen equivalent to the Bousfield-Friedlander [11] category of simplicial spectra.*

2.) *The category $\mathcal{S}$ has a closed symmetric monoidal smash product which descends to the usual smash product on the homotopy category; furthermore, with that monoidal structure, $\mathcal{S}$ is a monoidal model category.*

3.) *The smash product behaves well with respect to the simplicial structure; specifically, if S is the unit object of the smash product, then there is a natural monoidal isomorphism*

$$X \otimes K \xrightarrow{\cong} X \wedge (S \otimes K).$$

Note that Part 1 guarantees, among other things, that the homotopy category is the usual stable category.

Proof. Symmetric spectra in spaces is not immediately a simplicial model category, but a topological model category. But any topological model category is automatically a simplicial model category via the realization functor. The fact that we have a cellular model category follows from Remark 1.1.5. For example, the effective monomorphism condition follows from the fact the every Hurewicz cofibration of topological spaces is a closed inclusion and the "Cofibration Hypothesis", which is 5.3 in [33]. Parts 2 and 3 can be found in [33]. □

As with categories modeling the stable homotopy category one has to explicitly spell out what one means by some familiar terms.

1.2.2 Notation for Spectra. The following remarks and notation will be used throughout this paper.

1.) When referring to a spectrum, we will use the words cofibrant and cellular interchangeably. The generating cofibrations of $\mathcal{S}$ are inclusions of spheres into cells.

2.) We will write $[X, Y]$ for the morphisms in the homotopy category $\mathbf{Ho}(\mathcal{S})$. As usual, this is π_0 for some derived space of maps. See point (5) below.

3.) In the category $\mathcal{S}$ the unit object S for the smash product ("the zero-sphere") is not cofibrant. We will write S^k, $-\infty < k < \infty$ for a cofibrant model for the k-sphere unless we explicitly state otherwise. In this language the suspension functor on the homotopy category is induced by

$$X \mapsto X \wedge S^1.$$

Also the suspension spectrum functor from pointed simplicial sets to spectra is, by axiom 3, modeled by

$$K \mapsto S^0 \wedge K \overset{\text{def}}{=} \frac{S^0 \otimes K}{S^0 \otimes *}$$

Note that because the unit object S is not cofibrant, the functor $S \otimes (-)$ is not part of a Quillen pair.

4.) Let K be a simplicial set and $X \in \mathcal{S}$. We may write $X \wedge K_+$ for the tensor object $X \otimes K$. This is permissible by axiom 3 and in line with the geometry. The exponential object in $\mathcal{S}$ will be written X^K.

5.) We will write $\mathrm{map}(X, Y)$ or $\mathrm{map}_{\mathcal{S}}(X, Y)$ for the *derived* simplicial set of maps between two objects of $\mathcal{S}$. Thus, $\mathrm{map}(X, Y)$ is the simplicial mapping space between some fibrant-cofibrant models ("bifibrant") models for X and Y. This can be done functorially if necessary, as the category $\mathcal{S}$ is cofibrantly generated. Alternatively, we could use some categorical construction, such as the moduli spaces of Example 1.1.15. Note that with this convention

$$\pi_0 \,\mathrm{map}(X, Y) = [X, Y].$$

6.) We will write $F(X, Y)$ for the function spectrum of two objects $X, Y \in \mathcal{S}$. The closure statement in Axiom 2 of 1.2.1 amounts to the statement that

$$\mathrm{Hom}_{\mathcal{S}}(X, F(Y, Z)) \cong \mathrm{Hom}_{\mathcal{S}}(X \wedge Y, Z).$$

This can be derived:

$$\mathrm{map}(X, RF(Y, Z)) \simeq \mathrm{map}(X \wedge^L Y, Z)$$

where the R and L refer to the total derived functors and $\mathrm{map}(-, -)$ is the derived mapping space. In particular

$$\pi_k RF(Y, Z) \cong [\Sigma^k Y, Z].$$

7.) If X is cofibrant and Y is fibrant, then there is a natural weak equivalence

$$\mathrm{map}(X, Y) \simeq \mathrm{map}(S^0, F(X, Y))$$

and the functor $\mathrm{map}(S^0, -)$ is the total right derived functor of the suspension spectrum functor from pointed simplicial sets to $\mathcal{S}$. Thus we could write

$$\mathrm{map}(X, Y) \simeq \Omega^\infty F(X, Y).$$

In particular, $\mathrm{map}(X, Y)$ is canonically weakly equivalent to an infinite loop space.

We need a notation for iterated smash products. So, define, for $n \geq 1$,

$$X^{(n)} \overset{\text{def}}{=} \underset{\longleftarrow n \longrightarrow}{X \wedge \cdots \wedge X}\,.$$

Set $X^{(0)} = S$.

This paper is particularly concerned with the existence of A_∞ and E_∞-ring spectrum structures. Thus we must introduce the study of operads acting on spectra.

Let $\mathcal{O}$ denote the category of operads in simplicial sets. Our major source of results for this category is [38]. The category $\mathcal{O}$ is a cofibrantly generated simplicial model category where $C \to D$ is a weak equivalence or fibration if each of the maps $C(n) \to D(n)$ is a weak equivalence or fibration of Σ_n-spaces in the sense of equivariant homotopy theory. Thus, for each subgroup $H \subseteq \Sigma_n$, the induced map $C(n)^H \to D(n)^H$ is a weak equivalence or fibration. The existence of the model category structure follows from the fact that the forgetful functor from operads to the category with objects $X = \{X(n)\}_{n\geq 0}$ with each $X(n)$ a Σ_n-space has a left adjoint with enough good properties that the usual lifting lemmas apply.

If C is an operad in simplicial sets, then we have a category of $\mathbf{Alg}_C$ of algebras over C is spectra. These are exactly the algebras over the triple

$$X \mapsto C(X) \overset{\text{def}}{=} \vee_{n\geq 0} C(n) \otimes_{\Sigma_n} X^{(n)}.$$

Note that we should really write $X^{(n)} \otimes_{\Sigma_n} C(n)$, but we don't.

The object $C(*) \cong S \otimes C(0)$ is the initial object of $\mathbf{Alg}_C$. If the operad is *reduced* – that is, $C(0)$ is a point – then this is simply S itself.

If $f : C \to D$ is a morphism of operads, then there is a restriction of structure functor $f_* : \mathbf{Alg}_D \to \mathbf{Alg}_C$, and this has a left adjoint

$$f^* \overset{\text{def}}{=} D \otimes_C (-) : \mathbf{Alg}_C \to \mathbf{Alg}_D$$

The categories $\mathbf{Alg}_C$ are simplicial categories in the sense of Quillen and both the restriction of structure functor and its adjoint are continuous. Indeed, if $X \in \mathbf{Alg}_C$ and K is a simplicial set, and if X^K is the exponential object of K in $\mathcal{S}$, then X^K is naturally an object in $\mathbf{Alg}_C$ and with this structure, it is the exponential object in $\mathbf{Alg}_C$. Succinctly, we say the forgetful functor creates exponential objects. It also creates limits and reflexive coequalizers, filtered colimits, and geometric realization of simplicial objects.

Here is our second set of results about spectra. The numbering continues that of Theorem 1.2.1.

1.2.3 Theorem. *The category $\mathcal{S}$ of symmetric spectra in topological spaces has the following additional properties.*

4.) *For a fixed operad $C \in \mathcal{O}$, define a morphism of $X \to Y$ of C-algebras in spectra to be a weak equivalence or fibration if it is so in spectra. Then the category $\mathbf{Alg}_C$ becomes a cofibrantly generated simplicial model category. Furthermore, $\mathbf{Alg}_C$ has a generating set of cofibrations and a generating set of acyclic cofibrations with cofibrant source.*

5.) In the category $\mathbf{Alg}_C$*, every cofibration is a Hurewicz cofibration on the underlying spectra and, in particular, is a level-wise closed inclusion and an effective monomorphism.*

6.) Let $n \geq 1$ *and let* $K \to L$ *be a morphism of* Σ_n *spaces which is a weak equivalence on the underlying spaces. Then for all cofibrant spectra* X*, the induced map on orbit spectra*

$$K \otimes_{\Sigma_n} X^{(n)} \to L \otimes_{\Sigma_n} X^{(n)}$$

is a weak equivalence of spectra. If $K \to L$ *is a cofibration of simplicial sets, then this same map is a cofibration of spectra.*

Proof. First, part 4.) The argument goes exactly as in §15 of [33]. The argument there is only for the commutative algebra operad, but it goes through with no changes for the geometric realization of an arbitrary simplicial operad.

Part 5.) follows from the Cofibration Hypothesis, [33] 5.3.

Part 6.) follows from the observation that for cofibrant X (here is where the positive model category structure is required), the smash product $X^{(n)}$ is actually a free Σ_n-spectrum. See Lemma 15.5 of [33]. □

We wonder whether this result is also true for symmetric spectra in simplicial sets. This is not immediately obvious: many of the technical arguments of [33] use that the inclusion of a sphere into a disk is an NDR-pair.

The following result emphasizes the importance part 6.) of Theorem 1.2.3.

1.2.4 Theorem. *Let* $f : C \to D$ *be a morphism of operads in simplicial sets. Then the adjoint pair*

$$f^* : \mathbf{Alg}_C \rightleftarrows \mathbf{Alg}_D : f_*$$

is a Quillen pair. If, in addition, the morphism of operads has the the property that $C(n) \to D(n)$ *is a weak equivalence of spaces for all* $n \geq 0$*, this Quillen pair is a Quillen equivalence.*

Proof. The fact that we have a Quillen pair follows from the fact that the restriction of structure functor (the right adjoint) $f_* : \mathbf{Alg}_D \to \mathbf{Alg}_C$ certainly preserves weak equivalences and fibrations.

For the second assertion, first note that since f_* creates weak equivalences, we need only show that for all cofibrant $X \in \mathbf{Alg}_C$, the unit of the adjunction

$$X \to f_* f^* X = D \otimes_C X$$

is a weak equivalence. If $X = C(X_0)$ is actually a free algebra on a cofibrant spectrum, then this map is exactly the map induced by f:

$$C(X_0) = \bigvee_n C(n) \otimes_{\Sigma_n} X_0^{(n)} \to \bigvee_n D(n) \otimes_{\Sigma_n} X_0^{(n)} = D(X_0).$$

For this case, Axiom 6 of 1.2.3 supplies the result. We now reduce to this case.

Let $X \in \mathbf{Alg}_C$ be cofibrant. We will make use of an augmented simplicial resolution in $\mathbf{Alg}_C$

$$P_\bullet \longrightarrow X$$

with the following properties:

i.) the induced map $|P_\bullet| \to X$ from the geometric realization of $P_\bullet$ to X is a weak equivalence;

ii.) the simplicial C-algebra $P_\bullet$ is s-free on a set of C-algebras $\{C(Z_n)\}$ where each Z_n is a cofibrant spectrum. (The notion of s-free was defined in Definition 1.1.9.)

There are many ways to produce such a $P_\bullet$. For example, we could take an appropriate subdivision of a cofibrant model for X in the resolution model category for simplicial C-algebras based on the homotopy cogroup objects $C(S^n)$, $-\infty < n < \infty$. [1]

Given $P_\bullet$, consider the diagram

(1.2.1)
$$\begin{tikzcd}
{|P_\bullet|} \arrow[r] \arrow[d] & {|f_*f^*P_\bullet|} \arrow[d] \\
X \arrow[r] & f_*f^*X
\end{tikzcd}$$

For all n, we have an isomorphism

$$P_n \cong C\Big(\bigvee_{\phi:[n]\to[k]} Z_k \Big)$$

where ϕ runs over the surjections in the ordinal number category. Thus we can conclude that $P_n \to f_*f^*P_n$ is a (levelwise) weak equivalence and that both $P_\bullet$ and $f_*f^*P_\bullet$ are Reedy cofibrant. The morphism $|P_\bullet| \to X$ is a weak equivalence by construction, and $|P_\bullet| \to |f_*f^*P_\bullet|$ is a weak equivalence since geometric realization preserves weak equivalences between Reedy cofibrant objects. Thus we need only show that

$$|f_*f^*P_\bullet| \longrightarrow f_*f^*X$$

is a weak equivalence.

To see this, we note that since weak equivalences and geometric realizations are created in the underlying category of spectra, it is sufficient to show $|f^*P_\bullet| \to f^*X$ is a weak equivalence. However $|f^*P_\bullet| = f^*|P_\bullet|$ since f^* is a left adjoint. Finally, since f^* is part of a Quillen pair, it preserves weak equivalences between cofibrant objects (which is where that hypothesis is used). □

We now make precise the observation that Theorem 1.2.4 implies that the notion of, for example, an E_∞ ring spectrum is independent of which E_∞ operad we choose. Actually, even more is true. Let C be an operad so that for

---

[1] See Proposition 1.4.11. Resolution model categories are reviewed in section 1.4. The notion of Reedy cofibrant, used in the next paragraph, is discussed in the next section.

all n, the unique map to the one-point space $C(n) \to *$ is a weak equivalence (non-equivariantly). Then the obvious map $C \to \mathbf{Comm}$ from C to the commutative monoid operad satisfies the hypotheses of Theorem 1.2.4 and thus we may conclude that $\mathbf{Alg}_C$ is Quillen equivalent to the category of commutative S-algebras.

1.3 Simplicial spectra over simplicial operads

Simplicial objects are often used to build resolutions – and that is our main point here. However, given an algebra X in spectra over some operad, there are times when we will resolve not only X, but the operad as well. The main results of this section are that if X is a simplicial algebra over a simplicial operad T then the geometric realization $|X|$ is an algebra over the geometric realization $|T|$ and, furthermore, that geometric realization preserves level-wise weak equivalences between Reedy cofibrant objects, appropriately defined.

1.3.1 Remark. In what follows we are going to discuss the category $s\mathcal{O}$ of simplicial operads. These are bisimplicial operads is sets, but when we say simplicial operad, we will mean a simplicial object in $\mathcal{O}$, emphasizing the second (external) simplicial variable as the resolution variable. The first (internal) simplicial variable will be regarded as the geometric variable.

As mentioned in the previous section, the category of operads $\mathcal{O}$ is a simplicial model category. From this one gets the Reedy model category structure on simplicial operads $s\mathcal{O}$ ([37]), which are the simplicial objects in $\mathcal{O}$. Weak equivalences are level-wise and cofibrations are defined using the latching objects. The Reedy model category structure has the property that geometric realization preserves weak equivalences between cofibrant objects. It also has a structure as a simplicial model category; for example if T is a simplicial operad and K is simplicial set, then

$$T^K = \{T_n^K\}.$$

However, note that this module structure over simplicial sets is inherited from $\mathcal{O}$ and is not the simplicial structure arising externally, as in [35], §II.2.

Now fix a simplicial operad $T = \{T_n\}$. (At this point, T need not have any special properties.) The free algebra functor $X \mapsto C(X)$ is natural in X and the operad C; hence, for any simplicial spectrum X we can define a bisimplicial spectrum $\{T_q(X_q)\}$. We will denote the diagonal of this bisimplicial spectrum by $T(X)$. A *simplicial algebra* in spectra over T is a simplicial spectrum X equipped with a multiplication map

$$T(X) \longrightarrow X$$

so that the usual associativity and unit diagrams commute. In particular, if $X = \{X_n\}$, then each X_n is a T_n-algebra. Let $s\mathbf{Alg}_T$ be the category of simplicial T-algebras.

The category $s\mathbf{Alg}_T$ is a simplicial model category, and geometric realization behaves well with respect to this structure. The exact result we need is below in Theorem 1.3.4, but its complete statement requires some preliminaries.

Recall that given a morphism of operads $C \to D$, the restriction of structure functor $\mathbf{Alg}_D \to \mathbf{Alg}_C$ is continuous. This implies that if K is a simplicial set and $X \in s\mathbf{Alg}_T$, we may define $X \otimes K$ and X^K level-wise; for example,

$$X \otimes K = \{X_n \otimes K\}.$$

We could use this structure to define a geometric realization functor; however, we prefer to proceed as follows.

If $\mathcal{M}$ is a module category ([25], §4.2) over simplicial sets, then the geometric realization functor $|\cdot| : s\mathcal{M} \to \mathcal{M}$ has a right adjoint

$$Y \mapsto Y^{\Delta} = \{Y^{\Delta^n}\}.$$

where Δ^n is the standard n-simplex. In particular, this applies to simplicial operads, and we are interested in the unit of the adjunction $T \to |T|^{\Delta}$. If C is any operad and Y is a C-algebra, then for all simplicial sets K, the spectrum Y^K is a C^K algebra. From this it follows that Y^{Δ} is a simplicial C^{Δ} algebra. Setting $C = |T|$ and restricting structure defines a functor

$$Y \mapsto Y^{\Delta} : \mathbf{Alg}_{|T|} \longrightarrow s\mathbf{Alg}_T.$$

The result we want is the following.

1.3.2 Theorem. *Let T be a simplicial operad and $X \in s\mathbf{Alg}_T$ a simplicial T-algebra. Then the geometric realization $|X|$ of X as a spectrum has a natural structure as a $|T|$ algebra and, with this structure, the functor*

$$X \mapsto |X|$$

is left adjoint to $Y \mapsto Y^{\Delta}$.

Proof. We know that for an operad $C \in \mathcal{O}$ the forgetful functor from $\mathbf{Alg}_C$ to spectra creates geometric realization. Actually, what one proves is that if X is a simplicial spectrum and $C(X)$ is the simplicial C-algebra on X, then there is a natural (in C and X) isomorphism

$$C(|X|) \longrightarrow |C(X)|.$$

This uses a "reflexive coequalizer" argument; see Lemma II.6.6 of [18]. Now use a diagonal argument. If T is a simplicial operad and X is a simplicial spectrum, then, by definition,

$$T(X) = \mathrm{diag}\{T_p(X_q)\}.$$

Since the functor $Y \mapsto C(Y)$ is a continuous left adjoint, taking the realization in the p-variable yields a simplicial object

$$\{|\{T_\bullet(X_q)\}|\} \cong \{|T|(X_q)\}.$$

Now take the realization in the q variable and get

$$|T(X)| \cong |T|(|X|)$$

using the fact about the constant case sited above. The result now follows. $\square$

The next item to study is the homotopy invariance of the geometric realization functor in this setting. The usual result has been cited above: realization preserves level-wise weak equivalences between Reedy cofibrant objects. The same result holds in this case, but one must take some care when defining “Reedy cofibrant”. The difficulty is this: the definition of Reedy cofibrant involves the latching object, which is the colimit

$$L_n X = \operatorname*{colim}_{\phi:[n]\to[m]} X_m$$

where ϕ runs over the non-identity surjections in the ordinal number category. We must define this colimit if each of the X_m is an algebra over a different operad. The observation needed is the following. Let $S : I \to \mathcal{O}$ be a diagram of operads. Then an I-diagram of S-algebras is an I-diagram $X : I \to \mathcal{S}$ of spectra equipped with a natural transformation of I-diagrams

$$S(X) \to X$$

satisfying the usual associativity and unit conditions. For example if $I = \Delta^{op}$ one recovers simplicial S-algebras. Call the category of such $\mathbf{Alg}_S$.[2] Then one can form the colimit operad $\operatorname{colim} S = \operatorname{colim}_I S$ and there is a constant diagram functor

$$\mathbf{Alg}_{\operatorname{colim} S} \longrightarrow \mathbf{Alg}_S$$

sending X to the constant I-diagram $i \mapsto X$ where X gets an S_i structure via restriction of structure along

$$S_i \longrightarrow \operatorname*{colim}_I S.$$

1.3.3 Lemma. *This constant diagram functor has a left adjoint*

$$X \to \operatorname{colim}_I X.$$

Despite the notation, $\operatorname{colim}_I X$ is not the colimit of X as an I diagram of spectra; indeed, if $X = S(Y)$ where Y is an I-diagram of spectra

$$\operatorname{colim}_I X \cong (\operatorname{colim}_I S)(\operatorname{colim}_I Y).$$

If T is a simplicial operad we can form the latching object

$$L_n T = \operatorname*{colim}_{\phi:[n]\to[m]} T_m.$$

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[2] This is a slight variation on the notation $s\mathbf{Alg}_T$. If T is a simplicial operad, this new notation would simply have us write $\mathbf{Alg}_T$ for $s\mathbf{Alg}_T$. No confusion should arise.

There are natural maps $L_nT \to T_n$ of operads. If X is a simplicial T-algebra we extend this definition slightly and define

$$L_nX = T_n \otimes_{L_nT} \operatorname*{colim}_{\phi:[n]\to[m]} X_m$$

where, again, ϕ runs over the non-identity surjections in Δ. In short we extend the operad structure to make L_nX a T_n-algebra and the natural map $L_nX \to X_n$ a morphism of T_n-algebras.

With this construction on hand one can make the following definition. Let T be a simplicial operad and $f : X \to Y$ a morphism of simplicial T-algebras. Then f is a level-wise weak equivalence (or *Reedy weak equivalence*) if each of the maps $X_n \to Y_n$ is a weak equivalence of T_n-algebras – or, by definition, a weak equivalence as spectra. The morphism f is a Reedy cofibration if the morphism of T_n-algebras

$$L_nY \sqcup_{L_nX} Y_n \longrightarrow Y_n$$

is a cofibration of T_n-algebras. The coproduct here occurs in the category of T_n-algebras. (Fibrations are then determined; they have a description in terms of matching objects. See [23], §15.1.) The main result is then:

1.3.4 Theorem. *With these definitions, and the level-wise simplicial structure defined above, the category $s\mathbf{Alg}_T$ becomes a simplicial cellular model category. Furthermore,*

1. *the geometric realization functor $|-| : s\mathbf{Alg}_T \to \mathbf{Alg}_{|T|}$ sends level-wise weak equivalences between Reedy cofibrant objects to weak equivalences; and*
2. *any Reedy cofibration in $s\mathbf{Alg}_T$ is a Hurewicz cofibration in spectra at each simplicial level; in particular, it is an effective monomorphism.*

Proof. The standard argument for the existence of a Reedy model category structure (see [23] §15.6, for example) easily adapts to this situation; one need only take care with latching objects, and we have described these in some detail above. The same reference also supplies arguments to show that the model category structure is cellular. See [23] §15.7. That it is a simplicial model category is an easy exercise.

To prove point 1.), note that the right adjoint to geometric realization $Y \mapsto Y^{\Delta}$ preserves fibrations and weak equivalences when considered as a functor to $s\mathcal{S}$, hence it has the same properties when considered as a functor to $s\mathbf{Alg}_T$. Thus geometric realization is part of a Quillen pair. For point 2.), one checks that a Reedy cofibration $X \to Y$ in $s\mathbf{Alg}_T$ yields a (Quillen) cofibration of T_n-algebras $X_n \to Y_n$ for all n. This can be done by adapting the argument of Proposition 15.3.11 of [23]. Now apply Theorem 1.2.3. □

Now let us next spell out the kind of simplicial operads we want might want. One example is, obviously, the constant simplicial operad T on the commutative monoid operad or, perhaps, an E_∞-operad in $\mathcal{O}$. Then $s\mathbf{Alg}_T$ will simply be simplicial commutative algebras (or E_∞-algebras) in spectra. However, there are times when this might be too simplistic.

If E_* is the homology theory of a homotopy commutative ring spectrum and C is an operad in $\mathcal{O}$, one might like to compute $E_*C(X)$. This might be quite difficult, unless E_*X is projective as an E_* module and $\pi_0 C(q)$ is a free Σ_q-set for all q. Thus we'd like to resolve a general operad C using operads of this sort.

If T is a simplicial operad and E is a commutative ring spectrum in the homotopy category of spectra, then E_*T is a simplicial operad in the category of E_*-modules. The category of simplicial operads in E_*-modules has a simplicial model category structure in the sense of §II.4 of [35], precisely because there is a free operad functor. Cofibrant objects are retracts of diagrams which are "free" in the sense of [35]; meaning the underlying degeneracy diagram is a free diagram of free operads. Free operads are discussed in detail in the appendix to [38].

Given an operad $C \in \mathcal{O}$, we'd like to consider simplicial operads T of the following sort:

1.3.5 Theorem. *Let $C \in \mathcal{O}$ be an operad. Then there exists an augmented simplicial operad*

$$T \longrightarrow C$$

so that

1. *T is Reedy cofibrant as a simplicial operad;*
2. *For each $n \geq 0$ and each $q \geq 0$, $\pi_0 T_n(q)$ is a free Σ_q-set;*
3. *The map of operads $|T| \to C$ induced by the augmentation is a weak equivalence;*
4. *If $E_*C(q)$ is projective as an E_* module for all q, then E_*T is cofibrant as a simplicial operad in E_* modules and $E_*T \to E_*C$ is a weak equivalence of operads in that category.*

This theorem is not hard to prove, once one has the explicit construction of the free operad; for example, see the appendix to [38]. Indeed, here is a construction: first take a cofibrant model C' for C. Then, if $F_{\mathcal{O}}$ is the free operad functor on graded spaces, one may take T to be the standard cotriple resolution of C'. What this theorem does not supply is some sort of uniqueness result for T; nonetheless, what we have here is sufficient for our purposes.

Note that if C is the commutative monoid operad, then we can simply take T to be a cofibrant model for C in the category of simplicial operads and run it out in the simplicial (i.e., external in the sense of Remark 1.3.1) direction. Then T is, of course, an example of an E_∞-operad; furthermore, E_*T will be a simplicial E_∞-operad in E_*-modules in the sense of Definition 2.3.8.

1.4 Resolutions

Building on the results of the last section, we'd like to assert the following. Fix a homology theory E_*. Let X be a simplicial algebra over a simplicial operad T. Then, perhaps under hypotheses on T, we would like to assert there is a simplicial T-algebra Y and a morphism of T-algebras $Y \to X$ so that a.) $|Y| \to |X|$ is a weak equivalence and b.) E_*Y is cofibrant as an E_*T algebra. The device for this construction is an appropriate Stover resolution ([46],[16],[17]) and, particularly, the concise and elegant paper of Bousfield [10].[3] We explain some of the details in this section.

We begin by specifying the building blocks of our resolutions. We fix a spectrum E which is a commutative ring object in the homotopy category of spectra. Let $D(\cdot)$ denote the Spanier-Whitehead duality functor.

1.4.1 Definition. *A homotopy commutative and associative ring spectrum E satisfies* Adams's condition *if E can be written, up to weak equivalence, as a homotopy colimit of a filtered diagram of finite cellular spectra E_α with the properties that*

1. *E_*DE_α is projective as an E_*-module; and*
2. *for every module spectrum M over E the Künneth map*

$$[DE_\alpha, M] \longrightarrow \mathrm{Hom}_{E_*}(E_*DE_\alpha, M_*)$$

is an isomorphism.

This is the condition Adams (following Atiyah) wrote down in [1] to guarantee that the (co-)homology theory over E has Künneth spectral sequences. If M is a module spectrum over E, then so is every suspension or desuspension of M; therefore, one could replace the source and target of the map in part 2.) of this definition by the corresponding graded objects.

Many spectra of interest satisfy this condition; for example, if E is the spectrum for a Landweber exact homology theory, it holds. (This is implicit in [1], and made explicit in [39].) In fact, the result for Landweber exact theories follows easily from the example of MU, which, in turn, was Atiyah's original example. See [2]. Some spectra do not satisfy this condition, however – the easiest example is $H\mathbb{Z}$.

We want to use the spectra DE_α as detecting objects for a homotopy theory, but first we enlarge the scope a bit.

1.4.2 Definition. *Define $\mathcal{P}(E) = \mathcal{P}$ to be a set of finite cellular spectra so that*

1. *the spectrum $S^0 \in \mathcal{P}$ and E_*X is projective as an E_*-module for all $X \in \mathcal{P}$;*
2. *for each α there is finite cellular spectrum weakly equivalent to DE_α in $\mathcal{P}$;*

---

[3] Bousfield's paper is written cosimplicially, but the arguments are so categorical and so clean that they easily produce the simplicial objects we require.

3. $\mathcal{P}$ is closed under suspension and desuspension;

4. $\mathcal{P}$ is closed under finite coproducts (i.e, wedges); and

5. for all $X \in \mathcal{P}$ and all E-module spectra M the Künneth map

$$[X, M] \longrightarrow \mathrm{Hom}_{E_*}(E_*X, M_*)$$

is an isomorphism.

The E_2 or *resolution* model category which we now describe uses the set $\mathcal{P}$ to build cofibrations in simplicial spectra and, hence, some sort of projective resolutions.

Because the category of spectra has all limits and colimits, the category of simplicial spectra is a simplicial category in the sense of Quillen using external constructions as in §II.4 of [35]. However, the Reedy model category structure on simplicial spectra is not a simplicial model category using the external simplicial structure; for example, if $i : X \to Y$ is a Reedy cofibration and $j : K \to L$ is a cofibration of simplicial sets, then

$$i \otimes j : X \otimes L \sqcup_{X \otimes K} Y \otimes K \to Y \otimes L$$

is a Reedy cofibration, it is a level-wise weak equivalence if i is, but it is not necessarily a level-wise weak equivalence if j is.

The following ideas are straight out of Bousfield's paper [10].

1.4.3 Definition. *Let $\mathbf{Ho}(\mathcal{S})$ denote the stable homotopy category.*

1.) A morphism $p : X \to Y$ in $\mathbf{Ho}(\mathcal{S})$ is $\mathcal{P}$-epi if $p_* : [P, X] \to [P, Y]$ is onto for each $P \in \mathcal{P}$.

2.) An object $A \in \mathbf{Ho}(\mathcal{S})$ is $\mathcal{P}$-projective if

$$p_* : [A, X] \longrightarrow [A, Y]$$

is onto for all $\mathcal{P}$-epi maps.

3.) A morphism $A \to B$ of spectra is called $\mathcal{P}$-projective cofibration if it has the left lifting property for all $\mathcal{P}$-epi fibrations in $\mathcal{S}$.

The classes of $\mathcal{P}$-epi maps and of $\mathcal{P}$-projective objects determine each other; furthermore, every object in $\mathcal{P}$ is $\mathcal{P}$-projective. Note however, that the class of $\mathcal{P}$-projectives is closed under arbitrary wedges. The class of $\mathcal{P}$-projective cofibrations will be characterized below; see Lemma 1.4.7.

1.4.4 Lemma. *1.) The category $\mathbf{Ho}(\mathcal{S})$ has enough $\mathcal{P}$-projectives; that is, for every object $X \in \mathbf{Ho}(\mathcal{S})$ there is a $\mathcal{P}$-epi $Y \to X$ with Y a $\mathcal{P}$-projective.*

2.) Let X be a $\mathcal{P}$-projective object. Then E_*X is a projective E_*-module, and the Künneth map

$$[X, M] \longrightarrow \mathrm{Hom}_{E_*}(E_*X, M_*)$$

is an isomorphism for all E-module spectra M.

Proof. For part 1.) we can simply take

$$Y = \coprod_{P \in \mathcal{P}} \coprod_{f : P \to X} P$$

where f ranges over all maps $P \to X$ in $\mathbf{Ho}(\mathcal{S})$. Then, for part 2.), we note that the evaluation map $Y \to X$ has a homotopy section if X is $\mathcal{P}$-projective. Then the result follows from the properties of the elements of $\mathcal{P}$. □

We can now specify the $\mathcal{P}$-resolution model category structure. Recall that a morphism $f : A \to B$ of simplicial abelian groups is a weak equivalence if $f_* : \pi_* A \to \pi_* B$ is an isomorphism. Also $f : A \to B$ is a fibration if the induced map of normalized chain complexes $Nf : NA \to NB$ is surjective in positive degrees. The same definitions apply to simplicial R-modules or even graded simplicial R-modules over a graded ring R. A morphism is a cofibration if it is injective with level-wise projective cokernel.

1.4.5 Definition. *Let $f : X \to Y$ be a morphism of simplicial spectra. Then*

1.) *the map f is a $\mathcal{P}$-equivalence if the induced morphism*

$$f_* : [P, X] \longrightarrow [P, Y]$$

is a weak equivalence of simplicial abelian groups for all $P \in \mathcal{P}$;

2.) *the map f is a $\mathcal{P}$-fibration if it is a Reedy fibration and $f_* : [P, X] \longrightarrow [P, Y]$ is a fibration of simplicial abelian groups for all $P \in \mathcal{P}$;*

3.) *the map f is a $\mathcal{P}$-cofibration if the induced maps*

$$X_n \sqcup_{L_n X} L_n Y \longrightarrow Y_n, \qquad n \geq 0,$$

are $\mathcal{P}$-projective cofibrations.

Then, of course, the theorem is as follows.

1.4.6 Theorem. *With these definitions of $\mathcal{P}$-equivalence, $\mathcal{P}$-fibration, and $\mathcal{P}$-fibration, the category $s\mathcal{S}$ becomes a simplicial model category.*

The proof is given in [10]. We call this the $\mathcal{P}$-resolution model category structure. It is cofibrantly generated; furthermore there are sets of generating cofibrations and generating acyclic cofibrations with cofibrant source. An object is $\mathcal{P}$-fibrant if and only if it is Reedy fibrant. We will see below, in Theorem 1.4.9 – using the case where T is the identity operad – that this model category structure on $s\mathcal{S}$ is, in fact, cellular.

The next result gives a characterization of $\mathcal{P}$-cofibrations.

Call a morphism $X \to Y$ of spectra $\mathcal{P}$-*free* if it can be written as a composition

$$X \xrightarrow{i} X \amalg F \xrightarrow{q} Y$$

where i is the inclusion of the summand, F is cofibrant and $\mathcal{P}$-projective, and q is an acyclic cofibration. The following is also in [10]. Another characterization of cofibrations can be obtained from the Lemma 1.4.10, which displays a set of generating cofibrations.

1.4.7 Lemma. *A morphism $X \to Y$ of spectra is a $\mathcal{P}$-projective cofibration if and only if it is a retract of $\mathcal{P}$-free map.*

1.4.8 Remark. At this point we can explain one of the reasons for using the models $\mathcal{P}$ to define the resolution model category. Suppose $X \to Y$ is a weak equivalence between cofibrant objects in the $\mathcal{P}$-resolution model category. Then for each of the spectra DE_α we have an isomorphism

$$f_* : \pi_p[\Sigma^q DE_\alpha, X] \xrightarrow{\cong} \pi_p[\Sigma^q DE_\alpha, Y].$$

However, if $E_*(-)$ is our chosen homology theory

$$\begin{aligned}\pi_p E_q X &\cong \operatorname{colim}_\alpha \pi_p (E_\alpha)_q X \\ &\cong \operatorname{colim}_\alpha \pi_p[\Sigma^q DE_\alpha, X].\end{aligned}$$

In particular, if $X \to Y$ is a $\mathcal{P}$-equivalence of simplicial spectra, then

$$E_* X \longrightarrow E_* Y$$

is a weak equivalence of simplicial E_*-modules. Also note that if $X \to Y$ is a $\mathcal{P}$-cofibration, then $E_* X \to E_* Y$ is a cofibration of simplicial E_* modules. This follows from Lemma 1.4.7.

For a Reedy cofibrant simplicial spectrum X or, more generally a *proper*[4] simplicial object X, there is a spectral sequence

$$\pi_p E_q X \Longrightarrow E_{p+q}|X|. \tag{1.4.1}$$

This is, of course, the standard homology spectral sequence of a simplicial spectrum. If $X \to Y$ is an $\mathcal{P}$-equivalence of Reedy cofibrant simplicial spectra, then we get isomorphic E_* homology spectral sequences.

The $\mathcal{P}$-resolution model category structure can be promoted to a model category for simplicial algebras over a simplicial operad. Fix a simplicial operad T and let $s\mathbf{Alg}_T$ be the category of algebras over T. This category has an external simplicial structure; indeed, if K is a simplicial set and $X \in s\mathbf{Alg}_T$, one has

$$(X \otimes K)_n = \coprod_{K_n}{}^{T_n} X_n. \tag{1.4.2}$$

The superscript T_n is indicates that the coproduct is taken in the category of T_n algebras. The simplicial set of maps is defined again by

$$[n] \mapsto \operatorname{Hom}_{s\mathbf{Alg}_T}(X \otimes \Delta^n, Y).$$

We say that a morphism $X \to Y$ of simplicial T-algebras is a $\mathcal{P}$-fibration or $\mathcal{P}$-equivalence if the underlying morphism of simplicial spectra is. Then we have the *$\mathcal{P}$-resolution model category structure* on $s\mathbf{Alg}_T$. We will discuss cofibrations below when we have more hypthoses.

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[4] An object is proper if the inclusions of the latching objects $L_n X \to X_n$ are Hurewicz cofibrations.

1.4.9 Theorem. *With these definitions, the category $s\mathbf{Alg}_T$ becomes a simplicial cellular model category.*

Proof. The existence of the simplicial model category structure is the standard lifting argument. (See [21] §II.2 for the case of simplicial model categories, or [23] §11.3. for a more general statement.) Since $s\mathbf{Alg}_T$ is a simplicial category, in the sense of Quillen, the category$s\mathbf{Alg}_T$ has a functorial path object. Since the forgetful functor to $s\mathcal{S}$ creates filtered colimits in $s\mathbf{Alg}_T$, we need only supply a $\mathcal{P}$-fibrant replacement functor for $s\mathbf{Alg}_T$. However, every Reedy fibrant object in $s\mathbf{Alg}_T$ will be $\mathcal{P}$-fibrant, and the $s\mathbf{Alg}_T$ in its Reedy model category structure is cofibrantly generated, so we can choose a Reedy fibrant replacement functor. This will do the job. Note that this model category is cofibrantly generated, again by the standard lifting arguments.

To get that the model category is cellular, first note that since every Reedy weak equivalence is $\mathcal{P}$-equivalence and every Reedy acyclic fibration is a $\mathcal{P}$-acyclic fibration, every $\mathcal{P}$-cofibration will be Reedy cofibration, and hence a space-wise closed inclusion, by Theorem 1.3.4. Since $s\mathcal{S}$, in its $\mathcal{P}$-resolution model category structure has a set of generating cofibration $A \to B$ with cofibrant source, so does $s\mathbf{Alg}_T$; indeed, the generators will be of the form $T(A) \to T(B)$. To complete the argument, we apply Remark 1.1.5. □

We now give a set of generating cofibrations for $s\mathbf{Alg}_T$. This will be important when discussing the size of cell complexes in localization arguments. Recall that we have fixed our set $\mathcal{P}(E) = \mathcal{P}$ of projectives: see 1.4.2.

1.4.10 Lemma. *Fix a set of J of generating acyclic cofibrations for $\mathcal{S}$. The $\mathcal{P}$-model category structure on $s\mathbf{Alg}_T$ has, as a set I of generating cofibrations, the morphisms*

$$T(A_j \otimes \Delta^n \amalg_{A_j \otimes \partial\Delta^n} \otimes B_j \otimes \partial\Delta^n) \to T(B_j \otimes \Delta^n)$$

where $A_j \to B_j$ is a morphism in J and the morphisms

$$T(P \otimes \partial\Delta^n) \to T(P \otimes \Delta^n)$$

where $P \in \mathcal{P}$.

Proof. A morphism $X \to Y$ is an acyclic fibration if and only if it is a Reedy fibration and (by virtue of the spiral exact sequence, Theorem 3.1.4) the induced morphism of *underived* mapping spaces

$$s\mathbf{Alg}_T(T(P), X) \longrightarrow s\mathbf{Alg}_T(T(P), Y)$$

is an acyclic fibration of simplicial sets. The result follows by an adjointness argument. □

1.4.11 Proposition. *For each $X \in s\mathbf{Alg}_T$ there is a natural $\mathcal{P}$-equivalence*

$$P_T(X) \to X$$

so that

1.) $P_T(X)$ is cofibrant in the $\mathcal{P}$-resolution model category structure on $s\mathbf{Alg}_T$;

2.) the underlying degeneracy diagram of $P_T(X)$ is of the form $T(Z)$ where Z is free as a degeneracy diagram and each Z_n is a wedge of elements of $\mathcal{P}$.

Proof. The object $P_T(X)$ is produced by taking an appropriate subdivision (for example the big subdivision of [9] §XII.3, Example 3.4) of a cofibrant model for X. □

The following result has content because it is not at all obvious that a $\mathcal{P}$-cofibrant algebra in $s\mathbf{Alg}_T$ is Reedy cofibrant when regarded as a spectrum.

1.4.12 Corollary. *Suppose that T is a simplicial operad. Let X be a $\mathcal{P}$-cofibrant simplicial T-algebra in $s\mathbf{Alg}_T$. Then for any homology theory E_*, there is strongly convergent first quadrant spectral sequence*

$$\pi_p E_q X \Longrightarrow E_{p+q}|X|.$$

Proof. We may assume that X is of the form stipulated by Proposition 1.4.11. Then we claim that X is, in fact, Reedy cofibrant when regarded as a simplicial spectrum. This is routine, if tedious, and we leave the details to the reader. There are two key observations. First, if T is a Reedy cofibrant operad, then for each n, the bisimplicial set $T(n)$ is Reedy cofibrant. This is because all bisimplicial sets are Reedy cofibrant. Second, if C is any operad and Z_1 and Z_2 are spectra, then there is a decomposition formula

$$C(Z_1 \amalg Z_2) \cong \amalg\, C(n+m) \otimes_{\Sigma_m \times \Sigma_n} Z_1^{(m)} \wedge Z_2^{(n)}.$$

□

To make constructive use of the $\mathcal{P}$-resolution model category structure on $s\mathbf{Alg}_T$, we impose a further condition.

1.4.13 Definition. *An operad C is* adapted *to E_* if there is a triple C_E on E_*-modules so that*

1. *if X is a C-algebra in spectra, then E_*X is naturally a C_E-algebra in E_*-modules;*
2. *if Z is a cofibrant spectrum such that E_*Z is projective as an E_*-module, then the natural map of C_E-algebras*

   $$C_E(E_*Z) \longrightarrow E_*C(Z)$$

   is an isomorphism.

There is a simplicial version, also: a simplicial operad T is adapted to E_* if there is a triple T_E on simplicial E_*-modules so that

3. *if X is a simplicial T-algebra in spectra, then E_*X is naturally a T_E-algebra in simplicial E_*-modules;*

4. *if Z is a Reedy cofibrant spectrum such that E_*Z is a cofibrant simplicial E_*-module, then the natural map of T_E-algebras*

$$T_E(E_*Z) \longrightarrow E_*T(Z)$$

is an isomorphism.

Here are some basic examples. There are more below in Remark 1.4.17.

1.4.14 Example. 1.) If C is an operad adapted to E, then the C, regarded as a constant simplicial operad, is adapted as a simplicial operad to E.

2.) By the results of section 2.2 below, any E_∞-operad is adapted to p-complete K-theory.

3.) If C is any operad so that $\pi_0 C(k)$ is a free Σ_k set for all k, then C is adapted to any Adams-type homology theory. This means, specifically, that any A_∞-operad is adapted to E. More generally, if T is a simplicial operad so that for all k and n, the set $\pi_0 T_n(k)$ is a free Σ_k-set, then T is adapted as a simplicial operad to E.

In the following result, we make a cardinality statement about relative cell complexes. The generating set I of cofibrations is that of Lemma 1.4.10.

1.4.15 Lemma. *Suppose T is a simplicial operad adapted to E and suppose $f : X \to Y$ is a cofibration with cofibrant source in $s\mathbf{Alg}_T$ with its $\mathcal{P}$-resolution model category structure. The f is a retract of a morphism $g : X \to Z$ with the following property:*

(∗) *The underlying morphism of degeneracy diagrams for E_*g is isomorphic to a morphism of the form*

$$E_*X \xrightarrow{i} E_*X \amalg T_E(M)$$

where M is s-free on a projective E_*-module.

Furthermore, g has a presentation as a relative I-cell complex with γ cells, then M has a set of generators as an E_*-module of cardinality γ.

Proof. All acyclic cofibrations in spectra have a strong deformation retraction. This follows from Theorem 14.1 (see also Theorem 6.5) of [33]. This implies that if we have push-out diagram in simplicial T-algebras of the form

$$\begin{tikzcd}
T(A_j \otimes \Delta^n \amalg_{A_j \otimes \partial\Delta^n} \otimes B_j \otimes \partial\Delta^n) \arrow[r] \arrow[d] & X \arrow[d] \\
T(B_j \otimes \Delta^n) \arrow[r] & Y
\end{tikzcd}$$

then, at every simplicial level k, we have that $X_k \to Y_k$ is a homotopy equivalence. In particular $E_*X_k \cong E_*Y_k$. On the other hand, if we have push-out diagram of the form

$$\begin{tikzcd}
T(P \otimes \partial\Delta^n) \arrow[r] \arrow[d] & X \arrow[d] \\
T(P \otimes \Delta^n) \arrow[r] & Y
\end{tikzcd}$$

then, at every simplicial level k, we have that $Y_k \cong X_k \amalg T_k(\amalg_{I_k} P)$ for some finite indexing set I_k and this decomposition respects the degeneracies.

If $f : X \to Y$ is any cofibration with cofibrant source, then f is a retract of a cofibration $g : X \to Z$ built by the small object argument from the generating cofibrations. This, in turn, is a retract of a cofibration $g' : X' \to Z'$ so that X' is built by the small object argument from the initial object S and Z' is built from X' by the small object argument. The conclusion $(*)$ for $S \to X'$ and hence for g' by observations of the previous paragraph. Then $(*)$ holds for g because it is a retract of g'. □

To go further we have to assume that our the category of T_E-algebras has good homotopical behavior. This is encoded in the following definition.

1.4.16 Definition. *Let E_* be a homology theory so that E_*E is flat over E_* and let T be a simplicial operad adapted to E. Then we will say that T is* homotopically adapted *to E if:*

1. *the triple T_E on simplicial E_*-modules lifts to a triple on E_*E-comodules;*
2. *the category of simplicial T_E-algebras in E_*-modules supports the structure of a simplicial model category where a morphism is a weak equivalence or fibration if and only if it is so as as a simplicial E_*-module;*
3. *the category of simplicial T_E-algebras in the category of E_*E-comodules supports the structure of a simplicial model category such that the forgetful functor to T_E-algebras in E_*-modules creates weak equivalences and preserves fibrations.*

1.4.17 Remark. This definition is rather complicated; however, our three main examples will all produce homotopically adapted operads. But let us first say that what is needed in the next section is only part (2) of this definition. The rest becomes crucial later.

1. If E_* is any Adams-type homology theory with E_*E flat over E_*, then the associative monoid operad is homotopically adapted to E_*. Then T_E will be the simplicial associative algebra triple. The necessary model category structure on simplicial associative E_*-algebras is the one supplied by Quillen in [35]§II.4 and the model category structure on simplicial associative algebras in E_*E-comodules appeared in [19]. (See the beginning of section 2.5 for a more thorough review of the comodule case.)

2. Again let E_* be any Adams-type homology theory with E_*E flat over E_*. Let C be an E_∞ operad in the category of simplicial sets; thus $C(k)$ is contractible and has a free Σ_k-action. Then let T be the resulting simplicial operad obtained by running C out in the external[5] simplicial direction. Then $|T| \cong C$ and E_*T is an E_∞-operad in E_*-modules. (See Definition 2.3.8) and T_E is the free simplicial E_∞-algebra triple. Again the necessary model category structure on E_∞-algebras is the one supplied by Quillen in [35]§II.4 and the model category structure on E_∞-algebras in E_*E-comodules appeared in [19].

3. Let K_* be p-completed K-theory, and T the commutative monoid operad, so that T-algebras are simplical commutative S-algebras. Then T_E is the free theta-algebra functor. The details of this example, including the fact T is homotopically adapted to K_* appear in section 2.3.

The following result is an immediate consequence of Lemma 1.4.15 given Quillen's characterization ([35]§II.4) of cofibrations as retracts of "free" maps.

1.4.18 Corollary. *Suppose the simplicial operad T is homotopically adapted to E. Then the functor*

$$E_* : s\mathbf{Alg}_T \longrightarrow s\mathbf{Alg}_{T_E}$$

sends weak equivalences to weak equivalences and cofibrations with cofibrant source to cofibrations.

1.4.19 Example. Suppose we fix an operad $C \in \mathcal{O}$ and a simplicial resolution $T \to C$ of C as in Theorem 1.3.5. If X is an C-algebra, then X can be regarded as a constant object in $s\mathbf{Alg}_T$ and, hence, we have the resolution $P_T(X) \to X$ of Proposition 1.4.11. Then $P_T(X)$ is $\mathcal{P}$-cofibrant in $s\mathbf{Alg}_T$. Since Remark 1.4.8 implies that the augmentation $\pi_*E_*P_T(X) \to E_*X$ is an isomorphism, the previous result and Example 1.4.14.3 imply that $E_*P_T(X)$ is a cofibrant replacement for E_*X in simplicial E_*T-algebras. (Here we are using the model category structure on simplicial E_*-algebras of [35]§II.4.) Furthermore we can use the E_* homology spectral sequence of Corollary 1.4.12 to conclude

$$\pi_*E_*|P_T(X)| \cong E_*X.$$

1.5 Localization of the resolution model category

In the previous section, we developed the resolution model category of spectra, or simplicial T-algebras, based on some set of projectives $\mathcal{P}$. In particular, we were interested in the set $\mathcal{P} = \mathcal{P}(E)$ arising from an Adams-type homology theory, as in Definition 1.4.2. This resolution model category has the type of cofibrant objects we'd like, but – as the reader may have surmised – we are not

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[5] See Remark 1.3.1 for the meaning of "external".

primarily interested in the $\mathcal{P}$-equivalence classes of objects in simplicial spectra or simplicial T-algebras, but in certain types of E_*-equivalences. There does not appear to be a model category with these cofibrations and weak equivalences; therefore, we settle for a semi-model category, as in the next result. It is a localization of the one supplied in Theorem 1.4.9.

The material of this section developed out of some conversations with Phil Hirschhorn.

The rest of this section will be devoted to proving the following result. The notion of semi-model category was discussed in Section 1.1, and the definition of what it means for an operad to be homotopically adapted to a homology theory is in the Definitions 1.4.13 and 1.4.16.

1.5.1 Theorem. *Suppose that T is a simplicial operad homotopically adapted to the homology theory E. Then the category $s\mathbf{Alg}_T$ supports the structure of a cofibrantly generated simplicial semi-model category so that*

1.) *a morphism $f : X \to Y$ is an E_*-equivalence if*

$$\pi_* E_*(f) : \pi_* E_* X \longrightarrow \pi_* E_* Y$$

is an isomorphism;

2.) *a morphism is an E_*-cofibration if it is a $\mathcal{P}$-cofibration; and*

3.) *a morphism is an E_*-fibration if it has the right lifting property with respect to all morphisms which are at once an E_*-equivalence and an E_*-cofibration.*

Since, by Remark 1.4.8, every $\mathcal{P}$-equivalence in $s\mathbf{Alg}_T$ is an E_*-equivalence, this semi-model category structure can be produced using the localization technology of Bousfield, et al., with variations which have previously been confronted in [18], §VIII.1. There are many minute details, and we vary somewhat from the canonical path – as mapped out in [23] – but the route is familiar.

To begin, let E_* be our chosen Adams-type homology theory, and let $\mathbf{Ch}_{E_*}$ denote the category of non-negatively graded chain complexes over E_*. Then we have a functor

$$h_E \overset{\text{def}}{=} NE_*(-) : s\mathcal{S} \longrightarrow \mathbf{Ch}_{E_*}$$

given sending a simplicial spectrum X to the normalized complex $NE_*(X)$. Note that we have the $H_* h_E(X) = \pi_* E_* X$. The following is obvious, and included only to ground the argument.

1.5.2 Lemma. *The functor $h_E : s\mathcal{S} \longrightarrow \mathbf{Ch}_{E_*}$ has the following properties:*

i.) *If $X \to Y$ is a $\mathcal{P}$-equivalence, then $h_E(X) \to h_E(Y)$ is a homology isomorphism, and if $*$ is the initial object then $h_E(*) = 0$.*

ii.) *If $i \mapsto X_i$ is a filtered diagram of Reedy cofibrant objects, then*

$$\operatorname{colim} h_E(X_i) \to h_E(\operatorname{colim} X_i)$$

is an isomorphism.

iii.) If $A \to B$ is a $\mathcal{P}$-cofibration, then $h_E(A) \to h_E(B)$ is an injection. If $A \to X$ is any other map, then the resulting diagram

$$\begin{tikzcd}
h_E(A) \arrow[r] \arrow[d] & h_E(X) \arrow[d] \\
h_E(B) \arrow[r] & h_E(B \sqcup_A X)
\end{tikzcd}$$

is a push-out square.

As a remark on this result, we note that items ii.) and iii.) together imply that if $\{X_\alpha\}$ is any set of $\mathcal{P}$-cofibrant objects in $s\mathcal{S}$, then the evident map

$$\oplus_\alpha h_E(X_\alpha) \longrightarrow h_E(\coprod X_\alpha)$$

is an isomorphism. Note also that the hypothesis on the initial object is redundant; the empty diagram is filtered, so ii.) implies $h_E(*) = 0$.

The functor $\pi_* E_*(-)$ has some of the usual properties of a homology functor. For example, if $A \to B$ is a $\mathcal{P}$-cofibration with cofibrant source, we can define

$$\pi_* E_*(B, A) = H_*(h_E(B)/h_E(A))$$

and we have a long exact sequence of a pair, by Definition 1.5.2.iii. The same item also yields a Mayer-Vietoris sequence.

We now begin to set up the localization argument. In order for this to work, we need to know that intersections of subcomplexes exist. Here are the details.

Suppose we are given some category $\mathcal{C}$ and a set of maps I in $\mathcal{C}$. Then, in Definitions 1.1.1 and 1.1.2 we wrote down the definition of I-cell complexes and subcomplexes. Given two such subcomplexes $K, L \subseteq X$, we would like to define $K \cap L$ with the property that

$$T_\beta^{K \cap L} = T_\beta^K \cap T_\beta^L.$$

(This is called the combinatorial intersection in [18] §III.2 and simply the intersection in [23]). The difficulty is to show that $(K \cap L)_{(-)} : \lambda \to \mathcal{C}$ exists. Using transfinite induction, we can assume $(K \cap L)_\beta$ exists and to define $(K \cap L)_{\beta+1}$ we need to be able to complete the following diagram for every element of $T_\beta^K \cap T_\beta^L$:

(1.5.1)

$$\begin{tikzcd}
A \arrow[dr, dashed] \arrow[drr] \arrow[ddr] & & \\
& (K \cap L)_\beta \arrow[r] \arrow[d] & L_\beta \arrow[d] \\
& K_\beta \arrow[r] & X_\beta .
\end{tikzcd}$$

We will say that *intersections of subcomplexes exist* if for some set I of generating cofibrations of $\mathcal{C}$ we can solve this problem and produce $K \cap L$. The reason we went to all the trouble to specify that our various categories of simplicial spectra were cellular model categories was so that we could apply the following result.

1.5.3 Lemma. *Let $\mathcal{C}$ be a cellular model category. Then intersections of subcomplexes exist.*

Proof. See [23] §14.2. The proof is straightforward: the effective monomorphism condition and a diagram chase shows that the square of diagram 1.5.1 is a pullback diagram. ☐

To prove Theorem 1.5.1, we use a standard technique for constructing cofibrantly generated model categories: Theorem 2.1.19 of [25] (but see also the identical Theorem 13.4.1 of [23] which credits this result to Dan Kan). The exact statement will be incorporated in the proof below, but one begins by specifying a class of weak equivalences and sets of maps I and J which will generate the cofibrations and acyclic cofibrations respectively. Then one has to show these maps satisfy certain properties. In this case the class of weak equivalences will be the $\pi_*E_*(-)$ isomorphisms and, since $s\mathbf{Alg}_T$ (in the $\mathcal{P}$-resolution model category structure) is already cofibrantly generated, I will be a generating set for the cofibrations. The issue is to supply J, and for this we use an analog of the Bousfield-Smith argument (cf. [23] §4.5). This comes down to a cardinality argument, so we begin by spending a paragraph or so to specify some cardinals.

We choose, as our generating set $I \overset{\text{def}}{=} I_T$ of cofibrations of $s\mathbf{Alg}_T$, in the $\mathcal{P}$-resolution model category structure, the morphisms of Lemma 1.4.10. These are all of the form

$$T(A) \longrightarrow T(B)$$

where $A \to B$ are generating cofibrations for $s\mathcal{S}$ in its $\mathcal{P}$-resolution model category structure. By the properties of a cofibrantly generated model category (see Definition 2.1.3 of [25]), there is a cardinal number κ so that the domain of every morphism of I_T is κ-small relative to the class of cofibrations. This is the first cardinal we need.

We first record the following result. This is where the effective monomorphism condition on cofibrations in cellular model categories arises.

1.5.4 Lemma. *1.) Every I_T-cell of a relative I_T complex in $s\mathbf{Alg}_T$ is contained in a relative sub-I-cell complex of size at most κ.*

2.) Every I_T-complex of $s\mathbf{Alg}_T$ is the filtered colimit of its subcomplexes of size at most κ.

Proof. The first statement is Lemma 13.5.8 of [23]. The second statement follows from the first. ☐

The second cardinal we need is supplied by the following result.

We will assume for the rest of the section that we are working with a simplicial operad T homotopically adapted to E.

1.5.5 Lemma. *There is a cardinal η so that if X is I_T-cell complex of size γ in $s\mathbf{Alg}_T$, then $\pi_*E_*(X)$ has at most $\eta\gamma$ elements.*

Proof. By Lemma 1.4.15 the underlying degeneracy diagram of X has the property that

$$E_*X \cong T_E(M)$$

where M is s-free on a graded projective E_*-module with generating set of cardinality γ. Furthermore, the triple T_E has the property that

$$T_E(E_*Z) \xrightarrow{\cong} E_*T(Z)$$

whenever Z is Reedy cofibrant and E_*Z is level-wise projective. We use these two formulas to bound the cardinality of E_*X.

Since M is level-wise projective, it is a retract of a degeneracy diagram F of free E_*-modules with a generating set of the same cardinality γ. Thus we may assume M is actually s-free on a graded free E_*-modules. By fixing a set of generators we obtain an isomorphism of degeneracy diagrams $M \cong E_*Z$ where Z is itself s-free on a graded spectrum which is a wedge of spheres in each degree. Furthermore the cardinality of that set of spheres is γ. Thus we need only bound the cardinality of $E_*T(Z)$.

If U is a graded simplicial set, we denote the $\mathrm{card}(U)$ to be the cardinality of the union of all the sets that make up U. Since we are only trying to find a bound, we will assume all cardinals are infinite.

For any operad C in simplicial sets and any spectrum W with E_*W free as an E_*-module there is a first quadrant spectral sequence

$$H_*(\Sigma_k, E_*(C(k)) \otimes (E_*W)^{\otimes k}) \Longrightarrow E_*(C(k) \otimes_{\Sigma_k} W^{(k)}).$$

From this it follows that

$$\mathrm{card}[E_*(C(k) \otimes_{\Sigma_k} W^{(k)})] \leq \mathrm{card}(E_*(C(k)) \cdot \mathrm{card}(E_*W).$$

Thus, for our simplicial operad T and our chosen simplicial spectrum Z, we have

$$\mathrm{card}(E_*T(Z)) \leq \mathrm{card}(E_*T) \cdot \mathrm{card}(E_*Z).$$

But $\mathrm{card}(E_*Z) \leq \mathrm{card}(E_*) \cdot \gamma$. Thus we may take $\eta \geq \mathrm{card}(E_*(T)) \cdot \mathrm{card}(E_*)$. ☐

Now let ν be any infinite cardinal greater than $\eta\kappa$. Note that ν depends only on I_T, $E_*(-)$, and T. Here is our variant of Bousfield's key lemma. See Lemma X.3.5 of [21].

1.5.6 Lemma. *If $X \to Y$ is an inclusion of I_T-cell complexes in $s\mathbf{Alg}_T$ such that $\pi_*E_*(Y, X) = 0$, then there exists a subcomplex $D \subseteq Y$ satisfying the following conditions:*

1.) D is of size less than ν;

2.) D is not in X; and,

3.) $h_*(D, D \cap X) = 0$.

Proof. This argument is by now classic, and we won't repeat it. Bousfield's original argument goes through verbatim, using the existence of intersections of subcomplexes. See [23]. ☐

This immediately allows one to prove the following result:

1.5.7 Lemma. *Suppose q is a morphism in $s\mathbf{Alg}_T$ with the right lifting property with respect to any inclusion $j : A \to B$ of I_T-cell complexes with B of size at most ν and $\pi_* E_*(j)$ an isomorphism. Then q has the right lifting property with respect to any inclusion of I_T-cell complexes which is a $\pi_* E_*(-)$ isomorphism.*

Proof. This is a Zorn's lemma argument, and also classic. See [23], Lemma 2.4.8 or Lemma X.2.14 of [21]. ☐

Now let J_T be a set of representatives for the isomorphism classes of inclusions $A \to B$ of I_T-cell complexes with B of size at most ν and which induce an isomorphism on $\pi_* E_*(-)$. Recall that a J_T-cofibration in $s\mathbf{Alg}_T$ is a morphism in the class of maps containing J_T and closed under retract, coproduct, cobase change, and sequential colimits.

1.5.8 Lemma. *Suppose that $A \to B$ is a $\mathcal{P}$-cofibration with $\mathcal{P}$-cofibrant source in $s\mathbf{Alg}_T$ and a $\pi_* E_*(-)$-isomorphism. Then $A \to B$ is a J_T-cofibration.*

Proof. Recall (from [25]) that a *J_T-injective* is any morphism with the right lifting property with respect to all the elements of J_T. Suppose, for a moment, that we can show that $A \to B$ has the left lifting property with respect to all J_T-injectives. Then, using the small object argument, we can factor $A \to B$ as

$$\begin{tikzcd} A \arrow[r, "j"] & E \arrow[r, "p"] & B \end{tikzcd}$$

where j is a J_T-cofibration and p is a J_T-injective. A standard argument now shows $A \to B$ is a retract of j, which is all that is required.

We now must show that $A \to B$ has the left lifting property with respect to all J_T-injectives.

We start by choosing a cellular approximation $\tilde{A} \to \tilde{B}$ to $A \to B$. Thus, $\tilde{A} \to \tilde{B}$ is an inclusion of I_T-cell complexes and there is a commutative square

$$\begin{tikzcd} \tilde{A} \arrow[r] \arrow[d] & A \arrow[d] \\ \tilde{B} \arrow[r] & B \end{tikzcd}$$

with the horizontal maps weak equivalences. Note that $\tilde{A} \to \tilde{B}$ is a $\pi_* E_*(-)$ isomorphism. Now consider a lifting problem

$$\begin{tikzcd} \tilde{A} \arrow[r] \arrow[d] & A \arrow[r] \arrow[d] & X \arrow[d, "q"] \\ \tilde{B} \arrow[r] & B \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

where q is a J-injective. By the previous lemma, we can produce a map $\tilde{B} \to X$ solving the outer lifting problem, hence a map $\tilde{B} \sqcup_{\tilde{A}} A \to X$ solving the lifting problem under A. Since A is cofibrant, the induced map $\tilde{B} \sqcup_{\tilde{A}} A \longrightarrow B$ is a homotopy equivalence; hence we have a weak equivalence between cofibrant objects in the category of objects under A and over Y. Also, $q : X \to Y$ is a fibrant object in the same category, since any J_T-injective is a fibration. The original lifting problem is then solved by the following standard fact about model categories: if $C \to C'$ is a weak equivalence between cofibrant objects and $C \to E$ is a morphism to a fibrant object, then there is a morphism $C' \to E$ so that the composite $C \to C' \to E$ is homotopic to the original map. □

1.5.9 Remark. The model category $s\mathbf{Alg}_T$ is hardly ever left proper. If it were, we could immediately conclude that the map

$$\tilde{B} \sqcup_{\tilde{A}} A \longrightarrow B$$

was a weak equivalence for any A and, thus, drop the hypothesis that A be cofibrant. Then we would obtain a model category, rather than a semi-model category in Theorem 1.5.1. This will happen, for example, in the case of the identity operad; that is, when $s\mathbf{Alg}_T = s\mathcal{S}$.

Our final technical lemma is a closure property for $\pi_* E_*(-)$-equivalences.

1.5.10 Lemma. *Every J_T-cofibration with cofibrant source is an I_T-cofibration and a $\pi_* E_*(-)$-equivalence.*

Proof. Since every morphism in J_T is an I_T-cofibration, every J_T-cofibration is an I_T cofibration. So we must prove that every J_T-cofibration is a $\pi_* E(-)$-equivalence. It is sufficient to show that

1. an arbitrary coproduct of elements of J_T is a $\pi_* E_*(-)$-equivalence; and
2. if $X \longleftarrow A \xrightarrow{j} B$ is a two-source of T-algebras with A and X cofibrant and j a cofibration and a $\pi_* E_*(-)$-equivalence, then $X \to X \amalg_A B$ is a $\pi_* E_*(-)$ equivalence.

Then, since $\pi_* E_*(-)$ commutes with filtered colimits, the result will follow.

For (1), let $A \to B$ be a morphism in J_T. Since this is a cofibration with cofibrant source, Lemma 1.4.18 implies that $E_* A \to E_* B$ is a cofibration of T_E-algebras with cofibrant source. It is also, by assumption, a weak equivalence of T_E-algebras. Lemma 1.4.15 and the definition of what it means for an operad to be adapted (Definition 1.4.13) next imply that if $\{A_i \to B_i\}$ is a set of morphisms J_T, then

$$E_*(\amalg_i A) \longrightarrow E_*(\amalg_i B)$$

is isomorphic to

$$\amalg_i E_*(A) \longrightarrow \amalg_i E_*(B)$$

where the coproduct now is in T_E-algebras. Since the acyclic cofibrations are closed under coproduct, we have that $\amalg_i A \to \amalg_i B$ is a $\pi_* E_*(-)$-equivalence.

For (2), we note that the push-out $X \amalg_A B$ is homotopy equivalent to the homotopy push-out, which can be computed as the geometric realization of the bar construction $\mathcal{B}(X, A, B)$. Here $\mathcal{B}(X, A, B)$ is the simplicial T-algebra which, at level n is the coproduct

$$\mathcal{B}(X, A, B)_n = X \amalg \underset{\longleftarrow n \longrightarrow}{A \amalg \cdots \amalg A} \amalg B.$$

The geometric realization is created in spectra and, hence, is isomorphic to the diagonal of the bisimplicial spectrum $\mathcal{B}(X, A, B)$. We conclude that there is a spectral sequence

$$\pi_p \pi_q E_* \mathcal{B}(X, A, B) \Longrightarrow \pi_{p+q} E_*(X \amalg_A B).$$

Here we filter first by the external simplicial degree coming from the bar construction.

To finish, we assert that an argument very similar to that give for (1) implies that the natural map

$$\pi_* E_* \mathcal{B}(X, A, A)_n \longrightarrow \pi_* \mathcal{B}(X, A, B)$$

is an isomorphism. Then the spectral sequence just constructed shows $\pi_* E_* X \to \pi_* E_*(X \amalg_A B)$ is an isomorphism. □

1.5.11 Proof of Theorem 1.5.1. We specify the weak equivalences in $s\mathbf{Alg}_T$ to be the $\pi_* E_*$-isomorphisms. As above, we let I_T be a generating set for the cofibrations and we let J_T a set of representatives for the isomorphism classes of inclusions $A \to B$ of I_T-cell with B of size at most γ and which induce an isomorphism on $\pi_* E_*(-)$. We now must show

- both I_T and J_T permit the small object argument;
- every J_T-cofibration with cofibrant source is both an I_T-cofibration and an $\pi_* E_*(-)$-equivalence;
- every morphism with the right lifting property with respect to I_T has the right lifting property with respect to J_T and is a $\pi_* E_*(-)$-equivalence;
- every map with cofibrant source which is both an I_T-cofibration and a $\pi_* E_*(-)$-equivalence is a J_T-cofibration.

The first statement follows from the assumption that $s\mathbf{Alg}_T$ is cofibrantly generated, the second holds by Lemma 1.5.10, the third holds because $\pi_* E_*(-)$ takes $\mathcal{P}$-weak equivalences to isomorphisms, and the fourth point is Lemma 1.5.8.

Chapter 2

The Algebra of Comodules

2.1 Comodules, algebras, and modules as diagrams

In Section 1.4 we introduced the notion of a homology theory E_* which satisfied a condition developed by Atiyah and Adams. (See Definition 1.4.1.) This condition was the basis for the development of our simplicial resolutions. Now, if E_*E happens to be flat over E_*, then the pair (E_*, E_*E) forms a Hopf algebroid and, for any spectrum X, the module E_*X is a comodule over this Hopf algebroid. The purpose of section is to connect these two notions.

Specifically, we prove a variant of Giraud's Theorem (cf. [4] §6.8) to show that the category of comodules over a Hopf algebroid of Adams type is equivalent to a category of diagrams. In particular, we will embed comodules into a category of contravariant functors (i.e., presheaves) on some indexing category, and show that comodules are exactly those presheaves which satisfy a continuity (or sheaf) condition. We then use this to characterize various algebraic structures in comodules in terms of such structures on presheaves.

This is section is somewhat long, mostly because of a large number of routine – but not completely trivial – lemmas. It is included so we can discuss the kind of algebra and module structure supported by the spiral exact sequence in section 3.1. For this application, the key result is Theorem 2.1.13 and its analog for algebras and modules. See Corollary 2.1.21.

In this section and throughout this paper, (A, Γ) will be a graded Hopf algebroid and the category $\mathcal{C}_\Gamma$ will denote left Γ-comodules. But note that the conjugation in a Hopf algebroid induces an equivalence of categories between left and right comodules. As a bit of notation, if N is a comodule, then $\Sigma^k N$, $k \in \mathbb{Z}$, is the evident shifted comodule and

$$\mathrm{Hom}_{\mathcal{C}_\Gamma}(M, N) = \{\mathcal{C}_\Gamma(\Sigma^k, N)\}$$

will denote the *graded* A-module of comodule homomorphisms from M to N. Similarly, if we need it, we will write $\mathrm{Hom}_A(M, N)$ for the graded A-module of

A-module homomorphisms.

2.1.1 Comodules as product-preserving diagrams

2.1.1 Definition. *A Hopf algebroid (A, Γ) is of Adams-type if*

1.) The left unit $\eta_L : A \to \Gamma$ makes Γ a flat A-module;

2.) There is filtered system of sub-Γ-comodules $\Gamma_i \subset \Gamma$ which are finitely generated and projective over A and so that

$$\operatorname{colim} \Gamma_i {\longrightarrow} \Gamma$$

is an isomorphism.

2.1.2 Definition. *A* **generating system** *J of Γ-comodules is a diagram of comodule maps over Γ*

$$C_j \to \Gamma$$

so that the objects C_j are finitely generated and projective over A and the induced map $\operatorname{colim}_J\ C_j \to \Gamma$ is an isomorphism of comodules.

2.1.3 Example. Thus if (A, Γ) is of Adams type, then it has a generating system. Furthermore any diagram of comodules $C_j \to \Gamma$ over Γ so that each of the C_j is finitely generated and projective over A and which contains the diagram of inclusions $\Gamma_i \to \Gamma$ will be a generating system. For example, we could take as a generating system the diagram category which consists of one representative for each isomorphism class of comodule morphisms $C \to \Gamma$ with C finitely generated and projective over A. Morphisms would be commutative triangles. This generating system is maximal, in an obvious sense, and closed under the following tensor product operation. If $C_1 \to \Gamma$ and $C_2 \to \Gamma$ are in the system, then the composition

$$C_1 \otimes_A C_2 {\longrightarrow} \Gamma \otimes_A \Gamma \xrightarrow{m} \Gamma$$

where m is the Hopf algebroid multiplication.

If N is a Γ-comodule which is finitely generated over A, let

$$DN = \operatorname{Hom}_A(N, A)$$

be the dual comodule. The comodule structure is that associated to the right comodule structure of [36] Lemma A.1.16.

2.1.4 Remark. Let $J = \{C_j \to \Gamma\}$ be a generating system. Then, because the comodules C_j are finitely generated and projective as A-modules, the natural map $C_j \to D(DC_j)$ is an isomorphism of comodules. From this is follows that for all comodules M there are natural isomorphisms

$$\operatorname{colim}_J \operatorname{Hom}_\Gamma(DC_j, M) \cong \operatorname{colim}_{\mathcal{C}} \operatorname{Hom}_\Gamma(A, C_j \otimes_A M) \cong M. \tag{2.1.1}$$

The following Lemma explains the term "generating" system.

2.1.5 Lemma. *Let $C_j \to \Gamma$ be a generating system of Γ-modules. Then the comodules $\Sigma^k DC_j$ are projective as A-modules and generate the category of Γ comodules.*

Proof. In [19], §3 we showed that the comodules $\Sigma^k D\Gamma_i$ generate. The same argument works here. See also [27] for a cleaned-up version of this proof. The essential fact is Equation 2.1.1. ☐

2.1.6 Remark. In his paper on model category structures on categories of chain complexes in comodules [27], Mark Hovey has given a much more elegant discussion of the role of generating systems of comodules than we have given here. This ad hoc discussion predates his, however, and we're too tired to rewrite at this point. It will do for now.

Let $C_j \to \Gamma$ be any generating system of Γ-comodules and let $\mathcal{P}$ be the full subcategory of $\mathcal{C}_\Gamma$ which contains the objects $\Sigma^k DC_j$ and which is closed under *finite* direct sums. Now consider the category $\mathbf{Pre}(\mathcal{P})$ of contravariant functors

$$F : \mathcal{P}^{op} \longrightarrow \mathbf{Mod}_A.$$

Among all such functors, we single out the full-subcategory $\mathbf{Sh}(\mathcal{P})$ of functors which satisfy the following sheaf condition: if $Q \to P$ is a surjection, then

$$\begin{tikzcd} F(P) \arrow[r] & F(Q) \arrow[r, shift left] \arrow[r, shift right] & F(Q \times_P Q) \end{tikzcd} \tag{2.1.2}$$

is an equalizer diagram. We will call the objects of $\mathbf{Sh}(\mathcal{P})$ *sheaves.*[1] The inclusion functor $\mathbf{Sh}(\mathcal{P}) \to \mathbf{Pre}(\mathcal{P})$ has a left adjoint L; thus LF is the *associated sheaf.* We give a concrete description of LF in the proof of Lemma 2.1.8 below.

We are mainly concerned not so much with sheaves and presheaves as the following full subcategories.

2.1.7 Definition. *Let $\mathbf{Pre}_+(\mathcal{P})$ denote the contravariant functors*

$$F : \mathcal{P} \longrightarrow \mathbf{Sets}$$

which preserves finite products in the following sense: if $P \cong P_1 \oplus P_2$, then the natural map

$$F(P) \to F(P_1) \times F(P_2)$$

is an isomorphism. Morphisms in $\mathbf{Pre}_+(\mathcal{P})$ are morphisms of diagrams; hence $\mathbf{Pre}_+(\mathcal{P})$ is a full-subcategory of the category of $\mathbf{Pre}(\mathcal{P})$. Let $\mathbf{Sh}_+(\mathcal{P})$ be the be the full subcategory of $\mathbf{Pre}_+(\mathcal{P})$ of objects satisfying the sheaf condition of Equation 2.1.2; this, in turn, is a full-subcategory of $\mathbf{Sh}(\mathcal{P})$.

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[1] This nomenclature can be justified by introducing a suitable topology; however, we forebear.

Note that there is a Yoneda embedding

$$y_* : \mathcal{C}_\Gamma \longrightarrow \mathbf{Pre}_+(\mathcal{P})$$

sending a comodule M to the functor

$$P \mapsto \mathcal{C}_\Gamma(P, M).$$

This is, in fact, a sheaf. If this is not completely obvious see the next lemma.

2.1.8 Lemma. *1.) Every object in* $\mathbf{Pre}_+(\mathcal{P})$ *the graded set*

$$F(\Sigma^* P) = \{F(\Sigma^k P)\}$$

has a natural structure as an A*-module.*

2.) If $F \in \mathbf{Pre}_+(\mathcal{P})$*, and* LF *is the associated sheaf of* A*-modules, then* $LF \in \mathbf{Sh}_+(\mathcal{P})$*.*

3.) If $M \in \mathcal{C}_\Gamma$ *is comodules, then* $y_* M \in \mathbf{Pre}_+(\mathcal{P})$ *is sheaf.*

Proof. The first statement follows from the fact that, since $F \in \mathbf{Pre}_+(\mathcal{P})$ preserves products, $F(\Sigma^* P)$ is a right module over the graded ring $\mathcal{E}nd(P) = \mathrm{Hom}_{\mathcal{C}_\Gamma}(P, P)$, hence, an A-module. Furthermore, the actions on $\mathrm{Hom}_{\mathcal{C}_\Gamma}(P, Q)$ of $\mathrm{End}(P)$ and $\mathrm{End}(Q)$ on the left and right, respectively, give $\mathrm{Hom}_{\mathcal{C}_\Gamma}(P, Q)$ the identical structures as an A-module; hence, any morphism $P \to Q$ gives a morphism $F(Q) \to F(P)$ of A-modules.

For the second statement, let F be a presheaf. Define a new presheaf $L_0 F$ by

$$(L_0 F)(P) = \operatorname*{colim}_{Q \twoheadrightarrow P} F(Q)$$

where the colimit is over all epimorphisms in $\mathcal{P}$ and the colimit is in A-modules. If $P' \to P$ is a morphism in $\mathcal{P}$, then $(L_0 F)(P) \to (L_0 F)(P')$ is defined by using the maps $P' \times_P Q \to P'$. If $P = P_1 \oplus P_2$ and $Q \to P$ is an epimorphism, then

$$Q \cong (P_1 \times_P Q) \oplus (P_2 \times_P Q).$$

This equation and the fact that finite sums and products in A-modules are isomorphic, imply that if $F \in \mathbf{Pre}_+(\mathcal{P})$, then so is $L_0 F$. As usual, $LF = L_0(L_0 F)$.

For part 3, we use that colimits and finite limits in $\mathcal{C}_\Gamma$ are created in A-modules. Thus every epimorphism of comodules is, in fact, an effective epimorphism. In formulas, this means that if $Q \to P$ is an epimorphism of comodules, then

$$\begin{tikzcd} Q \times_P Q \arrow[r, shift left] \arrow[r, shift right] & Q \arrow[r] & P \end{tikzcd}$$

is a coequalizer diagram. □

The next result discusses limits and colimits in $\mathbf{Pre}_+(\mathcal{P})$.

Recall the a reflexive coequalizer in any category $\mathcal{C}$ is a coequalizer diagram

$$\begin{tikzcd} X_1 \arrow[r, shift left, "d_0"] \arrow[r, shift right, "d_1"'] & X_0 \arrow[r] & X \end{tikzcd}$$

which can be equipped with a "degeneracy" $s_0 : X_0 \to X_1$ so that $d_0 s_0 = d_1 s_0 = 1$.

2.1.9 Lemma. *1.) The categories* $\mathbf{Pre}_+(\mathcal{P})$ *and* $\mathbf{Sh}_+(\mathcal{P})$ *are complete and cocomplete.*

2.) Reflexive coequalizers in $\mathbf{Pre}_+(\mathcal{P})$ *are created in* $\mathbf{Pre}(\mathcal{P})$.

3.) The objects y_*P, *with* $P \in \mathcal{P}$, *generate* $\mathbf{Pre}_+(\mathcal{P})$ *and* $\mathbf{Sh}_+(\mathcal{P})$.

4.) The inclusions functors $\mathbf{Pre}_+(\mathcal{P}) \to \mathbf{Pre}(\mathcal{P})$ *and* $\mathbf{Sh}_+(\mathcal{P}) \to \mathbf{Sh}(\mathcal{P})$ *have left adjoints. In fact,* $\mathbf{Pre}_+(\mathcal{P})$ *is a category of algebras over a triple on* $\mathbf{Pre}(\mathcal{P})$.

Proof. Limits and colimits in $\mathbf{Pre}(\mathcal{P})$ are constructed object-wise or "point-wise". Since reflexive coequalizers in sets commute with products, point 2.) follows. For point 1.) note that limits and colimits in $\mathbf{Pre}_+(\mathcal{P})$ can be formed level-wise in A-modules; then limits in $\mathbf{Sh}_+(\mathcal{P})$ can are created in $\mathbf{Pre}_+(\mathcal{P})$ and colimits using sheafification and Lemma 2.1.8.2. For point 3, note that if $F \in \mathbf{Pre}_+(\mathcal{P})$, then the evident map

$$\bigoplus_P \bigoplus_{x \in F(P)} y_*P \longrightarrow F$$

is an epimorphism in $\mathbf{Pre}_+(\mathcal{P})$. If F is a sheaf, we can sheafify the source of this morphism. Finally point 4 follows from point 3 and the special adjoint functor theorem; the fact that we have a category of algebras follows from Beck's Theorem, Theorem 10 of [4] §3.3. □

2.1.10 Lemma. *The functor* $y_* : \mathcal{C}_\Gamma \to \mathbf{Pre}_+(\mathcal{P})$ *has a left adjoint* y^* *and this left adjoint restricts to a left adjoint to the induced functor* $y_* : \mathcal{C}_\Gamma \to \mathbf{Sh}_+(\mathcal{P})$.

Proof. This is formal. If M an A-module and P is a comodule, define a new comodule $M \otimes_A P$ as the evident A-module with coproduct

$$M \otimes_A P \xrightarrow{M \otimes \psi_P} M \otimes_A \Gamma \otimes_A P \xrightarrow{t \otimes P} \Gamma \otimes_A M \otimes_A P.$$

There is an adjoint isomorphism

$$\mathrm{Hom}_A(M, \mathrm{Hom}_{\mathcal{C}_\Gamma}(P, N)) \cong \mathrm{Hom}_{\mathcal{C}_\Gamma}(M \otimes_A P, N).$$

This immediately implies that y^* is the coend

$$y^*(F) = \int^P F(P) \otimes_A P$$

for F either a sheaf or a presheaf. □

2.1.11 Lemma. *The Yoneda embedding functor*

$$y_* : \mathcal{C}_\Gamma \to \mathbf{Sh}_+(\mathcal{P})$$

is exact.

Proof. It suffices to show that y_* preserves monomorphisms and epimorphisms. It clearly preserves monomorphisms. So let $q : M \to N$ be an epimorphism of comodules. The induced map of sheave $y_*M \to y_*N$ is an epimorphism if for all

$$f \in (y_*N)(P) = \mathcal{C}_\Gamma(P, N)$$

there is an epimorphism $j : Q \to P$ in $\mathcal{P}$ and an element

$$g \in (y_*M)(Q) = \mathcal{C}_\Gamma(Q, M)$$

so that

$$y_*(q)(g) = qg = fj = j^*(f) \in \mathcal{C}_\Gamma(Q, N) = (y_*N)(Q).$$

Form the pull-back $P \times_N M$ and note that the induced map $P \times_N M \to P$ is a surjection. Since the comodules DC_j generate the category of comodules, there is an epimorphism $Q' \to P \times_N M$ for some, possibly infinite, sum of comodules of the form DC_j. However, since P is finitely generated, there is a finite sub-sum $Q \subseteq Q'$ so that the composite

$$Q \longrightarrow P \times_N M \longrightarrow P$$

remains surjective. The resulting map

$$Q \longrightarrow P \times_N M \longrightarrow M$$

is the morphism g required. □

2.1.12 Proposition. *Let $C_j \to \Gamma$ be a generating system of comodules and let*

$$y_* : \mathcal{C}_\Gamma \to \mathbf{Sh}_+(\mathcal{P})$$

be the associated Yoneda embedding. Then y_* is an equivalence of categories.

Proof. Since all of the objects of $\mathcal{P}$ are finitely generated, the functor y_* commutes with sums. The previous lemma shows that y_* is exact. Next we show that y_* is full and faithful; that is, we need to show that

$$\mathcal{C}_\Gamma(Y, X) \to \mathbf{Sh}_+(y_*Y, y_*X) \tag{2.1.3}$$

is an isomorphism. Regard the source and target as functors of Y. If Y is an object in $\mathcal{P}$, this map is an isomorphism by the Yoneda Lemma. Since y_* preserves sums, it is an isomorphism if Y is sum of objects of $\mathcal{P}$. More generally, we can write Y as part of an exact sequence

$$Y_1 \longrightarrow Y_0 \longrightarrow Y \to 0$$

where Y_0 and Y_1 are sums of the generators DC_j, which are in $\mathcal{P}$. Since y_* is exact, Equation 2.1.3 is an isomorphism for Y as well.

To finish the argument, we need to know that for every sheaf F in $\mathbf{Sh}_+(\mathcal{P})$ there is an object $M \in \mathcal{C}_\Gamma$ and an isomorphism of sheaves $y_*M \cong F$. For this category Lemma 2.1.9 implies that every sheaf is a colimit of representable sheaves. Since y_* preserves products, this implies there is a short exact sequence of sheaves

$$y_*Y_1 \xrightarrow{f} y_*Y_2 \to F \to 0$$

where Y_1 is a sum of objects in $\mathcal{P}$. Since y_* is full and faithful, there is a morphism $g : Y_1 \to Y_2$ so that $y_*(g) = f$. Let M be the cokernel of g. Then the exactness of y_* implies $y_*M \cong F$. □

We can use Theorem 2.1.12 to give a formula for the left adjoint to the Yoneda embedding $y_* : \mathcal{C}_\Gamma \to \mathbf{Pre}_+(\mathcal{P})$.

2.1.13 Theorem. *Let J be our fixed generating system for Γ-comodules, regarded as a category of comodules over Γ. If $F \in \mathbf{Pre}_+(\mathcal{P})$ then there is a natural isomorphism of A-modules*

$$y^*(F) \cong \operatorname{colim}_{\mathcal{C}} F(DC_j). \tag{2.1.4}$$

Proof. We simply define a functor Ψ from $\mathbf{Pre}_+(\mathcal{P})$ to A-modules by the formula 2.1.4. This functor is exact, since the category $\mathcal{C}$ is filtered. By Remark 2.1.4, there is a natural isomorphism

$$\Psi(y_*M) \cong \operatorname{colim}_J \operatorname{Hom}_\Gamma(A, C_j \otimes_A M) \cong M.$$

Now, since y_* preserves sums, we can write any object $F \in \mathbf{Pre}_+(\mathcal{P})$ in a reflexive coequalizer diagram in $\mathbf{Pre}_+(\mathcal{P})$

$$\begin{tikzcd} y_*M_1 \arrow[r, shift left] \arrow[r, shift right] & y_*M_0 \arrow[r] & F \end{tikzcd} \tag{2.1.5}$$

where M_i is a sum of objects in $\mathcal{P}$. Since y_* is full and faithful, and reflexive coequalizers in $\mathcal{C}_\Gamma$ are created in sets (or A-modules), this implies there is a reflexive coequalizer diagram

$$\begin{tikzcd} M_1 \arrow[r, shift left] \arrow[r, shift right] & M_0 \arrow[r] & \Psi(F). \end{tikzcd}$$

Since reflexive coequalizers in $\mathcal{C}_\Gamma$ are created in A-modules (or even sets) this supplies $\Psi(F)$ with a natural structure as a Γ-comodule; furthermore, if $F = y_*M$, then this structure is isomorphic to the original structure on M.

Now, the fact the Ψ is a functor yields a natural map

$$\mathbf{Pre}_+(\mathcal{P})(F, y_*M) \longrightarrow \mathcal{C}_\Gamma(\Psi(F), M).$$

If $F = y_*N$ this is an isomorphism. Then an exactness argument using the reflexive coequalizer diagram 2.1.5 yields that this map is an isomorphism for all F. Thus, the uniqueness of adjoints supplies a natural isomorphism $\Psi(F) \cong y^*F$. □

2.1.14 Remark. The associated sheaf functor $L : \mathbf{Pre}_+(\mathcal{P}) \to \mathbf{Sh}_+(\mathcal{P})$ has a formula in terms of comodules. Indeed, if F is a presheaf

$$L(F) = y_* y^* F = y_* \operatorname{colim} F(\Sigma^* DC_i)$$

using Proposition 2.1.12 and Lemma 2.1.10.

2.1.2 Algebras as diagrams

We would now like to expand the notions of the previous subsection in order to encompass algebras and modules over algebras in comodules. This is the point of this theory, for we are working to put a module structure into the spiral exact sequence.

Let Φ be a triple in $\mathcal{C}_\Gamma$ and $\mathbf{Alg}_\Gamma^\Phi$ or simply $\mathbf{Alg}^\Phi$ as the category of algebras over Φ in comodules. We will assume that Φ preserves surjections. Let $C_j \to \Gamma$ be a generating system of of Γ-comodules and let $\Phi(\mathcal{P})$ be the full subcategory of $\mathbf{Alg}_\Gamma^\Phi$ which contains the objects $\Phi(\Sigma^k DC_j)$ and which is closed under finite coproducts and finite limits.

2.1.15 Definition. *1.) Let* $\mathbf{Pre}_+(\Phi(\mathcal{P}))$ *denote the contravariant functors*

$$F : \Phi(\mathcal{P})^{op} \longrightarrow \mathbf{Sets}$$

which preserve finite products; that is, which send finite coproducts to finite products.

2.) Let $\mathbf{Sh}_+(\Phi(\mathcal{P}))$ *be the full-subcategory of* $\mathbf{Pre}_+(\Phi(\mathcal{P}))$ *containing the functors which for which*

$$\begin{tikzcd} F(P) \arrow[r] & F(Q) \arrow[r, shift left] \arrow[r, shift right] & F(Q \times_P Q) \end{tikzcd}$$

is an equalizer for all surjections $Q \to P$ *in* $\Phi(\mathcal{P})$*).*

Note that there is a Yoneda embedding

$$y_* : \mathbf{Alg}_\Gamma^\Phi \longrightarrow \mathbf{Sh}_+(\Phi(\mathcal{P}))$$

sending B to the representable functor $\mathbf{Alg}_\Gamma^\Phi(-, B)$. Note also that the functor $\mathcal{P} \to \Phi(\mathcal{P})$ sending $P \to \Phi(P)$ defines a restriction functor

$$r_* : \mathbf{Pre}_+(\Phi(\mathcal{P})) \longrightarrow \mathbf{Pre}_+(\mathcal{P}).$$

2.1.16 Lemma. *Reflexive coequalizers and filtered colimits in* $\mathbf{Pre}_+(\Phi(\mathcal{P}))$ *exist and are created in* $\mathbf{Pre}_+(\mathcal{P})$*.*

Proof. Reflexive coequalizers in $\mathbf{Pre}_+(\mathcal{P})$ are constructed point-wise in sets. See Lemma 2.1.9. Thus, if we have parallel arrows $\begin{tikzcd} X_1 \arrow[r, shift left] \arrow[r, shift right] & X_0 \end{tikzcd}$ which can be given a degeneracy, we can form the equalizer X in $\mathbf{Pre}_+(\mathcal{P})$. Then we have, for each $f : \Phi(Q) \to \Phi(P)$ in $\Phi(\mathcal{P})$ a diagram

$$\begin{tikzcd}
X_1(\Phi(Q)) \arrow[r, shift left] \arrow[r, shift right] \arrow[d] & X_0(\Phi(Q)) \arrow[r, "\epsilon"] \arrow[d] & X(\Phi(Q)) \arrow[d, dashed] \\
X_1(\Phi(P)) \arrow[r, shift left] \arrow[r, shift right] & X_0(\Phi(P)) \arrow[r, "\epsilon"] & X(\Phi(P)).
\end{tikzcd}$$

The solid vertical arrows are induced by f and the fact that X_1 and X_0 are in $\mathbf{Pre}_+(\Phi(\mathcal{P}))$. The dotted vertical arrow exists because the horizontal rows are equalizer diagrams of sets. We have a functor X on $\Phi(\mathcal{P})^{op}$ because each of the maps ϵ is a surjection. Finally, X preserves products because it is the equalizer in $\mathbf{Pre}_+(\mathcal{P})$.

The same argument works for filtered colimits, which are also constructed point-wise in sets. □

2.1.17 Lemma. *The category* $\mathbf{Pre}_+(\Phi(\mathcal{P}))$ *has all coproducts. Furthermore, if* $\{A_\alpha = \Phi(P_\alpha)\}$ *is a set of free objects of* $\Phi(\mathcal{P})$*, then*

$$\sqcup y_* A_\alpha \cong y_*(\sqcup A_\alpha).$$

Proof. We first show that the Yoneda embedding preserves coproducts. This goes in several steps. First note that $y_*\Phi(P_1) \sqcup y_*\Phi(P_2) \cong y_*(\Phi(P_1) \sqcup \Phi(P_2))$. This is a consequence of the following isomorphism, where $F \in \mathbf{Pre}_+(\Phi(\mathcal{P}))$:

$$\begin{aligned}\mathbf{Pre}_+(\Phi(\mathcal{P}))(y_*(\Phi(P_1) \sqcup \Phi(P_2)), F) &\cong F(\Phi(P_1) \sqcup \Phi(P_2)) \\ &\cong F(P_1) \times F(P_2)\end{aligned}$$

Next, note that y_* commutes with filtered colimits, since each of the objects in $\Phi(\mathcal{P})$ is small. It follows immediately that y_* commutes with all coproducts.

To complete the existence of coproducts in $\mathbf{Pre}_+(\Phi(\mathcal{P}))$ we use that any object $F_\alpha \in \mathbf{Pre}_+(\Phi(\mathcal{P})$ fits into a reflexive coequalizer diagram

$$\begin{tikzcd}
\sqcup y_*\Phi(Q_{j,\alpha}) \arrow[r, shift left] \arrow[r, shift right] & \sqcup y_*\Phi(P_{i,\alpha}) \arrow[r] & F.
\end{tikzcd}$$

Taking the coproduct of such diagrams and applying Lemma 2.1.16 finishes the argument. □

2.1.18 Lemma. *1.) The categories* $\mathbf{Pre}_+(\Phi(\mathcal{P}))$ *and* $\mathbf{Sh}_+(\Phi(\mathcal{P}))$ *are complete and cocomplete.*

2.) The restriction functor $r_* : \mathbf{Pre}_+(\Phi(\mathcal{P})) \to \mathbf{Pre}_+(\mathcal{P})$ *has a left adjoint* r^* *with the property that if* $P \in \mathcal{P}$ *is a generating comodule, then there is a natural isomorphism*

$$r^* y_* P \cong y_*\Phi(P).$$

3.) The category of $\mathbf{Pre}_+(\Phi(\mathcal{P}))$ *is the category of algebras for some triple on* $\mathbf{Pre}_+(\mathcal{P})$.

4.) The Yoneda embedding $y_* : \mathbf{Alg}^{\Phi}_{\Gamma} \to \mathbf{Pre}_+(\Phi(\mathcal{P}))$ *has a left adjoint* y^*.

Proof. Part 1 follows from the previous two lemmas and the fact that limits are created in $\mathbf{Pre}_+(\mathcal{P})$.

For Part 2, note that if $F \in \mathbf{Pre}_+(\Phi(\mathcal{P}))$, then

$$\mathbf{Pre}_+(\Phi(\mathcal{P}))(y_*P, r_*F) \cong r_*F(P) = F(\Phi(P)).$$

We can take $r^*y_*P = y_*\Phi(P)$ as the definition. To define r^*G for general $G \in \mathbf{Pre}_+(\mathcal{P})$, write

$$G = \operatorname{colim} y_*P \to Gy_*P$$

and set

$$r^*G = \operatorname{colim} y_*P \to Gy_*\Phi(P).$$

Part 3 now follows from Lemma 2.1.16 and Beck's Theorem. See [4], §3.3. The triple has underlying functor r_*r^*.

For Part 4, the adjoint y^* can be written down as a coend; compare Lemma 2.1.10. ☐

The first part of this last lemma implies that the category $\mathbf{Pre}_+(\Phi(\mathcal{P}))$ has an initial object. In fact, one can take that object to be $y_*\Phi_0$, where Φ_0 is the is the initial object in $\mathbf{Alg}_\Gamma^\Phi$.

We now turn to the relationship between the category of sheaves and the category of algebras in comodules.

2.1.19 Lemma. *The Yoneda embedding*

$$y_* : \mathbf{Alg}_\Gamma^\Phi \longrightarrow \mathbf{Sh}_+(\Phi(\mathcal{P}))$$

preserves reflexive coequalizers and coproducts.

Proof. It is a consequence of Lemma 2.1.16 that the reflexive coequalizers in $\mathbf{Sh}_+(\Phi(\mathcal{P}))$ are created in $\mathbf{Sh}_+(\mathcal{P})$. Now apply Lemma 2.1.11 to get the first half of the statement. For the part about coproducts, use that every object $A \in \mathbf{Alg}_\Gamma^\Phi$ fits into a reflexive coequalizer diagram

$$\begin{tikzcd} X_1 \arrow[r, shift left] \arrow[r, shift right] & X_0 \arrow[r] & A \end{tikzcd}$$

where X_i is a coproduct of objects of the form $\Phi(P) \in \Phi(\mathcal{P})$. This is because those objects generate the category $\mathbf{Alg}_\Gamma^\Phi$. Now apply Lemma 2.1.17 and the fact the y_* preserves reflexive coequalizers. ☐

2.1.20 Proposition. *The Yoneda embedding functor*

$$y_* : \mathbf{Alg}_\Gamma^\Phi \longrightarrow \mathbf{Sh}_+(\Phi(\mathcal{P}))$$

is an equivalence of categories.

Proof. The argument is essentially the same as that for Theorem 2.1.12, although the two arguments there using exact sequences must replaced by arguments using reflexive coequalizers. The details are routine. ☐

As in Lemma 2.1.13, this result can be used to give a formula for the adjoint to the Yoneda embedding $y_* : \mathbf{Alg}_\Gamma^\Phi \to \mathbf{Pre}_+(\Phi(\mathcal{P}))$.

2.1.21 Corollary. *Let J be our fixed generating system for Γ-comodules, regarded as a category of comodules over Γ. Then, if $F \in \mathbf{Pre}_+(\Phi(\mathcal{P}))$ then there is a natural isomorphism of A-modules*

$$y^*(F) \cong \operatorname{colim}_J F(\Sigma^*\Phi(DC_j)). \tag{2.1.6}$$

Proof. The argument is the same as for Lemma 2.1.13; one defines an auxiliary functor Ψ by the formula of 2.1.21 and uses a succession of reflexive coequalizer arguments to show that is must be the adjoint. □

2.1.3 Modules as diagrams

In this section we talk about modules over algebras and how they can be described in terms of the presheaves.

We fix an object F in $\mathbf{Pre}_+(\Phi(\mathcal{P}))$. Then an abelian object over F is a morphism in $G \to F$ in $\mathbf{Pre}_+(\Phi(\mathcal{P}))$ so that the functor

$$\mathrm{Hom}_{\mathbf{Pre}_+(\Phi(\mathcal{P}))/F}(-, G) : \mathbf{Pre}_+(\Phi(\mathcal{P}))^{op} \to \mathbf{Sets}$$

has a chosen lift to abelian groups. As usual, this means that there are specified maps

$$\mu : G \times_F G \longrightarrow G \qquad \text{and} \qquad e : F \to G$$

over F satisfying the evident associative, commutative, and unital diagrams. Let $\mathbf{Abpre}_+(\Phi(\mathcal{P}))/F$ be the evident category of abelian objects over F.

2.1.22 Example. 1.) Let $\Lambda \in \mathbf{Alg}_\Gamma^\Phi$; then the notion of an abelian object $q : E \to \Lambda$ over Λ can be defined the same way. However, if M is the kernel of q, then $E \cong M \oplus \Lambda$ as A-modules, and we may as well write $M \rtimes \Lambda$ for the Φ-algebra E. We will call M an Λ-module. Note that $y_*(M \rtimes \Lambda) \to y_*\Lambda$ is an abelian group object over $y_*\Lambda$.

In the same way, abelian objects over a fixed object $F \in \mathbf{Pre}_+(\Phi\mathcal{P}))$ can be identified with modules of the following sort.

2.1.23 Definition. *Let $F \in \mathbf{Pre}_+(\Phi(\mathcal{P}))$. Then we specify an F-module M by the following data:*

1.) an object $M \in \mathbf{Pre}_+(\mathcal{P})$; and

2.) for each $f : \Phi(Q) \to \Phi(P)$ a map of sets

$$\phi_f : M(P) \times F(\Phi(P)) \longrightarrow M(Q)$$

subject to the conditions that

a.) if $f = \Phi(f_0)$, then $\phi_f(x, a) = M(f_0)x$;

b.) for any composable pair of arrows in $\Phi(\mathcal{P})$,

$$\phi_{gf}(x,a) = \phi_f(\phi_g(x,a), F(g)a);$$

c.) for all $a \in F(\Phi(P))$, the function $\phi_f(-,a)$ is a homomorphism of abelian groups.

The F-modules form a category $\mathbf{Mod}_F(\mathcal{P})$ in the obvious way.

2.1.24 Remark. If M is an F-module, we form a new object $M \rtimes F$ of $\mathbf{Pre}_+(\Phi(\mathcal{P}))$ by setting

$$(M \rtimes F)(\Phi(P)) = M(P) \times F(\Phi(P))$$

and for any morphism $f : \Phi(Q) \to \Phi(P)$, we set

$$(M \rtimes F)(f)(x,a) = (\phi_f(x,a), F(f)a).$$

The fact that M preserves coproducts and conditions a.) and b.) guarantee that we do indeed have an object in $\mathbf{Pre}_+(\Phi(\mathcal{P}))$. We define a multiplication and unit for $M \rtimes F$

$$m(x,y,a) = (x+y,a)$$

and $e(a) = (0,a)$. Then condition c.) implies that these give natural transformations of functors and yield an abelian object over F.

2.1.25 Lemma. *The functor*

$$(-) \rtimes F : \mathbf{Mod}_F(\mathcal{P}) \longrightarrow \mathbf{Abpre}_+(\Phi(\mathcal{P}))/F$$

is an equivalence of categories.

Proof. We write down the inverse functor. If $G \to F$ is an abelian object, let $M \in \mathbf{Pre}_+(\mathcal{P})$ be defined by the split short exact sequence of A modules

$$0 \to M(P) \to G(\Phi(P)) \to F(\Phi(P)) \to 0$$

and, for $f : \Phi(Q) \to \Phi(P)$, let ϕ_f be defined by the composite

$$M(P) \times F(\Phi(P)) \cong G(\Phi(P)) \xrightarrow{G(f)} G(\Phi(Q)) \to M(Q).$$

Then the evident isomorphisms $G(\Phi(P)) \to M(P) \times F(\Phi(P))$ assemble to give an isomorphism of abelian objects over F. $\square$

We define $\mathbf{ShMod}_F(\mathcal{P})$ to be the full sub-category of those modules M for which $M \rtimes F \in \mathbf{Sh}_+(\Phi(\mathcal{P}))$.

The following result is now a more-or-less evident consequence of Theorem 2.1.12 and Proposition 2.1.20.

2.1.26 Proposition. *Fix an algebra $\Lambda \in \mathbf{Alg}_\Gamma^\Phi$. Then the Yoneda embedding*

$$y_* : \mathbf{Mod}_\Lambda^\Phi \longrightarrow \mathbf{Mod}_{y_*\Lambda}(\Phi(\mathcal{P}))$$

defines an equivalence of categories from $\mathbf{Mod}_\Lambda^\Phi$ to $\mathbf{ShMod}_{y_*\Lambda}(\Phi(\mathcal{P}))$.

2.2 Theta-algebras and the p-adic K-theory of E_∞-ring spectra

In this section we define and discuss the concept of a theta-algebra, which is the algebraic model for the p-adic K-theory of an E_∞-ring spectrum. We also discuss the appropriate notion of modules over such rings. The key point for our obstruction theory is that the p-adic K-theory of E_∞-ring spectra can be made algebraic in the following sense. There is a forgetful functor from theta-algebras to (certain) continuous $\mathbb{Z}_p^\times$-modules, and it has a left adjoint S_θ. Furthermore, if X is a cofibrant spectrum such that K_*X is torsion free, and C is an operad weakly equivalent to the commutative monoid operad, then the natural map

$$S_\theta(K_*X) \to K_*(C(X))$$

is an isomorphism.

Let K denote the p-complete K-theory spectrum. If X is any spectrum, we define the p-adic K-theory of X by the equation

$$K_*X = \pi_* L_{K(1)}(K \wedge X).$$

Under favorable circumstances, which will nearly always apply here,

$$K_*X = \lim K_*(X \wedge M(p^k))$$

where $M(p^k)$ is the evident Moore spectrum. Thus, we should really adorn K_* with some sort of completion symbol, but it is the only kind of K-theory that we will have, so we forebear.

Note that K_*X is not really a homology theory: it does not take coproducts to direct sums of abelian groups. However, it is the appropriate analog for homology when discussing $K(1)$-local spectra, where $K(n)$ is the nth-Morava K-theory. This sort of phenomenon discussed at length in [29] and we draw freely from that source.

As with all 2-periodic homology theories, we may either regard K_*X as $\mathbb{Z}$-graded or $\mathbb{Z}/2\mathbb{Z}$-graded. The latter is often more convenient, but the former can be important, for example, when keeping track of behavior under suspension.

To talk about the structure of K_*X, we first need a definition. Let L_0 be the zeroth derived functor of p-completion. Then a $\mathbb{Z}_p$-module is *L-complete* if the natural map $M \to L_0M$ is an isomorphism. If M is torsion free, this is equivalent to being p-complete.

If X is any spectrum, K_*X is L-complete. Furthermore, K_*X has a continuous action by the group $\mathbb{Z}_p^\times$ of units in the p-adics. If $k \in \mathbb{Z}_p^\times$ the action of k is by the kth Adams operation ψ^k:

$$\psi^k \wedge X : L_{K(1)}(K \wedge X) \to L_{K(1)}(K \wedge X).$$

However, not every continuous action can arise as the K_* homology of some spectrum.

2.2.1 Definition. *Let C_{K_*K} denote the category of L-complete $\mathbb{Z}/2\mathbb{Z}$-graded $\mathbb{Z}_p^\times$-modules M with the property that the quotient $\mathbb{Z}/p\mathbb{Z} \otimes M = M/pM$ is a discrete $\mathbb{Z}_p^\times$-module. We will call this the category of K_*-***Morava modules** *or simply* **Morava modules**.

2.2.2 Proposition. *The p-completed K-theory $K_*(-) = \pi_* L_{K(1)}(K \wedge (-))$ takes any spectrum to a Morava module.*

Proof. This follows from the facts that $K_1 K = 0$ and

$$\begin{aligned} K_0 K &= \mathrm{Hom}_{\mathrm{cont}}(\mathbb{Z}_p^\times, \mathbb{Z}_p) \\ &\cong \lim_n \operatorname{colim}_j \mathrm{Hom}(\mathbb{Z}_p^\times / U_j, \mathbb{Z}/p^n\mathbb{Z}) \end{aligned}$$

where U_j runs over a sequence of normal subgroups so that $\cap U_j = \{1\}$. $\square$

An elementary example of a Morava module we will use often is the following: if $u \in K_2 = [S^2, K] = \tilde{K}^0(S^2)$ is the Bott element, then

$$\psi^k(u) = ku. \tag{2.2.1}$$

2.2.3 Definition. *A* **theta-algebra** *is a $\mathbb{Z}/2\mathbb{Z}$-graded continuous commutative $\mathbb{Z}_p$-algebra A so that*

1. *For $i = 0, 1$, the module A_i is a Morava-module. Write the action of k on A_i as Adams operations $\psi^k : A_i \to A_i$.*
2. *The Adams operations $\psi^k : A \to A$ are linear and*

$$\psi^k(xy) = \begin{cases} \psi^k(x)\psi^k(y) & |x| = 0 \quad \text{or} \quad |y| = 0 \\ \frac{1}{k}\psi^k(x)\psi^k(y) & |x| = 1 = |y|. \end{cases}$$

3. *There is a continuous operation $\theta : A_i \to A_i$ so that $\theta\psi^k = \psi^k\theta$ for all $k \in \mathbb{Z}_p^\times$ and*

$$\theta(x + y) = \begin{cases} \theta(x) + \theta(y) - \sum_{s=1}^{p-1} \frac{1}{p}\binom{p}{s} x^s y^{p-s} & |x| = 0 = |y|; \\ \theta(x) + \theta(y) & |x| = 1 = |y|. \end{cases}$$

4. *$\theta(1) = 0$, where $1 \in A_0$ is the multiplicative identity and*

$$\theta(xy) = \begin{cases} \theta(x)y^p + x^p\theta(y) + p\theta(x)\theta(y) & |x| = 0 \quad \text{or} \quad |y| = 0; \\ \theta(x)\theta(y) & |x| = 1 = |y|. \end{cases}$$

Theta-algebras form an obvious category $\mathbf{Alg}_\theta$.

The following result explains the origin of this definition; it is implied by the work of McClure [12], Chapter IX.

2.2.4 Theorem. *Suppose that X is an E_∞-ring spectrum. Then K_*X is naturally a theta-algebra.*

2.2.5 Remark. If A is a theta-algebra, define an operation $\psi : A \to A$ by the equation

$$\psi(x) = x^p + p\theta(x).$$

If the degree of x is 1, then $\psi(x) = p\theta(x)$. The operation ψ is a continuous, linear endomorphism of A that commutes with the Adams operations; furthermore

$$\psi(x)\psi(y) = \begin{cases} \psi(xy), & |x| = 0 \text{ or } |y| = 0; \\ \\ p\psi(xy), & |x| = 1 = |y|. \end{cases} \tag{2.2.2}$$

The operation ψ is also a lift of the Frobenius in the sense that $\psi(x) = x^p$ modulo p. If A is torsion free, then the operation ψ also determines θ; indeed, any lift of the Frobenius that commutes with the Adams operations and has the multiplicative properties of Equation 2.2.2 will then determine an operation θ with desired properties.

2.2.6 Example. Suppose X is a finite CW complex. Let $D(X_+) = F(X_+, S^0)$ denote the Spanier-Whitehead dual of X with a disjoint basepoint added. Then $D(X_+)$ is naturally an E_∞ ring spectrum and there is a natural duality isomorphism

$$\tau : K_*D(X_+) = K^*(X) \overset{\text{def}}{=} \lim K^*(X, \mathbb{Z}/p^k\mathbb{Z})$$

given by applying homotopy to the homotopy inverse limit of the evident maps

$$M(p^k) \wedge K \wedge F(X_+, S^0) \to F(X_+, M(p^k) \wedge K).$$

Note that in degree 1 this defines an isomorphism

$$\tau : K_1D(X_+) \xrightarrow{\cong} K^{-1}(X).$$

The morphism τ is an isomorphism of graded $\mathbb{Z}_p$-algebras that commutes with Adams operations ψ^k and

$$\tau(\theta(x)) = \theta^p(\tau(x))$$

where θ^p is in the unstable cohomology operation so that $\psi^p(x) = x^p + p\theta^p(x)$. This allows for the following easy, but crucial calculation: as a theta-algebra

$$K_*D(S^1_+) \cong \mathbb{Z}_p[\epsilon]$$

where $|\epsilon| = 1$, $\psi^k(\epsilon) = k\epsilon$ and $\theta(\epsilon) = \epsilon$.

In fact, the element ϵ is defined to be the element which goes to the Bott element u under the isomorphisms

$$K_1D(S^1_+) \cong K^{-1}S^1 \cong \tilde{K}^0(S^2).$$

and we can apply Equation 2.2.1.

We now come to the notion of a module over a theta-algebra.

2.2.7 Definition. *Let A be a theta-algebra. Then an A-***module** *is a continuous $\mathbb{Z}/2\mathbb{Z}$-graded module M over the commutative ring A equipped with continuous homomorphisms $\psi^k : M \to M$, $k \in \mathbb{Z}_p^\times$ and $\theta : M \to M$ so that M is a Morava module and*

1. *if $k \in \mathbb{Z}_p^\times$, $a \in A$, and $x \in M$, then*
$$\psi^k(ax) = \psi^k(a)\psi^k(x);$$
2. *if $a \in A$ and $x \in M$, then*
$$\theta(ax) = \begin{cases} a^p\theta(x) + p\theta(a)\theta(x) & |a| = 0 \text{ or } |x| = 0; \\ \theta(a)\theta(x) & |a| = 1 = |x|. \end{cases}$$

If A is a theta-algebra, there is an evident abelian category of A-modules.

2.2.8 Remark. Suppose that A is a theta-algebra and that M is an A-module. Then we can define a new theta-algebra $M \rtimes A$ as follows. As a module, this algebra is $M \oplus A$ and we give it the usual infinitesimal multiplication:

$$(x, a)(y, b) = (ay + xb, ab).$$

Define $\psi^k(x, a) = (\psi^k(x), \psi^k(a))$ and

$$\theta(x, a) = (\theta(x) - a^{p-1}x, \theta(a)).$$

One easily checks this yields a theta-algebra. Furthermore, there is an evident short exact sequence of modules

$$\begin{tikzcd} 0 \arrow[r] & M \arrow[r] & M \rtimes A \arrow[r, shift left, "q"] & A \arrow[l, shift left, "s"] \arrow[r] & 0 \end{tikzcd}$$

so that s and q are theta-algebra maps, the inclusion $M \to M \rtimes A$ commutes with the Adams operations and θ, and $M^2 = 0$. We will call such a diagram a *split square-zero extension* or *split infinitesimal extension* of theta-algebras.

This process can be reversed. If $q : B \to A$ is an abelian group object in the category of theta-algebras over A, then there is a split square-zero extension

$$\begin{tikzcd} 0 \arrow[r] & M \arrow[r] & B \arrow[r, shift left, "q"] & A \arrow[l, shift left, "s"] \arrow[r] & 0 \end{tikzcd}$$

where M is the kernel of q. This diagram gives M the structure of a module over the theta-algebra A and defines an isomorphism $B \cong M \rtimes A$. Thus, the functor $M \mapsto M \rtimes A$ is an equivalence of categories between A-modules and abelian theta-algebras over A.

2.2.9 Example. If A is theta-algebra, then A is not a module over itself, as $\theta : A \to A$ is not linear. However, one can define a new module ΩA with

$$[\Omega A]_n = A_{n+1}.$$

If $x \in A_{n+1}$, let us write ϵx for the corresponding element in $[\Omega A]_n$. (If it's not clear already, see Equation 2.2.3 for a reason to choose this notation.) Then we define the action of the Adams operations by

$$\psi^k(\epsilon x) = \begin{cases} k\epsilon\psi^k(x), & |x| = 0; \\ \\ \epsilon\psi^k(x), & |x| = 1; \end{cases}$$

and the action of θ by

$$\theta(\epsilon x) = \begin{cases} \epsilon\psi(x), & |x| = 0; \\ \\ \epsilon\theta(x); & |x| = 1. \end{cases}$$

Recall that $\psi(x) = x^p + p\theta(x)$ is linear in x. The action of A on ΩA is the obvious one:

$$a(\epsilon x) = \epsilon(ax).$$

The resulting split square-zero extension can be written

$$\Omega A \rtimes A \overset{\text{def}}{=} A[\epsilon]$$

where $|\epsilon| = 1$ and with $\psi^k(\epsilon) = k\epsilon$, $\theta(\epsilon) = \epsilon$, and $\epsilon^2 = 0$. This mimics K-theory: if X is an E_∞ ring spectrum, then there is a natural isomorphism of theta-algebras

$$K_*F(S^1, X) \cong (K_*X)[\epsilon]. \tag{2.2.3}$$

Indeed, the natural pairing $F(S^1, S^0) \wedge X \to F(S^1, X)$ defines the isomorphism

$$K^*(S^1)\hat{\otimes}K_*X \xrightarrow{\cong} K_*F(S^1, X).$$

Compare Example 2.2.6.

2.2.10 Example. The functor $\Omega(-)$ can be extended to modules as well. If A is a theta-algebra and M is an A-module, define ΩM to be the shifted graded $\mathbb{Z}_p$ module with

$$\psi^k(\epsilon x) = \begin{cases} k\epsilon\psi^k(x), & |x| = 0; \\ \\ \epsilon\psi^k(x), & |x| = 1; \end{cases}$$

and

$$\theta(\epsilon x) = \begin{cases} \epsilon p\theta(x), & |x| = 0; \\ \\ \epsilon\theta(x); & |x| = 1. \end{cases}$$

Of course, if $a \in A$ and $x \in M$, then $a(\epsilon x) = \epsilon(ax)$.

This definition of $\Omega(-)$ and the one given in the previous example dovetail in the following way. There is a split short exact sequence of $M \rtimes A$ modules

$$\begin{tikzcd} 0 \arrow[r] & \Omega M \arrow[r] & \Omega(M \rtimes A) \arrow[r, shift left] & \Omega A \arrow[l, shift left] \arrow[r] & 0 \end{tikzcd}$$

and the action of $M \rtimes A$ on ΩM factors through A.

We can iterate the functor Ω to form a functor Ω^k. For example, if M is an A-module, then $\Omega^{2n} M \cong M$ as an ordinary $\mathbb{Z}/2\mathbb{Z}$-graded A-module, but we write $\epsilon_{2n} x$ for x under this identification, then

$$\psi^k(\epsilon_{2n} x) = k^n \epsilon_{2n} \psi^k(x) \qquad \text{and} \qquad \theta(\epsilon_{2n} x) = p^n \epsilon_{2n} \theta(x).$$

We now show that we have listed all the possible operations in the p-complete K-theory of E_∞ ring spectra. As in Definition 2.2.1, let C_{K_*K} denote the category of Morava modules. The following also follows from results of McClure in Chapter IX of [12]. Let C be any operad weakly equivalent to the commutative monoid operad. Then $C(-)$ is a model for the free E_∞-algebra functor on spectra. (See Theorem 1.2.4.)

2.2.11 Theorem. *The forgetful functor* $\mathbf{Alg}_\theta \to C_{K_*K}$ *sending a theta-algebra to the underlying module over the Adams operations has a left adjoint* S_θ*. Furthermore, if* X *is a cofibrant spectrum so that* K_*X *is torsion free, then the natural map*

$$S_\theta(K_*X) \longrightarrow K_*(CX)$$

is an isomorphism.

2.2.12 Remark. It is possible to write down a formula for S_θ. There is a category $\mathbf{Alg}^0_\theta$ of continuous graded $\mathbb{Z}_p$-algebras equipped with an operation θ satisfying such conditions that there is a forgetful functor $\mathbf{Alg}_\theta \to \mathbf{Alg}^0_\theta$. The forgetful functor from $\mathbf{Alg}^0_\theta$ all the way down to continuous $\mathbb{Z}_p$ modules has a left adjoint which, by abuse of notation, we also call S_θ. The abuse is not great: if M is a continuous $\mathbb{Z}_p$-module the two obvious meanings of $S_\theta(M)$ in $\mathbf{Alg}^0_\theta$ agree up to natural isomorphism. Calculations can now be made using two basic facts. First, there is a natural isomorphism

$$S_\theta(M_1) \hat{\otimes} S_\theta(M_2) \xrightarrow{\cong} S_\theta(M_1 \oplus M_2).$$

The source of this isomorphism is the completed tensor product. Second, if $M = \mathbb{Z}_p$ with generator x we have a completed polynomial algebra

$$S_\theta(\mathbb{Z}_p) \cong \mathbb{Z}_p[x, \theta(x), \theta^2(x), \ldots]^\wedge_p$$

if M is concentrated in degree 0, and a completed exterior algebra

$$S_\theta(\mathbb{Z}_p) \cong \Lambda[x, \theta(x), \theta^2(x), \ldots]^\wedge_p$$

if M is concentrated in degree 1.

For our applications, we would like to write down a model category structure on simplicial theta-algebras so that the cofibrant objects are s-free on a set of objects of the form $S_\theta(M)$, where M is a free continuous $\mathbb{Z}_p$-module. This can be done using the arguments used in [19]. We will give an outline here.

2.2.13 Lemma. *Let $A = \{M_\alpha\}$ be a set with one representative for each isomorphism class of Morava modules which are free and finitely generated as $\mathbb{Z}_p$-modules. Then the elements of the set A generate the category C_{K_*K} of Morava modules.*

Proof. We reduce to a simpler case. There is an isomorphism of topological groups $\mathbb{Z}_p^\times \cong G \times \mathbb{Z}_p$ where G is a finite cyclic group. Let $C^0_{K_*K}$ be the category of continuous modules over the profinite group ring $\mathbb{Z}_p[[\mathbb{Z}_p]]$ modules M so that M/p^nM is discrete for all n. Then there is a forgetful functor $C_{K_*K} \to C^0_{K_*K}$ with a left adjoint given by inducing up along G. We will show C_{K_*K} has a set of generators $\{N_\alpha\}$ where with each element free and finitely generated as a $\mathbb{Z}_p$-module. Since our set A includes the classes of modules obtained by inducing up the modules N_α, the result will follow.

By choosing a topological generator $\gamma \in \mathbb{Z}_p$, we obtain an isomorphism $\mathbb{Z}_p[[t]] \to \mathbb{Z}_p[[\mathbb{Z}_p]]$ sending t to $\gamma - 1$. (This is an old result of Serre, and easy to prove.) So we can translate our problem as follows. Let M be a $\mathbb{Z}_p[[t]]$-module with the property that every element in M/pM has a non-trivial annihilator ideal in $\mathbb{F}_p[[t]]$. Let $x \in M$. Then we show there is a $\mathbb{Z}_p[[t]]$-module N which is free and finitely generated as a $\mathbb{Z}_p$-module and a morphism $N \to M$ of $\mathbb{Z}_p[[t]]$-modules so that x is in the image. Note that we may assume that M is cyclic as a $\mathbb{Z}_p[[t]]$-module and generated by x.

Let $I \subseteq \mathbb{Z}_p[[t]]$ be the annihilator ideal of x. Since the annihilator ideal of $x + pM \in M/pM$ must be of the form $(t^n) \subseteq \mathbb{F}_p[[t]]$ for some n, $1 \leq n < \infty$, I is non-trivial; in fact, there is a sequence of surjections

$$I \longrightarrow I/pI \longrightarrow (t^n).$$

In particular, there is an element $g(t) \in I$ so that $g(t)$ is congruent to t^n mod p. If we apply the Weierstrass preparation theorem to $g(t)$, we see we may assume that $g(t)$ is the of the form

$$t^n + a_{n-1}t^{n-1} + \cdots + a_1 t + a_0$$

where $a_i = 0$ mod p. Then we set $N = \mathbb{Z}_p[[t]]/(g(t))$, and the result follows. □

2.2.14 Remark. From the previous proof it is easy to see that each of the elements M_α of the set of generators A of C_{K_*K} is a cyclic $\mathbb{Z}_p[[\mathbb{Z}_p^\times]]$-module with a preferred generator x_α. Define a diagram of these generators by specifying a morphism of Morava modules $M_\beta \to M_\alpha$ if and only if $x_\beta \mapsto x_\alpha$. Then we immediately have that for all Morava modules M, evaluation at the generators yields a natural isomorphism

$$\operatorname{colim}_\alpha \operatorname{Hom}_{C_{K_*K}}(M_\alpha, M) \xrightarrow{\cong} M.$$

Note that every object in the set A of generators is small in C_{K_*K}. Then the arguments in [19], §§3 and 4 immediately imply the following. Give sC_{K_*K} the standard structure of a simplicial category: see [35], §II.2.

2.2.15 Proposition. *The category sC_{K_*K} of simplicial Morava modules supports the structure of a cofibrantly generated simplicial model category where $f : X \to Y$ is*

1. *a weak equivalence if $\pi_* f$ is an isomorphism; and*
2. *a cofibration if it is a retract of a morphism which is s-free on set $\{Z_n\}$ of Morava modules with each Z_n a coproduct of objects in the generating set A.*

Furthermore, the cofibrations are generated by the set I of morphisms

$$M_\alpha \otimes \partial\Delta^n \to M_\alpha \otimes \Delta^n$$

with $n \geq 0$ and $M_\alpha \in A$.

2.2.16 Remark. This model category is the localization of an auxiliary model category created from the generators M_α. Compare Remark 2.5.1.

This result and the standard lifting lemmas (in [25], for example) imply the result we want. Similar arguments appear in [19]. Again give $s\mathbf{Alg}_\theta$ the standard structure of a simplicial category.

2.2.17 Theorem. *The category $s\mathbf{Alg}_\theta$ of simplicial theta-algebras supports the structure of a cofibrantly generated simplicial model category where $f : X \to Y$ is*

1. *a weak equivalence if $\pi_* f$ is an isomorphism; and*
2. *a cofibration if it is a retract of a morphism which is s-free on set $\{S_\theta(Z_n)\}$ of Morava modules with each Z_n a coproduct of objects in the generating set A.*

We can immediately write down the following consequence of the fact that every object in the generating set is free as a continuous $\mathbb{Z}_p$-module. Give the category $s\mathbf{Alg}_{\mathbb{Z}_p}$ of simplicial commutative continuous $\mathbb{Z}_p$ algebras the usual simplicial model category structure of [35] §II.4.

2.2.18 Corollary. *The forgetful functor from the category $s\mathbf{Alg}_\theta$ of simplicial theta-algebras to $s\mathbf{Alg}_{\mathbb{Z}_p}$ preserves cofibrations.*

2.3 Homotopy push-outs of simplicial algebras

The category of simplicial algebras over a simplicial operad is often not left proper, and we seek to give a condition which serves as an acceptable substitute. We will state this condition in Definition 2.3.3 and then show the condition is satisfied when the operad is E_∞ or A_∞. Mandell has related results in the E_∞ case. See [31].

Recall that the category of simplicial algebras over an operad supports in the standard simplicial model category structure. Thus, we let $C = C_\bullet$ be a simplicial operad in R-modules and $s\mathbf{Alg}_C$ the category of simplicial algebras over C. This is a simplicial category in the external simplicial structure; for example, if K is a simplicial set and $X \in s\mathbf{Alg}_C$ then

$$(A \otimes K)_n = \coprod_{K_n} A_n$$

with the coproduct in C_n-algebras. Also, among the morphisms of $s\mathbf{Alg}_C$ we single out the *free* maps: a morphism $X \to Y$ is free if the underlying morphism of degeneracy diagrams is isomorphic to a map of the form

$$X \to X \sqcup C(Z)$$

where Z is a s-free diagram on a free R-module. The definition of s-free is in Definition 1.1.9.

The main theorem of [35] §II.4 immediately implies the following:

2.3.1 Proposition. *The category $s\mathbf{Alg}_C$ has the structure of a simplicial model category with a morphism $f : X \to Y$*

1. *a weak equivalence if $\pi_* f : \pi_* X \to \pi_* Y$ is an isomorphism;*
2. *a fibration if the induced map $Nf : NX \to NY$ of normalized chain complexes in R-modules is surjective in positive degrees;*
3. *a cofibration if it is a retract of a free map.*

Recall that a model category is *left proper* if whenever there is a push-out square

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "j"'] & B \arrow[d] \\ X \arrow[r, "g"] & Y \end{tikzcd}$$

with j a cofibration and f a weak equivalence, then g is a weak equivalence. For example, the category of simplicial associative algebras is not left proper; see Example 2.3.4. This lends teeth to the following example.

2.3.2 Example. Let R be a commutative ring. Then the category of simplicial commutative R-algebras is left proper. Suppose we are given a two-source $B \leftarrow A \to X$ with $A \to X$ a cofibration. Then, by [35], §II.6, there is a spectral sequence

$$\mathrm{Tor}^{\pi_* A}(\pi_* B, \pi_* X) \Rightarrow \pi_* B \otimes_A X.$$

Since $B \otimes_A X$ is the push-out in simplicial R-algebras, the claim follows. Exactly the same argument shows that the category $s\mathbf{Alg}_\theta$ of simplicial theta-algebras is left proper.

2.3.3 Definition. *Fix a commutative ring R and a simplicial operad C of R-modules. The model category $s\mathbf{Alg}_C$ of simplicial C-algebras is* relatively left proper *if*

1. *whenever W is cofibrant simplicial R-module and $f : A \to B$ is a weak equivalence between simplicial C-algebras which are cofibrant as simplicial R-modules, then*
$$A \sqcup C(W) \to B \sqcup C(W)$$
is a weak equivalence, and
2. *any cofibrant $X \in s\mathbf{Alg}_C$ is cofibrant as a simplicial R-module.*

2.3.4 Example. The category of simplicial associative algebras is not left proper, but is relatively left proper. For if C is the free associative algebra functor, W is an R-module and A any associative algebra, then

$$A \sqcup C(W) \cong \bigoplus_{n\geq 0} A \otimes W \otimes A \cdots A \otimes W \otimes A$$

where W appears n times and A appear $(n+1)$ times in the nth summand. This follows from the fact that $A \sqcup C(W)$ is the free algebra under A on the A-bimodule $A \otimes W \otimes A$.

In order to explore the implications of this relative notion of properness, we will use the following standard observation.

2.3.5 Lemma. *Let $X \in s(s\mathbf{Alg}_C)$ be a simplicial object in the category of simplicial C-algebras. Then the geometric realization of X is the diagonal:*

$$|X| \simeq \operatorname{diag} X = \{X_{n,n}\}.$$

The nomenclature “relatively left proper” is justified by the next result.

2.3.6 Lemma. *Let*

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "j"'] & B \arrow[d] \\ X \arrow[r, "g"] & Y \end{tikzcd}$$

be a push-out square in $s\mathbf{Alg}_C$ with j a cofibration and f a weak equivalence between objects which are cofibrant as simplicial R-modules. Then g is a weak equivalence.

Proof. In the simplicial model category $s\mathbf{Mod}_R$, define objects $R \wedge \Delta^n/\partial\Delta^n$, $n \geq 0$, by the push-out diagram

$$\begin{tikzcd} R \cong R \otimes * \arrow[r] \arrow[d] & R \otimes \Delta^n/\partial\Delta^n \arrow[d] \\ 0 \arrow[r] & R \wedge \Delta^n/\partial\Delta^n. \end{tikzcd}$$

This forms a collection of cofibrant cogroup objects in $s\mathcal{M}_R$; hence the objects

$$C(R \wedge \Delta^n/\partial\Delta^n) \in s\mathbf{Alg}_C$$

form a set of cofibrant cogroup objects. Let

$$A \to W_\bullet \to X$$

be a factorization of $A \to X$ as a cofibration followed by a weak equivalence in the resolution model category on $s(s\mathbf{Alg}_C)$ determined by these objects. (The bullet ($\bullet$) here refers to the new, external, simplicial degree.) Then $|W_\bullet| \to X$ is a weak equivalence of cofibrant objects in the under category $A/s\mathbf{Alg}_C$. Since every object of this under category is fibrant, this map is necessarily a homotopy equivalence under A. It follows that

$$B \sqcup_A |W_\bullet| \to B \sqcup_A X \cong Y$$

is a homotopy equivalence. Thus we need only show

$$|W_\bullet| \to B \sqcup_A |W_\bullet| \cong |B \sqcup_A W_\bullet|$$

is a weak equivalence. By the previous lemma, it is enough to show

$$W_n \to B \sqcup_A W_n$$

is a weak equivalence. This is a retract of a morphism of the form

$$A \sqcup C(Z) \to B \sqcup C(Z)$$

where $Z \cong \bigoplus_\alpha R \wedge \Delta^{n_\alpha}/\partial\Delta^{n_\alpha}$. This map is a weak equivalence by the definition of relatively left proper. □

We now prove:

2.3.7 Proposition. *Suppose* $s\mathbf{Alg}_C$ *is relatively left proper and*

$$\begin{tikzcd} A \arrow[r] \arrow[d, "j"'] & B \arrow[d] \\ X \arrow[r] & Y \end{tikzcd}$$

is a push-out diagram. If j *is a cofibration and* A, B *are cofibrant in* $s\mathbf{Mod}_R$*, then* Y *is weakly equivalent to the homotopy push-out in* $s\mathbf{Alg}_C$*.*

Proof. Choose a surjective weak equivalence $A_0 \to A$ with A_0 cofibrant and, as in the proof of Lemma 2.3.6, let

$$A_0 \to W_\bullet \to X$$

be a factorization of $A_0 \to X$ as a cofibration followed by a weak equivalence in the resolution model category structure on $s(s\mathbf{Alg}_C)$. By Lemma 2.3.6,

$|W_\bullet| \to X$ is a weak equivalence; furthermore $A_0 \to |W_\bullet|$ is a cofibration. Factor $A_0 \to B$ as $A_0 \to B_0 \to B$ where the first map is a cofibration and the second a weak equivalence – now simply in $s\mathbf{Alg}_C$. Then

$$B_0 \sqcup_{A_0} |W_\bullet| \cong |B_0 \sqcup_{A_0} W_\bullet|$$

is a model for the homotopy push-out.

There is a homotopy equivalence under A_0

$$W_n \simeq A_0 \sqcup C(Z_n) \tag{2.3.1}$$

where Z_n is a sum of copies of $R \wedge \Delta^k/\partial\Delta^k$. From this it follows that there is a homotopy equivalence under A

$$A \sqcup_{A_0} W_n \simeq A \sqcup C(Z_n)$$

and, hence,

$$W_n \to A \sqcup_{A_0} W_n$$

is a weak equivalence. Thus, the natural map

$$|A \sqcup_{A_0} W_n| \cong A \sqcup_{A_0} |W_n| \to A \sqcup_{A_0} X \cong X$$

is a weak equivalence. The last isomorphism uses that $A \to A_0$ is surjective and that we already have the map $j : A \to X$. Since $A \to A \sqcup_{A_0} |W_n|$ is a cofibration and every object of $s\mathbf{Alg}_C$ is fibrant we have (as in Lemma 2.3.6) that

$$|B \sqcup_{A_0} W_\bullet| = B \sqcup_A (A \sqcup_{A_0} |W_\bullet|) \to Y$$

is a weak equivalence. By (2.3.1) we have that for each n

$$B_0 \sqcup_{A_0} W_n \to B \sqcup_{A_0} W_n$$

is a homotopy equivalent to

$$B_0 \sqcup C(Z_n) \to B \sqcup C(Z_n).$$

This map is a weak equivalence and the result follows. □

It is possible to prove that any *cofibrant* simplicial operad in R-modules is relatively left proper. We will make a remark on this below, but this result is tangential to our project here, so we won't elaborate. More important is the case of an E_∞-operad; we want any such operad to be relatively left proper. This result can be obtained from [31], but we will give an outline here as well.

We begin a decomposition which we learned from Charles Rezk. For each simplicial operad C, each C-algebra A, and each $k \geq 0$, we claim there is a $R[\Sigma_k]$ module $D^k_C A$ so that $D^0_T A = A$ and there is an isomorphism of simplicial R-modules

$$A \sqcup C(W) \cong \bigoplus_k D^k_C A \otimes_{R[\Sigma_k]} W^{\otimes k}. \tag{2.3.2}$$

The isomorphism is natural in A, C, and W. We have $D^0_T A \cong A$.

To see this, first note that if $A = C(A_0)$ for some simplicial R-module A_0, then

$$A \sqcup C(W) = \oplus_k[\oplus_n C(n+k) \otimes_{R[\Sigma_n]} A_0^{\otimes n}] \otimes_{R[\Sigma_k]} W^{\otimes k}$$

which gives

$$D^k_C C(A_0) = \oplus_n C(n+k) \otimes_{R[\Sigma_n]} A_0^{\otimes n}.$$

For more general A, we write down a coequalizer diagram

$$D^k_C C^2(A) \rightrightarrows D^k_C C(A) \longrightarrow D^k_C A. \tag{2.3.3}$$

The parallel arrows

$$\oplus_n C(n+k) \otimes_{R[\Sigma_n]} C(A)^{\otimes n} \rightrightarrows \oplus_n C(n+k) \otimes_{R[\Sigma_n]} A^{\otimes n}$$

are given respectively by the evaluation $C(A) \to A$ and the partial operad maps

$$C(n+q) \otimes C(m_1) \otimes \cdots \otimes C(m_n) \to C(m_1 + \cdots m_n + q). \tag{2.3.4}$$

2.3.8 Definition. *Let R be a commutative ring. Then a simplicial E_∞-operad over R is an augmented simplicial operad $C \to$ **Comm** with the properties that*

1.) *The augmentation induces an isomorphism $\pi_* C \to$ **Comm**; and*

2.) *for all n, the simplicial $R[\Sigma_n]$-module $C(n)$ is cofibrant.*

The last requirement implies that $C(n)$ is level-wise projective as a $R[\Sigma_n]$-module.

2.3.9 Lemma. *Let C be an E_∞-operad in simplicial R-modules and let A be any C-algebra. Then there is a natural zig-zag of homotopy equivalences of $R[\Sigma_k]$-modules between $D^k_C A$ and $C(k) \otimes A$.*

Proof. The operad multiplication

$$\mu : C(2) \otimes C(n) \otimes C(k) \longrightarrow C(n+k)$$

supplies a weak equivalence between cofibrant simplicial $R[\Sigma_k]$-modules. As a result, μ has a homotopy inverse in this category. From this we obtain, for all simplicial R-modules A_0, a homotopy equivalence of simplicial R-modules

$$[\oplus_n C(2) \otimes C(n) \otimes C(k)] \otimes_{R[\Sigma_n]} A_0^{\otimes n} \longrightarrow \oplus_n C(n+k) \otimes_{R[\Sigma_n]} A_0^{\otimes n} = D^k_C C(A_0).$$

The equalizer diagram in Equation 2.3.3 – and the description below that equation of the two maps to be equalized – now yields a homotopy equivalence of simplicial R-modules

$$C(2) \otimes C(k) \otimes A \to D^k_C A$$

for any A. Since this a morphism of simplicial $R[\Sigma_k]$-modules, it is a weak equivalence of simplicial $R[\Sigma_k]$-modules. To complete the zig-zag, take the projection

$$C(2) \otimes C(k) \otimes A \longrightarrow R \otimes C(k) \otimes A.$$

Since $C(2) \otimes C(k) \to C(k)$ is a weak equivalence of cofibrant simplicial $R[\Sigma_k]$-modules, we obtain a homotopy equivalence. □

2.3.10 Remark. If C is a cofibrant simplicial operad a more delicate argument using the language of trees analyzes $D^k_C A$ and shows that $s\mathbf{Alg}_C$ is also relatively left proper.

The following is immediately obvious from Lemma 2.3.9 and Equation 2.3.2.

2.3.11 Proposition. *Let C be an E_∞-operad in simplicial R-modules and A a C-algebra. Then:*

1.) If W is any simplicial R-module, there is a natural zig-zag of weak equivalences between the simplicial R-modules $A \sqcup C(W)$ and $A \otimes C(W)$.

2.) Let B be a cofibrant C-algebra. Then there is a natural zig-zag of weak equivalences between the simplicial R-modules $A \sqcup B$ and $A \otimes B$.

3.) The model category $s\mathbf{Alg}_C$ is relatively left proper.

Proof. For the first statement, the previous lemma supplies a natural zig-zag of homotopy equivalences between $A \sqcup C(W)$ and

$$[\oplus_k C(k) \otimes_{R[\Sigma_k]} W^{\otimes k}] \otimes A.$$

For the second statement, take resolution $W_\bullet \to B$ of B is $s(s\mathbf{Alg}_C)$ using the objects $C(R \wedge \Delta^n \partial \Delta^n)$ as the homotopy cogroup objects. (See the proof of Lemma 2.3.6.) Then $|W_\bullet| \to B$ is a weak equivalence between cofibrant C-algebras, hence a homotopy equivalence. Now part (1) supplies a homotopy equivalence of simplicial R-modules between $A \sqcup |W_\bullet|$ and $A \otimes |W_\bullet|$. The third statement follows immediately from the first. □

2.3.12 Corollary. *Let C be an E_∞-operad and suppose we are given a two-source*

$$\begin{tikzcd} X & A \arrow[l, "j"'] \arrow[r, "f"] & B \end{tikzcd}$$

in $s\mathbf{Alg}_C$ with j a cofibration and A and B cofibrant as simplicial R-modules. Then there is a spectral sequence

$$\mathrm{Tor}_p^{\pi_* A}(\pi_* X, \pi_* B)_q \Longrightarrow \pi_{p+q}(X \sqcup_A B).$$

Proof. This follows immediately from Lemma 2.3.7, Proposition 2.3.11.2 and the fact that we can use the bar construction to calculate the homotopy push-out. □

Let $f : A \to B$ a morphism of simplicial R-modules. Define $\pi_*(f)$ to be the homotopy groups of the morphism. If f is a cofibration, then this is simply the homotopy groups of the pair; more generally, it can be computed by replacing f by a cofibration. As always, we will write $\pi_*(B, A)$ when f is understood.

The following result is almost proved many places. See, for example, [6], §I.C.4 or [43]. The wrinkle here is that we have may have a simplicial operad.

2.3.13 Theorem. *Let $s\mathbf{Alg}_F$ be either the category of simplicial algebras over a simplicial E_∞-operad C, the category of simplicial theta-algebras, or the category* *of associative R-algebras. Suppose we are given a homotopy push-out diagram in $s\mathbf{Alg}_C$*

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "j"'] & X \arrow[d] \\
B \arrow[r] & Y
\end{tikzcd}$$

and, furthermore, that $\pi_i(B,A) = 0$ for $i < m$ and $\pi_i(X,A) = 0$ for $i < n$. Then

$$\pi_i(B,A) \longrightarrow \pi_i(Y,X)$$

is an isomorphism for $i \leq n+m-2$ and onto for $i = n+m-1$.

Proof. The for the category of simplicial algebras over an E_∞-operad, we apply the spectral sequence of Corollary 2.3.12; similarly, for simplicial theta-algebras, apply the spectral sequence of Example 2.3.2. The case of simplicial associative algebras is covered by [6], §I.C.4. Alternatively, we could use a bar complex argument and the decomposition result of Example 2.3.4. □

2.3.14 Remark. The previous result is actually true for an arbitrary simplicial operad. This can be proved by adapting the methods of [7], Section 5. Indeed, these methods make it clear that this result really follows from very general considerations about functors from sets to itself.

2.3.15 Corollary. *Let $s\mathbf{Alg}_F$ be either the category of simplicial algebras over a simplicial E_∞-operad C, the category of simplicial theta-algebras, or the category of associative R-algebras. Suppose we are given a push-out diagram in $s\mathbf{Alg}_C$*

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "i"'] & X \arrow[d] \\
B \arrow[r] & Y
\end{tikzcd}$$

and, furthermore, that $\pi_i(B,A) = 0$ for $i < m$ and $\pi_i(X,A) = 0$ for $i < n$. Then there is a partial long exact sequence

$$\pi_{m+n-2}(B) \oplus \pi_{m+n-2}(X) \to \pi_{m+n-2}(Y) \to \pi_{m+n-3}(A) \to \cdots \to \pi_0(Y) \to 0.$$

Proof. Given any commutative square (not necessarily a push-out), we can define two modules D_n and K_n by the formulas

$$D_m = (j_*)^{-1}\mathrm{Im}(\pi_m(B,A) \to \pi_m(Y,X)) \subseteq \pi_m Y$$

where $j_* : \pi_m(Y) \to \pi_m(Y,X)$ is the natural map and

$$K_{m-1} = \pi_{m-1}A/\delta(\mathrm{Ker}(\pi_m(B,A) \to \pi_m(Y,X)).$$

where $\delta : \pi_m(B,A) \to \pi_{m-1}A$ is connecting map. Note that D_* and K_* are functors of the square; furthermore, $D_m = \pi_m(Y)$ if $\pi_m(B,A) \to \pi_m(Y,X)$ is onto and $K_{m-1} = \pi_{m-1}(A)$ if $\pi_m(B, A) \to \pi_m(Y, X)$ is one-to-one. A diagram chase shows there is a long exact sequence

$$\cdots \to D_{m+1} \to K_m \to \pi_m(B) \oplus \pi_m(X) \to D_m \to K_{m-1} \to \cdots \to D_0 \to 0.$$

The result now follows easily. □

2.4 André-Quillen cohomology

If A is a commutative algebra over a commutative ring R, M an A-module and $X \to A$ a morphism of R-algebras, then the André-Quillen cohomology of X with coefficients in M is the non-abelian right derived functors of the functor

$$X \mapsto \mathrm{Der}_R(X, M)$$

which assigns to X the A-module of R-derivations from X to M. This cohomology has natural generalization to algebras over operads and their modules; it also has a generalization to theta-algebras and their modules. Indeed, much of the formalism of Quillen's paper [34] goes through without difficulty – in the theta-algebra case the formalism is nearly identical. This section outlines the details and gives an example of an application to the computation of the homotopy type of the space of maps between between $K(1)$-local E_∞-ring spectra.

2.4.1 Cohomology of algebras over operads

This first part is written algebraically. We fix a commutative ring R, possibly graded, and we consider R-modules (again possibly graded), operads in R-modules, and so on. All tensor products will be over R. In our applications R will be E_* for some homotopy commutative ring spectrum E. Any omitted details can be found in [19].

Let C be an operad in R-modules and suppose A is a C algebra. We define what it means for M to be an A-module. Let $\Phi(A, M)$ to be the graded R-module with

$$\Phi(A, M)_n = \bigoplus_i A \otimes \cdots \otimes A \otimes \underset{i}{M} \otimes A \otimes \cdots \otimes A$$

with each summand having n-terms, M appearing once in each summand and then in the ith slot. Note that $\Phi(A, M)_n$ has an obvious action of the symmetric group Σ_n. Define

$$C(A, M) = \bigoplus_n C(n) \otimes_{k\Sigma_n} \Phi(A, M)_n = \bigoplus_n C(n) \otimes_{R\Sigma_{n-1}} A^{\otimes(n-1)} \otimes M.$$

It is an exercise to show that there is a natural isomorphism of bifunctors

$$C(C(A), C(A, M)) \cong (C \circ C)(A, M)$$

where $(\cdot) \circ (\cdot)$ is the composition of operads. The R-module M is an *A-module over C* (or simply an A-module) if there is a morphism of k-modules $\eta : C(A, M) \to M$ which fits into a coequalizer diagram

$$C(C(A), C(A, M)) \cong (C \circ C)(A, M) \overset{d_0}{\underset{d_1}{\rightrightarrows}} C(A, M) \xrightarrow{\eta} M$$

where the maps d_0 and d_1 are induced by the operad multiplication of C, and by η and the algebra structure on A respectively. Furthermore, the unit $\mathbf{1} \to \mathcal{C}$ defines a morphism of R-modules $M = \mathbf{1}(A, M) \to C(A, M)$ which is required to be a section of η.

If A is a commutative R-algebra, and M is an A-module, we can can form a new commutative algebra over A called $M \rtimes A$, which as an R-module is simply $M \oplus A$, but with algebra multiplication

$$(x, a)(y, b) = (xb + ay, ab).$$

The algebra $M \rtimes A$ is an *infinitesimal extension* and an abelian object in the category of algebras over A; all abelian group objects in this category have this form.

Now let $k \to A$ be a morphism of commutative R-algebras. Then $A \rtimes M$ represents the functor that assigns to an algebra over A the A-module of k-derivations from A to M:

$$\mathrm{Der}_k(X, M) \cong (\mathbf{Alg}_k/A)(A, M \rtimes A)$$

where we write $\mathbf{Alg}_k/A$ is the category of k-algebras over A; that is, $\mathbf{Alg}_k$ is the category of algebras over the commutative algebra operad over A and under k.

These concepts easily generalize. If C is an operad, A a C-algebra and M an A-module, define a new C-algebra over A called $M \rtimes A$ as follows: as a R-module $M \rtimes A$ is simply $M \oplus A$, but the C-algebra structure is defined by noting that there is a natural decomposition

$$C(M \oplus A) \cong E(A, M) \oplus C(A, M) \oplus C(A)$$

where $E(A, M)$ consists of those summands of $C(M \oplus A)$ with more than one M term. Since M is an A-module we get a composition

$$C(M \oplus A) \to C(A, M) \oplus C(A) \to M \oplus A$$

which defines the C-algebra structure on $M \rtimes A$. Again we obtain an abelian object in the category of algebras over A; again, all abelian objects have this form. This last observation makes it possible to *define* the category of A-modules over C to be the category of abelian C-algebras over A. For comparison, see Remark 2.2.8.

Note that if we are in a graded setting and M is an A-module, then the graded object $\Omega^t M$ with

$$(\Omega^t M)_k = M_{k+t}$$

is also an A-module. In this operadic setting, an obvious example of an A-module is A itself.

The object $M \rtimes A$ in the category of C-algebras over A represents an abelian group valued functor which we might as well call *derivations.* If $k \to A$ is a morphism of C-algebras and M is an A-module, we define

$$\mathrm{Der}_k(A, M) \overset{\mathrm{def}}{=} \mathbf{Alg}_k/A(A, M \rtimes A). \tag{2.4.1}$$

Such a derivation is determined by a R-module homomorphism $d : A \to M$ which fits into an appropriate diagram which reduces to the usual definition of derivation in the commutative or associative algebra case. We invite the reader to fill in the details.

Note that the definition of derivations in Equation 2.4.1 depends on the operad C; thus we might want C in the notation somewhere. But we hope that C will always be implicit from the discussion, so we leave it out.

Cohomology in this context should be derived functors of derivations; for this we need the model category structure on $s\mathbf{Alg}_C$ discussed in Proposition 2.3.1. We now allow ourselves the generality of a simplicial operad C in R-modules.

If $A \in s\mathbf{Alg}_C$ then $\pi_0 A$ is a $\pi_0 C$-algebra. If M is a $\pi_0 A$-module (over the operad $\pi_0 C$) then M is an A_n-module (over C_n) for all $n \geq 0$. Then we can form the simplicial module $K(M, n)$ over A whose normalization $NK(M, n) \cong M$ concentrated in degree n. From this object we can form the simplicial C-algebra $K_A(M, n) = K(M, n) \rtimes A$ over A. In following definition, we will use the notion of relatively left proper, which appeared in Definition 2.3.3.

2.4.1 Definition. *Suppose that C is a simplicial operad in R-modules so that the model category $s\mathbf{Alg}_C$ is relatively left proper. Let $k \to A$ be a morphism of simplicial C-algebras. Let X be a C-algebra under k and over A. Then* André-Quillen cohomology *of X with coefficients in M is defined by*

$$H^n_C(X/k, M) \overset{\mathrm{def}}{=} [X, K_A(M, n)]_{s\mathbf{Alg}_k/A} \cong \pi_0 \operatorname{map}_{s\mathbf{Alg}_k/A}(X, K_A(M, n)).$$

Here we are writing $s\mathbf{Alg}_k/A$ for simplicial C-algebras over A and under k and in this formula we mean, as always, the derived mapping space. If C is understood, we will write $H^*(X/k, M)$; if k is the initial object in $s\mathbf{Alg}_C$, we may write simply $H^*(X, M)$.

We note immediately that there are natural isomorphisms

$$H^{n-i}_C(X/k, M) \cong \pi_i \operatorname{map}_{s\mathbf{Alg}_k/A}(X, K_A(M, n))$$

and that, in fact, the collection of spaces $\operatorname{map}_{s\mathbf{Alg}_k/A}(X, K_A(M, n))$, $n \geq 0$, assemble into a spectrum $\hom_{s\mathbf{Alg}_k/A}(X, K_A M)$ so that

$$H^n_C(X/k, M) \cong \pi_{-n} \hom_{s\mathbf{Alg}_k/A}(X, K_A M).$$

2.4.2 Remark. We have defined a relative André-Quillen cohomology for a morphism $k \to A$ of simplicial C-algebras. At this level of generality, it may actually be necessary to resolve the source k as well as the target A to get a good theory. By this we mean that we ought to choose a cofibrant model $k' \to k$ for k as a simplicial C-algebra and set

$$H^n_C(X/k, M) = \pi_0 \operatorname{map}_{s\mathbf{Alg}_{k'}/A}(X, K_A(M, n)),$$

again using the derived mapping space. Only in this way, for example, do we get a transitivity sequence for this cohomology theory. (See Remark 2.4.3 below.) However, we have assumed that the category $s\mathbf{Alg}_C$is relatively left proper. Then if k is projective as R-module, Lemma 2.3.7, implies for all weak equivalences $f : k' \to k$ restriction yields an adjoint pair

$$f^* = k \sqcup_{k'} (-) s\mathbf{Alg}_{k'} \rightleftarrows s\mathbf{Alg}_k : f_*$$

which is part of a Quillen equivalence. Hence the more naïve definition of André-Quillen cohomology given above agrees with the definition wherein one also resolves k. The more general situation is discussed in [19].

2.4.3 Remark (Transitivity Sequence). As defined, there is a long exact sequence, or *transitivity sequence* for André-Quillen cohomology. Suppose we have a sequence of C-algebras $k \to X \to A$ and suppose that M is a $\pi_0 A$-module. Then there is a homotopy pull-back square

(2.4.2)
$$\begin{tikzcd}
\operatorname{map}_{s\mathbf{Alg}_X/A}(A, K_A(M, n)) \arrow[r] \arrow[d] & \operatorname{map}_{s\mathbf{Alg}_k/A}(A, K_A(M, n)) \arrow[d] \\
\{s\} \arrow[r] & \operatorname{map}_{s\mathbf{Alg}_k/A}(X, K_A(M, n))
\end{tikzcd}$$

where s is composition $X \to A \to K_A(M, n)$ induced by the zero-section. Hence there is a long exact sequence

$$\begin{aligned} \cdots \to H^n_C(A/X, M) \to H^n_C(A/k, M) \to & H^n_C(X/k, M) \\ & \to H^{n+1}_C(A/X, M) \to \cdots . \end{aligned}$$

To get the fiber sequence 2.4.2, choose a commutative square

$$\begin{tikzcd}
X' \arrow[r, "j"] \arrow[d, "\simeq"'] & A' \arrow[d, "\simeq"] \\
X \arrow[r] & A
\end{tikzcd}$$

where the vertical maps are weak equivalences, X' is cofibrant in $s\mathbf{Alg}_k$ and j is a cofibration in $s\mathbf{Alg}_k$. Then, because we have assumed that $s\mathbf{Alg}_C$ is relatively left proper, the induced map

$$X \sqcup_{X'} A' \longrightarrow A$$

is a weak equivalence and the source is cofibrant in $s\mathbf{Alg}_X$. Then we have a pull-back

$$\begin{tikzcd}
\mathrm{map}_{s\mathbf{Alg}_X/A}(X \sqcup_{X'} A', K_A(M,n)) \arrow[r] \arrow[d] & \mathrm{map}_{s\mathbf{Alg}_k/A}(A', K_A(M,n)) \arrow[d] \\
\{s\} \arrow[r] & \mathrm{map}_{s\mathbf{Alg}_k/A}(X', K_A(M,n))
\end{tikzcd}$$

as needed.

2.4.4 Remark. In all the applications we have in mind both k and A will be *constant* simplicial C-algebras, or equivalently, k and A will be $\pi_0 C$-algebras, regarded as constant simplicial C-algebras. In this case, we have a natural isomorphism

$$H^0_C(A/k, M) = \mathrm{Der}_k(A, M).$$

Also, André-Quillen cohomology can be written down as the cohomology of a chain complex.

To do this, suppose $k \to A$ a morphism of constant C-algebras. Let M be a $A = \pi_0 A$-module. Then for any simplicial C-algebra Y under k and over A, we have abelian groups

$$\mathrm{Der}_k(Y_n, M) = (\mathbf{Alg}_k/A)(Y_n, M \rtimes A).$$

Furthermore, if $\phi : [n] \to [m]$ is a morphism in the ordinal number category, the Y_n is a C_m-algebra by restriction of structure along $\phi^* : C_m \to C_n$ and then

$$\phi^* : Y_m \longrightarrow Y_n$$

is a morphism of C_m-algebras. Hence we get a map

$$\mathrm{Der}_k(Y_n, M) \longrightarrow \mathrm{Der}_k(Y_m, M)$$

and, in fact, $\mathrm{Der}_k(Y, M)$ becomes a cosimplicial abelian k-module. Then, if $X \in s\mathbf{Alg}_k/A$, we have

$$H^n_C(X/k, M) = H^n N\,\mathrm{Der}_C(Y, M) \tag{2.4.3}$$

where Y is some cofibrant model for X and N is the normalization functor from cosimplicial k-modules to cochain complexes of k-modules. This concept is important enough that we will write

$$\mathbb{D}_C(X/k, M) \in \mathbf{Ho}(\mathrm{Ch}^* k) \tag{2.4.4}$$

for the well-defined object in the derived category of cochain complexes defined by $N\,\mathrm{Der}_k(Y, M)$, with Y a cofibrant model for A.

2.4.5 Example (Cohomology of associative algebras). This discussion applies to the case where $k \to A$ is a morphism of associative algebras over our ground ring R. If k is *commutative* and k is central in A, then $H^*(A/k, M)$ is, by the results of [34], closely related to the Hochschild cohomology of the k-algebra A. In this case, an A-module is an A-bimodule and there are isomorphisms

$$H^s(A/k, M) \cong HH^{s+1}(A/k, M), \quad s \geq 1$$

and an exact sequence

$$\begin{tikzcd} 0 \arrow[r] & Z(M) \arrow[r, "\subseteq"] & M \arrow[r, "f"] & H^0(A/k, M) \arrow[r] & HH^1(A/k, M) \arrow[r] & 0 \end{tikzcd}$$

where

$$Z(M) = HH^0(A/k, M) = \{x \in M \mid ax = xa \text{ for all } a \in A\ \}$$

and f sends $x \in M$ to the derivation $\partial_x \in \mathrm{Der}_k(A, M) = H^0(A/k, M)$ given by

$$\partial_x(a) = ax - xa.$$

2.4.6 Example (Cohomology over an E_∞ operad). Recall that we defined an E_∞-operad to be a simplicial operad C of R-modules so that each $C(k)$ is a cofibrant $R[\Sigma_k]$-module and so that there is a weak equivalence of operads $C \to \mathbf{Comm}$ to the commutative algebra operad.

If A is a commutative R-algebra and M is an A-module, we can – by using the augmentation – regard A as a constant C-algebra and M as an A-module over C. Hence we may form the André-Quillen cohomology groups $H^*_C(A/k, M)$ for any morphism $k \to A$ of commutative R-algebras. These groups turn out to be a independent of the choice of C, and are naturally isomorphic to almost any other version of E_∞-algebra cohomology of A one might possible contrive. In particular, by work of Mandell [32] $H^*_C(A/k, M)$ is isomorphic to the topological André-Quillen cohomology of the Eilenberg-MacLane spectrum HA regarded as an Hk-algebra and, combining this with work of Basterra and McCarthy [5], $H^*_C(A/k, M)$ is also isomorphic to the Γ-cohomology of the k-algebra A as defined by Robinson and Whitehouse in [42].

2.4.2 Cohomology of algebras in comodules

In our applications we will have a homology theory $E_*(\cdot)$ and $R = E_*$. We will also have a simplicial operad T – that is, a simplicial object in the category $\mathcal{O}$ of simplicial operads – so $C = E_*T$ and a typical C-algebra will be of the form E_*X where $X \in s\mathbf{Alg}_T$. If E_*E if flat over E_*, this will imply that we are actually working with operads, algebras and so forth in the category of E_*E-comodules, rather than simply in the more basic category of E_*-modules. Under appropriate hypotheses – for example, if E satisfies the Adams condition of Definition 1.4.1 – the E_*E-comodule version of Proposition 2.4.7 is true, and one can use this to define André-Quillen cohomology in the category of E_*E-comodules.

To do this requires a little care, as we are forced to resolve not only algebras, but also the modules; the short reason for this technical difficulty is that not every chain complex of comodules is fibrant. The same problem arose in [30] and our solution is not much different.

To get started, fix a simplicial operad C in E_*E-comodules and a $\pi_0 C$ algebra A, also all in E_*E-comodules.

To ease notation, let us abbreviate the extended comodule functor by

$$\Gamma(M) = E_*E \otimes_{E_*} M.$$

The functor Γ also induces a right adjoint to the forgetful functor from A-modules in E_*E-comodules to A-modules. Indeed, if M is an A-module, the module structure on $\Gamma(A)$ is determined by the top split row of the diagram

$$\begin{tikzcd}
\Gamma(M) \arrow[r] \arrow[d, "="'] & \Gamma(M) \rtimes A \arrow[r, shift left] \arrow[d] & A \arrow[l, shift left] \arrow[d, "\psi_A"] \\
\Gamma(M) \arrow[r] & \Gamma(M \rtimes A) \arrow[r, shift left] & \Gamma(A), \arrow[l, shift left]
\end{tikzcd}$$

where the right square is a pull-back and where ψ_A is the comodule structure map, which, by assumption, is a morphism of algebras. The functor $\Gamma(-)$ thus becomes the functor of a triple on A-modules in E_*E-comodules.

Let $k \to A$ be a morphism of $\pi_0 C$-algebras in E_*E-comodules and let Y be a simplicial C-algebra under k and over A in E_*E comodules. Then we can form the bicosimplicial E_*-module

$$\begin{aligned}
\mathrm{Der}_k(Y, \Gamma^\bullet(M)) &= \{\mathrm{Der}_k(Y_p, \Gamma^{q+1}(M))\} \\
&= \{\mathbf{Alg}_k/A(Y_q, \Gamma^{q+1}(M) \ltimes A)\}.
\end{aligned}$$

where $\mathbf{Alg}_k/A$ is the category of C-algebras under k and over A. If X is a simplicial C-algebra in E_*E comodules under k and over A, we now write

$$\mathbb{D}_{C/E_*E}(X/k, M) \in \mathbf{Ho}(\mathrm{Ch}^* E_*E) \tag{2.4.5}$$

for the object in the derived category of comodules defined by taking Y to be some cofibrant model for A in simplicial C-algebras under k and then taking the total complex of the double normalization of the cosimplicial object $\mathrm{Der}_k(Y, \Gamma^\bullet(M))$. Then, still assuming that the $s\mathbf{Alg}_C$ is relatively left proper, we define the André-Quillen cohomology by

$$H^n_{C/E_*E}(X/k, M) = H^n \mathbb{D}_{C/E_*E}(X/k, M). \tag{2.4.6}$$

However, with luck, one can reduce the calculation of the comodule cohomology to the case of module cohomology. Here is the result we will use. The definitions should make the following results plausible; the proof is in [19].

2.4.7 Proposition. *Let C be a simplicial operad in E_*E comodules and $k \to A$ a morphism of $\pi_0 C$-algebra in E_*E-comodules. If M is a A-module in E_*-modules, then the extended comodule $\Gamma(M) = E_*E \otimes_{E_*} M$ is an A-module in E_*E-comodules and there is a natural isomorphism*

$$H^*_{C/E_*E}(X/k, E_*E \otimes_{E_*} M) \cong H^*_C(X/k, M).$$

A stronger assertion is true: there is an isomorphism

$$\mathbb{D}_{C/E_*E}(X/k, E_*E \otimes_{E_*} M) \cong \mathbb{D}_C(X/k, M)$$

in the derived category of E_*-modules.

2.4.8 Remark (Comodule transitivity sequence). In this setting there is also a transitivity sequence identical to that of Remark 2.4.3. The argument remains the same.

2.4.3 The cohomology of theta-algebras

Another variant on the cohomology of a commutative algebras occurs in the context of theta-algebras and their modules. Here we use the model category structure on simplicial theta algebras developed at the end of §2.2. See, in particular, Theorem 2.2.17.

Let k be a theta-algebra and let $\mathbf{Alg}_\theta^k$ be the category of θ-algebras under k. If A is an object in $\mathbf{Alg}_\theta^k$ and M is a θ-module over A. In this case, we simply define

$$H^n_\theta(A/k, M) = \pi_0 \operatorname{map}_{s\mathbf{Alg}_\theta^k/A}(A, K_A(M, n))$$

where, as always, we are taking the derived mapping space. So in particular, for computations, we will have to choose a cofibrant replacement $X \to A$ for A as a simplicial object in $\mathbf{Alg}_\theta^k$. As before there is well-defined object

$$\mathbb{D}_\theta(A/k, M) \in \mathbf{Ho}(\mathrm{Ch}^* k)$$

whose cohomology is $H^*_\theta(A/k, M)$. There is a mild wrinkle here: $\mathbf{Ho}(C^*k)$ is the derived category of continuous k-modules.

2.4.9 Remark. One example of a theta-algebra is the algebra $\mathbb{Z}_p = K_*S^0$. This is the initial object in the category of theta-algebras and we will abbreviate $H^*_\theta(A/\mathbb{Z}_p, M)$ as $H_\theta(A, M)$.

2.4.10 Remark (Theta-algebra transitivity sequence). The cohomology of theta-algebras also has a transitivity sequence. The proof in [34] goes through verbatim, but we could also use the arguments of Remark 2.4.3.

This example is very closely related to the standard André-Quillen cohomology of A as a commutative k-algebra. If $k \to A$ is a morphism θ-algebras and M is module over A, then we have a module $\mathrm{Der}_k^\theta(A, M)$ of continuous

k-derivations $\partial : A \to M$ which commute with the Adams operations and so that

$$\partial\theta(x) = \begin{cases} \theta(\partial x) - x^{p-1}dx & |x| = 0 \\ \\ \theta(\partial x) & |x| = 1. \end{cases}$$

This formula is obtained by viewing the natural isomorphism

$$\mathrm{Der}_k^\theta(A, M) \cong \mathbf{Alg}_\theta^k/A(A, M \rtimes A)$$

In the end $H_\theta^*(A/k, M)$ are the right derived functors of Der_k^θ.

The functor $\mathrm{Der}_k^\theta(A, -)$ of A-modules is representable by the A-module on $\Omega_{A/k}$ of continuous A-differentials. This inherits a natural structure as a θ-module over A and the universal derivation $d : A \to \Omega_{A/k}$ is a derivation for the theta-algebra A. As always, one derives this functor by taking a cofibrant resolution of $X \to A$ as a simplicial θ-algebra under k and setting

$$\mathbb{L}_\theta(A/k) = A \otimes_X \Omega_{X/k}$$

where $\otimes$ should be interpreted as a completed tensor product. Then there is a composite functor spectral sequence

$$(2.4.7) \qquad R\mathrm{Hom}^s_{\mathbf{Mod}_A^\theta}(H_t\mathbb{L}_\theta(A/k), M) \Longrightarrow H_\theta^{s+t}(A/k, M)$$

where RHom denotes the derived functors of Hom in the category of θ-modules over the theta-algebra A. More is true. Since free θ-algebras are free commutative $\mathbb{Z}_p$-algebras, there is a natural isomorphism

$$(2.4.8) \qquad H_*\mathbb{L}_\theta(A/k) \cong H_*\mathbb{L}_{A/k}$$

where $\mathbb{L}_{A/k}$ is the ordinary cotangent complex of the the completed algebra A. In particular, if A is smooth as complete graded k-algebra, then

$$H_t\mathbb{L}_\theta(A/k) = \begin{cases} \Omega_{A/k} & t = 0 \\ \\ 0 & t > 0 \end{cases}$$

regardless of the action of θ and the module $\Omega_{A/k}$ is projective as a continuous A-module. (Although not a projective A-module in the category of theta-modules.) In particular, the spectral sequence of 2.4.7 collapses and we have

$$(2.4.9) \qquad R\mathrm{Hom}^s_{\mathbf{Mod}_A^\theta}(\Omega_{A/k}, M) \cong H_\theta^s(A/k, M).$$

If, in addition, M is an induced θ-module – which in this case means it is of the form $\mathrm{Hom}_c(\mathbb{Z}_p^\times, M_0)$ where M_0 is some continuous A-module – then we have a further reduction

$$(2.4.10) \qquad R\mathrm{Hom}^s_{\mathbf{Mod}_A^\theta}(\Omega_{A/k}, M) \cong \mathrm{Ext}^s_{A[\theta]}(\Omega_{A/k}, M_0)$$

where the target Ext group is the derived functors of continuous homomorphisms over the ring $A[\theta]_p^\wedge$. Then, since $\Omega_{A/k}$ is a projective A-module

$$(2.4.11) \qquad H^s(A/k, M) \cong \mathrm{Ext}^s_{A[\theta]}(\Omega_{A/k}, M_0) = 0, \qquad s > 1.$$

2.4.4 Computing mapping spaces – the $K(1)$-local case

In this part, we show how to construct a Bousfield-Kan spectral sequence for the mapping space of E_∞-ring spectrum morphisms from an E_∞-ring spectrum X to $K(1)$-local E_∞-ring spectrum Y. A similar spectral sequence for simplicial T-algebras in another setting was constructed in [20].

In this subsection, our E_∞-ring spectra will be algebras over the commutative monoid operad – that is, we will work with commutative S-algebras (or simply "S-algebras", for short). This is so we have a simple description of the coproduct in this category. By Theorem 1.2.4, this is not a loss of generality.

We begin with some preliminary results. Let K be the p-adic complex K-theory spectrum. Note that for any spectrum Y there is a homotopy pairing

$$\mu : K \wedge L_{K(1)}(K \wedge Y) \to L_{K(1)}(K \wedge Y)$$

obtained as the unique completion of the diagram

$$\begin{tikzcd}
K \wedge K \wedge Y \arrow[r, "m"] \arrow[d, "K\wedge\eta"'] & K \wedge Y \arrow[r] & L_{K(1)}(K \wedge Y) \\
K \wedge L_{K(1)}(K \wedge Y) \arrow[urr, dashed] & &
\end{tikzcd}$$

obtained by from the multiplication m of K and the fact that $K \wedge \eta$ is a $K(1)_*$-equivalence. This yields, for any two spectra X and Y, a Künneth map

$$\pi_0 \operatorname{map}(X, L_{K(1)}(K \wedge Y)) \to \operatorname{Hom}_{K_*}(K_*X, K_*Y)$$

sending a morphism f to the map obtained by applying homotopy to the composite

$$L_{K(1)}(K \wedge X) \xrightarrow{K\wedge f} L_{K(1)}(K \wedge (L_{K(1)}(K \wedge Y)) \xrightarrow{\mu} L_{K(1)}(K \wedge Y).$$

Here is a continuous version of one of the key items in the definition of Adams's condition on ring spectra. See Definition 1.4.1. Here and below we will specify that K_*Y be p-complete. *A priori* K_*Y is only L-complete. See the material before Definition 2.2.1. However K_*Y will be p-complete if $K(1)_*Y$ is in even degrees or even if K_*Y is torsion-free.

2.4.11 Lemma. *Let X be a finite CW complex with cells in even degrees and let Y be any spectrum so that K_*Y is p-complete. Then the Künneth map*

$$\pi_0 \operatorname{map}(X, L_{K(1)}(K \wedge Y)) \to \operatorname{Hom}_{K_*}(K_*X, K_*Y)$$

is an isomorphism.

Proof. The result is obvious if X is a sphere. Now induct over the number of cells. □

If X and Y are commutative S-algebras, then their coproduct as a commutative S-algebra is isomorphic to $X \wedge Y$. (See [18], Proposition II.3.7; the proof there works in any of the models of spectra with a symmetric monoidal smash product.) In particular, if Y is an S-algebra, so is $K \wedge Y$. Also, if X is an S-algebra, there is a model for $L_{K(1)}X$, which is also an S-algebra. (See [18], §VIII.2; again, the argument is very general.) Thus we may conclude that if Y is an S-algebra, so is $L_{K(1)}(K \wedge Y)$. More than that, we can form the augmented cobar construction

$$Y \to L_{K(1)}(K^{(\cdot)} \wedge Y) \tag{2.4.12}$$

obtained from the usual cobar construction by applying the localization functor; this will be a cosimplicial S-algebra. We will use this cosimplicial S-algebra to build our spectral sequence.

Lemma 2.4.11 has the following obvious consequence. Let C be the free commutative S-algebra functor. As a bit of notation, if Z is a commutative S-algebra, write $\mathrm{map}_Z(-,-)$ for the (underived) space of Z-algebra maps. Similarly, write $\mathrm{Hom}_{K_*Z}(-,-)$ for the set theta-algebra maps under K_*Z.

2.4.12 Proposition. *Let Y be a $K(1)$-local commutative S-algebra so that K_*Y is p-complete. Let X be a finite CW-spectrum concentrated in even (or in odd) degrees. Then the natural map*

$$\pi_0 \, \mathrm{map}_{S-alg}(C(X), Y) \to \mathrm{Hom}_{\mathbf{Alg}_\theta}(K_*C(X), K_*Y)$$

is an isomorphism. More generally, let $Z = \vee Z_\alpha$ be any spectrum which is wedge of spectra Z_α with cells in even (or odd) degrees Then

$$\pi_0 \, \mathrm{map}_{C(Z)}(C(Z) \amalg C(X), Y) \to \mathrm{Hom}_{K_*C(Z)}(K_*(C(Z) \amalg C(X)), K_*Y)$$

is an isomorphism.

Proof. This is routine, using Lemma 2.4.11 and Theorem 2.2.11. □

We also have the following convergence fact.

2.4.13 Lemma. *Let Y be a $K(1)$-local S-algebra so that K_*Y is p-complete. Then the natural map*

$$Y \to \mathrm{holim}_\Delta \, L_{K(1)}(K^{(\cdot)} \wedge Y)$$

is a weak equivalence of commutative S-algebras.

Proof. The natural map is a morphism of S-algebras, so we need only show it is a weak equivalence. Under the hypotheses listed, we have from [29] Proposition 7.10(e) that there is a natural weak equivalence

$$L_{K(1)}(K^{(\cdot)} \wedge Y) \xrightarrow{\simeq} \mathrm{holim}_n[(K^{(\cdot)} \wedge Y) \wedge M(p^n)]$$

where $M(p^n)$ is the mod p Moore space. Now the arguments at the end of the proof Proposition 7.4 of [24] imply the result. □

Putting this all together, we have the following result.

2.4.14 Theorem. *Let Z be a commutative S-algebras and let X be a commutative Z-algebra. Let Y a $K(1)$-local commutative Z-algebra with K_*Y p-complete. Fix a morphism $\phi : X \to Y$ of Z-algebras. Then there is a second quadrant spectral sequence abutting to*

$$\pi_{t-s}(\mathrm{map}_Z(X,Y);\phi)$$

with E_2-term

$$E_2^{0,0} = \mathrm{Hom}_{K_*Z}(K_*X, K_*Y)$$

and

$$E_2^{s,t} = H^s_\theta(K_*X/K_*Z, \Omega^t K_*Y), \qquad t > 0.$$

Proof. Since p-completed K-theory is Landweber exact, we can use the resolution model category structure of Theorem 1.4.9, with $T = C$, the commutative monoid operad. We use Lemma 1.4.15 to compute the effect of K_* on cofibrant objects.

In the category $s\mathbf{Alg}_C$, form a commutative diagram

$$\begin{tikzcd} Z^{cf} \arrow[r, "j"] \arrow[d, "\simeq"'] & X^{cf} \arrow[d, "\simeq"] \\ Z \arrow[r] & X \end{tikzcd}$$

where $(-)^{cf}$ denotes a simplicial $\mathcal{P}$-cofibrant replacement and the morphism j is a $\mathcal{P}$-cofibration. Now form the cosimplicial space

$$M^\bullet = \mathrm{diag}\,\mathrm{map}_{Z^{cf}}(X^{cf}, L_{K(1)}(K^{(\cdot)} \wedge Y)).$$

The morphism $\phi : X \to Y$ supplies this with the basepoint. Since the geometric realization of Z^{cf} is weakly equivalent to Z, the geometric realization of X^{cf} is weakly equivalent to X, and using Lemma 2.4.13, the total space of this cosimplicial space will be weakly equivalent to $\mathrm{map}_Z(X,Y)$. We now identify the E_2-term.

First, since $\pi_0 K_* X^{cf} \cong K_*X$ and $\pi^0 K_* L_{K(1)}(K^{(\cdot)} \wedge Y) \cong K_*Y$, Proposition 2.4.12 implies that

$$\pi^0\pi_0 M^\bullet = \mathrm{Hom}_{K_*Z}(K_*X, K_*Y).$$

For the rest of the E_2-term we use a bicomplex argument.

There is a spectral sequence converging to $\pi^{p+q}\pi_t M^\bullet$ with

$$E_1^{p,q} = \pi^q \pi_t \,\mathrm{map}_{Z_p^{cf}}(X_p^{cf}, L_{K(1)}(K^{(\cdot)} \wedge Y)).$$

Since $t > 0$, Proposition 2.4.12 implies that

$$\begin{aligned} &\pi_t \,\mathrm{map}_{Z_p^{cf}}(X_p^{cf}, K_*L_{K(1)}(K^{(q+1)} \wedge Y)) \\ &\qquad\cong \mathrm{Der}_{K_*Z_p^{cf}}(K_*X_p^{cf}, \Omega^t K_* L_{K(1)}(K^{(q+1)} \wedge Y)) \\ &\qquad\cong \mathrm{Der}_{K_*Z}(K_*Z \otimes_{K_*Z_p^{cf}} K_*X_p^{cf}, \Omega^t K_* L_{K(1)}(K^{(q+1)} \wedge Y)). \end{aligned}$$

The augmented cosimplicial K_*Y-module

$$\Omega^t K_*Y \to \Omega^t K_* L_{K(1)}(K^{(q+1)} \wedge Y)$$

has a cosimplicial retraction as K_*Y modules and, thus, as $K_*X_p^{cf}$-modules. It follows that

$$E^1_{p,q} = \begin{cases} \mathrm{Der}_{K_*Z}(K_*Z \otimes_{K_*Z_p^{cf}} K_*X_p^{cf}, \Omega^t K_*Y) & q = 0 \\ 0 & q > 0 \end{cases}.$$

Since $K_*Z \otimes_{K_*Z^{cf}} K_*X^{cf} \to K_*X$ is a cofibrant resolution of K_*X as a K_*Z-algebra in theta-algebras, the result follows. $\Box$

Bousfield's work [8] on obstructions in the total tower of a cosimplicial space, implies the following result:

2.4.15 Corollary. *Let Z be a commutative S-algebras and let X be a commutative Z-algebra. Let Y a $K(1)$-local commutative Z-algebra with K_*Y p-complete. Then there are successively defined obstructions to realizing a map $f \in \mathrm{Hom}_{K_*Z}(K_*X, K_*Y)$ in the groups*

$$H^{s+1}_\theta(K_*X/K_*Z, \Omega^s K_*Y) \qquad s \geq 1.$$

In particular, if these groups are all zero, then the Hurewicz map

$$\pi_0(\mathrm{map}_Z(X, Y)) \to \mathrm{Hom}_{K_*Z}(K_*X, K_*Y) \tag{2.4.13}$$

is surjective. If, in addition, the groups

$$H^s_\theta(K_*X/K_*Z, \Omega^s K_*Y) = 0$$

for $s \geq 1$, the Hurewicz map of Equation 2.4.13 is a bijection.

2.5 Postnikov systems for simplicial algebras

In this section we supply a detailed description of the Postnikov systems of a simplicial algebra. We are particular interested in simplicial algebras in simplicial comodules over some Adams-type Hopf algebroid (A, Γ); therefore, we will concentrate on this case. However, the theory is very general and will apply, for example, to the case of simplicial theta-algebras, as discussed in §2.2. The primary technical input in this case will be supplied by Lemma 2.2.13, Remark 2.2.14, and Theorem 2.2.17.

The discussion parallels section 5 of [7] very closely.

Let $\mathcal{C}_\Gamma$ be the category of comodules over our fixed Adams-type Hopf algebroid (A, Γ) and let $\{C_j\}$ be an arbitrary, but fixed, generating system of Γ-comodules. (See Definition 2.1.2.) Let $D(-)$ be the duality functor on comodules which are finitely generated and projective as A-modules. (See Lemma

2.1.5.) Since our Hopf algebroid and comodules will be graded, let us write $M[k]$ for the shifted comodule obtained from M with$M[k]_n = M_{k+n}$. Thus, in the language of Example 2.2.10, we might also write $M[k] = \Omega^k M$; however, the bracket notation is simpler for this section.

We now consider the category $s\mathcal{C}_\Gamma$ of simplicial objects in $\mathcal{C}_\Gamma$. In [19] we supplied the category $s\mathcal{C}_\Gamma$ with the structure of a simplicial model category so that

1. a morphism $f : X \to Y$ is a weak equivalence if $\pi_* X \to \pi_* Y$ is an isomorphism; and
2. a morphism $f : X \to Y$ is a cofibration if it is in the class of morphisms generated by the set of maps

   $$DC_j[k] \otimes \partial\Delta^n \to DC_j[k] \otimes \Delta^n.$$

   for all j, all integers k and all positive integers n.

The fibrations are determined by the lifting property and a localization argument. They are not easily otherwise described.[2]

2.5.1 Remark. More specifically, there is an auxiliary model category structure on $s\mathcal{C}_\Gamma$ with the cofibrations above, but we specify that $f : X \to Y$ is a weak equivalence or fibration if for all j and k, the induced morphism of underived simplicial mapping spaces

$$\mathrm{map}_{s\mathcal{C}_\Gamma}(DC_j[k], X) \longrightarrow \mathrm{map}_{s\mathcal{C}_\Gamma}(DC_j[k], Y)$$

is a weak equivalence or fibration. Any such weak equivalence is automatically induces an isomorphism $\pi_* X \to \pi_* Y$, and it is this auxiliary model category that gets localized.

These technicalities not withstanding, we can ground the model category structure on $s\mathcal{C}_\Gamma$ with the following comparison result. Give the category $s\mathbf{Mod}_A$ of simplicial A-modules its standard simplicial model category structure [35].

2.5.2 Lemma. *1.) The forgetful functor from $s\mathcal{C}_\Gamma$ to $s\mathbf{Mod}_A$ preserves weak equivalences and cofibrations. The extended comodule functor*

$$\Gamma \otimes_A (-) : s\mathbf{Mod}_A \longrightarrow s\mathcal{C}_\Gamma$$

preserves fibrations and weak equivalences.

2.) The forgetful functor from $s\mathcal{C}_\Gamma$ to $s\mathbf{Mod}_A$ preserves fibrations.

---

[2] A similar, but perhaps more elegant model category structure could be obtained using the techniques of [27].

Proof. 1.) The statements about the forgetful functor follow from the definition of weak equivalence and the fact that each of the C_j is a projective A-module. The statements about the extended comodule functor follow from the fact that Γ is flat over A and an adjointness argument.

2.) Let $X \to Y$ be a fibration in $s\mathcal{C}_\Gamma$. For each j and k, and each s and t, the map of simplicial sets

$$\begin{tikzcd}
\operatorname{map}_{s\mathcal{C}_\Gamma}(DC_j[k] \otimes \Delta^s, X) \arrow[d] \\
\operatorname{map}_{s\mathcal{C}_\Gamma}(DC_j[k] \otimes \Delta^s_t, X) \times_{\operatorname{map}_{s\mathcal{C}_\Gamma}(DC_j[k]\otimes\Delta^s_t, Y)} \operatorname{map}_{s\mathcal{C}_\Gamma}(DC_j[k] \otimes \Delta^s, Y)
\end{tikzcd}$$

is an acyclic fibration. (Here we are *not* using the derived simplicial mapping spaces, but the usual mapping spaces for a simplicial category.) If K is a finite simplicial set, then there are natural isomorphisms

$$\begin{aligned}
\operatorname{colim}_j \operatorname{map}_{s\mathcal{C}_\Gamma}(DC_j[k] \otimes K, X) &\cong \operatorname{colim}_j \operatorname{map}_{s\mathcal{C}_\Gamma}(A[k] \otimes K, C_j \otimes_A X) \\
&\cong \operatorname{map}_{s\mathcal{C}_\Gamma}(A[k] \otimes K, \Gamma \otimes_A X) \\
&\cong \operatorname{map}_{s\mathbf{Mod}_A}(A[k] \otimes K, X).
\end{aligned}$$

The filtered colimit of fibrations of simplicial sets is a fibration and the result follows. □

We will be interested in various categories of algebras in comodules. Let F be a triple on $s\mathcal{C}_\Gamma$. We are thinking of the triple T_E which arises from a homotopically adapted operad T; see Definition 1.4.16. In particular, we could have either the free simplicial E_∞-algebra functor (for a general Hopf algebroid) or the prolonged free θ-algebra (for p-complete K-theory). Let $s\mathbf{Alg}_F$ be the category of F-algebras and will assume that the forgetful functor

$$s\mathbf{Alg}_F \longrightarrow s\mathcal{C}_\Gamma$$

creates a simplicial model category structure on $s\mathbf{Alg}_F$. This model category will automatically be cofibrantly generated and the cofibrations will be generated by

$$F(DC_j[k] \otimes \partial\Delta^n) \to F(DC_j[k] \otimes \Delta^n).$$

2.5.3 Remark. In Remark 2.5.1 we noted that the model category structure on $s\mathcal{C}_\Gamma$ is the localization of an auxiliary model category structure with fewer weak equivalences. This auxiliary structure also lifts to an auxiliary model category structure on $s\mathbf{Alg}_F$ and again we have a localization, at least in all our examples. Compare [19].

2.5.4 Lemma. *Suppose the triple F is a lift of a triple F_0 on $s\mathbf{Mod}_A$, and suppose the forgetful functor $s\mathbf{Alg}_{F_0} \to s\mathbf{Mod}_A$ creates a simplicial model category structure on $s\mathbf{Alg}_{F_0}$. Then there is a forgetful functor*

$$s\mathbf{Alg}_F \longrightarrow s\mathbf{Alg}_{F_0}$$

which preserves cofibrations and weak equivalences.

Proof. This follows immediately from Lemma 2.5.2. ☐

The hypotheses of this result are satisfied in both the examples we are interested in.

We now come to Postnikov towers.

2.5.5 Definition. *Let $X \in s\mathbf{Alg}_F$. Then an nth Postnikov section of X is a morphism $f : X \to Y$ in $s\mathbf{Alg}_F$ so that $\pi_k Y = 0$ for $k > n$ and f induces an isomorphism $\pi_k X \cong \pi_k Y$ for $k \leq n$. A Postnikov tower for X is a tower under X*

$$X \to \cdots \to X_n \to X_{n-1} \to \cdots X_1 \to X_0$$

so that $X \to X_n$ is an nth Postnikov section.

Note that in a Postnikov tower for X, X_n is an nth Postnikov section of X_k for $k \geq n$.

We will see below that functorial Postnikov towers and functorial k-invariants exist in $s\mathbf{Alg}_F$. We begin with the towers.

2.5.6 Proposition. *The category $s\mathbf{Alg}_F$ has functorial Postnikov towers: for all $X \in s\mathbf{Alg}_F$ there is a natural tower under X*

$$X \to \cdots \to P_n X \to P_{n-1} X \to \cdots \to P_1 X \to P_0 X$$

so that for all n, $P_n X$ is an nth Postnikov section for X.

Proof. The argument here is the standard one, but with the twist that we begin with the auxiliary model category mentioned above in Remark 2.5.3. We will say that $X \to Y_n$ is a Γ-Postnikov section if

$$\pi_k \operatorname{map}_{s\mathcal{C}_\Gamma}(DC_j[k], X) \to \pi_k \operatorname{map}_{s\mathcal{C}_\Gamma}(DC_j[k], Y_n)$$

is an isomorphism for $k \leq n$ and if the target homotopy group is zero for $k > n$. There is an associated notion of a Γ-Postnikov tower and we first claim that functorial Γ-Postnikov towers exits. This is the standard argument:

$$P_n X = \operatorname{colim}_i P_{n,i} X$$

where $P_{n,i}X = X$ for $i \leq n$ and, for $i > n$, $P_{n,i}X$ fits into a push-out diagram

$$\begin{tikzcd}
\coprod_W F(DC_j[k] \otimes \partial\Delta^i) \arrow[r] \arrow[d] & P_{n,i-1}X \arrow[d] \\
\coprod_W F(DC_j[k] \otimes \Delta^i) \arrow[r] & P_{n,i}X.
\end{tikzcd}$$

Here W is the set of all maps $F(DC_j[k] \otimes \partial\Delta^i) \to P_{n,i-1}X$. Then Corollary 2.3.15 and the fact that $\operatorname{map}_{s\mathcal{C}_\Gamma}(DC_j[k], -)$ commutes with filtered colimits implies that $X \to P_nX$ is a natural Γ-Postnikov section. There is an evident inclusion $P_nX \to P_{n-1}X$ induced from the inclusions $P_{n,i}X \to P_{n-1,i}X$, and we obtain the natural tower.

We would now like to claim that the same tower is actually a Postnikov tower. This follows immediately from the formula

$$\pi_k X = \operatorname{colim} \pi_k \operatorname{map}_{s\mathcal{C}_\Gamma}(DC_j[*], X).$$

$\square$

We next write down k-invariants. For this we will need our triple F on $s\mathcal{C}_\Gamma$ to have an augmentation $F \to \Phi$ to a triple on $\mathcal{C}_\Gamma$. Here is the definition of that concept.

2.5.7 Definition. *Let F be triple on sC_Γ. Then an augmentation for F is a triple on C_Γ equipped with a natural isomorphism*

$$d_X = d : \pi_0 FX \longrightarrow \Phi\pi_0 X$$

so that there are commutative diagrams

$$\begin{tikzcd}
\pi_0 X \arrow[r, "="] \arrow[d, "\eta_\Phi"'] & \pi_0 X \arrow[d, "\eta_F"] \\
\pi_0 FX \arrow[r, "d"'] & \Phi\pi_0 X
\end{tikzcd}$$

and

$$\begin{tikzcd}
\pi_0 F^2 X \arrow[r, "p"] \arrow[d, "d_{FX}"'] & \pi_0 F(\pi_0 FX) \arrow[r, "\pi_0 F(d)"] & \pi_0 F(\pi_0 \Phi X) \arrow[d, "d_{\Phi\pi_0 X}"] \\
\Phi(\pi_0 X) \arrow[rr, "\Phi d"'] & & \Phi^2(\pi_0 X)
\end{tikzcd}$$

where p in induced by the augmentation $FX \to \pi_0 FX$ and

$$\begin{tikzcd}
\pi_0 F^2 X \arrow[r, "d"] \arrow[d, "\pi_0 \epsilon_F"'] & \Phi\pi_0 FX \arrow[r, "\Phi d"] & \Phi^2 \pi_0 X \arrow[d, "\epsilon_\Phi"] \\
\pi_0 FX \arrow[rr, "d"'] & & \Phi\pi_0 X.
\end{tikzcd}$$

Here η and ϵ are the unit and multiplication of the respective triples. As an abuse of notation we may write that there is an augmentation of triples $F \to \Phi$.

This concept fits closely with all our major examples.

2.5.8 Example. If F is the triple induced by a simplicial operad sC_Γ then we may take Φ to be the triple induced by the operad $\pi_0 F$. The augmentation is then the observation that there is a natural isomorphism $\pi_0 F(X) \cong \pi_0 F(X)$.

Indeed, the forgetful functor from F-algebras to to sC_Γ creates reflexive coequalizers.

In particular, if F is a simplicial E_∞-operad (see Definition 2.3.8), then $\pi_0 F$ is the commutative algebra operad. If F is the constant associative algebra operad, then $\pi_0 F$ is simply the associative algebra operad.

The other case of interest in theta-algebras. In this case, F is the free theta-algebra triple, prolonged to the simplicial setting and Φ is also the free theta-algebra triple.

The following result is an exercise in diagrams.

2.5.9 Proposition. *Suppose $F \to \Phi$ is an augmentation from a triple on sC_Γ to a triple on C_Γ. Then*

1. *if A is a Φ-algebra in C_Γ, then the constant simplicial comodule A is an F-algebra in sC_Γ with structure morphism*

$$\begin{tikzcd} FA \arrow[r, "p"] & \pi_0 FA \arrow[r, "d"] & \Phi A \arrow[r, "\epsilon_A"] & A; \end{tikzcd}$$

2. *if X is an F-algebra in sC_Γ, then $\pi_0 X$ is a Φ-algebra in C_Γ with structure morphism*

$$\begin{tikzcd} \Phi(\pi_0 X) \arrow[r, "d_X^{-1}"] & \pi_0 FX \arrow[r, "\pi_0 \epsilon_X"] & \pi_0 X; \end{tikzcd}$$

3. *the functor $X \mapsto \pi_0 X$ from F-algebras to Φ-algebras is left adjoint to the functor that assigns to any Φ-algebra A the constant simplicial F-algebra A.*

The existence of an augmentation not only has implications for π_0, but for the higher homotopy groups as well. In fact, if X is an F-algebra, $\pi_i X$ will be a $\pi_0 X$ module. For any triple Φ and any Φ-algebra A, an A-*module* M is determined by a split extension of Φ-algebras

$$\begin{tikzcd} M \arrow[r] & B \arrow[r, shift left] & A \arrow[l, shift left] \end{tikzcd} \tag{2.5.1}$$

with the further additional property that B is an abelian Φ-algebra over A with unit given by the splitting.

2.5.10 Proposition. *Suppose $F \to \Phi$ is an augmentation from a triple on sC_Γ to a triple on C_Γ and suppose that X is an F-algebra. Then for all $i \geq 1$, $\pi_i X$ is a module over the Φ-algebra $\pi_0 X$.*

Proof. If K is a simplicial set and $X \in sC_\Gamma$, let $\hom(K, X)$ denote the internal exponential object in sC_Γ. Since the forgetful functor $s\mathbf{Alg}_F \to sC_\Gamma$ creates the simplicial model category structure on $s\mathbf{Alg}_F$, if X is a fibrant F-algebra, so is $\hom(K, X)$. If K is pointed, then let $\hom_*(K, X)$ be defined by fiber at 0 of the morphism $\hom(K, X) \to \hom(*, X) = X$. To obtain the result, apply $\pi_0(-)$ to the split extension of F-algebras

$$\begin{tikzcd} \hom_*(\Delta^i/\partial\Delta^i, X) \arrow[r] & \hom(\Delta^i/\partial\Delta^i, X) \arrow[r, shift left] & X \arrow[l, shift left] \end{tikzcd}$$

and apply Proposition 2.5.9.2. □

Finally, for our proofs, we are going to have to assume that push-outs in the category $s\mathbf{Alg}_F$ are quite regular. Thus, for the rest of this monograph, we make the following assumption. It is satisfied for all our main examples by Theorem 2.3.13 and, in fact, for many other examples as well. See Remark 2.3.14.

2.5.11 Assumptions. *The category $s\mathbf{Alg}_F$ satisfies the following Blakers-Massey Excision property: Suppose we are given a homotopy push-out diagram in $s\mathbf{Alg}_F$*

$$\begin{tikzcd} A \arrow[r, "f"] \arrow[d, "j"'] & X \arrow[d] \\ B \arrow[r] & Y \end{tikzcd}$$

and, furthermore, that $\pi_i(B,A) = 0$ for $i < m$ and $\pi_i(X,A) = 0$ for $i < n$. Then

$$\pi_i(B,A) \longrightarrow \pi_i(Y,X)$$

is an isomorphism for $i \leq n+m-2$ and onto for $i = n+m-1$.

When this assumption is satisfied, there is a truncated Mayer-Vietoris sequence in homotopy, as in Corollary 2.3.15.

We can now introduce our Eilenberg-MacLane objects.

2.5.12 Definition. *1.) Let A be a Φ-algebra. Then $X \in s\mathbf{Alg}_F$ is of type K_A if $\pi_0 X \cong A$ and the augmentation $X \to A$ is a weak equivalence of simplicial F-algebras. In particular $\pi_i X = 0$ for $i > 0$.*

2.) Let M be an A-module and let $n \geq 1$. Then a morphism $X \to Y$ in $s\mathbf{Alg}_F$ is of type $K_A(M,n)$ if X is of type K_A, the morphism $\pi_0 X \to \pi_0 Y$ is an isomorphism and

$$\pi_i Y \cong \begin{cases} M & i = n \\ 0 & i \neq n,\ i > 0 \end{cases}$$

This isomorphism should be as A-modules. If the morphism $X \to Y$ is understood, we will simply call Y an object of type $K_A(M,n)$.

Collectively, we will call the objects of type K_A and $K_A(M,n)$ *Eilenberg-MacLane objects.* As would be expected such objects exist; indeed, A itself, regarded as a constant object is of type K_A and if M is an A-module, the twisted object

$$K(M,n) \ltimes A$$

yields a morphism of type $K_A(M,n)$. Here $K(M,n)$ is the simplicial module whose normalization is M is degree n; this is naturally a simplicial A-module, and $K(M,n) \ltimes A$ is the simplicial infinitesimal extension.

In fact, Proposition 2.5.19 below says that the moduli space of all Eilenberg-MacLane objects is a space of the form BG where G is a discrete group of automorphisms. Before proving that however, we state and prove the result about k-invariants and pull-backs.

Suppose we are given a morphism $X \to Y$ in $s\mathbf{Alg}_F$ for which $\pi_i X \to \pi_i Y$ is an isomorphism for $i < n$, $n \geq 1$. Write A for $\pi_0 X \cong \pi_0 Y$ and let $M = \pi_n F$ where F is the homotopy fiber of $X \to Y$. Then let C be the homotopy push-out of

$$\begin{tikzcd} Y & X \arrow[l] \arrow[r] & P_0 X. \end{tikzcd}$$

Then Assumption 2.5.11 (or Corollary 2.3.15 in our main examples) implies that $P_0 X \to P_{n+1} C$ is of type $K_A(M, n+1)$. A calculation of homotopy groups now implies the following result.

2.5.13 Proposition. *If Z is the homotopy fiber of $X \to Y$ and $\pi_i Z = 0$ for $i \neq n$, then the natural diagram*

$$\begin{tikzcd} X \arrow[r] \arrow[d] & P_0 X \arrow[d] \\ Y \arrow[r] & P_{n+1} C \end{tikzcd}$$

is a homotopy pull-back diagram.

In other words, we have a *natural* formulation of the fact that there is a homotopy cartesian square

$$\begin{tikzcd} P_n X \arrow[r] \arrow[d] & K_A \arrow[d] \\ P_{n-1} X \arrow[r] & K_A(\pi_n X, n+1). \end{tikzcd} \tag{2.5.2}$$

2.5.14 Remark. The construction we used in Proposition 2.5.13 will be repeated throughout later sections, so we will give it a name. Given a morphism $f : X \to Y$, let us write

$$\delta_n(f) : P_0 X \longrightarrow P_{n+1} C$$

for the resulting morphism, and call it the nth *difference construction.* It is natural in the morphism f.

2.5.15 Remark (The relative version). In our applications we will need to consider the relative case where we have fixed a morphism $k \to A$ of Φ-algebras. In order to do this, we will assume that the category $s\mathbf{Alg}_F$ is relatively left proper (as in Definition 2.3.3) and that k is projective as an R-module. This is to avoid the question of whether or not we have to resolve the algebra k or not. See Remark 2.4.2. Then all of the constructions we have made so far are valid not simply in $s\mathbf{Alg}_F$, but in the relative category $s\mathbf{Alg}_k$ of simplicial F-algebras under k. Thus we have Postnikov towers under A, for example, and we can require that our Eilenberg-MacLane objects K_A and $K_A(M, n)$ be objects in $s\mathbf{Alg}_k$ as well. The difference construction and Proposition 2.5.13 also remains valid in $s\mathbf{Alg}_k$ as homotopy pull-backs in $s\mathbf{Alg}_k$ are created in $s\mathbf{Alg}_F$. Keeping this in mind, we will work, for the rest of this section in this relative case. Note that a *simplicial k-algebra* will be an object in $s\mathbf{Alg}_F$ under k.

Proposition 2.5.13 has a continuous version that is phrased in terms of moduli spaces. Let $k \to A$ be a morphism of Φ-algebras and let Y be a simplicial k-algebra. Suppose $\pi_i Y = 0$ for $i > n$. Let M be a $\pi_0 Y = A$ module and write $\mathcal{M}(Y \oplus (M, n))$ for the moduli space of all simplicial k-algebras so that $P_{n-1}X \simeq Y$ and $\pi_n X \cong M$ as an A-module. (Neither the weak equivalence nor the isomorphism are part of the data.) The notation using the arrows $\looparrowright$ was defined in Example 1.1.19.

Note that we might write $\mathcal{M}(Y \oplus (M, n))$ as $\mathcal{M}_k(Y \oplus (M, n))$ if we want to emphasize the role of k; however, we hope that k normally remains clear from the context.

2.5.16 Theorem. *The difference construction defines a natural weak equivalence*

$$\mathcal{M}(Y \oplus (M, n)) \xrightarrow{\simeq} \mathcal{M}(Y \looparrowright K_A(M, n+1) \looparrowleft K_A).$$

Proof. The difference construction is natural and provides a functor from the category whose nerve defines $\mathcal{M}(Y \oplus (M, n))$ to the category whose nerve defines $\mathcal{M}(Y \looparrowright K_A(M, n+1) \looparrowleft K_A)$. A natural version of homotopy pull-back defines the functor back. Then Proposition 2.5.13 – which remains true in the relative case – supplies the natural transformations needed to make these functors into an equivalence on nerves. □

There is a variant of these results which can be used to analyze Eilenberg-MacLane objects. Let $k \to A$ be a morphism of Φ-algebras and M an A-module. If $X \to Y$ is a morphism of type $K_A(M, n)$, then the difference construction and Proposition 2.5.13 supplies a homotopy cartesian diagram in $s\mathbf{Alg}_k$

$$\begin{tikzcd}
Y \arrow[r] \arrow[d] & P_0 Y \arrow[d] \\
P_0 Y \arrow[r] & P_{n+1} C
\end{tikzcd}$$

and the morphism $P_0 Y \to P_{n+1} Y$ is of type $K_A(M, n+1)$. Write $\mathcal{M}_{A/k}(M, n)$ for the moduli space of all morphisms of type $K_A(M, n)$ in $s\mathbf{Alg}_k$.

2.5.17 Lemma. *Let $n \geq 1$. The assignment*

$$\{f : X \to Y\} \mapsto \{\delta_n(f) : P_0 Y \to P_{n+1} C\}$$

yields a weak equivalence of moduli spaces

$$\mathcal{M}_{A/k}(M, n) \xrightarrow{\simeq} \mathcal{M}_{A/k}(M, n+1).$$

Proof. The functor back takes sends a morphism $f : X \to Y$ of type $K_A(M, n+1)$ to the homotopy pullback of the two-sink

$$X \xrightarrow{f} Y \xleftarrow{f} X.$$

□

We now analyze the uniqueness of Eilenberg-MacLane objects. We assuming we have an augmented triple $F \to \Phi$ and we are keeping mind the results of Propositions 2.5.9 and 2.5.10. Anything labeled "Eilenberg-MacLane" should represent cohomology and that is indeed the case here. Let $k \to A$ be a morphism of Φ-algebras and M an A module. Let X be a simplicial F-algebra under k equipped with an augmentation $X \to A$. Recall that we can define the André-Quillen cohomology of X with coefficients in M by the formula

$$\begin{aligned} H^n_F(X/k, M) &= \pi_0 \operatorname{map}_{s\mathbf{Alg}_k/A}(X, K_A(M,n)) \\ &\cong \pi_t \operatorname{map}_{s\mathbf{Alg}_k/A}(X, K_A(M, n+t)). \end{aligned} \tag{2.5.3}$$

Here $s\mathbf{Alg}_k/A$ is the category of simplicial F-algebras under k and over A. The following is now immediately obvious.

2.5.18 Lemma. *Let $k \to A$ be a morphism of Φ-algebras and M an A module. Let X be a simplicial F-algebra under k. Then there is a natural isomorphism*

$$\pi_0 \operatorname{map}_{s\mathbf{Alg}_k}(X, K_A(M,n)) = \coprod_{f:\pi_0 X \to A} H^n_F(X/k, M).$$

Only slightly more complicated is the following result. If A is an algebra and M is an A-module, the group $\operatorname{Aut}(A, M)$ of automorphisms of the pair (A, M) is defined to be the group of automorphisms in the category of algebras of the diagram

$$M \ltimes A \rightleftarrows A.$$

For example, if A is a commutative algebra, this is equivalent to specifying an algebra automorphism $f : A \to A$ and an isomorphism of abelian groups $\phi : M \to M$ so that $\phi(ax) = f(a)\phi(x)$ for all $a \in A$ and $x \in M$.

In the following result, recall that $s\mathbf{Alg}_k$ is the category of F-algebras under a fixed F-algebra k.

2.5.19 Proposition. *1.) Let $k \to A$ be a morphism of Φ-algebras and $\operatorname{Aut}_k(A)$ the group of automorphisms of A under k as a Φ-algebra. If $\mathcal{M}_A$ is the moduli space of all objects in $s\mathbf{Alg}_k$ of type K_A, then there is a weak equivalence*

$$\mathcal{M}_A \simeq B \operatorname{Aut}_k(A).$$

2.) Let $k \to A$ be a morphism of Φ-algebras and M an A-module. Let $\operatorname{Aut}_k(A, M)$ denote the group of automorphisms of the pair (A, M) under k. If $\mathcal{M}_{A/k}(M, n)$ is the moduli space of all morphisms in $s\mathbf{Alg}_k$ of type $K_A(M, n)$, then there is a weak equivalence

$$\mathcal{M}_{A/k}(M, n) \simeq B \operatorname{Aut}_k(A, M).$$

In particular, this moduli space is connected and any object of $\mathcal{M}_{A/k}(M, n)$ represents André-Quillen cohomology.

Proof. The first claim follows immediately from the definition of type K_A, Proposition 2.5.9.3, and the case $M = 0$ of the previous lemma.

For the second claim, let us write $\mathcal{M}_{A/k}(M, n) = \mathcal{M}_n$ for $n \geq 1$. Because of Lemma 2.5.17 we need only calculate $\mathcal{M}_1$. Let $\mathcal{M}_0$ be the moduli space of pairs of the form $K_{M \ltimes A} \rightleftarrows K_A$; that is diagrams of the form

$$Y \rightleftarrows X$$

of simplicial F-algebras under k so that Y and X have trivial higher homotopy there is an isomorphism of Φ-algebras from $\pi_0 Y \rightleftarrows \pi_0 X$ to $M \ltimes A \rightleftarrows A$. An easy calculation shows $\mathcal{M}_0 \simeq B\operatorname{Aut}_k(A, M)$. We establish a weak equivalence $\mathcal{M}_1 \simeq \mathcal{M}_0$.

If $X \to Y$ is a morphism of type $K_A(M, 1)$, we take the homotopy pull-back of $X \to Y \leftarrow X$ to get a morphism $Y' \to X$ – with the evident section – of the form $K_{M \ltimes A} \rightleftarrows K_A$. This gives the map $\mathcal{M}_1 \to \mathcal{M}_0$. To get map back let $Y \rightleftarrows X$ be a morphism of the form $K_{M \ltimes A} \rightleftarrows K_A$ and let M' be the kernel of $\pi_0 Y \to \pi_0 X$ and form

$$K_{\pi_0 X} \to K(M', 1) \ltimes K_{\pi_0 X}$$

That these two functors have natural transformations to the identity is an exercise left to the reader. Or see the proof of Proposition 6.5 of [7]. □

2.5.20 Remark. This last result provides an equivalence of moduli spaces

$$\mathcal{M}_{A/k}(M, n) \xrightarrow{\simeq} \mathcal{M}(K_A(M, n) \looparrowleft K_A). \tag{2.5.4}$$

In particular, $\mathcal{M}_{A/k}(M, n)$ is connected and any morphism of k-algebras of type $K_A(M, n)$ is weakly equivalent (although not canonically) to $K_A \to K_A(M, n)$. Combining this statement with the pull-back diagram of Proposition 2.5.13 and the isomorphism of Lemma 2.5.18 we have the following statement: if X is a simplicial F-algebra under the constant simplicial F-algebra k, then the k-invariants of the Postnikov tower of X lie in

$$H_F^{n+1}(P_n X/k, \pi_n X).$$

By Proposition 2.5.10 we know that $\pi_n X$ is, in fact, a $\pi_0 X$-module.

We record the following result for later use. Recall that all the moduli spaces we are considering are built in the category $s\mathbf{Alg}_k$ of F-algebras under k.

2.5.21 Lemma. *Let $k \to A$ be a morphism of Φ-algebras, let M be an A-module and let $m \geq 1$. Then there is a commutative square with horizontal maps weak equivalences*

$$\begin{tikzcd}
\mathcal{M}(K_A \looparrowright K_A(M, n)) \arrow[r, "\simeq"] \arrow[d] & \mathcal{M}(K_A \looparrowright K_A(M, n+1)) \arrow[d] \\
\mathcal{M}(K_A \oplus (M, n)) \arrow[r, "\simeq"] & \mathcal{M}(K_A \looparrowright K_A(M, n+1) \looparrowleft K_A).
\end{tikzcd}$$

The left vertical map sends $X \to Y$ to Y and the right vertical map sends a morphism $X \to Y$ to $X \to Y \leftarrow X$.

Proof. This is a combination of Theorem 2.5.16, Lemma 2.5.17, and the equivalence 2.5.4. □

We now investigate the homotopy type of the one of the spaces that arises here.

Let $k \to A$ be a morphism Φ-algebras, M an A-module and B a simplicial F-algebra under k.

2.5.22 Proposition. *There is a homotopy fiber sequence*

$$\coprod_f \mathcal{H}^n_F(B/k, M_f) \longrightarrow \mathcal{M}(B \hookrightarrow K_A(M,n) \hookleftarrow A) \xrightarrow{p} \mathcal{M}(B) \times B\,\mathrm{Aut}(A,M)$$

where $f : \pi_0 B \to A$ runs over all Φ-algebra isomorphisms under k and M_f indicates the $\pi_0 B$-module induced by f

Proof. We will identify the fiber of the arrow p as

$$\mathrm{map}_{s\mathbf{Alg}_k}(X, K_A(M,n))$$

and then apply Proposition 2.5.22.

As in the proof of Proposition 1.1.17, the fiber of the morphism p is the nerve of the category of diagrams

$$\begin{tikzcd}
 & & K_A(M,n) \arrow[d, "\simeq"'] & A \arrow[l] \arrow[d, "\simeq"] \\
B & U \arrow[l, "\simeq"'] \arrow[r] & V & W \arrow[l]
\end{tikzcd}$$

However, the functor that takes such a diagram to the diagram

$$\begin{tikzcd}
B & U \arrow[l, "\simeq"'] \arrow[r] & V & K_A(M,n) \arrow[l, "\simeq"']
\end{tikzcd}$$

induces an equivalence of categories and the result follows from Example 1.1.15. □

Finally, we specialize to the case where $B = A$. The following is an easy consequence of the previous result, the fact that $\mathcal{M}(A) = B\,\mathrm{Aut}_k(A)$, and the fact that $\mathrm{Aut}_k(A)$ acts freely on $\pi_0\,\mathrm{map}^0(A, K_A(M,n))$. Recall that

$$\hat{\mathcal{H}}^n_F(A/k, M) = E\,\mathrm{Aut}(A,M) \times_{\mathrm{Aut}(A,M)} \mathcal{H}^n(A,M)$$

is the Borel construction of the natural action of $\mathrm{Aut}_k(A,M)$ on the André-Quillen cohomology space.

2.5.23 Corollary. *There is a homotopy fiber sequence*

$$\mathcal{H}^n_F(A/k, M) \longrightarrow \mathcal{M}(A \looparrowright K_A(M,n) \looparrowleft A) \xrightarrow{p} B\,\mathrm{Aut}_k(A, M)$$

and the induced action of $\mathrm{Aut}_k(A, M)$ *on* $\mathcal{H}^n_F(A/k, M)$ *is the natural action on the André-Quillen cohomology space. Furthermore, there is a weak equivalence*

$$\mathcal{M}(A \looparrowright K_A(M,n) \looparrowleft A) \simeq \hat{\mathcal{H}}^n_F(A/k, M).$$

Chapter 3

Decompositions of Moduli Spaces

3.1 The spiral exact sequence

The spiral exact sequence displays the relationship between two different sets of homotopy groups that can be defined on a simplicial T-algebra in spectra. The existence of this exact sequence and its properties are discussed in [17] and [7] and this section is an amalgamation of those two papers. The added value here and the whole reason for running through these ideas once again is so that we can prove Corollary 3.1.18, which displays a localized version of the more traditional spiral exact sequence. This version is at the heart of our computations.

3.1.1 Natural homotopy groups and the exact sequence

We give ourselves a model category $\mathcal{C}$ and a set $\mathcal{P}$ of small projectives, in the sense of Bousfield – all as discussed in section 1.4. We also assume enough that we get the $\mathcal{P}$-resolution model category on $s\mathcal{C}$; this is a simplicial model category. Given $P \in \mathcal{P}$, there are two notions of homotopy groups for objects in $s\mathcal{C}$. First, if $X \in s\mathcal{C}$, we can form the simplicial abelian group $[P, X]$, where $[-, -]$ denotes the morphisms in the homotopy category of $\mathcal{C}$. We can then take the homotopy groups of this simplicial abelian group:

$$\pi_i[P, X] \overset{\text{def}}{=} \pi_i \pi_P(X).$$

These are the homotopy groups used to define the weak equivalences in the $\mathcal{P}$-model category structure. On the other hand, we can form the simplicial mapping space $\text{map}(P, X)$, where we now regard P as a constant simplicial object in $s\mathcal{C}$ and, as always, we either assume that X is $\mathcal{P}$-fibrant or we take the derived mapping space. Because the objects of $\mathcal{P}$ are homotopy cogroup objects, this mapping space has a basepoint given by the morphism

$$P \to \phi \to X$$

where ϕ is the initial object. Define the *natural* homotopy groups by

$$\pi_{i,P}X \overset{\text{def}}{=} \pi_i \operatorname{map}(P, X).$$

These natural homotopy groups are representable. If K is any pointed simplicial set and $P \in \mathcal{P}$, define $P \wedge K$ by the push-out diagram

$$\begin{tikzcd} P \otimes * \arrow[r] \arrow[d] & P \otimes K \arrow[d] \\ \phi \otimes * \arrow[r] & P \wedge K. \end{tikzcd}$$

Then there is a natural isomorphism

$$\pi_{i,P}X \cong [P \wedge \Delta^i/\partial\Delta^i, X]_{\mathcal{P}}$$

where the symbol $[-,-]_{\mathcal{P}}$ means homotopy classes of maps in the $\mathcal{P}$-resolution model category structure. In contrast, the homotopy groups $\pi_1\pi_P(X)$ do not seem to be representable. (The groups $\pi_i\pi_P(X)$ are representable if $i \neq 1$. See [17].)

The representability of $\pi_{i,P}(-)$ suggests a construction. Let K be a pointed simplicial set and let $\mathcal{C}/\phi$ be the arrow category of objects in $\mathcal{C}$ equipped with an augmentation $Z \to \phi$ to the initial object. Then we have defined a functor

$$(-) \wedge K : \mathcal{C}/\phi \longrightarrow s\mathcal{C}.$$

This functor has a right adjoint $C_K(-)$. Indeed the functor from $\mathcal{C}$ to $s\mathcal{C}$ which assigns $Z \otimes K$ to Z has a right adjoint given by the zeroth object in the exponential object

$$M_K X \overset{\text{def}}{=} \hom(K, M)_0. \tag{3.1.1}$$

If $*$ is "one-point" simplicial set, then $C_K X$ is defined by the pull-back diagram

$$\begin{tikzcd} C_K X \arrow[r] \arrow[d] & M_K X \arrow[d] \\ \phi \arrow[r] & M_* X = X_0. \end{tikzcd} \tag{3.1.2}$$

The construction of $C_K X$ is natural in K; in other words, we have a bifunctor

$$C_{(-)}(-) : s\mathbf{Sets}_*^{op} \times s\mathcal{C} \longrightarrow \mathcal{C}/\phi.$$

where $s\mathbf{Sets}_*$ is the category of pointed simplicial sets. Note also that if $K \to L$ is a cofibration of simplicial sets, then there is pull-back diagram

(3.1.3)
$$\begin{tikzcd}
C_{L/K}X \arrow[r] \arrow[d] & M_L X \arrow[d] \\
\phi \arrow[r] & M_K X.
\end{tikzcd}$$

An important aspect of this construction is the following:

3.1.1 Lemma. *1.) Let $A \to B$ be an acyclic cofibration in $\mathcal{C}$ and $K \to L$ a fibration of simplicial sets. Then*

$$A \otimes L \sqcup_{A \otimes K} B \otimes K \to B \otimes L$$

and

$$A \wedge L \sqcup_{A \wedge K} B \wedge K \to B \wedge L$$

are acyclic Reedy cofibrations.

2.) Suppose $X \in s\mathcal{C}$ is Reedy fibrant and $K \to L$ is a fibration of pointed simplicial sets. Then the morphism

$$M_L X \longrightarrow M_K X$$

is a fibration in $\mathcal{C}$ and the morphism

$$C_L X \longrightarrow C_K X$$

is a fibration in $\mathcal{C}/\phi$ with the fiber at $\phi \to C_K X$ naturally isomorphic to $C_{K/L}X$.

Proof. The first statement is simply a matter of inspection. The second statement follows from an adjointness argument, using the first statement. Alternatively, combine the diagrams 3.1.2 and 3.1.3. □

To shorten notation, we define

$$C_n X \overset{\text{def}}{=} C_{\Delta^n/\Delta^n_0} X$$
$$Z_n X \overset{\text{def}}{=} C_{\Delta^n/\partial\Delta^n} X.$$

Then the morphism

$$d_0 : \Delta^{n-1}/\partial\Delta^{n-1} \to \Delta^n/\Delta^n_0$$

and Lemma 3.1.1 define – at least for X Reedy fibrant – a fibration sequence in $\mathcal{C}/\phi$

(3.1.4)
$$\begin{tikzcd}
Z_n X \arrow[r] & C_n X \arrow[r, "d_0"] & Z_{n-1} X.
\end{tikzcd}$$

The following lemma starts the calculations. If A is a simplicial abelian group, let NA be its normalized chain complex.

3.1.2 Lemma. *Let X be a Reedy fibrant object in $s\mathcal{C}$.*

1. *(1) For all $P \in \mathcal{P}$ projective, there is a natural isomorphism $[P, C_nX] \cong N_n[P, X]$;*
2. *(2) If $P \in \mathcal{P}$ is a projective, then there is a natural exact sequence*

$$[P, C_{n+1}X] \xrightarrow{d_0} [P, Z_nX] \longrightarrow \pi_n \operatorname{map}(P, X) \to 0$$

Proof. The cofiber sequence

$$\Delta^n_0 \to \Delta^n \to \Delta^n/\Delta^n_0$$

of simplicial sets yields, using Lemma 3.1.1, a fibration sequence

$$C_nX \to X_n \to M_{\Delta^n_0}X. \tag{3.1.5}$$

Furthermore

$$[P, M_{\Delta^n_0}X] \to M_{\Delta^n_0}[P, X]$$

is an isomorphism, by the standard induction argument. (See [21], VIII.1.8, for the cosimplicial analog.) The fibration sequence of 3.1.5 yields a short exact sequence

$$0 \to [P, C_nX] \to [P, X_n] \to [P, M_nX] \to 0$$

and part (1) now follows.

For (2), note that the adjoint isomorphism

$$\mathrm{Hom}_{\mathcal{C}}(P, Z_nX) \to \mathrm{Hom}_{s\mathcal{C}}(P \wedge \Delta^n/\partial\Delta^n, X)$$

and Lemma 3.1.1.1 yields a well defined map

$$[P, Z_nX] \longrightarrow \pi_n \operatorname{map}(P, X).$$

Since any element in $\pi_n \operatorname{map}(P, X)$ is represented by an element $P \wedge \Delta^n/\partial\Delta^n \to X$, this morphism is onto. If $P \wedge \Delta^n/\partial\Delta^n \to X$ represents the zero object in $\pi_n \operatorname{map}(P, X)$, then it automatically extends over $P \wedge \Delta^{n+1}/\Delta^{n+1}_0$. □

3.1.3 Corollary. *There is a natural isomorphism*

$$\pi_0\pi_P(X) \xrightarrow{\cong} \pi_{0,P}X$$

Proof. This is case $n = 0$ of Lemma 3.1.2.2. □

We now get a set of long exact sequences

$$\cdots \to [\Sigma P, Z_{n-1}X] \to [P, Z_nX] \to [P, C_nX] \to [P, Z_{n-1}X]$$

which can be spliced together into an exact couple

(3.1.6)
$$\begin{tikzcd}
{[\Sigma^{q+1}P, Z_{n-1}X]} \arrow[rr, dashed] & & {[\Sigma^{q+1}P, Z_nX]} \arrow[dl] \\
& {[\Sigma^{q+1}P, C_nX]} \arrow[ul] &
\end{tikzcd}$$

Using Lemma 3.1.2 we immediately see that the first derived long exact sequences of this exact couple yield the spiral exact sequence:

3.1.4 Proposition. *For all $P \in \mathcal{P}$ and all Reedy fibrant X in $s\mathcal{C}$ there is a long exact sequence*

$$\cdots \to \pi_{i+1}\pi_P(X) \to \pi_{i-1,\Sigma P}X \to \pi_{i,P}X \to \pi_i\pi_P X \to$$
$$\cdots \to \pi_{0,\Sigma P}X \to \pi_{1,P}X \to \pi_1\pi_P X \to 0.$$

For the rest of the section, it is convenient to write

$$\pi_*(X; P) \overset{\text{def}}{=} \pi_*\pi_P(X)$$
$$\pi^\natural_*(X; P) \overset{\text{def}}{=} \pi_{*,P}(X)$$

in order to avoid very complicated subscripts.

The long exact sequences of Proposition 3.1.4 can be spliced together to give a spectral sequence

(3.1.7)
$$\pi_p(X; \Sigma^q P) \Longrightarrow \operatorname{colim}_k \pi^\natural_k(X; \Sigma^{p+q-k}P).$$

using the triangles

(3.1.8)
$$\begin{tikzcd}
\pi^\natural_{p-1}(X; \Sigma^{q+1}P) \arrow[rr] & & \pi^\natural_p(X; \Sigma^q P) \arrow[dl] \\
& \pi_p(X; \Sigma^q P) \arrow[ul, dashed] &
\end{tikzcd}$$

as the basis for an exact couple. Here and below the dotted arrow means a morphism of degree -1. In the basic case when $\mathcal{C} = \mathcal{S}$ is the category of spectra and $\mathcal{P} = \mathcal{P}_E$ is the set of projective arising from an Adams-type homology theory (see Definition 1.4.2), this is actually a very familiar spectral sequence in disguise, as we now explain.

So let us assume we are working with spectra and simplicial spectra and that $\mathcal{P} = \mathcal{P}_E$.

We may assume that X is Reedy cofibrant spectrum, and let $\mathrm{sk}_n X$ denote the nth skeleton of X as a simplicial spectrum. Then geometric realization makes $\{|\mathrm{sk}_n X|\}$ into a filtration of $|X|$ and the standard spectral sequence of the geometric realization of a simplicial spectrum is gotten by splicing the together the long exact sequences obtained by apply the functor $[\Sigma^{p+q}P, -]$ to the cofibration sequence

$$|sk_{p-1}X| \longrightarrow |sk_pX| \longrightarrow \Sigma^p(X_p/L_pX).$$

If we let

$$[\Sigma^{p+q}P, |sk_pX|]^{(1)} = \mathrm{Im}\{[\Sigma^{p+q}P, |sk_pX|] \longrightarrow [\Sigma^{p+q}P, |sk_{p+1}X|]\}$$

then the first derived long exact sequence of this exact couple is

(3.1.9)
$$\begin{tikzcd}
{[\Sigma^{p+q}P, |sk_{p-1}X|]^{(1)}} \arrow[rr] & & {[\Sigma^{p+q}P, |sk_pX|]^{(1)}} \arrow[dl] \\
& \pi_p[\Sigma^qP, X] \arrow[ul, dashed] &
\end{tikzcd}$$

and we obtain the usual spectral sequences

(3.1.10)
$$\pi_p(X; \Sigma^qP) = \pi_p[\Sigma^qP, X] \Longrightarrow [\Sigma^{p+q}P, |X|].$$

Thus the two spectral sequences have isomorphic E^2-terms. More is true. The next result says that the two exact couples obtained from the triangles of 3.1.8 and 3.1.9 are isomorphic; hence, we have isomorphic spectral sequences and we can assert that geometric realization induces an isomorphism

$$\mathrm{colim}_k\, \pi^\natural_k(X; \Sigma^{p+q-k}P) \xrightarrow{\cong} [\Sigma^{p+q}P, |X|].$$

3.1.5 Lemma. *Geometric realization induces as isomorphism between the spiral exact sequence*

$$\cdots \to \pi^\natural_{p-1}(X; \Sigma^{q+1}P) \to \pi^\natural_p(X; \Sigma^qP) \to \pi_p(X; \Sigma^qP) \to \cdots$$

and the derived exact sequence

$$\cdots \to [\Sigma^{p+q}P, |sk_{p-1}X|]^{(1)} \longrightarrow [\Sigma^{p+q}P, |sk_pX|]^{(1)} \longrightarrow \pi_p[\Sigma^qP, X] \to \cdots$$

Proof. We construct a map between the exact sequences which induces an isomorphism $\pi_p(X; \Sigma^qP) \cong \pi_p[\Sigma^qP, X]$. Once that is in place, the five lemma and an induction argument show that we must have an isomorphism. To do this, we write down the map

$$\mathrm{Hom}_{\mathcal{S}}(Z, C_KX) \cong \mathrm{Hom}_{s\mathcal{S}}(Z \wedge K, X) \longrightarrow \mathrm{Hom}_{\mathcal{S}}(Z \wedge |K|, |X|).$$

This does not induce a map out of the triangle of 3.1.6; however, after taking first derived triangles, we get a morphism from the triangle of 3.1.8 to the triangle 3.1.9, as required. $\square$

3.1.6 Remark. Lemma 3.1.5 implies that we have a spectral sequence

$$\pi_p[\Sigma^q DE_\alpha, X] \Longrightarrow [\Sigma^{p+q} DE_\alpha, |X|]$$

where the E_α are the finite cellular spectra so that $\operatorname{colim} E_\alpha \simeq E$. Taking the colimit of α, as in Remark 1.4.8 we get a spectral sequence

$$\pi_p E_q(X) \Longrightarrow E_{p+q}|X|.$$

Lemma 3.1.5 implies that this is the usual homology spectral sequence of a simplicial spectrum.

3.1.2 The module structure

The spiral exact sequence is natural in X and P and the naturality in P leads to the module structure of the exact sequence. To be concrete, we will limit ourselves to the situation which will arise here, but there are possibilities for almost infinite generalization. Thus in our basic case we will work with spectra and $\mathcal{P} = \mathcal{P}_E$ as in Definition 1.4.2.

Thus we will have a simplicial operad T that is homologically adapted to E_* and so that the resulting triple T_E on E_*E has an augmentation $T_E \to \Phi$. The notion of homologically adapted was defined in Definitions 1.4.13 and 1.4.16. The notion of an augmented triple was defined in Definition 2.5.7. In particular, we have a triple Φ on E_*E-comodules so that if X is a T-algebra, then $\pi_0 E_* T$ is a Φ-algebra. See Example 2.5.8 and Propositions 2.5.9 and 2.5.10.

3.1.7 Example. Here are the main examples:

1. In the case where T is the constant simplicial commutative monoid operad (so that a T-algebra is a simplicial E_∞-ring spectrum) and $E_* = K_*$ (p-completed K-theory), then Φ is the free θ-algebra functor.
2. In the case when T is a simplicial E_∞-operad and E_* is arbitrary, then Φ is simply the graded commutative algebra functor. Recall that T is a simplicial E_∞-operad if for all k the space $T(k)$ is contractible and if the action of Σ_k on $T(k)$ is level-wise free.
3. In the case when T is the constant simplicial associative monoid operad (so that T-algebras are simplicial A_∞-ring spectra), we can take Φ to be the associative algebra operad.

Now let $T(\mathcal{P})$ be the category with objects the simplicial T-algebras $T(P)$, $P \in \mathcal{P}$ (regarded as constant objects) and morphisms all classes of morphisms of T-algebras in the $\mathcal{P}$-resolution homotopy category obtained from Theorem 1.4.9. Let $\mathbf{Pre}_+(T(\mathcal{P}))$ be the product preserving presheaves of sets on $T(\mathcal{P})$ (there are no sheaves).

3.1.8 Example. The main example we have of an object in $\mathbf{Pre}_+(T(\mathcal{P}))$ is

$$T(P) \mapsto \pi_0 \operatorname{map}_T(T(P), X) \cong \pi_{0,P} X$$

when X is a (fibrant) simplicial T-algebra. Let $\pi_{0,*}X$ denote this object in $\mathbf{Pre}_+(T(\mathcal{P}))$

If we let $\mathcal{P}$ stand (by abuse of notation) for the category with objects $\mathcal{P}$ and morphisms all homotopy classes in spectra. There is a forgetful functor

$$\mathbf{Pre}_+(T(\mathcal{P}))\longrightarrow\mathbf{Pre}_+(\mathcal{P}).$$

given by restricting along the functor $T : \mathcal{P} \to T(\mathcal{P})$. In particular, we see that for each $P \in \mathcal{P}$ and each object $F \in \mathbf{Pre}_+(T(\mathcal{P}))$, the set $F(T(P))$ is actually an abelian group. However, not every transition function $F(T(P)) \to F(T(Q))$ need be a homomorphism of abelian groups.

We would like to regard the objects of $\mathbf{Pre}_+(T(\mathcal{P}))$ as algebras of a certain sort. In Section 2.1.2 we showed that there was an equivalence of categories

$$y_* : \mathbf{Alg}^{\Phi}_{E_*E}\longrightarrow\mathbf{Sh}_+(\Phi(E_*\mathcal{P}))$$

where $\mathbf{Sh}_+(\Phi(E_*\mathcal{P})) \subseteq \mathbf{Pre}_+(\Phi(E_*\mathcal{P}))$ was a full-subcategory satisfying a descent (or sheaf) condition. The functor y_* is the Yoneda embedding

$$A \mapsto \mathrm{Hom}_{\Phi}(-, A).$$

Less formally, the left adjoint to this equivalence was given by

$$y^*G = \mathrm{colim}_{\alpha}\, G(E_*\Sigma^* DE_{\alpha}).$$

See Lemma 2.1.21 for an exact statement. This functor extends to a functor $y^* : \mathbf{Pre}_+(\Phi(\mathcal{P})) \to \mathbf{Alg}^{\Phi}_{E_*E}$.

The functor

$$\pi_0 E_*(-) : T(\mathcal{P})\longrightarrow\Phi(E_*\mathcal{P})$$

guaranteed by our assumptions defines a restriction functor

$$\mathbf{Pre}_+(\Phi(E_*\mathcal{P})) \to \mathbf{Pre}_+(T(\mathcal{P}))$$

which has a left adjoint given by left Kan extension. This yields a composable pair of functors

$$\mathbf{Pre}_+(T(\mathcal{P})) \xrightarrow{L_{\mathrm{Kan}}} \mathbf{Pre}_+(\Phi(\mathcal{P})) \xrightarrow{y^*} \mathbf{Alg}^{\Phi}_{E_*E}$$

By abuse of notation we write $y^* : \mathbf{Pre}_+(T(\mathcal{P})) \to \mathbf{Alg}^{\Phi}_{E_*E}$ for this composite functor as well; it is left adjoint to the functor

$$A \mapsto \mathrm{Hom}_{\Phi}(\pi_0 E_*(-), A).$$

3.1.9 Lemma. *This composite functor* $y^* : \mathbf{Pre}_+(T(\mathcal{P})) \to \mathbf{Alg}^{\Phi}_{E_*E}$ *is isomorphic to the functor*

$$F \mapsto \mathrm{colim}_{\alpha}\, F(T(\Sigma^* DE_{\alpha})).$$

Proof. Let us drop the suspensions from the notation. After dissecting the definitions, we find that the composite is given by the coend[1]

$$\int^{T(\mathcal{P})} F(T(P)) \otimes \Phi(E_*P).$$

Since $\Phi(E_*P) \cong \pi_0 E_* T(P)$, we can write

$$\begin{aligned}\Phi(E_*P) &\cong \operatorname{colim}_i \pi_0 \operatorname{map}(DE_i, T(P)) \\ &\cong \operatorname{colim}_i \pi_0 \operatorname{map}_T(T(DE_i), T(P)).\end{aligned}$$

Thus, evaluation gives a map

$$\epsilon : \int^{T(\mathcal{P})} F(T(P)) \otimes \Phi(E_*P) \to \operatorname{colim}_i F(T(DE_i)).$$

The claim is that this natural map is an isomorphism. It clearly is if F is a representable of the form

$$F(-) = \pi_0 \operatorname{map}_T(-, T(P)).$$

Since the coend and the colimit commute all colimits in F, this implies that ϵ is an isomorphism if F is a coproduct of representables. The general case follows, since every F is the coequalizer of a pair of maps between coproducts of representables. □

Modules over algebras can be defined as abelian objects in an over category and a similar definition applies to the objects in $\mathbf{Pre}_+(T(\mathcal{P}))$; see Proposition 3.1.11 below. However, we can offer a more concrete definition exactly as in Definition 2.1.23. Only the base category on which our contravariant functors has changed.

3.1.10 Definition. *Let $F \in \mathbf{Pre}_+(T(\mathcal{P}))$. Then we specify an F-module M by the following data:*

1.) *an object $M \in \mathbf{Pre}_+(\mathcal{P})$; and*

2.) *for each $f : T(Q) \to T(P)$ a map of sets*

$$\phi_f : M(P) \times F(T(P)) \longrightarrow M(Q)$$

subject to the conditions that

a.) *if $f = T(f_0)$, then $\phi_f(x, a) = M(f_0)x$;*

b.) *for any composable pair of arrows in $T(\mathcal{P})$,*

$$\phi_{gf}(x, a) = \phi_f(\phi_g(x, a), F(g)a);$$

---

[1] If X is set and A is any category with coproducts, then $X \otimes A = \coprod_X A$.

c.) for all $a \in F(T(P))$, the function $\phi_f(-, a)$ is a homomorphim of abelian groups.

The F-modules form a category $\mathbf{Mod}_F(\mathcal{P})$ in the obvious way.

If M is an F-module, we form a new object $M \rtimes F$ of $\mathbf{Pre}_+(T(\mathcal{P}))$ exactly as in Remark 2.1.24 and then we have the analog of Proposition 2.1.25. The proof remains the same.

3.1.11 Proposition. *The functor*

$$(-) \ltimes F : \mathbf{Mod}_F(\mathcal{P}) \longrightarrow \mathbf{Abpre}_+(T(\mathcal{P}))/F$$

is an equivalence of categories.

3.1.12 Remark. If M is an F-module, then then there is a split projection of Φ-algebras

$$\begin{tikzcd} y^*(M \ltimes F) \arrow[r, shift left] & y^*F \arrow[l, shift left] \end{tikzcd}$$

which defines a module y^*M over the Φ-algebra y^*F. In our examples, this will actually be an ordinary module over the ring y^*F, perhaps with some additional structure if the operation θ is present. Lemma 3.1.9 implies that the module y^*M has a simple formula

$$y^*M = \operatorname{colim}_i M(\Sigma^* DE_i).$$

3.1.13 Example. Let $F \in \mathbf{Pre}_+(T(\mathcal{P}))$. Then F is not a module over itself, but there are modules $\Omega^n F$, for $n \geq 1$ and these modules play a very important part in this discussion. For any spectrum X set $\Sigma^n_+ X$ denote the spectrum $S^n_+ \wedge X$, where S^n_+ is the topological n-sphere with a disjoint basepoint. If $P \in \mathcal{P}$, then $\Sigma^n_+ P \in \mathcal{P}$. Then, if $F \in \mathbf{Pre}_+(T(\mathcal{P}))$, we define a new object $\Omega^n_+ F \in \mathbf{Pre}_+(T(\mathcal{P}))$ by the formula

$$\Omega^n_+ F(T(\mathcal{P})) = F(T(\Sigma^n_+ P)).$$

The evident split short exact sequence

$$\begin{tikzcd} 0 \arrow[r] & \Omega^n F \arrow[r] & \Omega^n_+ F \arrow[r, shift left] & F \arrow[l, shift left] \arrow[r] & 0 \end{tikzcd}$$

defines $\Omega^n F$ and its module structure over F. Note that as an abelian group $\Omega^n F(P) = F(T(\Sigma^n P))$.

If M is an E_*E-comodule we can define the shifted E_*E comodule $\Omega^n M$ by the formula $[\Omega^n M]_k = M_{n+k}$. (In Section 2.5 we called this module $M[n]$.) If M is a module over the Φ-algebra A, then so it $\Omega^n M$. Now one easily checks that

$$y^*\Omega^n F \cong \Omega^n(y^*F)$$

as a module over Φ-algebra y^*F.

3.1.14 Example. 1.) If X is a simplicial T-algebra, then

$$T(P) \mapsto \pi_n \operatorname{map}_T(T(P), X) = \pi_{n,P}X$$

is a $\pi_{0,*}X$ module which we will call $\pi_{n,*}X$. In fact, the natural split cofibration sequence of simplicial T-algebras

$$\begin{tikzcd} T(P) \arrow[r, shift left] & T(P \otimes \Delta^n/\partial\Delta^n) \arrow[l, shift left] \arrow[r] & T(P \wedge \Delta^n/\partial\Delta^n) \end{tikzcd}$$

yields the abelian object over $\pi_{0,*}X$ necessary to display $\pi_{n,*}X$ as a module:

$$\begin{tikzcd} \pi_{n,*}X \arrow[r] & \pi_0 \operatorname{map}_T(T(* \wedge \Delta^n/\partial\Delta^n), X) \arrow[r, shift left] & \pi_{0,*}X \arrow[l, shift left] \end{tikzcd}$$

An immediate consequence of these observations is that $y^*\pi_{n,*}X = \pi_{n,*}EX$ has a natural structure over the Φ-algebra $y^*\pi_{0,*}X = \pi_0 E_*X$.

2.) Slightly less obvious is that $\pi_n\pi_*X = \pi_n[-, X]$ is also a module over $\pi_{0,*}X \cong \pi_0\pi_*X$, for $n > 0$. To see this, let $T(\mathcal{P})_{\text{Reedy}}$ denote the category with objects $T(P)$, $P \in \mathcal{P}$ and morphisms the Reedy homotopy classes of maps in simplicial T-algebras. Then $C_0[-, X] \in \mathbf{Pre}_+(T(\mathcal{P})_{\text{Reedy}})$ and Lemma 3.1.2 implies that the functor $C_n[-, X]$ is an object in $\mathbf{Mod}_{C_0[-,X]}(T(\mathcal{P})_{\text{Reedy}})$. The projection functor $T(\mathcal{P})_{\text{Reedy}} \longrightarrow T(\mathcal{P})$ gives a restriction functor

$$\mathbf{Pre}_+(T(\mathcal{P})) \longrightarrow \mathbf{Pre}_+(T(\mathcal{P})_{\text{Reedy}})$$

and this gives $\pi_0\pi_*X$ the structure of an object in $\mathbf{Pre}_+(T(\mathcal{P})_{\text{Reedy}})$. The fact that the categories of modules have kernels and cokernels now imply that $\pi_n\pi_*X$ is an object in $\mathbf{Mod}_{\pi_0\pi_*X}(T(\mathcal{P})_{\text{Reedy}})$. We now have to argue that it actually descends to an object in $\mathbf{Mod}_{\pi_0\pi_*X}(T(\mathcal{P}))$. Because the morphisms $f : T(Q) \to T(P)$ in $T(\mathcal{P})_{\text{Reedy}}$ (or $T(\mathcal{P})$ for that matter) form an abelian group, it is sufficient to show that if f descends to the trivial morphism $T(Q) \to T(*) \to T(P)$ in $T(\mathcal{P})$, then the induced morphism on $\pi_n\pi_*X$ is trivial. But we have a factoring

$$\begin{tikzcd} T(Q \wedge \Delta^n/\partial\Delta^n) \arrow[r, "d^0"] \arrow[d, "f \wedge \Delta^n/\partial\Delta^n"'] & T(Q \otimes \Delta^{n+1}/\Delta_0^{n+1}) \arrow[d] \\ T(P \otimes \Delta^n/\partial\Delta^n) \arrow[r] & T(P \otimes \Delta^n/\partial\Delta^n)' \end{tikzcd}$$

where $(-)'$ means some functorial fibrant replacement. The claim follows.

An immediate consequence of these observations is that $y^*\pi_n\pi_*X = \pi_n E_*X$ has a natural structure as a module over the Φ-algebra $\pi_0 E_*X$.

The main result on module structures is the following:

3.1.15 Theorem. *Let $X \in s\mathbf{Alg}_T$ be a fibrant simplicial T-algebra. Then the isomorphism*

$$\pi_{0,*}X \to \pi_0\pi_*X$$

is an isomorphism of objects in $\mathbf{Pre}_+(T(\mathcal{P}))$ and the the spiral exact sequence is naturally an exact sequence of $\pi_{0,*}X$-modules.

The proof is exactly the same as for Proposition 7.13 of [7]. Since it is tedious we won't give it here.

We now come to the main result. In order to state it, we need a bit of notation.

3.1.16 Definition. *If X is a simplicial spectrum and E_* is a homology theory with representing spectrum E, form the new simplicial spectrum $EX = E \wedge X$ and define its* **bigraded homotopy groups** *by the equation*

$$\pi_{p,q}EX = \pi_p \operatorname{map}_{s\mathcal{S}}(S^q, EX)$$

The mapping space here is the external mapping space defined using the standard simplicial structure on a category of simplicial objects and we derived the mapping space, if necessary, using the resolution model category structure based on the set of projectives $\{S^q\}$, $q \in \mathbb{Z}$. See Theorem 1.4.6.

3.1.17 Example. From Example 3.1.14 we immediately have that $\pi_{p,*}EX$ and $\pi_p E_*(X)$ are modules over the Φ-algebra $\pi_{0,*}EX = \pi_0 E_* X$.

The following now immediately follows from Theorem 3.1.15 by applying the functor y^*; that is, by passing to a colimit.

3.1.18 Corollary. . *Let $X \in s\mathbf{Alg}_T$ be a fibrant simplicial T-algebra. Then the isomorphism*

$$\pi_{0,*}EX \cong \pi_0 E_* X$$

is an isomorphism in $\mathbf{Alg}^{\Phi}_{E_*E}$ and the spiral exact sequence

$$\begin{aligned} \cdots \to \Omega\pi_{n-1,*}EX \to \pi_{n,*}EX \to \pi_n E_* X \to \\ \Omega\pi_{n-2,*}EX \to \cdots \to \pi_{1,*}EX \to \pi_1 E_* X \to 0 \end{aligned}$$

is an exact sequence of $\pi_{0,*}EX$-modules.

3.2 Postnikov systems for simplicial algebras in spectra

This section sets up a theory of Postnikov towers for simplicial T-algebras, where T is one of our simplicial operads. The important correspondence to the theory for simplicial algebras constructed in Section 2.5 is provided by the k-invariants and the Eilenberg-MacLane objects, which will represent André-Quillen cohomology. In order to make this correspondence explicit, we must make some assumptions. The following holds for this rest of this monograph, and we note that most of this has come up before. The notion of homologically adapted was defined in Definitions 1.4.13 and 1.4.16. The notion of an augmented triple was defined in Definition 2.5.7.

3.2.1 Assumptions. *Let T be a simplicial operad and $s\mathbf{Alg}_T$ the category of simplicial algebras in spectra. Fix an Adams-type homology theory E_* and give $s\mathbf{Alg}_T$ the $\mathcal{P}_E = \mathcal{P}$-resolution model category structure. Furthermore*

1. *The simplicial operad T is homotopically adapted to E_*;*
2. *the resulting triple T_E on simplicial E_*E-comodules has an augmentation $T_E \to \Phi$. In particular, we have a triple Φ on E_*E-comodules so that if X is a T-algebra, then $\pi_0 E_*X$ is a Φ-algebra.*
3. *the zeroth simplicial set $T(0)$ of the simplicial operad T is a point; in particular, the sphere spectrum is the initial object in $s\mathbf{Alg}_T$;*
4. *the category $s\mathbf{Alg}_{T_E}$ satisfies Blakers-Massey Excision, as in 2.5.11.*

3.2.2 Example. There are three examples we have in mind. The following statement collect the results of Example 2.5.8, Propositions 2.5.9 and 2.5.10, and Theorem 2.3.13.

1. Let T be the associative monoid operad, regarded as a constant simplicial operad. The $s\mathbf{Alg}_T$ is the category of simplicial associative algebras in spectra – that is, simplicial A_∞-ring spectra. We can let F and Φ be the associative algebra triple on E_*E-comodules.
2. Let T be a simplicial E_∞-operad. Then we can let $F = E_*T$ regarded as triple and we can let Φ be the commutative algebra triple.
3. For this example, we specialize to the case of $E_* = K_*$, p-completed K-theory. Then we can let T be constant commutative monoid operad, so that $s\mathbf{Alg}_T$ is the category of simplicial commutative algebras in spectra – that is, simplicial E_∞-ring spectra. Then we can let $F = \Phi$ be the free theta-algebra triple.

The question of whether these operads are relatively left proper and satisfied Blakers-Massey excision was settled in Example 2.3.2, Example 2.3.4, and Proposition 2.3.11.

3.2.3 Remark (Notation for André-Quillen Cohomology). In the rest of this paper were are going to work with André-Quillen cohomology of simplicial E_*-algebras.

Suppose k is Φ-algebra and Y is a simplicial T-algebra equipped with a weak equivalence of E_*T-algebras $E_*Y \to k$. Equivalently, we could require that $\pi_n E_*Y = 0$ for $n > 0$ and $\pi_0 E_*Y \cong k$ as Φ-algebras. (In the context of the three examples just given, we are thinking of the example where Y is the constant simplicial algebra on some E_∞-ring spectrum.) Suppose we are given a morphism of $k \to A$ of Φ-algebras and an A-module M. Now let $Y \to X$ be a morphism of simplicial T-algebras so that X is equipped with a morphism of Φ-algebras $\pi_0 X \to A$ so that the composite

$$k \cong \pi_0 E_*Y \to \pi_0 E_*X \to A$$

is our chosen morphism $k \to A$. Then we will be concerned with the André-Quillen cohomology groups

$$H^n_{T_E/E_*E}(E_*X/k, M).$$

This is a bit of a mouthful, so we will write $H^n(E_*X/k, M)$ for these groups, or even $H^n(E_*X, M)$ if $k = E_*S = E_*$ with the Φ-algebra structure obtained from Assumptions 3.2.1.

We now get down to our construction of Postnikov towers. Recall that we have two homotopy theories on simplicial T-algebras. First, there is the $\mathcal{P}$-resolution model category structure where $\mathcal{P}$ is a fixed set of finite CW-spectra closed under coproducts and containing the spectra $\Sigma^k DE_i$. This simplicial model category structure was defined and discussed in Section 1.4 and figured in the Assumptions 3.2.1. Second, there is the localization of this category, where we define a morphism $f : X \to Y$ to be an $\pi_*E_*(-)$-equivalence if

$$\pi_*E_*X \longrightarrow \pi_*E_*Y$$

is an isomorphism. This yielded only a semi-model category (See Definition 1.1.6.); the cofibrations remained the same as in $\mathcal{P}$-resolution model category. While the latter is the one that is ultimately important, the former is the key to constructions, and we will take care to keep them straight.

3.2.4 Definition. *Let $X \in s\mathbf{Alg}_T$ be a simplicial T-algebra in spectra. Then an nth* **Postnikov section** *for X is a morphism of simplicial T-algebras $q : X \to Y$ so that there is an isomorphism*

$$f_* : \pi_{i,P}X \xrightarrow{\cong} \pi_{i,P}Y, \qquad i \leq n$$

for all $P \in \mathcal{P}$ and so that $\pi_{i,P}Y = 0$ for $i > n$. More succinctly, we will say that $f_* : \pi_{i,*}X \to \pi_{i,*}Y$ is an isomorphism for $i \leq n$ and that $\pi_{i,*}Y = 0$ for $i > n$. The asterisk $(*)$ is a placeholder for $P \in \mathcal{P}$. A Postnikov tower for X is a tower of simplicial T-algebras under X

$$X \to \cdots \to X_n \to X_{n-1} \to \cdots \to X_0$$

so that $X \to X_n$ is an nth Postnikov section.

The reader will have noticed that this definition depends on $\mathcal{P}$ and, perhaps, that $\mathcal{P}$ should be included in the notation at some point. However, since $\mathcal{P}$ will be fixed throughout, we forebear.

3.2.5 Lemma. *Let X be a simplicial T-algebra in spectra. Then there exists a natural Postnikov tower for X*

$$X \to \cdots \to P_nX \to P_{n-1}X \to \cdots \to P_0X$$

Proof. The only wrinkle on the standard construction is that not every object in $s\mathbf{Alg}_T$ is Reedy fibrant. We let $X \to X'$ denote some functorial acyclic cofibration from X to a fibrant object. Then $P_n X = \operatorname{colim} P_{n,t} X$ where $P_{n,0}X = X'$ and $P_{n,t+1}X = Y'$ with Y defined by the push-out diagram

$$\begin{tikzcd}
\coprod_{P,k>n} \coprod_{f: P\wedge \Delta^k/\partial\Delta^k \to P_{n,t}X} P \wedge \Delta^k/\partial\Delta^k \arrow[r] \arrow[d] & P_{n,t}X \arrow[d] \\
\coprod_{P,k>n} \coprod_{f: P\wedge \Delta^k/\partial\Delta^k \to P_{n,t}X} P \wedge \Delta^{k+1}/\Delta^{k+1}_0 \arrow[r] & Y.
\end{tikzcd}$$

□

Recall that $\mathbf{Pre}_+(T(\mathcal{P}))$ is the category of functors

$$F : T(\mathcal{P})^{op} \longrightarrow \mathbf{Sets}$$

which preserve products.

3.2.6 Definition. *1.) Let $F \in \mathbf{Pre}_+(T(\mathcal{P}))$. Then we say that a simplicial T-algebra is of type B_F if $\pi_{0,*}X \cong F$ and $\pi_{i,*}X = 0$ for $i > 0$.*

2.) Suppose further that M is an F-module. Then we say a morphism $X \to Y$ is of T-algebras is of type $B_F(M, n)$, $n \geq 1$, if X is of type B_F, the morphism

$$\pi_{0,*}X \longrightarrow \pi_{0,*}Y$$

is an isomorphism, $\pi_{n,*}Y \cong F$ as an F-module, and $\pi_{i,*}Y = 0$ if $i \neq 0$ or n. As a shorthand, we may say Y is of type $B_F(M, n)$, leaving the morphism $X \to Y$ understood.

Note that $X \to Y$ is of type $B_F(M, n)$, then the composition

$$X \longrightarrow Y \longrightarrow P_0 Y$$

is a weak equivalence. This observation, the spiral exact sequence, and Theorem 3.1.15 immediately imply the following lemma.

3.2.7 Lemma. *1.) Let X be of type B_F. Then $\pi_0\pi_*X \cong F$, $\pi_i\pi_*F = 0$ if $i \neq 0, 2$ and*

$$\pi_2\pi_*X \cong \Omega F$$

as an F-module.

2.) Let $X \to Y$ be of type $B_F(M, n)$. Then there is an isomorphism

$$\pi_i\pi_*Y \cong \pi_i\pi_*X \times \begin{cases} M & i = n; \\ \Omega M & i = n+2; \\ 0 & \text{otherwise.} \end{cases}$$

If $i \geq 1$, this is an isomorphism of F-modules.

3.2.8 Example. Let $A \in \mathbf{Alg}_\Phi$ and N an A-module. Recall that the triple Φ on E_*E-comodules is built into our Assumptions 3.2.1. Then we have the associated object $F \in \mathbf{Pre}_+(T(\mathcal{P}))$

$$F(-) = \mathrm{Hom}_{\mathbf{Alg}_\Phi}(\pi_* E_*(-), A)$$

and the F-module M

$$M(-) = \mathrm{Hom}_{E_*E}(E_*(-), N).$$

The previous result and a colimit argument as in Remark 1.4.8 show that if X is of type B_F, then

$$\pi_i E_* X \cong \begin{cases} A & i = 0; \\ \Omega A & i = 2; \\ 0 & \text{otherwise} \end{cases}$$

and, by Corollary 3.1.18 this is an isomorphism of A-modules for $i \geq 1$. Furthermore, if $X \to Y$ is of type $B_F(M, n)$, then

$$\pi_i E_* Y \cong \pi_i E_* X \times \begin{cases} M & i = n; \\ \Omega M & i = n+2; \\ 0 & \text{otherwise.} \end{cases}$$

Again this is an isomorphism of A-modules in positive degrees. Note, in particular, that E_*Y is not of type $K_A(M, n)$. Compare Definition 2.5.12.

We now come to a functorial construction of k-invariants. Let $f : X \to Y$ be any morphism in $s\mathbf{Alg}_T$ and let C be the pushout of the two-source

$$Y' \longleftarrow X' \longrightarrow (P_0 X)'$$

where use the symbol $(-)'$ to denote some functorial construction to replace X be a $\mathcal{P}$-cofibrant simplicial algebra and the two maps by $\mathcal{P}$-cofibrations. Then, applying the Postnikov section functor of Lemma 3.2.5, we obtain a commutative diagram

(3.2.1)
$$\begin{tikzcd}
X \arrow[d, "f"'] & X' \arrow[l, "\simeq"'] \arrow[r] \arrow[d] & (P_0 X)' \arrow[d, "\delta_n(f)"] \\
Y & Y' \arrow[l, "\simeq"'] \arrow[r] & P_{n+1} C.
\end{tikzcd}$$

We will refer to the morphism $\delta_n(f)$ as the *difference construction* applied to f.

3.2.9 Proposition. *Let $f : X \to Y$ be a morphism of simplicial T-algebras and suppose there is an $n \geq 1$ so that*

1. *$f_* : \pi_{i,*} EX \to \pi_{i,*} EY$ is an isomorphism for $i < n$, and*
2. *$f_* : \pi_{n,*} EX \to \pi_{n,*} EY$ is surjective.*

Let $M = \pi_{n+1,*}(EY, EX)$. Then M is naturally an $A = \pi_{0,*}EX \cong \pi_0 E_*X$ module and there in an $\pi_*E_*(-)$-equivalence from $\delta_n(f)$ to a morphism of type $B_A(M, n+1)$. If $\pi_{i,*}(Y, X) = 0$ for $i \neq n+1$, then the right hand square of 3.2.1 induces an $\pi_*E_*(-)$-equivalence

$$X' \to \operatorname{holim}\{Y' \to P_{n+1}C \leftarrow (P_0X)\}.$$

Proof. There is a homotopy push-out in simplicial E_*T-algebras

$$\begin{tikzcd}
E_*X' \arrow[r] \arrow[d] & E_*(P_0X)' \arrow[d] \\
E_*Y' \arrow[r] & E_*C.
\end{tikzcd}$$

This is because the functor $E_*(-) : s\mathbf{Alg}_T \to s\mathbf{Alg}_{E_*T}$ preserves cofibrations, weak equivalences, and push-outs along free cofibrations. By the five-lemma and the spiral exact sequence, we have that

$$\pi_i E_*X \to \pi_i E_*Y$$

is a surjection for $i \leq n$ and an isomorphism for $i < n$. Furthermore,

$$\pi_{n+1}E_*(Y, X) \cong M$$

as an A-module. Then, Corollary 2.3.15 implies that $\pi_i E_*(C, P_0X) = 0$ for $i \leq n$ and

$$\pi_i E_*(C, P_0X) \cong M$$

as A-modules. This and using the spiral exact sequence in reverse proves that $\delta_n(f)$ is as claimed. It is then straightforward to check the final claim. □

3.2.10 Remark. There is a stronger result than the one we just proved. Indeed, let $f : X \to Y$ be a morphism of simplicial T-algebras and suppose there is an $n \geq 1$ so that

1. $f_* : \pi_{i,*}X \to f_{i,*}Y$ is an isomorphism for $i < n$, and
2. $f_* : \pi_{n,*}X \to \pi_{n,*}Y$ is a pointwise surjective

Let $M = \pi_{n+1,*}(Y, X)$. The M is naturally a $F = \pi_{0,*}X$ module and $\delta_n(f)$ is a morphism of type $B_F(M, n+1)$. If $\pi_{i,*}(Y, X) = 0$ for $i \neq n+1$, then the right hand square of 3.2.1 is a homotopy pull-back square.

This can be proved exactly as the comparable result in section 8 of [7]. However, this would mean developing the homotopy theory of $\mathbf{Pre}_+(T(\mathcal{P}))$ and we haven't done that. Since this is not relevant for our main applications, we will be content with the previous result.

The next question is whether Eilenberg-MacLane objects exist. Again we concentrate on the case where A is the kind of algebra which can arise as $\pi_0 E_*X$, where X is a simplicial T-algebra. Thus we will have a simplicial operad T that is homologically adapted to E_* and so that the resulting triple T_E on E_*E has an augmentation $T_E \to \Phi$. See Assumptions 3.2.1 and Examples 3.2.2.

3.2.11 Proposition. *Let A be a Φ-algebra and M a Φ-module over A. Then there is a simplicial T-algebra of type B_A and for each $n \geq 1$ there is a morphism of simplicial T-algebras of type $B_A(M,n)$. Furthermore, for $X \in s\mathbf{Alg}_T$ there are natural isomorphisms*

$$\begin{aligned} \pi_0 \operatorname{map}(X, B_A) &\cong \pi_0 \operatorname{map}_{s\mathbf{Alg}_{E_*T}}(E_*X, A) \\ &\cong \operatorname{Hom}_{\mathbf{Alg}_\Phi}(\pi_0 E_*X, A). \end{aligned}$$

and

$$\begin{aligned} \pi_0 \operatorname{map}(X, B_A(M,n)) &\cong \pi_0 \operatorname{map}_{s\mathbf{Alg}_{E_*T}}(E_*X, K_A(M,n)) \\ &\cong \coprod_{\pi_0 E_*X \to A} H^n(E_*X, M). \end{aligned}$$

Proof. This can be done by a generator and relations argument. (See [7].) Alternatively, we could use a Brown representability argument. (See [22].) We need to show certain functors are representable – namely, the targets of the isomorphisms listed in the statement of the result. The argument given in [44] certainly works, where we use as our spheres the objects $T(P \otimes \Delta^k/\partial\Delta^k)$. We leave the details to the reader. □

It is worth recording immediately that the Eilenberg-MacLane object B_A constructed in this result has a strong homotopy discreteness property.

3.2.12 Lemma. *Let B_A be an Eilenberg-MacLane object so that*

$$\pi_0 \operatorname{map}(X, B_A) \cong \operatorname{Hom}_{\mathbf{Alg}_\Phi}(\pi_0 E_*X, A)$$

for all simplicial T-algebras A. Then all of the components of $\operatorname{map}(X, B_A)$ *are contractible.*

Proof. This follows from the fact that if $* \to \Delta^k/\partial\Delta^k$ is the inclusion of the basepoint, then the induced map

$$X \cong X \otimes * \to X \otimes \Delta^k/\partial\Delta^k$$

induces an isomorphism on $\pi_0 E_*(-)$. □

We next turn to the project of identifying the homotopy type of the mapping space $\operatorname{map}(X, B_A(M,n))$.

By taking the class of the identity in $\pi_0 \operatorname{map}(B_A(M,n), B_A(M,n))$ and using the isomorphism supplied by the second part of Proposition 3.2.11, we have a universal morphism $u : E_*B_A(M,n) \longrightarrow K_A(M,n)$ and a diagram

(3.2.2)
$$\begin{tikzcd} E_*B_A(M,n) \arrow[r, "u"] \arrow[d] & K_A(M,n) \arrow[d] \\ E_*B_A \arrow[r] & A. \end{tikzcd}$$

Now Example 3.2.8 implies that if $X \to Y$ is of type $B_A(M,n)$, then

$$\delta_n(E_*f) : P_0^{alg}(E_*X) \to P_n^{alg}C$$

is of type $K_A(M,n)$. Here P_n^{alg} denote the algebraic Postnikov section of Proposition 2.5.6 (there simply called P_n) and C is the homotopy push-out in $s\mathbf{Alg}_{E_*T}$ of

$$\begin{tikzcd} P_0^{alg}E_*X & E_*X \arrow[l] \arrow[r] & E_*Y. \end{tikzcd}$$

Applying this observation to the universal morphism u we get a diagram

$$\begin{tikzcd} E_*B_A(M,n) \arrow[r] \arrow[d] & P_{n+1}^{alg}C \arrow[r, "v"] \arrow[d] & K_A(M,n) \arrow[d] \\ E_*B_A \arrow[r] & A \arrow[r, "="'] & A. \end{tikzcd}$$

3.2.13 Lemma. *The induced map*

$$v : P_{n+1}^{alg}C \longrightarrow K_A(M,n)$$

is a weak equivalence of simplicial T_E-algebras in E_*E-comodules.

Proof. Let $X = T(P \wedge \Delta^n/\partial\Delta^n)$. Then we get, by examining the definition of u, a commutative diagram

$$\begin{tikzcd} \pi_{n,P}B_A(M,n) \arrow[r, "\cong"] \arrow[d, "\cong"'] & \pi_0 \operatorname{map}_{sC_{E_*E}}(E_*P \wedge \Delta^n/\partial\Delta^n, K_A(M,n)). \\ \pi_n\pi_P B_A(M,n) \arrow[ur] & \end{tikzcd}$$

The horizontal map is an isomorphism by construction and the vertical map is an isomorphism by the spiral exact sequence. In the end, we get an isomorphism

$$\pi_n\pi_P B_A(M,n) \xrightarrow{\cong} \pi_0 \operatorname{map}_{sC_{E_*E}}(E_*P \wedge \Delta^n/\partial\Delta^n, K_A(M,n)).$$

Letting $P = \Sigma^k DE_i$, taking the colimit over i and letting k vary gives an isomorphism

$$\pi_n E_*B_A(M,n) \xrightarrow{\cong} \pi_n K_A(M,n).$$

The result follows. □

We now give a continuous version of the statement that Eilenberg-MacLane objects represent cohomology, and we also take a moment to present a relative version. If M is some A-module, let

$$\mathcal{H}^n(E_*X/k, M) = \operatorname{map}_{s\mathbf{Alg}_{T_E/A}}(E_*X, K_A(M,n))$$

denote the André-Quillen cohomology *space*. This is the derived space of maps of simplicial T_E-algebras over A. Of course,

$$\pi_i \mathcal{H}^n(E_*X/k, M) \cong H^{n-i}(E_*X/k, M).$$

We should really write $\mathcal{H}^n_{T_E/E_*E}(E_*X/k, M)$, but in keeping with Remark 3.2.3 we shorten the notation. If $k = E_*$, we write will continue to write $H^*(E_*X, M)$ for $H^*(E_*X/E_*, M)$.

First we have an absolute result.

3.2.14 Proposition. *Let A be a Φ-algebra, M an A-module and let $B_A(M, n)$ be an Eilenberg-MacLane object which represents Andre-Quillen cohomology as in 3.2.11.2. Let $n \geq 2$. Then functor which sends*

$$X \leftarrow U \to V \to B_A(M, n)$$

to

$$E_*X \leftarrow E_*U \to E_*V \to E_*B_A(M, n) \xrightarrow{u} K_A(M, n)$$

defines a natural weak equivalence

$$f_X : \mathrm{map}_{s\mathbf{Alg}_T/B_A}(X, B_A(M, n)) \to \mathcal{H}^n(E_*X, M).$$

Proof. In this proof we will write

$$\mathrm{map}_{s\mathbf{Alg}_T/B_A}(-,-) = \mathrm{map}_{B_A}(-,-)$$

to make some of our more cluttered calculations easier on the eye. The morphism f_X is a morphism of H-spaces, so it is sufficient to show that f_X induces an isomorphism on homotopy groups. We choose as basepoint of the mapping space $\mathrm{map}_{B_A}(X, B_A(M, n))$ the "constant" map

$$X \to B_A \to B_A(M, n).$$

This maps to the corresponding constant map

$$E_*X \to A \to K_A(M, n).$$

We have an isomorphism on π_0 by Proposition 3.2.11.

To examine what happens in higher homotopy groups, we make a construction. Let $\mathcal{C}$ be any simplicial category. If K is a simplicial set and Y is in $\mathcal{C}$ let $\mathrm{hom}(K, Y)$ be the internal mapping (or exponential) object. We may fix an object U and consider the category $\mathcal{C}/U$ of objects over U. If K is a simplicial set and $Y \to U$ is in $\mathcal{C}/U$, we define the mapping object $\mathrm{hom}^U(K, Y)$ by the pull-back diagram

$$\begin{tikzcd}
\mathrm{hom}^U(K, Y) \arrow[r] \arrow[d] & \mathrm{hom}(K, Y) \arrow[d] \\
U = \mathrm{hom}(*, U) \arrow[r] & \mathrm{hom}(K, U).
\end{tikzcd}$$

If $Y \to U$ has a section and K is pointed, we may define the pointed mapping object by making a further pull-back

$$\begin{tikzcd}
\hom_*^U(K, Y) \arrow[r] \arrow[d] & \hom^U(K, Y) \arrow[d] \\
U \arrow[r] & \hom^U(*, Y) = Y.
\end{tikzcd}$$

Note that the section on Y induces a section $U \to \hom_*^U(K, Y)$. One now checks that we have a commutative square

$$\begin{tikzcd}
\pi_p \operatorname{map}_{B_A}(X, B_A(M, n)) \arrow[r, "\cong"] \arrow[d] & \pi_0 \operatorname{map}_{B_A}(X, \hom_*^{B_A}(\Delta^p/\partial\Delta^p, B_A(M, n))) \arrow[d] \\
\pi_p \operatorname{map}_A(E_*X, K_A(M, n)) \arrow[r, "\cong"'] & \pi_0 \operatorname{map}_A(E_*X, \hom_*^{K_A}(\Delta^p/\partial\Delta^p, K_A(M, n))).
\end{tikzcd}$$

The result follows once one checks that $B_A \to \hom_*^{B_A}(\Delta^k/\partial\Delta^k, B_A(M, n))$ and $A \to \hom_*^A(\Delta^k/\partial\Delta^k, K_A(M, n))$ are of type $B_A(M, n-k)$ and $K_A(M, n-k)$ respectively, and that

$$E_* \hom_*^{B_A}(\Delta^k/\partial\Delta^k, B_A(M, n)) \to \hom_*^A(\Delta^k/\partial\Delta^k, K_A(M, n))$$

is a model for the universal morphism. This is easy and left to the reader. □

We are now going to prove two results about the homotopy types of various moduli spaces of Eilenberg-MacLane objects. It is important for the next section that we have a relative version of the results here. Choose a morphism $k \to A$ of Φ-algebras and suppose we have an E_∞-ring spectrum Y so that $E_*Y \cong k$ as Φ-algebras. We may regard Y as a constant object in $s\mathbf{Alg}_T$ and then choose a $\mathcal{P}$-equivalence $Y_c \to Y$ with Y_c cofibrant in the $\mathcal{P}$-resolution model category. In particular, $\pi_n E_* Y_c = 0$ for $n > 0$ and we have an isomorphism of Φ-algebras, $\pi_0 E_* Y_c \cong k$. Corollary 1.4.12 implies that the induced map $|Y_c| \to Y$ is an E_* equivalence.

In this section and the next we are going to be working with the category $s\mathbf{Alg}_{Y_c}$ of simplicial T-algebras under Y_c. Because of our assumptions 3.2.1, this category is independent of the choice of Y_c; specifically, we have the following result.

3.2.15 Lemma. *Suppose $f : Y_0 \to Y_1$ is a $\pi_* E_*(-)$ equivalence of $\mathcal{P}$-cofibrant objects in $s\mathbf{Alg}_T$. Then the adjoint pair*

$$f^* = Y_1 \sqcup_{Y_0} (-) s\mathbf{Alg}_{Y_0} \rightleftarrows s\mathbf{Alg}_{Y_1} : f_*$$

is a Quillen equivalence of semi-model categories.

Proof. Recall that we are using the $\pi_* E_*(-)$ isomorphisms as our weak equivalences. The functor f_* sends a morphism $Y_1 \to X$ to the composition

$$Y_0 \xrightarrow{f} Y_1 \longrightarrow X.$$

The functor f_* preserves all $\pi_* E_*(-)$-equivalences and fibrations; the functor f^* preserves cofibrations for formal reasons and $\pi_* E_*(-)$-equivalences between cofibrant objects by Lemma 1.5.10 – or, more exactly, by the argument given for the second part of the proof of that result.

The lemma here now follows as any two $\mathcal{P}$-cofibrant replacements can be connected by a chain $\pi_* E_*(-)$-equivalences. $\square$

Now select the model for an Eilenberg-MacLane object of type B_A constructed in Proposition 3.2.11. The morphism $k \to A$ of Φ-algebras yields a unique homotopy class of T-algebra maps $Y_c \to B_A$; by fixing a representative, we may assume that B_A is a T-algebra under Y_c. Similarly, we may construct Eilenberg-MacLane objects of type $B_A(M, n)$ under Y_c.

3.2.16 Proposition. *Let $k \to A$ be a morphism of Φ-algebras, Y an E_∞-ring spectrum so that $E_* Y \cong k$ as Φ-algebras and $Y_c \to Y$ a $\mathcal{P}$-cofibrant model for Y in simplicial T-algebras. Let B_A and $B_A(M, n)$ be the Eilenberg-MacLane objects of 3.2.11.*

1. *Evaluation at $\pi_0 E_*(-)$ defines a natural isomorphism*

   $$\pi_0 \operatorname{map}_{Y_c}(X, B_A) \cong \operatorname{Hom}_k(\pi_0 E_* X, A)$$

   where map_{Y_c} is the derived space of morphisms of simplicial T-algebras under Y_c and Hom_k means homomorphisms of Φ-algebras under k. In addition, the components of $\operatorname{map}_{Y_c}(X, B_A)$ are contractible.

2. *If $n \geq 2$, the universal element $u : E_* B_A(M, n) \to K_A(M, n)$ defines a natural weak equivalence*

   $$\operatorname{map}_{Y_c/B_A}(X, B_A(M, n)) \simeq \mathcal{H}^n(E_* X/k, M)$$

   where $\operatorname{map}_{Y_c/B_A}$ denotes the derived space of morphisms of simplicial T-algebras under Y_c and over B_A.

Proof. The first statement follows from a pull-back argument using Proposition 3.2.11.1 and Lemma 3.2.12. The second statement follows from a pull-back argument, Proposition 3.2.14, and Remarks 2.4.3, 2.4.8, and 2.4.10. $\square$

All our moduli spaces will be formed in the category $s\mathbf{Alg}_{Y_c}$. In order to specify these moduli spaces we need to specify a class of weak equivalences. In both Proposition 3.2.17 and Proposition 3.2.16 we will mean the $\pi_* E_*(-)$-equivalences of simplicial T-algebras. Recall that $\mathcal{M}(K_A \looparrowright K_A(M, n))$ is the moduli morphisms of simplicial $E_* T$-algebras which induce an isomorphism in π_0. This is exactly the moduli space of all *algebraic* Eilenberg-MacLane objects of type $K_A(M, n)$. See Definition 2.5.12 and Proposition 2.5.19. Even algebraically, we are still working in a relative situation; for example, K_A will be an object in the category $s\mathbf{Alg}_k$ of simplicial F-algebras under k and $\mathcal{M}(K_A)$ is formed in $s\mathbf{Alg}_k$.

3.2.17 Proposition. *Let $k \to A$ be a morphism of Φ-algebras and M a Φ-module over A. Furthermore, let Y be an E_∞-ring spectrum so that $E_*Y \cong k$ as Φ-algebras and suppose $Y_c \to Y$ is a $\mathcal{P}$-cofibrant model for Y as a simplicial T-algebra.*

1. *Let $\mathcal{M}(A)$ be moduli space of all simplicial T-algebras of type B_A under Y_c. Then the functor $X \mapsto P_0^{alg} E_*X$ defines a weak equivalence*

$$\mathcal{M}(A) \to \mathcal{M}(K_A) \simeq B\,\mathrm{Aut}_k(A).$$

2. *Let $\mathcal{M}_A(M, n)$ be the moduli space of all morphisms of type $B_A(M, n)$ in simplicial T-algebras under Y_c. Then the functor $f \mapsto \delta_{n-1}(E_*)$ defines a weak equivalence*

$$\mathcal{M}_A(M, n) \to \mathcal{M}(K_A \looparrowright K_A(M, n)) \simeq B\,\mathrm{Aut}_k(A, M)$$

In particular, these spaces are connected and any Eilenberg-MacLane object in $s\mathbf{Alg}_T$ will represent André-Quillen cohomology.

Proof. Both of these statements follow from examining the functor that the object in question represents. We begin with first point. Choose a fixed bifibrant simplicial T-algebra Z under Y_c which represents $\mathrm{Hom}_k(\pi_0 E_*(-), A)$. See Proposition 3.2.16. Then if X is any simplicial T-algebra of type B_A under Y_c, the isomorphism $\pi_0 E_*X \to A$ defines a morphism $X \to Z$ under Y_c which is $\mathcal{P}$-equivalence. Thus $\mathcal{M}(A) \cong B\mathrm{Aut}(X)$. Now an easy calculation shows that

$$\pi_0 \mathrm{Aut}(X) \cong \mathrm{Aut}_k(A).$$

via $f \mapsto \pi_0 E_* f$. To complete the argument, use Proposition 3.2.16 to show that $\mathrm{Aut}(X)$ is homotopically discrete.

The second point is proved similarly. Choose a bifibrant model Z for B_A and a cofibration $g : Z \to W$ of type $B_A(M, n)$ so that W represents

$$X \mapsto \coprod_{\pi_0 E_* X \to A} H^n(E_*X/k, M).$$

Then if we have any morphism f of type $B_A(M, n)$, there is an evident map $E_*f \to \delta_{n-1} E_* f \cong E_* g$, which – using the strong representability result of Proposition 3.2.16 – defines an E_*-equivalence from f to g. This shows that $\mathcal{M}(A, n)$ is connected, and now we need only show that $\mathrm{Aut}(g)$ is homotopically discrete. But this is a simple calculation. Compare the corresponding result in [7]. □

3.2.18 Remark. Combining Proposition 3.2.9 with Proposition 3.2.16 we can identify where k-invariants for simplicial T-algebras live. Indeed, when $X \in s\mathbf{Alg}_{Y_c}$ is a simplicial T-algebra under Y_c and $\pi_0 X \cong A$ as Φ-algebras, the Postnikov tower becomes a tower under Y_c and the nth k-invariant determines an equivalence class of elements in the group

$$H^{n+1}(E_* P_{n-1} X/k, \pi_n E_* X).$$

3.3 The decomposition of the moduli spaces

Let us recall the basic setup. We have a simplicial operad T so that the assumptions of 1.4.16 and 3.2.1 hold. In particular, there is a fixed homology theory E_* and a triple Φ on E_*E-comodules so that for all simplicial T-algebras X, $\pi_0 E_* X$ is naturally a Φ-algebra. In our two main examples, Φ is the free commutative algebra functor or the free theta-algebra functor.

The arguments and ideas of this section also apply to the case of associative algebras. These are considerably easier, and left to the reader.

Note that if Y is simply an E_∞-ring spectra, then Y may be regarded as a constant object in $s\mathbf{Alg}_T$; hence, our assumptions imply that E_*X is Φ-algebra. In the case where Φ is the free commutative algebra functor, this amounts to regarding E_*Y simply as a commutative algebra and forgetting any other structure that might be present – for example, any Dyer-Lashof operations.

If A is a Φ-algebra in E_*E-comodules, then we have a moduli space $\mathcal{TM}(A)$ of realizations of A. This is the nerve of the category $\mathcal{R}(A)$ with objects the commutative ring spectra X so that $E_*X \cong A$ as Φ-algebras; the morphisms are E_*-equivalences. The Dwyer-Kan decomposition theorem of Proposition 1.1.12 gives a weak equivalence

$$\mathcal{TM}(A) \simeq \coprod_{[X]} B\mathrm{Aut}(X)$$

where $[X]$ runs over the E_*-equivalence class of objects in $\mathcal{R}(A)$, and $\mathrm{Aut}(X)$ is the (derived) space of self-equivalences of X in the E_*-local category of E_∞-ring spectra. The point of this section is give a decomposition of $\mathcal{TM}(A)$ in terms of algebraic data.

We will actually work out a more general relative case. Fix a cofibrant E_∞-ring spectrum Y and let $k = E_*Y$. Choose a Φ-algebra morphism $k \to A$ and let $\mathcal{TM}(A/k)$ be the moduli space of realizations of the Φ-algebra A under k. This is the nerve of the category $\mathcal{R}(A/k)$ with objects the morphisms of commutative ring spectra $Y \to X$ so that there is an isomorphism from $E_*Y \to E_*X$ to the chosen morphism $k \to A$. The morphisms in $\mathcal{R}(A/k)$ are morphisms under Y which induce an isomorphism on E_*. Again there is a decomposition

$$\mathcal{TM}(A/k) \simeq \coprod_{[X]} B\mathrm{Aut}_Y(X)$$

where $[X]$ runs over the E_*-equivalence class of objects in $\mathcal{R}(A)$, and $\mathrm{Aut}_Y(X)$ is the (derived) space of self-equivalences of X under Y in the E_*-local category of E_∞-ring spectra.

For our decomposition results, we will work with the $\pi_*E_*(-)$ localization of the $\mathcal{P}$-resolution model category structure on simplicial T-algebras in spectra, where $\mathcal{P}$ is the fixed set of projectives defined in Definition 1.4.2. For a simplicial spectrum X, we are writing $\pi_{i,*}X = \{\pi_{i,P}X\}$ where P runs over the elements of $\mathcal{P}$.

Regard our fixed E_∞-ring spectrum Y as a constant object in the category $s\mathbf{Alg}_T$ of simplicial T-algebras, and choose a $\mathcal{P}$-equivalence $Y_c \to Y$ so that Y_c is $\mathcal{P}$-cofibrant; thus Y_c is a $\mathcal{P}$-resolution of Y. The reason for making this replacement is so that we can apply Lemma 3.2.15, which will imply that any moduli space we construct out of the category $s\mathbf{Alg}_{Y_c}$ will be independent of the choice of Y_c.

3.3.1 Definition. *Let Y be an E_∞-ring spectrum and let $k = E_*Y$ be the resulting Φ-algebra in E_*E-comodules. Let A be a Φ-algebra under k in E_*E-comodules. A potential n-stage for A is a simplicial T-algebra X under Y_c so that the following three conditions hold*

1. *$\pi_0E_*X \cong A$ as Φ-algebra under k;*
2. *$\pi_{i,*}X = 0$ for $i > n$; and*
3. *$\pi_iE_*X = 0$ for $1 \le i \le n+1$.*

The partial moduli space $\mathcal{TM}_n(A/k)$ is defined to be the moduli space of all simplicial T-algebras under Y_c which are potential n-stages for A. Morphisms are the $\pi_*E_*(-)$ equivalences under Y_c.

It follows from the spiral exact sequence that if X is a potential n-stage for A, then

$$\pi_iE_*X \cong \begin{cases} A & i = 0; \\ \Omega^{n+1}A & i = n+2; \\ 0 & \text{otherwise.} \end{cases} \tag{3.3.1}$$

Furthermore, the structure of $\pi_{n+2}E_*X$ as a π_0E_*X-module is the standard one. See Examples 2.2.10 and 3.1.13.

Definition 3.3.1 makes sense for $n = \infty$. If X is a potential ∞-stage for A, then

$$\pi_iE_*X \cong \begin{cases} A & i = 0; \\ 0 & i \neq 0. \end{cases}$$

Let $\mathcal{TM}_\infty(A/k)$ be the resulting moduli space.

Here are two preliminary decomposition results.

3.3.2 Theorem. *The geometric realization functor induces a weak equivalence*

$$\mathcal{TM}_\infty(A/k) \to \mathcal{TM}(A/k).$$

Proof. The spaces $\mathcal{TM}_\infty(A/k)$ and $\mathcal{TM}(A/k)$ are the nerves of categories $\mathcal{R}_\infty(A/k)$ and $\mathcal{R}(A/k)$ respectively. Therefore, it is sufficient to define functors $F : \mathcal{R}_\infty(A/k) \to \mathcal{R}(A/k)$ and $G : \mathcal{R}(A/k) \to \mathcal{R}_\infty(A/k)$ so that the two composites FG and GF are connected to the respective identity functors by chains of natural transformations which are $\pi_*E_*(-)$ equivalences. The functor G is easy: given a morphism $Y \to X$ we may regard X as a constant T-algebra under Y and, hence, under Y_c; this is a tautological potential ∞-stage.

We now define the functor F. If $Y_c \to X$ is a potential ∞-stage, form a functorial factorization

$$\begin{tikzcd} Y_c \arrow[r, "i"] & X' \arrow[r, "p"] & X \end{tikzcd}$$

where X' is a $\mathcal{P}$-cofibrant simplicial T-algebras and the morphisms p is a $\pi_*E_*(-)$-equivalence. Apply geometric realization to the top map in this diagram and form the push-out in E_∞-algebras

$$\begin{tikzcd} {|Y_c|} \arrow[r, "i"] \arrow[d, "\epsilon"'] & {|X'|} \arrow[d] \\ Y \arrow[r] & Y \sqcup_{|Y_c|} |X'|. \end{tikzcd}$$

We now apply Corollary 1.4.12 to the top row and use that $|Y_c| \to Y$ is an E_*-equivalence between cofibrant E_∞-algebras to conclude that the bottom row is in $\mathcal{R}(A/k)$. Then

$$F(Y \to X) = Y \to Y \sqcup_{|Y_c|} |X'|.$$

We leave it to the reader to connect FG and GF by weak equivalences to the respective identities. □

3.3.3 Theorem. *The nth-Postnikov stage functor P_n induces a map of moduli spaces*

$$P_n : \mathcal{TM}_k(A/k) \longrightarrow \mathcal{TM}_n(A/k), \qquad n \le k \le \infty$$

and the resulting map

$$\mathcal{TM}_\infty(A/k) \longrightarrow \operatorname*{holim}_{n<\infty} \mathcal{TM}_n(A/k)$$

is a weak equivalence.

Proof. This follows from [14], §4.6. □

Because of the these results, we next address the homotopy type of the space $\mathcal{TM}_n(A/k)$.

3.3.4 Theorem. *The functor $\pi_0E_*(-)$ induces a natural weak equivalence*

$$\mathcal{TM}_0(A/k) \simeq B\,\mathrm{Aut}_k(A)$$

where $\mathrm{Aut}_k(A)$ *is the group of automorphisms of the Φ-algebra A over k. In particular, $\mathcal{TM}_0(A/k)$ is non-empty and connected.*

Proof. A potential 0-stage for A is nothing more nor less than a simplicial T-algebra of type B_A under Y_c. The result now follows from Proposition 3.2.17. $\square$

The main theorem of this section and, indeed, of this paper now identifies how to pass up the layers of the tower. If A is Φ-algebra and M is an A-module, then we have defined

$$\mathcal{H}^n(A/k, M) \overset{\text{def}}{=} \mathcal{H}^k_{T_E/E_*E}(A/k, M) = \operatorname{map}_{s\mathbf{Alg}_k/A}(A, K_A(M, n))$$

and

$$\hat{\mathcal{H}}^n(A/k, M) \overset{\text{def}}{=} \hat{\mathcal{H}}^n_{T_E/E_*E}(A/k, M) = E\operatorname{Aut}_\Phi(A, M) \times_{\operatorname{Aut}(A,M)} \mathcal{H}^n(A/k, M).$$

See Remark 3.2.3 for more on this notation.

3.3.5 Theorem. *Let $n \geq 1$, then there is a natural homotopy pull-back diagram*

$$\begin{tikzcd}
\mathcal{TM}_n(A/k) \arrow[r] \arrow[d, "P_{n-1}"'] & B\operatorname{Aut}_\Phi(A/k, \Omega^n A) \arrow[d] \\
\mathcal{TM}_{n-1}(A/k) \arrow[r] & \hat{\mathcal{H}}^{n+2}(A/k, \Omega^n A).
\end{tikzcd}$$

The proof will occupy the rest of the section. We begin with an analysis of how to pass from potential $(n-1)$-stages to n-stages.

Suppose that X is a potential n-stage for A. Then $\pi_n E_* X \cong \Omega^n A$ as an A-module, by the spiral exact sequence. Then $Z = P_{n-1}X$ is a potential $(n-1)$-stage for A and Proposition 3.2.9 implies that there is a homotopy pull-back square in the E_*-local category under Y_c

(3.3.2)
$$\begin{tikzcd}
X \arrow[r] \arrow[d, "p"'] & B_A \arrow[d, "q"] \\
Z \arrow[r, "f"] & B_A(\Omega^n A, n+1).
\end{tikzcd}$$

Note that all the maps in this diagram induce an isomorphism on $\pi_0 E_*(-)$. The next result shows how to reverse this process. Recall from Proposition 3.2.16 that the simplicial T-algebra $B_A(M, n)$ represents André-Quillen cohomology; that is,

(3.3.3)
$$\pi_0 \operatorname{map}_{s\mathbf{Alg}_{Y_c}}(Z, B_A(M, n)) \cong \pi_0 \operatorname{map}_k(E_* Z, K_A(M, n)).$$

3.3.6 Proposition. *Suppose that Z is a potential $n-1$-stage for A and that $n \geq 1$. Suppose further that X lies in a homotopy fiber square of the form displayed in 3.3.2. Then X is a potential n-stage if and only if the map*

$$g : E_* Z \longrightarrow K_A(\Omega^n A, n+1)$$

induced by f is a weak equivalence of simplicial T_E-algebras.

Proof. This is a simple calculation, using that there is a Mayer-Vietoris sequence in $\pi_{*,*}(-)$ – and hence in $\pi_* E(-)$ – for homotopy pull-backs. Compare Proposition 9.11 of [7]. $\square$

3.3.7 Remark (Obstructions to realization). Given a potential $(n-1)$-stage Z for A, then E_*Z, as an F-algebra, has exactly two non-vanishing homotopy groups; thus, taking algebraic Postnikov sections, we obtain a homotopy pull-back square in F-algebras under k

$$\begin{tikzcd}
E_*Z \arrow[r] \arrow[d] & A \arrow[d] \\
A \arrow[r, "\chi"] & K_A(\Omega^n A, n+2).
\end{tikzcd}$$

The previous result implies that there exists a potential n-stage X so that $P_{n-1}X \simeq Z$ if and only if

$$0 = \chi \in H^{n+2}(A/k, \Omega^n A).$$

Thus we see the obstructions to realizing A as elements of André-Quillen cohomology. The next result extends this observation to a statement about moduli spaces.

We continue to work in the category $s\mathbf{Alg}_{Y_c}$ of simplicial T-algebras under Y_c. If X and Z are two T-algebras under Y_c, then recall from Example 1.1.19 that $\mathcal{M}(X \looparrowright Z)$ means the moduli space of all arrows $X \to Z$ which induce an isomorphism on *non-zero* homotopy groups. In this case, we would have $\pi_m E_*(X) \to \pi_m E_*(Z)$ is an isomorphism when both source and target are non-zero. If Z is a potential $(n-1)$-stage for A, let $\mathcal{M}(Z \oplus (\Omega^n A, n))$ denote the moduli space of potential n-stages X for A under Y_c so that there is some $\pi_* E_*$-weak equivalence $P_{n-1}X \to Z$. This weak equivalence is not part of the data, we are simply assuming we can find one.

3.3.8 Remark (Labeling of moduli spaces). In the rest of this section we will be working with *relative* moduli spaces; that is, moduli spaces built from objects either under Y_c (on the topological side) or under E_*Y_c (on the algebraic side). We could adorn our space to indicate this; for example, in the previous paragraph we could have written $\mathcal{M}_{Y_c}(X \looparrowright Z)$ and in the statement of the next result we could write $\mathcal{M}_{E_*Y_c}(E_*Z)$ for the moduli space of the object E_*Z under E_*Y_c. However, since this will be completely universal, we won't add this extra bit of notation, but leave it understood.

3.3.9 Proposition. *Suppose that Z is potential $(n-1)$-stage for A under Y_c and that $n \geq 1$. Then there is a natural homotopy fiber square*

$$\begin{tikzcd}
\mathcal{M}(Z \oplus (\Omega^n A, n)) \arrow[r] \arrow[d, "P_{n-1}"'] & \mathcal{M}(E_*Z \looparrowright K_A(\Omega^n A, n+1) \looparrowleft K_A) \arrow[d] \\
\mathcal{M}(Z) \arrow[r, "E_*"'] & \mathcal{M}(E_*Z).
\end{tikzcd}$$

Note that the space $\mathcal{M}(Z \oplus (\Omega^n A, n))$ may be empty. By Proposition 3.3.6 this will happen if and only if there is no weak equivalence

$$E_*Z \to K_A(\Omega^n A, n+1)$$

under E_*Y_c. In this case, the space $\mathcal{M}(E_*Z \looparrowright K_A(\Omega^n A, n+1) \looparrowleft K_A)$ will also be empty.

Proof. Let $M = \Omega^n A$. The difference construction supplies a map

$$\mathcal{M}(Z \oplus (M, n)) \to \mathcal{M}(Z \xrightarrow{\oplus} B_A(M, n+1) \looparrowleft B_A)$$

where the symbol $\oplus$ in the target means morphisms $Z \to B_A(M, n+1)$ under Y_c which correspond to weak equivalences $E_*Z \to K_A(M, n+1)$ under E_*Y_c. See Proposition 3.2.16. Then Proposition 3.3.6 implies that this map is a weak equivalence; thus we have a homotopy pull-back square

$$\begin{tikzcd}
\mathcal{M}(Z \oplus (M, n)) \arrow[r] \arrow[d, "P_{n-1}"'] & \mathcal{M}(Z \xrightarrow{\oplus} B_A(M, n+1) \looparrowleft B_A) \arrow[d] \\
\mathcal{M}(Z) \arrow[r, "="] & \mathcal{M}(Z).
\end{tikzcd}$$

Now applying homology and composing with the universal map of Diagram 3.2.2

$$u : E_*B_A(M, n+1) \to K_A(M, n+1)$$

supplies a commutative diagram

$$\begin{tikzcd}
\mathcal{M}(Z \xrightarrow{\oplus} B_A(M, n+1) \looparrowleft B_A) \arrow[r] \arrow[d] & \mathcal{M}(E_*Z \looparrowright K_A(M, n+1) \looparrowleft K_A) \arrow[d] \\
\mathcal{M}(Z) \arrow[r, "E_*"'] & \mathcal{M}(E_*Y).
\end{tikzcd}$$

To complete the proof, we show that this is a homotopy pull-back square. To do this, note that Proposition 3.2.17 yields a weak equivalence

$$\mathcal{M}(B_A(M, n+1) \looparrowleft B_A) \longrightarrow \mathcal{M}(K_A(M, n+1) \looparrowleft K_A).$$

Therefore it is sufficient to prove that there is a homotopy pull-back square

$$\begin{tikzcd}
\mathcal{M}(Z \xrightarrow{\oplus} B_A(M, n+1) \looparrowleft B_A) \arrow[r] \arrow[d] & \mathcal{M}(E_*Z \looparrowright K_A(M, n+1) \looparrowleft K_A) \arrow[d] \\
\mathcal{M}(Z) \times \mathcal{M}(B_A(M, n+1) \looparrowleft B_A) \arrow[r] & \mathcal{M}(E_*Z) \times \mathcal{M}(K_A(M, n+1) \looparrowleft K_A).
\end{tikzcd}$$

Note that the two spaces at the bottom of this diagram are connected. The induced map on fibers is

$$\mathrm{map}^w_{Y_c}(Z, B_A(M, n+1)) \to \mathrm{map}^w_{E_*Y_c}(E_*Z, K_A(M, n+1)).$$

Here the superscript w means, on the right, the subspace of the space of all maps which are weak equivalences and, on the left, those maps which correspond to weak equivalences. Then Proposition 3.2.16 shows this morphism is a weak equivalence. The result follows. ☐

We can now supply the proof of our core result.

3.3.10 Proof of the Theorem 3.3.5. For any morphism $k \to A$ of Φ-algebras, any A-module M, and any $m \geq 1$, there is a commutative square

(3.3.4)
$$\begin{tikzcd}
\mathcal{M}(K_A(M,m) \looparrowleft K_A) \arrow[r, "\simeq"] \arrow[d] & \mathcal{M}(K_A(M,m+1) \looparrowleft K_A) \arrow[d] \\
\mathcal{M}(K_A \oplus (M,m)) \arrow[r, "\simeq"] & \mathcal{M}(K_A \looparrowright K_A(M,m+1) \looparrowleft K_A).
\end{tikzcd}$$

(Recall that these are all moduli spaces of morphisms under E_*Y_c and that E_*Y_c is weakly equivalent to k.) As indicated the horizontal maps are weak equivalences, as demonstrated by the analysis of Postnikov sections given in Proposition 2.5.16. In particular, we have a pull-back square. If Y is a potential $(n-1)$-stage for A, we take $M = \Omega^n A$ and $m = n+1$. Then $\mathcal{M}(E_*Z)$ is one component of $\mathcal{M}(K_A \oplus (M,m))$. There are two cases.

The first case is that there is no weak equivalence of simplicial algebras $E_*Z \to K_A(\Omega^n A, M)$ under E_*Y_c. With that assumption Proposition 3.3.6 shows that $\mathcal{M}(Y \oplus (\Omega^n A, n))$ is empty. We also have that the component $\mathcal{M}(E_*Y)$ is not in the image of

$$\mathcal{M}(K_A(M,m) \looparrowleft K_A) \to \mathcal{M}(K_A \oplus (M,m)).$$

Together with the pull-back 3.3.4, these facts imply that

(3.3.5)
$$\begin{tikzcd}
\mathcal{M}(Z \oplus (\Omega^n A, n)) \arrow[r] \arrow[d] & \mathcal{M}(K_A(M,m+1) \looparrowleft K_A) \arrow[d] \\
\mathcal{M}(Z) \arrow[r] & \mathcal{M}(K_A \looparrowright K_A(M,m+1) \looparrowleft K_A)
\end{tikzcd}$$

is a pull-back square – rather trivially, in fact.

For the second case we assume that there is some weak equivalence of simplicial algebras $E_*Z \to K_A(\Omega^n A, M)$. Then we assert that there is is a weak equivalence

(3.3.6)
$$f : \mathcal{M}(K_A(M,m) \looparrowleft K_A) \to \mathcal{M}(E_*Z \looparrowright K_A(\Omega^n A, m) \looparrowleft K_A).$$

To see this recall that source and target are given by nerves of categories of arrows. The morphism f sends $U \leftarrow V$ to

$$U \xrightarrow{=} U \longleftarrow V;$$

the homotopy inverse sends $W \to U \leftarrow V$ to $U \leftarrow V$. Then Proposition 3.3.9 implies that the square of 3.3.5 is a homotopy pull-back square in this case also.

Finally taking the coproduct over all weak equivalence classes of potential $(n-1)$-stages Z yields a pull-back square

$$\begin{tikzcd}
\mathcal{TM}_n(A/k) \arrow[r] \arrow[d] & \mathcal{M}(K_A(M,m+1) \hookleftarrow K_A) \arrow[d] \\
\mathcal{TM}_{n-1}(A/k) \arrow[r] & \mathcal{M}(K_A \hookrightarrow K_A(M,m+1) \hookleftarrow K_A).
\end{tikzcd}$$

and the result follows. Indeed, the identification

$$\mathcal{M}(K_A(M,m+1) \hookleftarrow K_A) \simeq B\,\mathrm{Aut}(A, \Omega^n A)$$

follows from Proposition 2.5.19 and the identification

$$\mathcal{M}(K_A \hookrightarrow K_A(M,m+1) \hookleftarrow K_A) \simeq \hat{\mathcal{H}}^{n+2}(A/k, \Omega^n A)$$

follows from Corollary 2.5.23.

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