ON THE SIMS CONJECTURE AND DISTANCE TRANSITIVE GRAPHS

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In this paper we use the recently announced *classification of finite simple groups* to prove the well-known conjecture of Sims on finite permutation groups:

THEOREM 1. *There exists an integral function f having the property that whenever G is a primitive permutation group on a finite set Ω, if G_α is the stabilizer of the point α in Ω and if d is the length of any G_α-orbit in $\Omega - \{\alpha\}$ then $|G_\alpha| \leqslant f(d)$.*

Charles Sims suggested that this should be true in [**11**]. Much work has been done since—see for instance the papers of Thompson [**15**], Wielandt [**17**] and Knapp [**6**]. In particular, Thompson proved in [**15**] the existence of a function g such that for any G as above there is a prime p and a normal p-subgroup P of G with $|G_\alpha : P| \leqslant g(d)$. Nevertheless, the conjecture has remained open until now.

As a consequence we obtain a result on distance transitive graphs:

THEOREM 2. *There are only finitely many finite connected distance transitive graphs of any given valency greater than* 2.

The paper is divided into five sections. In Section 1 we use the O'Nan–Scott theorem on the structure of primitive permutation groups to reduce the problem to groups with simple socles. Section 2 contains some further reductions. In Sections 3 and 4 we deal with groups of Lie type; our approach differs depending on whether or not the Thompson prime p mentioned above is equal to the characteristic of the group. Finally, in Section 5 we prove Theorem 2.

It is perhaps worth mentioning that while our proof depends crucially on the classification of finite simple groups, we do not need the full strength of the classification theorem. We only need the fact that there exist just finitely many isomorphism types of sporadic groups.

We have not attempted to get the best possible function f, but our arguments show that one can take $f(d)$ of the form $\exp(d^2 o(d))$. This is comparable with the function $g(d)$ obtained by Thompson and Wielandt.

1. *Reduction to groups with simple socles*

Throughout this section we shall assume that the Sims conjecture holds for all primitive groups with simple socles. Let then h be an increasing integral function such that whenever G is a primitive group with a simple socle and d is a non-trivial subdegree of G then $|G_\alpha| \leqslant h(d)$.

Let now Ω be an arbitrary set of n points, let G be an arbitrary primitive

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permutation group on Ω. Let d be a non-trivial subdegree of G on Ω. Let N be the socle of G, say $N = T_1 \times \ldots \times T_m$ with $m > 1$ and each T_i simple. Then all the T_i are isomorphic, say to T (see [3, p. 5]).

Assume first that N is regular on Ω. Then the G_α-actions on $N - \{1\}$ by conjugation and on $\Omega - \{\alpha\}$ are isomorphic. Now G is primitive, so there are no non-trivial G_α-invariant subgroups of N. Hence any G_α-orbit on $N - \{1\}$ generates N. It follows that G_α is faithful on each orbit, and so $|G_\alpha| \leqslant d!$. We shall therefore assume that N is not regular on Ω.

By the O'Nan–Scott theorem ([1] and [9, Appendix] or [3, Theorem 4.1]), we now have one of

(a) wreath action: T is the socle of a primitive group H of degree s, and G is a subgroup of the wreath product H wr S_m with the product action, where $n = s^m$,

(b) diagonal action: $N_\alpha = D_1 \times \ldots \times D_l$ where $m = kl$ for some integer k and D_i is a diagonal subgroup of $T_{(i-1)k+1} \times \ldots \times T_{(i-1)k+k}$, with $n = |T|^{(k-1)l}$.

Remark. In the statement of (ii)(a) of [3, Theorem 4.1], "the socle" should be changed to "a normal subgroup"; but if T is not the socle of H then it is a regular normal subgroup and hence N is a regular normal subgroup of G. (See also [1].)

We now consider these cases separately.

Case (a)—*wreath action.* Here $N_\alpha = R_1 \times \ldots \times R_m$ with all R_i isomorphic to a subgroup R of T and with the coset spaces $\Gamma_i = (T_i : R_i)$ isomorphic to $\Gamma = (T : R)$. Then $G \leqslant H \operatorname{wr} \operatorname{Sym} \Delta$, where $\Delta = \{\Gamma_1, \ldots, \Gamma_m\}$ and H is the group obtained as follows: If $g \in G$ then $g = (s_1, \ldots, s_m)\sigma$ with $s_i \in \operatorname{Sym} \Gamma_i$ and $\sigma \in \operatorname{Sym} \Delta$. The s_i correspond to elements of $\operatorname{Sym} \Gamma$. Then H is the group of all $s \in \operatorname{Sym} \Gamma$ that appear as s_1 in some g in G which fixes Γ_1. Note that H is primitive on Γ and has simple socle T. Identify Ω with Γ^m.

Let $\alpha \in \Omega$ with $\alpha = (\mu, \mu, \ldots, \mu)$ and consider the G_α-orbits on $\Omega - \{\alpha\}$. Choose $\beta \in \Omega$ with $|\beta G_\alpha| = d$. Let d_1 be the minimum non-trivial subdegree of H on Γ. By our assumption in this section, $|H_\mu| \leqslant h(d_1)$, and $|G_\alpha| \leqslant |H_\mu|^m m! \leqslant h(d_1)^m m!$.

It remains to show that m is bounded by a function of d. Let d_2 be the minimal non-trivial subdegree of T on Γ and let t be the number of coordinates where α differs from β. Then $|\beta N_\alpha| \geqslant d_2^t$. As G is primitive on Ω, we have G_α transitive on Δ (cf. [3, p. 5]), so for each index i there is a point βG_α with i-component different from μ. Hence $d = |\beta G_\alpha| \geqslant d_2^t m/t \geqslant m$. (In fact, refining the argument slightly, $d \geqslant d_1 m$.)

Case (b)—*diagonal action.* First consider the case where $l = 1$, so that $N_\alpha = D$, a diagonal subgroup of $T_1 \times \ldots \times T_m$. We can take $D = \{(t, t, \ldots, t) : t \in T\}$. We have $G \leqslant T^m(\operatorname{Out} T \times S_m)$ and $D \leqslant G_\alpha \leqslant \operatorname{Aut} D \times S_m$. Note that G_α induces a primitive group of degree m on $\{T_1, \ldots, T_m\}$ (cf. [3, Remark 2 on p. 6]). Identify Ω with the set of cosets of D in T^m.

Let K be the kernel of the action of G_α on the suborbit Γ of size d. We assume that $K \neq 1$. Notice first that $D \nleqslant K$, for otherwise $D(g_1, \ldots, g_m)D = D(g_1, \ldots, g_m)$ for all $D(g_1, \ldots, g_m) \in \Gamma$; taking $g_1 = 1$ we have $t^{-1} g_i t = g_i$ for all $t \in T$, so that $g_2, \ldots, g_m \in Z(T) = 1$ and hence Γ is trivial, a contradiction. Hence $D \cap K = 1$, so that $|D| \leqslant d!$, and also $[D, K] = 1$, so that $K \leqslant S_m$.

Now set $i \sim j$ if $g_i = g_j$ for all $D(g_1, \ldots, g_m) \in \Gamma$. This is a G_α-invariant equivalence relation. It is not the universal relation, so it is the relation of equality. So, if $i \neq j$, there is some coset $D(g_1, \ldots, g_m)$ in Γ with $g_i \neq g_j$. If now $\pi \in K$ and π fixes 1 then $i\pi \neq j$. It follows that K is semiregular, and in fact regular (being a normal subgroup of a primitive group) of degree m. We may therefore identify indices and elements of K.

Fix now an element $D(g_1, \ldots, g_m)$ of Γ. For any $\pi \in K$ we have $D(g_{1\pi}, \ldots, g_{m\pi}) = D(g_1, \ldots, g_m)$, whence $g_i g_{i\pi}^{-1}$ is an element of T independent of i. Define $\theta: K \to T$ by $\pi\theta := g_i g_{i\pi}^{-1}$. For any $\pi_1, \pi_2 \in K$ we then have $\pi_1\pi_2\theta = g_i g_{i\pi_1\pi_2}^{-1} = g_i g_{i\pi_1}^{-1} g_{i\pi_1} g_{i\pi_1\pi_2}^{-1} = \pi_1\theta \cdot \pi_2\theta$, so that θ is a homomorphism. Of course θ depends on the choice of $D(g_1, \ldots, g_m)$ in Γ. In this way we obtain d normal subgroups of K. But the intersection of all these normal subgroups is trivial: for if π lies in this intersection then $(g_{1\pi}, \ldots, g_{m\pi}) = (g_1, \ldots, g_m)$ for all $D(g_1, \ldots, g_m)$ in Γ; but, if $i \neq j$, choose $D(g_1, \ldots, g_m)$ in Γ with $g_i \neq g_j$. Hence $\pi = 1$. Moreover, the image of each θ is a subgroup of T. Hence K is a subdirect product of d subgroups of T, and so $|K| \leqslant (d!)^d$ and $|G_\alpha| \leqslant (d!)^{d+1}$.

Let now $l > 1$, so $N_\alpha = D_1 \times \ldots \times D_l$. As in case (a), we can write Ω as a product $\Gamma_1 \times \ldots \times \Gamma_l$ of l copies of a set Γ, and $G \leqslant H \text{ wr } S_l$, where H is obtained as before and is primitive on Γ with diagonal action with $l = 1$ (cf. [**3**, Remark 3 on p. 6]). By the above, $|H_\mu|$ is bounded by a function of d, and so, as in case (a), the same holds for $|G_\alpha|$.

2. *Some further reductions*

From now on we shall consider the case where G has a simple socle X. By the theorem of Thompson mentioned in the introduction, we may assume that G_α is the normalizer of a non-trivial p-group P for some prime p. Moreover, $|G_\alpha/O_p(G_\alpha)| \leqslant g(d)$. Now $p \leqslant d$ by [**16**, 18.4], so we can consider the relevant primes separately and then take f to be the maximum of the relevant functions. This strategy, incidentally, will be used throughout: whenever there are finitely many cases to be considered, we shall treat these separately and then take f to be the maximum of the functions thus obtained.

Now the *sporadic groups* have among them only finitely many primitive representations, so that these can be ignored until the end, when they are easily handled by adjusting the coefficients of our function f.

Let now X be an *alternating group* A_c for some c. We may assume (as above) that $c > 16$. Then G is A_c or S_c. If G_α is intransitive in its natural action on c points then, as G_α is maximal in G, for some k with $0 < k \leqslant c/2$ we have $G_\alpha = (S_k \times S_{c-k}) \cap G$, a subgroup of index at most 2 in $S_k \times S_{c-k}$. Since G_α is p-local, it follows that $2 \leqslant k \leqslant 4$. Hence $|P| \leqslant 4$, so the order of P is certainly bounded and hence so is $|G_\alpha|$. If G_α is transitive but imprimitive on the c points then $G_\alpha = (S_k \text{ wr } S_{c/k}) \cap G$ with $2 \leqslant k \leqslant c/2$. Since $1 \neq P \lhd G_\alpha$, we have $2 \leqslant k \leqslant 4$ and $|P| \leqslant 2^{c/2}$ or $|P| \leqslant 3^{c/3}$. On the other hand, as $c > 16$, the $A_{c/k}$ acts non-trivially on each suborbit [**16**, 18.2], so $d \geqslant c/k \geqslant c/4$ and so $|P|$ is bounded by a function of d. Finally, if G_α is primitive in the natural action of degree c then P is transitive and $c = p^a$ for some integer a. Then $Z(P)$ is a regular normal abelian subgroup of G_α, so $\mathrm{ASL}(a, p) \leqslant G_\alpha \leqslant \mathrm{AGL}(a, p)$. If $a = 1$ then $|P| = p$. If $a \geqslant 2$ then $|O_p(G_\alpha)| < |G_\alpha : O_p(G_\alpha)|$, so $|O_p(G_\alpha)| \leqslant g(d)$ by [**15**].

NOTATION. In what follows the socle X of G will be a group $X_l(q)$ of Lie type of rank l over the field of q elements, where $q = r^a$ for some prime number r. Then

$X = O^{r'}(\bar{X}_\sigma)$ with $\bar{X}$ a simple adjoint algebraic group over the field $\overline{\mathrm{GF}(r)}$ and with σ an endomorphism of X. The group $Y = \bar{X}_\sigma$ is X together with all diagonal automorphisms of X. We choose a field endomorphism δ of $\bar{X}$ (or the square root of a field endomorphism in the case where X is a Suzuki or a Ree group) such that σ is δ^a or $\delta^a\tau$ with τ a graph automorphism of $\bar{X}$. Then δ restricts to X inducing an automorphism μ of order a or $2a$ (and $3a$ if $X = {}^3D_4(q)$), respectively. The group $Z = Y\langle\mu\rangle$ is normal in $\mathrm{Aut}\,X$ of index dividing 6 (in fact 2 unless $\bar{X}$ is of type D_4). We regard G as a subgroup of $\mathrm{Aut}\,X$.

We shall write $P = O_p(G_\alpha)$, $Q = P \cap Y$, $F = P \cap Z$. Primitivity of G implies that $G_\alpha = N_G(A)$ whenever A is a non-trivial normal subgroup of G_α.

To conclude this section we shall now show that if $|Q|$ is bounded by a function of d then so is $|P|$. Assume then that $|Q| = p^m \leqslant k(d)$. Then $|P| \leqslant 3p^m \log_2 q$, so if q is also bounded then so is $|P|$. We may assume that $q > 9$. Let $D = \mho_m(C_F(Q))$, where $\mho_m(H)$ is as usual the group generated by all the p^m-powers of elements of the p-group H. As F/Q is cyclic, we conclude that D is a cyclic normal subgroup of G_α. If $x \in F$ then $x^{p^m} \in C_F(Q)$ and so $x^{p^{2m}} \in D$; hence $|P:D| \leqslant p^{3m}$. Let $D = \langle h\rangle$, $D \cap Y = \langle y\rangle$ and $\Omega_1(D) = \langle g\rangle$.

Assume first that $y \neq 1$. Suppose that p is the characteristic of G. Then $y \in X$ and it follows from the Borel–Tits theorems (3.12 of [2]) that Y_α is a parabolic subgroup of Y. It then follows that q^l divides $|Q|$, so in fact $|G|$ is bounded. Next consider the case where p is different from r. Using [13, II.1.1] we choose a maximal torus T containing y. Then $T \leqslant N(D)$, since $T \leqslant C_G(y) \leqslant G_\alpha$ and $D \lhd G_\alpha$, so $[D, T] \leqslant D \cap Y = \langle y\rangle \leqslant T \cap C(h)$. It follows that $h^{p^m} \in C(T)$. But then $h^{p^m} \in Y$ (by 2.8 of [10]), so $h^{p^m} \in Q$. Since $|Q| \leqslant p^m$, we have $|D| \leqslant p^{2m}$ and $|P| \leqslant p^{5m}$.

Now assume that $D \cap Y = 1$. Since $C_X(g) \leqslant G_\alpha$, it will even be enough to show that $|C_X(g)| \geqslant q^{1/p}$: for then $q^{1/p} \leqslant k(d)g(d)$, so $|P| \leqslant 3dk(d)\log_2\big(k(d)g(d)\big)$. Now the possible actions of g on X are known; we proceed to sketch the argument. Suppose that ζ is an endomorphism of $\bar{X}$ satisfying $\zeta^p = \sigma$ and such that $g \in Y\langle\zeta_1\rangle$, where ζ_1 is the restriction of ζ to X. Changing generators of $\Omega_1(D)$ if necessary we may assume that $g = y_1\zeta_1$ for some $y_1 \in Y$. Regarding $y_1\zeta$ as an element of the semidirect product $Y\langle\zeta\rangle$, the condition that the order of g is p is equivalent to $\sigma = (y_1\zeta)^p$. Lang's theorem (10.1 of [14]) gives the existence of an element $\bar{x} \in \bar{X}$ such that $y_1 = \bar{x}\zeta\bar{x}^{-1}\zeta^{-1}$. Hence $y_1\zeta = \bar{x}\zeta\bar{x}^{-1}$, and so $\sigma = (y_1\zeta)^p = (\bar{x}\zeta\bar{x}^{-1})^p = \bar{x}\sigma\bar{x}^{-1}$. Hence $\bar{x} \in \bar{X}_\sigma = Y$. Taking restrictions, g is Y-conjugate to ζ_1 and $C_X(g) \simeq C_X(\zeta_1)$.

If $\sigma = \delta^a$ then setting $\zeta = \delta^{a/p}$ we have the above configuration with $C_X(g)$ of Lie type (that of X) over a field of size $q^{1/p}$. Hence certainly $|C_X(g)| \geqslant q^{1/p}$. Suppose that $\sigma = \delta^a\tau$. Here $Z = \mathrm{Aut}\,X$, so $\mathrm{Aut}\,X/Y$ is cyclic. If the order of τ is not p then $\zeta = \delta^{a/p}\tau$ (or possibly $\delta^{a/p}\tau^2$ in case of $X = {}^3D_4(q)$) satisfies $\zeta^p = \sigma$. Then $g \in Y\langle\zeta_1\rangle$, where ζ_1 is the restriction of ζ to X, and we continue as before. Finally we suppose that τ has order p. If $p = r$, let U be a g-invariant r-subgroup of X. Then $Z(U)$ is a long root subgroup of X. Hence $|Z(U)| \geqslant q$, and so $|C_X(g)| \geqslant |C_{Z(U)}(g)| \geqslant q^{1/p}$, as required. Assume then that $p \neq r$. We have $g \in Y\langle\tau\rangle$, so g is a semisimple automorphism of $\bar{X}$ (see [14]). By 7.5 of [14], there is a g-invariant maximal torus $\bar{T}$ of $\bar{X}$ and a g-invariant Borel subgroup $\bar{B}$ with $\bar{T} \leqslant \bar{B} \leqslant \bar{X}$, so that g acts on the unipotent radical $\bar{U}$ of $\bar{B}$. Considering the action of g on the $\bar{T}$-root subgroups in $\bar{U}$ and using the fact that $\bar{X} \neq \mathrm{PSL}(2, \overline{\mathrm{GF}(q)})$ it is easily checked that $C_{\bar{U}}(g) \neq 1$. By 8.1 of [14], $C_{\bar{G}}(g)^\circ$ is reductive, hence $(C_{\bar{G}}(g)^\circ)'$ is a non-trivial, σ-invariant, semisimple group, and taking fixed points we have a commuting product of groups of Lie type over extension fields of GF (q) and contained in $C_X(g)$. This completes the argument.

Hence in the rest of the proof it will be enough to show that $|Q|$ is bounded by a function of d.

3. *Groups of Lie type of characteristic p*

Assume now that X has characteristic p. Then $Q \leqslant X$, since Y/X is a p'-group. Then the theorem of Borel and Tits (Proposition 3.12 of [2]) gives the existence of a canonical parabolic subgroup of X containing $N_X(Q)$. (Indeed, setting $U_1 = Q$, $N_1 = N_X(U_1)$, $U_2 = O_p(N_1)$, $N_2 = N_X(U_2)$ etc., we have $U_1 \leqslant U_2 \leqslant \ldots$ and $N_1 \leqslant N_2 \leqslant \ldots$, and the terminal element in the sequence $\{N_i\}$ is a parabolic subgroup of X.) Now $G_\alpha \cap X$ is a parabolic subgroup of X, since $G_\alpha = N_G(Q)$ is maximal in G. Moreover, $G_\alpha \cap X$ is a maximal G_α-invariant subgroup of X.

Let J be a maximal parabolic subgroup of X with $G_\alpha \cap X \leqslant J$. Since $G_\alpha \cap Z$ normalizes all parabolic subgroups containing $G_\alpha \cap X$, we see that J^G has cardinality at most 3 (with equality only if $X = D_4(q)$). Let J_0 be the intersection of the parabolic subgroups of J^G. Then J_0 is G_α-invariant and we must have $J_0 = G_\alpha \cap X$. By the theorem of Thompson in the introduction, the order of the Levi factor L of J_0 is bounded by $g(d)$. Hence both q and the Lie rank of L are bounded, and therefore so is $|G|$, since rank $(X) \leqslant$ rank $(L)+3$.

4. *Groups of Lie type of characteristic r prime to p*

Let X now be as in the heading of this section. We consider first the classical case and show that there the Lie rank l is bounded. Then we deal with the case of bounded Lie rank.

4a. *Classical groups*

Let V be the vector space corresponding to X, let $\mathscr{I}(V)$ be the full isometry group of V. Then $M = \mathscr{I}(V)/Z(\mathscr{I}(V))$ is a subgroup of Aut X containing Y as a subgroup of index at most 2 (and $M = Y$ unless Y is an orthogonal group). Let $P = O_p(G_\alpha)$ and (varying slightly from previous notation) let $Q = P \cap X$ or $Q = P \cap Y$, according to whether or not X is an orthogonal group. Whenever S is a subgroup of Q, we shall write $\hat{S}$ for the Sylow p-subgroup of its pre-image in $\mathscr{I}(V)$.

Consider first the situation where G_α has no abelian normal p-subgroup A for which $\hat{A}$ is abelian. Then $\hat{Q} = Q_0 C$, where $C = \hat{Q} \cap Z(\mathscr{I}(V))$ and Q_0 is an extraspecial subgroup (cf. [7, III, 13.10]). If $|Q_0| = p^{2m+1}$ then we can write $Q_0 = AB$ with A, B maximal normal abelian subgroups of Q_0 and $A \cap B = Z(Q_0)$. In fact, $A = A_0 \times A_1 \times \ldots \times A_k$ with the A_i cyclic of order p for $i > 0$ and either $A_0 = Z(Q_0)$, $k = m$, or A_0 cyclic order p^2 with $A_0 > Z(Q_0)$, $k = m-1$, and similarly $B = B_0 \times B_1 \times \ldots \times B_k$. Moreover, if $A_i = \langle a_i \rangle$ and $B_i = \langle b_i \rangle$, we can make sure that $[a_i, b_j] = \delta_{ij} z$ with z a fixed non-identity element of $Z(Q_0)$. Then there is a symmetric group S_k acting naturally on the sets $\{a_1, \ldots, a_k\}$ and $\{b_1, \ldots, b_k\}$, and acting trivially on A_0, B_0 and C. We can embed this S_k into $N_{\mathscr{I}(V)}(\hat{Q})$ as follows: Applying the process of simultaneous diagonalization to $a_1, \ldots, a_k$ on V we obtain a decomposition of V as $V = W_1 \perp \ldots \perp W_{p^k}$, where each a_i acts as a scalar on each W_j (note that $p \mid q-1$, or $p \mid q+1$ if X is unitary). Also, $B_1 \times \ldots \times B_k$ acts regularly on the set $\{W_1, \ldots, W_{p^k}\}$: if b in $B_1 \times \ldots \times B_k$ fixes W_i setwise then b commutes with all the a_j

on W_i, which forces $b = 1$ since $[a_j, b_j] = z$ and z acts as a scalar on V and hence fixes no non-zero vectors. Choose the notation so that W_1 is fixed pointwise by $A_1 \times \ldots \times A_k$. For $w \in W_i$ there is now a unique $w_1 \in W_1$ and $b \in B_1 \times \ldots \times B_k$ so that $w = w_1 b$, and we define $w\sigma = w_1 b^\sigma$. Then it is easy to check that this embeds our S_k into $N_{\mathscr{I}(V)}(\hat{Q})$. It follows that $N_G(Q)$ has A_k as a section, and hence so does G_α. But then $k \leqslant d$ and so $|Q|$ is bounded by a function of d.

Hence we can take it that G_α has a normal elementary abelian p-subgroup A contained in Q with $\hat{A}$ abelian. Let $A_0 = \Omega_1(\hat{A})$. (If $A_0 \leqslant Z(\mathscr{I}(V))$ for any suitable A then we see as above that $\hat{Q}$ is cyclic; in this case we take i minimal such that $\Omega_i(\hat{Q})$ is not central, put $A_0 = \Omega_i(\hat{Q})$ and adjust the e below slightly in the obvious way.) Consider now the Wedderburn decomposition of V with respect to A_0. The non-trivial homogeneous components can be labelled $W_1, W'_1, \ldots, W_k, W'_k, W_{k+1}, W_{k+2}, \ldots, W_m$ in such a way that for $1 \leqslant j \leqslant k$, the representation of A_0 on W'_j is a sum of irreducibles each contragredient to the irreducibles in W_j, and so that

$$V = C_V(A_0) \perp (W_1 \oplus W'_1) \perp \ldots \perp (W_k \oplus W'_k) \perp W_{k+1} \perp \ldots \perp W_m.$$

(If $\mathscr{I}(V) = \mathrm{GL}(V)$, we take $k = 0$ and replace $\perp$ by $\oplus$.) Then

$$C := C_{\mathscr{I}(V)}(A_0) = \mathscr{I}(C_V(A_0)) \times C_{\mathscr{I}(W_1 \oplus W'_1)}(A_0|_{W_1 \oplus W'_1}) \times \ldots \times C_{\mathscr{I}(W_m)}(A_0|_{W_m}).$$

Let e be minimal subject to $p|q^e - 1$ (or $p|q^{2e} - 1$ if X is unitary). If $1 \leqslant j \leqslant k$ then $C_{\mathscr{I}(W_j \oplus W'_j)}(A_0|_{W_j \oplus W'_j}) = \mathrm{GL}(s_j, q^e)$ (resp. $\mathrm{GL}(s_j, q^{2e})$) for some s_j. Consequently $s_j \leqslant d$ (and in fact $s_j \leqslant c \log d$ by [**5**] and its predecessors). Let now $B_0 = \Omega_1(O_p(C)) \cap \mathscr{I}(V)'$. Then the image of B_0 is normal in G_α and we replace A by this image—so $A_0 = B_0$. Suppose (changing the notation slightly) that the first r_1 of the s_i are all equal to s_1, etc. Then $N_{\mathscr{I}(V)'}(B_0)$ contains $\mathrm{SL}(s_j, q^e)^{r_j} A_{r_j}$. Then $r_j \leqslant d$ and so $\sum r_j s_j \leqslant c^2 d \log^2 d$, and since e is also clearly bounded, the dimension of $W_1 \oplus W'_1 \oplus \ldots \oplus W_k \oplus W'_k$ is bounded. Essentially the same argument shows that the dimension of $W_{k+1} \oplus \ldots \oplus W_m$ is also bounded: Each $C_{\mathscr{I}(W_j)}(A_0|_{W_j})$ with $k+1 \leqslant j \leqslant m$ is a classical group over $\mathrm{GF}(q^e)$ of dimension s_j, say. As before, each s_j is bounded. Moreover, if $s_{j_1} = \ldots = s_{j_r}$ then at least half of the corresponding spaces $W_{j_1}, \ldots, W_{j_r}$ are of the same type, so A_s is a section of G, where $s = [r/2]$. Finally, the dimension of $C_V(A_0)$ is clearly also bounded.

Hence l is bounded by a function of d. In fact, $l \leqslant d^{1+o(1)}$.

4b. *Bounded rank*

We now have the Lie rank l of X bounded. We can assume that $q > 9$. Let $L = Q \cap X$. Since l is bounded, it will be enough to show that $|L|$ is bounded.

Now L is a r'-group, so $L \leqslant N_X(T)$ for some maximal torus T of X (see II, 5.16 of [**13**]). Let $\bar{T}$ be a σ-stable maximal torus of $\bar{X}$ such that $T = X \cap \bar{T}$. Then $L \leqslant N_{\bar{X}}(\bar{T})$ by 2.7 of [**10**], so $|LT/T|$ is bounded by the order of the Weyl group of $\bar{X}$. Now $O_p(T)$ has rank at most l. Hence we may take it that for suitably large k we have $1 \neq \mho_k(L) \leqslant T$, so there is an abelian normal subgroup A of G_α with $A \leqslant L$ and A is contained in a maximal torus T of X.

Let $\bar{T}$ be as above and let $\bar{C} = \langle \bar{U}_\alpha : [A, \bar{U}_\alpha] = 1 \rangle$, where the groups $\bar{U}_\alpha$ are the $\bar{T}$-root subgroups of $\bar{X}$. Then $\bar{C}$ is a semisimple group and from the Bruhat decomposition of $\bar{X}$ we have $C_{\bar{X}}(A)^0 = \bar{C}\bar{T}$. Then $(C_{\bar{X}}(A))' = \bar{C}$ is σ-invariant, so it follows (as in 2.9 of [**10**]) that either $\bar{C} = 1$ or $O^{r'}(\bar{C}_\sigma)$ is a commuting product of groups of Lie type over extension fields of GF(q). In the latter case we have $q \leqslant d$ (note that any such group contains PSL$(2, q)$ or $Sz(q)$). Hence we may assume that $\bar{C} = 1$. Then $C_{\bar{X}}(A) \leqslant N(\bar{T})$ and taking fixed points we have $C_X(A) \leqslant N(T)$. For $g \in G_\alpha$ we have $T^g \leqslant C_X(A)$, so that $T^g \leqslant N(T)$. But then $T^g = T$ by 6.3 of [**10**], so that $G_\alpha \leqslant N_G(T)$. Hence $G_\alpha = N_G(T)$ by primitivity of G.

Let now $g \in G$ so that αg is in a G_α-orbit of size d. Let also $R = T \cap T^g$. Now $|G_\alpha : G_\alpha \cap G_\alpha^g| = d$ and we have seen that $|G_\alpha : T|$ is bounded (by the order of the Weyl group of $\bar{X}$). Hence $|T : R|$ is bounded by a function of d.

As above, consider $C_{\bar{X}}(R)^0 = \bar{D}\bar{T}$ with $\bar{D}$ semisimple. Since $T^g \leqslant C_X(R)^0$, we have $\bar{D} \neq 1$. Hence $D = O^{r'}(\bar{D}_\sigma)$ is a non-trivial commuting product of groups of Lie type over extension fields of GF(q). Now $\bar{T} \cap \bar{D}$ is a maximal torus of $\bar{D}$, so $T \cap D$ is a maximal torus of D. Hence $\frac{1}{2}(q-1) \leqslant |T \cap D|$ (see 2.4 of [**10**]). On the other hand, $|T \cap D : R \cap D|$ and $|R \cap D|$ are both bounded above by functions of d (note that $R \cap D \leqslant Z(D)$ and that $|Z(D)|$ is bounded by the order of the Weyl group of X, for example). Hence q is bounded by a function of d. But then, since $G_\alpha = N(T)$, we have $|G_\alpha|$ bounded as required.

This completes the proof of Theorem 1.

5. *On distance transitive graphs*

Suppose that, for some integer $k > 2$, there exist infinitely many (finite connected) distance transitive graphs of valency k.

If such a graph Γ is bipartite, then a non-bipartite graph Γ_1 is obtained by choosing as vertex set one of the bipartite blocks of Γ, and joining two vertices in Γ_1 if they lie at distance 2 in Γ. If Γ_1 has valency l then $2 < l \leqslant k(k-1)$. Moreover, a given graph Γ_1 can arise from only finitely many bipartite graphs Γ in this way. So there must exist infinitely many non-bipartite distance transitive graphs of valency l, for some integer $l > 2$.

A graph of diameter d is called antipodal if the relation of being equal or at distance d is an equivalence relation on the vertex set. If Γ is an antipodal distance transitive graph of valency l, then a non-antipodal graph Γ_2 is obtained by identifying antipodal vertices in Γ. The graph Γ_2 is distance transitive and has valency l. Any given graph Γ_2 arises from only finitely many antipodal graphs Γ in this way, since the diameter of Γ is at most $1 + 2\,\mathrm{diam}(\Gamma_2)$. So there exist infinitely many distance transitive graphs of valency l which are neither antipodal nor bipartite. A theorem of Smith [**12**] asserts that the automorphism group G of such a graph Γ acts primitively on the vertex set of Γ. Clearly G_x has an orbit of length l, consisting of the neighbours of x. By our Theorem 1 we have $|G_x| \leqslant f(l)$. Now G_x acts transitively on the set of vertices at distance i from x, for any given $i \leqslant d = \mathrm{diam}(\Gamma)$. So the number k_i of such vertices satisfies $k_i \leqslant f(l)$.

In a suitable first-order language we may write a set Σ of formulae asserting that Γ is a graph of valency l with a chosen vertex x, the number of vertices at distance i from x is non-zero and at most $f(l)$ for each i, and G is a group acting on Γ in such a way that for any two pairs of vertices at the same finite distance there is an element of G sending one pair to the other. By assumption, any finite subset of Σ has a model. So the Compactness Theorem shows that Σ has an infinite model. The connected component of the chosen vertex x is then an infinite distance transitive graph of valency l. However, such graphs have been completely determined by

Macpherson [8]. It follows from his result that for any i, the number of vertices at distance i from x is $(t+1)t^{i-1}s^i$ for some integers s, t with $l = (t+1)s$. Taking i large now leads to a contradiction. This establishes Theorem 2.

Further work on this question is reported in [4]. It includes an explicit bound for the diameter d of a vertex-primitive distance transitive graph of valency $k > 2$, say, $d \leqslant 8k \log f(k) + 11$, so that $d \leqslant k^{3+\varepsilon}$ for $k > 2$, and an outline of a possible proof of Theorem 2 not relying on Theorem 1.

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