Stochastic Evaluation and Analysis of Free Vibrations in Simply Supported Piezoelectric Bimorphs

Piezoelectric bimorph benders are a particular class of piezoelectric devices, which are characterized by the ability to produce flexural deformation greatly larger than the length or thickness deformation of a single piezoelectric layer. Due to extensive dimensional reduction of devices and to the high accuracy and repeatability requested, the effect of erroneous parameter estimation and the fluctuation of parameters due to external reasons, sometimes, cannot be omitted. As such, we consider mechanical, electrical and piezoelectric parameters as uniformly distributed around a nominal value and we calculate the distribution of natural frequencies of the device. We consider an analytical model for the piezoelectric bimorph proposed in literature. The results show how the parameters errors are reflected on the natural frequencies and how an increment of the error is able to change the shape of the frequencies distribution. [DOI: 10.1115/1.4007721]

1 Introduction

The authors are interested in piezoelectric bimorphs because they can be easily used as actuators to generate mechanical movements or, specifically, vibrations; for this reason, the paper is devoted to the analysis of the kinematical behavior of benders in terms of natural frequencies under different conditions. A piezoelectric bimorph bender is a structure composed by two superimposed active piezoelectric beam shaped layers. Under the effect of an electric field, the polarization of piezoelectric layers is able to generate a bending of the beam, so the proposed structure can be classified as a mechanical actuator, or a device able to produce mechanical power. These sorts of devices are diffusely studied in literature [1–6] and applied in different industrial sectors. We are interested in kinematical and dynamical characteristics of the system [7–10], especially in natural frequencies [10–13], considering the effect of uncertainty [14] on the knowledge of electric, piezoelectric, and mechanical parameters, to evaluate how this uncertainty is reflected on the natural frequencies evaluation.

The model combines an equivalent single-layer theory for the mechanical displacements (without relative sliding between electrical layers) with layer-wise-type approximation for the electric potential. First-order Timoshenko shear deformation theory kinematics and quadratic electric potentials are assumed in developing the analytical solution. Mechanical displacement and electric potential Fourier-series amplitudes are treated as fundamental variables, and full electromechanical coupling is maintained. Numerical analysis of simply supported bimorphs under free vibration conditions are presented for different length-to-thickness ratios (i.e., aspect ratio), and the results are verified by those obtained from the exact 2D solution. According to the Timoshenko theory, a shear correction factor is introduced with a value proposed by Timoshenko [7] and by Cowper [12]. The free vibration problem is investigated for simply supported piezoelectric bimorphs with parallel arrangement for the closed circuit condition, and the results for different length-to-thickness ratios are compared with those obtained from the exact 2D solution. Numerical examples are presented on bimorphs constituted by two orthotropic piezoceramic layers (PZT-5A material). The calculation of natural frequencies is based on a Weibull distribution, because it is capable to properly model a large class of stochastic behaviors. The effect of errors on the Weibull distribution of the natural frequencies is shown in terms of change of the Weibull parameters.

We tested the proposed algorithm for a length to thickness ratio \( \lambda \) from 200 to 10, we introduce a reasonable uncertainty on some parameters of the systems to test a wide range of situations, then results are shown in a graphical form and in a statistical formulation to suggest some scientific and practical considerations.

2 Model Definition

In the undeformed initial condition, the neutral axis of the bar is coincident with the \( x_1 \) axis of the Cartesian reference system (0), that is composed by the origin \( O \) and three orthogonal axes \( x_1, x_2 \) and \( x_3 \); furthermore, under appropriate hypotheses, the proposed problem is bidimensional, so the bar is studied only in the plane identified by the axes \( x_1 \) and \( x_3 \). This bar is composed by two piezoelectric layers, mechanically and electrically connected, the bar has a length \( l \) and each layer has a thickness \( h \). Under mechanical and/or electric actions, the bar can be deformed so a cross section of the bar is subjected to a translation \( w \) along the axis \( x_3 \) and to a rotation \( \Psi \) around the axis \( x_2 \) (Fig. 1).

An element of beam is shown in Fig. 2 and, under plane stress and small strain hypotheses, with a symmetric stress tensor, the

Fig. 1 Simply supported piezoelectric bimorph
equal to 2, under zero normal stress in the
direction and of $\mathbf{r}$.

For an orthotropic piezoelectric material, the constitutive equations in concise notation are
approximated with Timoshenko theory (Eq. (5)), on the elec-
tric side; the field intensity is related to electric potential (Eq.(6))
as proposed in Ref. [6].

The strain tensor is related to displacements (Eq.(4)) and these
parameters $\sigma_{ij}$ and $\varepsilon_{ij}$ could be expressed by Eq.(10). Then, expanding
$w$, $\psi$, and $\Phi$ with the Fourier series Eq. (11) and substituting in the motion equation,
avoiding the effect of electromechanical loads, we can find the
Fourier equation, Eq. (12), where the matrix $A$ is defined and
completely derived in Ref. [6]. When the determinant of $A$ is set to
zero, the free vibration frequencies can be calculated.

\[
\begin{align*}
\sigma_{11} + \sigma_{33} + f_{i}^{0} &= \rho \cdot \ddot{u}_{i}, \quad i = 1, 3 \\
D_{11} + D_{33} &= 0 \\
\sigma_{ij} &= C_{ij} \varepsilon_{j} - (1)\varepsilon_{ij} E_{j}, \quad D_{i} = (1)\varepsilon_{ij} S_{ij} + \varepsilon_{ij} E_{j} \\
S_{ij} &= u_{ij}, \quad S_{ij} = u_{ij} + u_{ij}, \quad i \neq j \\
u_{3}(x_{1}, t) &= w(x_{1}, t) \quad u_{1}(x_{1}, x_{3}, t) = -x_{3} \psi(x_{1}, t) \\
E_{i} &= -\Phi_{j} \\
\psi(x_{1}, x_{3}, t) &= g(x_{3}) V(x_{1}, t) + f(x_{3}) \Phi(x_{1}, t) \\
g(x_{3}) &= (-1)^{j}[x_{3}]/h \\
f(x_{3}) &= (-1)^{j} [1 - (1 - 2|x_{3}|/h)^{2}] \\
\sigma_{11} \big|_{x_{1}=0} &= 0, \quad \sigma_{33} \big|_{x_{1}=0} = 0, \quad \psi \big|_{x_{1}=0} = 0
\end{align*}
\]

Under the considered hypotheses (Eq. (13)) on the distribution
of the material parameters, the nominal values are expressed in
Table 1, therefore the frequencies calculated with Eq. (12) are ran-
dom variables. The cumulative frequency distribution is numeri-
cally reconstructed with a median ranking on an equally grouped
histogram. Then, under the hypothesis of Weibull distribution
[14], the localization parameter $\gamma$ is set to 99.9% of the minimum
absissa of the histogram (to avoid numerical errors), so the shape
parameter $\beta$ and the characteristic life $\eta$ are calculated with a
linear fitting on a bilogarithmic scaled Weibull distribution.

The computed Weibull parameters $\gamma$, $\beta$, and $\eta$ allow the evaluation
of the mean and median values $\mu$ and $m$, and of the Fisher pa-
ters $\gamma_{1}$ and $\gamma_{2}$ (see Appendix), which are useful to describe
the profile of the distribution (Table 2). With the proposed
approach, we are able to evaluate the effects of stochastic errors
$\Delta_{r}$ on natural frequencies $\omega_{r}$, varying the geometrical ratio and
the shear coefficient $\kappa$. The results, in terms of maximum relative
error $\Delta_{r}$, on the first natural frequency, approximately computed
as the difference between the 99th and the 1st percentiles related
to the mean value. It can be used, i.e., to evaluate the robustness
of the procedure (Eq. (12)), to correlate the quality of the material
with the desired dynamical performance of the actuator, or to fore-
cast an opportune calibrability of the electric feeding system.
high frequencies are less significant than low frequencies to forecast a frequency change due to input errors will be proposed in this study. A method to forecast a frequency change due to input errors will be proposed in Sec. 5 and it can be applied only to fundamental frequencies, but proper generalizations can be investigated to extend this approach also to high vibrational modes.

### Table 1: Nominal values of material parameters

<table>
<thead>
<tr>
<th>Param.</th>
<th>N. val.</th>
<th>Unit</th>
<th>Param.</th>
<th>N. val.</th>
<th>Unit</th>
</tr>
</thead>
<tbody>
<tr>
<td>$c_{11}$</td>
<td>110</td>
<td>GPa</td>
<td>$c_{12}$</td>
<td>53.1</td>
<td>GPa</td>
</tr>
<tr>
<td>$c_{13}$</td>
<td>87.6</td>
<td>GPa</td>
<td>$c_{13}$</td>
<td>52.1</td>
<td>GPa</td>
</tr>
<tr>
<td>$c_{44}$</td>
<td>21.7</td>
<td>GPa</td>
<td>$\rho$</td>
<td>7700</td>
<td>kg/m$^3$</td>
</tr>
<tr>
<td>$c_{31}$</td>
<td>$-9.72$</td>
<td>C/m$^2$</td>
<td>$e_{33}$</td>
<td>13.2</td>
<td>C/m$^2$</td>
</tr>
<tr>
<td>$c_{33}$</td>
<td>12.0</td>
<td>C/m$^2$</td>
<td>$e_{11}$</td>
<td>16.7</td>
<td>nF/m</td>
</tr>
<tr>
<td>$c_{55}$</td>
<td>15.5</td>
<td>nF/m</td>
<td></td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

### Table 2: Description of a Weibull profile with Fisher parameters

<table>
<thead>
<tr>
<th>Parameter $\gamma_1$</th>
<th>Value</th>
<th>Profile</th>
</tr>
</thead>
<tbody>
<tr>
<td>Positive</td>
<td>Right skewed</td>
<td></td>
</tr>
<tr>
<td>Zero</td>
<td>Symmetric</td>
<td></td>
</tr>
<tr>
<td>Negative</td>
<td>Left skewed</td>
<td></td>
</tr>
<tr>
<td>Parameter $\gamma_2$</td>
<td>Value</td>
<td>Profile</td>
</tr>
<tr>
<td>Positive</td>
<td>Positive</td>
<td></td>
</tr>
<tr>
<td>Zero</td>
<td>Zero</td>
<td></td>
</tr>
<tr>
<td>Negative</td>
<td>Negative</td>
<td></td>
</tr>
</tbody>
</table>

### Table 3: 500 different tested conditions

<table>
<thead>
<tr>
<th>Param.</th>
<th>Tested values</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Delta_{\text{input}}$</td>
<td>0.01, 0.02, 0.03, 0.04, 0.05</td>
</tr>
<tr>
<td>$\lambda$</td>
<td>5, 10, 20, 50, 100</td>
</tr>
<tr>
<td>$\Delta_{\text{cor}}$</td>
<td>0.8333, 0.8472, 0.8611, 0.8750, 0.8889</td>
</tr>
<tr>
<td>Input</td>
<td>$C$, $e$, $\varepsilon$, $C$, $e$, $\varepsilon$ together</td>
</tr>
</tbody>
</table>

### Fig. 4: $\lambda$ versus $\lambda$ for mechanical (dC), piezoelectric (de), electric (deps), and mixed mechanical, piezoelectric and electric (dttot) parametric errors with $\Delta_{\text{input}} = 0.01$ and shear coefficient $k = 5/6$

The choice of the shear correction factor $k$ is really significant, as just shown in literature [6,18] and a range of values is proposed in Table 3 for $k$ with the limits proposed by Timoshenko [7] and by Cowper [12].

With the aim of Fig. 4, we can evaluate the effect of mechanical, piezoelectric, and electric errors on natural frequency error, for different geometrical ratios, with a fixed relative input error and a fixed shear coefficient. Based on this diagram we can affirm that the effect of piezoelectric and electric errors on natural frequency error are negligible compared with mechanical errors; we can also observe that, when $\lambda$ is equal to 10, the effect of mechanical error is more significant than the effect of total error—this is due only to stochastic phenomena.

On the algorithmic viewpoint, Fig. 14 shows how the shear coefficient affects results, as a matter of fact the selection of a Cowper or a Timoshenko value can influence numerical results [6,12], but the effect of mechanical errors is predominant on the algorithmic selections of shear coefficient.

### Fig. 5: $\Delta_{\text{cor}}$ versus $\lambda$ for different values of mechanical relative error $\Delta_{\text{cor}}$ with shear coefficient $k = 5/6$

Figures 5–7 can be considered to evaluate the effects of mechanical errors; Fig. 5 shows that $\Delta_{\text{cor}}$ do not depend on $\lambda$, and $\Delta_{\text{cor}}$ increases with an increment of $\Delta_{\text{cor}}$. Fig. 6 illustrates that incrementing $\Delta_{\text{cor}}$, the Weibull distribution of the output error becomes right skewed. Observing Fig. 7, we can affirm that an increment of the mechanical error produces a steepening of the Weibull distribution of the natural frequencies of the system.

Comparing Figs. 5–7 with Figs. 8–10, we do not highlight important differences between the effect of mechanical errors and the effect of mixed errors on natural frequencies: we believed that some differences, i.e., $\lambda = 50$, $\Delta_{\text{input}} = 0.03$, can probably be associated with numerical or stochastic errors; however, further investigations are necessary to properly dispel this doubt.

Finally, comparing Figs. 8–10 with Figs. 11–13, we observe an oscillation of the diagrams when the shear coefficient is set to 8/9, but further investigations are necessary to evaluate the possibility of a correlation between the stability of the numerical solver and the value of shear coefficient.
5 Empirical Deductions

With the purpose of producing some simplified indications, starting from the previous observations and considering especially the trials illustrated in Fig. 8, we will try to calibrate a simplified one-degree-of-freedom model, which can be used without the help of complex algorithms. The calculation of the unique natural frequency of a simplified one-degree-of-freedom model can be easily performed with the well-known expression (Eq. (15)), as the square ratio of stiffness and mass.

Fig. 6 $\gamma_1$ versus $\lambda$ for different values of mechanical relative error $\Delta_c$ with shear coefficient $k = 5/6$

Fig. 7 $\gamma_2$ versus $\lambda$ for different values of mechanical relative error $\Delta_c$ with shear coefficient $k = 5/6$

Fig. 8 $\gamma_n$ versus $\lambda$ for different values of mixed relative error $\Delta_{c,e}$ with shear coefficient $k = 5/6$

Fig. 9 $\gamma_1$ versus $\lambda$ for different values of mixed relative error $\Delta_{c,e}$ with shear coefficient $k = 5/6$

Fig. 10 $\gamma_2$ versus $\lambda$ for different values of mixed relative error $\Delta_{c,e}$ with shear coefficient $k = 5/6$

Fig. 11 $\gamma_n$ versus $\lambda$ for different values of mixed relative error $\Delta_{c,e}$ with shear coefficient $k = 5/6$

Fig. 12 $\gamma_1$ versus $\lambda$ for different values of mixed relative error $\Delta_{c,e}$ with shear coefficient $k = 5/6$

Fig. 13 $\gamma_2$ versus $\lambda$ for different values of mixed relative error $\Delta_{c,e}$ with shear coefficient $k = 5/6$
Differentiating Eq. (15) and dividing by the natural frequency, we can easily find, with opportune hypotheses and with a proper abstraction, Eq. (16), which express the relative maximum error on the natural frequency in terms of relative maximum (mechanical) stiffness error.

\[ \Delta_{\omega} = \Delta_C / 2 \]  

(16)

Empirically, Eq. (16) is corrected with the help of two empirical coefficients, \( \rho_0 \) and \( \rho_1 \), which respectively correct the slope and the intercept of Eq. (16) and, for the considered conditions, Eq. (17) represents an approximate formulation to forecast \( \Delta_{\omega} \).

\[ \Delta_{\omega} = (\rho/\rho_0) \cdot \Delta_C - (\rho/\rho_1) \]
\[ \rho_0 = 2 \times 10^3 \text{ kg/m}^3; \quad \rho_1 = 4 \times 10^5 \text{ kg/m}^3 \]  

(17)

To verify the validity of Eq. (17), the trials illustrated in Fig. 8 are examined and, for each mechanical error, the average value of the output error over the geometrical ratio domain is considered and compared with Eq. (17), as shown in Fig. 15.

Figure 15 shows that Eq. (17) is able to forecast average \( \Delta_{\omega} \) with a desirable slight overestimation; to properly verify the prediction ability of Eq. (17), the average estimated \( \Delta_{\omega} \) is directly compared with \( \Delta_{\omega} \) associated to trials illustrated in Fig. 8, as shown in Fig. 16, where the overestimation is more visible in every condition except for the smallest input error (0.01); as a matter of fact, the proposed formulation, Eq. (17), is capable of forecasting the maximum relative output error in the geometrical ratio domain for a considered input error. Higher order fitting can be considered to improve forecasting performances of Eq. (17), i.e., power fitting (Eq. (18)), or exponential fitting (Eq. (19)), logarithmic fitting (Eq. (20)), sinusoidal fitting (Eq. (21)), or logistic fitting (Eq. (22)), with parametric values proposed in Table 4, as represented in Fig. 17.

According to Figs. 18–19, between the proposed higher order formulations (Eqs. (18)–(22)) with the associated values in Table 4, the best fitting is performed by exponential function, Eq. (19).
errors on mechanical stiffness matrix and relative errors on natural frequencies, only mechanical stiffness errors can be considered, for a correct evaluation of the relative maximum error of natural frequencies, which are predominant on the effects of relative errors of piezoelectric and/or electric matrices; so, for a correct evaluation of the relative maximum error of natural frequencies, only mechanical stiffness errors can be considered. The second result is that the correlation between relative errors on mechanical stiffness matrix and relative errors on natural frequency can be represented with a simple empirical expression.

6 Conclusions

The problem of calculation of natural frequencies of a simply supported bending piezoelectric bimorph was considered under the presence of uncertainty on mechanical, piezoelectric, and electric parameters. The analysis of the system was carried out for PZT-5A material, for parametric conditions indicated in Table 1 and in Table 3, and only for the first natural frequency of the system, so eventual generalizations out of the considered domains cannot be guaranteed. Under these limitations on the validity of the analysis two results are suggested. The first result is that relative errors of mechanical stiffness produce effects on relative errors of the natural frequencies, which are predominant on the effects of relative errors of piezoelectric and/or electric matrices; so, for a correct evaluation of the relative maximum error of natural frequencies, only mechanical stiffness errors can be considered. The second result is that the correlation between relative errors on mechanical stiffness matrix and relative errors on natural frequency can be represented with a simple empirical expression.

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Nomenclature

\[ (0) = \text{cartesian reference system composed by the origin } O \]
\[ \{x, y, z\} = \text{orthogonal axes } x_i, x_j, x_k \]
\[ x_i = \text{coordinate measured on the axis } x_j \]
\[ l, h = \text{length and thickness of a piezoelectric layer} \]
\[ w, \Psi = \text{translation and rotation of a cross section} \]
\[ F_i = \text{partial derivative of } F \text{ with respect to } x_j \]
\[ \sigma = \text{stress tensor}; \]
\[ f^b = \text{body force} \]
\[ u = \text{displacement of a bar element} \]
\[ \rho = \text{mass density of the bar} \]
\[ D = \text{electric displacement} \]
\[ E = \text{electric field} \]
\[ C = \text{elastic stiffness matrix} \]
\[ e = \text{piezoelectric matrix} \]
\[ \varepsilon = \text{dielectric matrix} \]
\[ S = \text{strain tensor} \]
\[ s = \text{strain tensor organized in a vector} \]
\[ r = \text{poling direction coefficient} \]
\[ \phi = \text{electric potential} \]
\[ V = \text{amplitude of the applied electric potential} \]
\[ g = \text{thickness distribution of } V \]
\[ \Phi = \text{electric potential amplitude on the layer midline} \]
\[ f = \text{thickness distribution of } \Phi \]
\[ k = \text{shear coefficient} \]
\[ \omega_n = \text{natural frequencies of the system} \]
\[ \delta = \text{matricial rectangular probability density function} \]
\[ \mu, m = \text{matricial mean, median values of a probability density function} \]
\[ \Delta = \text{maximum relative error of a probability density function} \]
\[ \varsigma = \text{variance of a probability density function} \]
\[ \gamma, \beta, \eta = \text{localization, shape, life parameters of a Weibull distribution} \]
\[ \gamma_1, \gamma_2 = \text{Fisher parameters of a Weibull distribution} \]
\[ \lambda = \text{geometrical ratio} \]
\[ m_{\text{eq}} = \text{equivalent mass of simplified model} \]
\[ \rho_{0}f_1 = \text{corrective coefficients of empirical model} \]

Appendix

The Fisher parameters \( \gamma_1 \) and \( \gamma_2 \) are expressed in Eq. (A1) where \( \mu_2, \mu_3, \mu_4 \) are defined in Eq. (A2), while \( \Gamma_1 \) is defined in Eq. (A3), in terms of an Euler \( I \) function.

\[ \gamma_1 = \frac{\mu_3}{\mu_2^{3/2}}; \quad \gamma_2 = \frac{\mu_4}{\mu_2^2} - 3 \]  \hspace{1cm} (A1)
\[ \mu_2 = \Gamma_2 - \Gamma_1^2; \quad \mu_3 = 3\Gamma_2^2\Gamma_1 - 2\Gamma_1^3 \]  \hspace{1cm} (A2)
\[ \mu_4 = 4\Gamma_2^3\Gamma_1 + 6\Gamma_2\Gamma_1^2 - 3\Gamma_1^4 \]
\[ \Gamma_1 = \Gamma(i/\beta + 1) \]  \hspace{1cm} (A3)

References