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Supraconvergence of a finite difference scheme for solutions in $H^s(0, L)$

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In this paper we study the convergence of a centred finite difference scheme on a non-uniform mesh for a 1D elliptic problem subject to general boundary conditions. On a non-uniform mesh, the scheme is, in general, only first-order consistent. Nevertheless, we prove for $s \in (1/2, 2]$ order $\mathrm{O}(h^s)$-convergence of solution and gradient if the exact solution is in the Sobolev space $H^{1+s}(0, L)$, i.e. the so-called supraconvergence of the method. It is shown that the scheme is equivalent to a fully discrete linear finite-element method and the obtained convergence order is then a superconvergence result for the gradient. Numerical examples illustrate the performance of the method and support the convergence result.

Keywords: finite difference scheme; fully discrete linear finite element method; non-uniform grid; stability; superconvergence; supraconvergence.

1. Introduction

We consider the discretisation of the differential equation

$$Au := -(au')' + (bu)' + cu = f, \quad \text{in } (0, L) \subset \mathbb{R}, \tag{1.1}$$

subject to either Dirichlet boundary conditions

$$u(0) = \gamma_0, \quad u(L) = \gamma_L, \tag{1.2}$$

or third kind boundary conditions

$$-(au')(0) + \beta_0 u(0) = \gamma_0, \quad (au')(L) + \beta_L u(L) = \gamma_L. \tag{1.3}$$

A combination of the two types of boundary conditions is not explicitly considered for ease of presentation. Our scheme can be written as a finite difference approximation on the (in general) non-uniform grid

$$\mathbb{I}_h := \{0 = x_0 < x_1 < \cdots < x_{N-1} < x_N = L\},$$

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where h is the vector of mesh-sizes $h_j := x_{j+1} - x_j,\ j = 0, \dots, N-1$. By $W_h := \{u_h, v_h, w_h, \dots\}$, we denote the space of complex-valued grid functions defined on $\mathbb{I}_h$ and we introduce the centred divided finite differences

$$(\delta v_h)_j := \frac{v_{j+1} - v_{j-1}}{x_{j+1} - x_{j-1}}, \quad (\delta^{(1/2)} v_h)_{j+1/2} := \frac{v_{j+1} - v_j}{x_{j+1} - x_j}, \quad (\delta^{(1/2)} v_h)_j := \frac{v_{j+1/2} - v_{j-1/2}}{x_{j+1/2} - x_{j-1/2}},$$

where $v_j := v_h(x_j)$, $x_{j+1/2} := x_j + h_j/2$ and $v_{j+1/2}$ is used as far as it makes sense. Our scheme has the form

$$A_h u_h := -\delta^{(1/2)}(a\delta^{(1/2)} u_h) + \delta(b u_h) + c u_h = f_h, \quad \text{in } \mathbb{I}'_h, \tag{1.4}$$

together with the discretised boundary conditions

$$u_0 = \gamma_0, \quad u_N = \gamma_L, \tag{1.5}$$

or

$$\begin{aligned} -(a\delta^{(1/2)} u_h)_{1/2} + \frac{h_0}{2}[\delta^{(1/2)}(b u_h)_{1/2} + c_0 u_0 - f_0] + \beta_0 u_0 = \gamma_0, \\ (a\delta^{(1/2)} u_h)_{N-1/2} + \frac{h_{N-1}}{2}[\delta^{(1/2)}(b u_h)_{N-1/2} + c_N u_N - f_N] + \beta_L u_N = \gamma_L, \end{aligned} \tag{1.6}$$

in the case (1.2) or (1.3), respectively. Here, we have introduced the set $\mathbb{I}'_h := \mathbb{I}_h \backslash \{x_0, x_N\}$ of inner grid points. In (1.4), the grid function f_h approximating the right-hand side of (1.1) is given by

$$f_j := \frac{2}{h_{j-1} + h_j} \int_{x_{j-1/2}}^{x_{j+1/2}} f(x)\,\mathrm{d}x, \quad j = 0, \dots, N, \tag{1.7}$$

where for a simpler notation, we have introduced the additional mesh-sizes $h_{-1} := h_N := 0$ and points $x_{-1/2} := 0,\ x_{N+1/2} := L$.

There are no restrictions made on the non-uniformity of the grid $\mathbb{I}_h$. We consider the behaviour of the scheme for a sequence of grids $\mathbb{I}_h,\ h \in H$, with maximal mesh-size $h_{\max} := \max\{h_j,\ j = 0, \dots, N-1\}$ tending to zero. The scheme (1.4) is, in general, first-order consistent only (it is of second order on uniform grids). One purpose of the present paper is to show that, nevertheless, the solutions u_h are second-order accurate, a fact that has been called supraconvergence (see Kreiss *et al.*, 1986). An additional feature of the scheme is that the first-order forward divided differences with respect to the grid are also second-order accurate. These results hold under the optimal regularity assumption $u \in H^3(0, L)$ for the continuous solution u, while earlier results for similar schemes needed $u \in C^4[0, L]$ (see Grigorieff, 1986; de Hoog & Jackett, 1985; Kreiss *et al.*, 1986; Manteuffel & White, 1986; Samarskij, 1984). The convergence result is stated in Theorem 3.1 of Section 3, where the more general case of $u \in H^{1+s}(0, L)$ with $s \in (1/2, 2]$ and corresponding order $\mathrm{O}(h^s)$-convergence is considered. Here, $H^t(0, L)$ denotes the usual Sobolev space with fractional order t. The corresponding norm is written as $\|\cdot\|_t$.

For the numerical solution of boundary value ordinary differential equations (BVODEs), there exists a variety of well-established efficient codes that are easily available to the user. So the present code is of interest for special cases where the normally higher regularity assumptions for the application of those codes are not met. Also, for parabolic equations in one space dimension, finite difference methods are frequently used. The results of this paper can be directly transferred to this situation (see Ferreira, 1994; Levermore *et al.*, 1987; Thomée, 1997).

Another aspect of this paper is the new way of analysing supraconvergence that can be generalised to elliptic problems in two dimensions. This can be seen from Section 2, where we show the useful relation that the scheme (1.4) with discretised boundary conditions (1.5) or (1.6) is equivalent to a linear finite-element method with quadrature. This relation simplifies the analysis. The direct finite difference analysis would require some technical effort for our problems that are subject to general boundary conditions and are not positive definite and also for obtaining error estimates in fractional Sobolev spaces. Also, the error estimate in Theorem 3.1 as a sum of local error contributions is a useful consequence of our method, e.g. for implementing adaptivity of the scheme.

Supraconvergence of finite difference schemes for BVODEs has found some attention in the literature (see Garcia-Archila, 1992; Garcia-Archila & Sanz-Serna, 1991; Grigorieff, 1986; de Hoog & Jackett, 1985; Kreiss *et al.*, 1986; Manteuffel & White, 1986; Samarskij, 1984). Different methods of proof have been used by the various authors. Our approach was inspired by the method in Jovanović *et al.* (1987) for the equidistant case. The phenomenon of supraconvergence in more than one space dimension has also been studied in the literature (see, e.g. Ferreira & Grigorieff, 1998; Forsyth & Sammon, 1988; Levermore *et al.*, 1987; Marletta, 1988). A paper extending our techniques to such problems is under preparation. Superconvergence results for the gradient have been obtained in Bojović & Jovanović (2001), Ferreira & Grigorieff (1998), Jovanović *et al.* (1987), Jovanović (2001) and Lesaint & Zlámal (1979). Another direction of interest lies in setting up discretisation schemes that work for coefficients with low smoothness in the differential equation (1.1). The corresponding assumptions in Godev *et al.* (1988), Jovanović (1993), Lazarov *et al.* (1984) and Süli *et al.* (1985) are weaker than ours as stated at the end of Section 2.

2. The variational formulation

The linear finite-element scheme that is equivalent to (1.4) and (1.5) or (1.6) can be written with the aid of the piecewise linear interpolation of a grid function v_h with breakpoints in $\mathbb{I}_h$ that we denote by $P_h v_h$. We restrict ourselves to presenting the case of boundary conditions (1.3) only, the Dirichlet case (1.2) being similar but slightly simpler. Let $a_h(\cdot,\cdot)$ denote the sesquilinear form

$$\begin{aligned} a_h(v_h, w_h) := \sum_{j=0}^{N-1} h_j a_{j+1/2} (P_h v_h)'_{j+1/2} (P_h \overline{w}_h)'_{j+1/2} - (b v_h, (P_h w_h)')_h \\ + (c v_h, w_h)_h + (\beta_0 - b(0)) v_0 \overline{w}_0 + (\beta_L + b(L)) v_N \overline{w}_N. \end{aligned} \tag{2.1}$$

Here, $\overline{w}$ denotes the complex conjugate of w and

$$(v_h, w_h)_h := \sum_{j=0}^{N-1} \frac{h_j}{2} (v_j \overline{w}_j + v_{j+1} \overline{w}_{j+1}), \quad v_h, w_h \in W_h, \tag{2.2}$$

which has the form of the composite trapezoidal rule. Note that in (2.1), the function $(P_h w_h)'$ may have jumps at the breakpoints. For such functions, (2.2) has to be evaluated in the natural way of a sum of basic trapezoidal rules over each single subinterval (x_j, x_{j+1}) invoking the limits of $(P_h w_h)'$ from the interior in the breakpoints. It is not difficult to see that the finite difference schemes (1.4) and (1.6) are equivalent to the variational problem of finding $u_h \in W_h$ such that

$$a_h(u_h, v_h) = (f_h, v_h)_h + \gamma_0 \overline{v}_0 + \gamma_L \overline{v}_N \quad \forall v_h \in W_h. \tag{2.3}$$

The formulation (2.3) corresponds to the variational formulation of the continuous problem (1.1) and (1.3) of finding $u \in H^1(0, L)$ such that

$$a(u, v) = (f, v)_0 + \gamma_0 \overline{v}(0) + \gamma_L \overline{v}(L) \quad \forall\, v \in H^1(0, L), \tag{2.4}$$

where $(\cdot, \cdot)_0$ denotes the standard inner product in $L^2(0, L)$ and

$$a(v, w) := (av', w')_0 - (bv, w')_0 + (cv, w)_0 + (\beta_0 - b(0))v(0)\overline{w}(0) + (\beta_L + b(L))v(L)\overline{w}(L),$$

for $v, w \in H^1(0, L)$. The coefficients in the differential equation are assumed to be smooth enough, e.g. $a \in C[0, L]$ and $b, c \in W^{2,\infty}(0, L)$, and to satisfy $a(x) \geqslant \underline{a} > 0$, $x \in (0, L)$. Even less restrictively, it is sufficient that a, b and c satisfy the smoothness conditions only piecewise provided the corresponding breakpoints belong to the mesh $\mathbb{I}_h'$ for $h \in H$.

3. The main result

The main result of this paper in Theorem 3.1 relies on the following inverse stability result.

PROPOSITION 3.1 Assume that the variational problem belonging to (1.1) and (1.2) or (1.3) is uniquely solvable. Then, there exists a positive constant C such that for $h \in H$ with $h_{\max}$ small enough

$$\|P_h v_h\|_1 \leqslant C \sup_{0 \neq w_h \in W_h} \frac{|a_h(v_h, w_h)|}{\|P_h w_h\|_1} \quad \forall\, v_h \in W_h. \tag{3.1}$$

The proof is similar to the one of Theorem 2 in Ferreira & Grigorieff (1998) and we do not reproduce it here.

By $R_h v$, we denote the pointwise restriction of a function v to the grid $\mathbb{I}_h$. If it is clear from the context, we write only v in place of $R_h v$. We set $I_j := (x_j, x_{j+1})$. It is well known that $H^s(0, L)$ is continuously imbedded in $C[0, L]$ for $s > 1/2$ (in fact, the imbedding is compact) and, consequently, pointwise evaluation of the derivative of functions in $H^{1+s}(0, L)$ makes sense for $s > 1/2$.

THEOREM 3.1 Assume that the variational problem belonging to (1.1) and (1.2) or (1.3) is uniquely solvable. Then, the discretised problem (1.4) and (1.5) or (1.6) has a unique solution u_h for $h \in H$ with $h_{\max}$ sufficiently small. Let $s \in (1/2, 2]$ and assume that the solution u of (1.1) and (1.2) or (1.3) lies in $H^{1+s}(0, L)$. Then, there holds the error estimate

$$\|P_h(R_h u - u_h)\|_1 \leqslant C \left(\sum_{j=0}^{N-1} h_j^{2s} \|u\|^2_{H^{1+s}(I_j)} \right)^{1/2} \leqslant C h^s_{\max} \|u\|_{1+s}. \tag{3.2}$$

Proof. Let u_h be the solution of (1.4) and (1.6) that exists uniquely for sufficiently small $h_{\max}$ due to the stability inequality (3.1). Since u_h solves the variational problem (2.3), an estimate of $\|P_h(R_h u - u_h)\|_1$ will be obtained with the aid of (3.1) by bounding

$$a_h(R_h u - u_h, v_h) = a_h(R_h u, v_h) - (f_h, v_h)_h - \gamma_0 \overline{v}_0 - \gamma_L \overline{v}_N. \tag{3.3}$$

Invoking the definition of f_h in (1.7), we obtain after an integration and a summation by parts (recall that $x_{-1/2} = 0$ and $x_{N+1/2} = L$)

$$(f_h, v_h)_h = -\sum_{j=0}^{N-1}[(au')(x_{j+1/2}) - (bu)(x_{j+1/2})](\overline{v}_{j+1} - \overline{v}_j) + \sum_{j=0}^{N}\int_{x_{j-1/2}}^{x_{j+1/2}} cu\,\mathrm{d}x\overline{v}_j$$
$$+ [(au')(0) - (bu)(0)]\overline{v}_0 - [(au')(L) - (bu)(L)]\overline{v}_N.$$

Taking the definition of $a_h(\cdot,\cdot)$ and the boundary conditions (1.3) into account, an estimate of (3.3) will be obtained by the sum of bounds for the quantities

$$\begin{aligned} T_a &:= \sum_{j=0}^{N-1} h_j a_{j+1/2}[(P_h u)'_{j+1/2} - u'(x_{j+1/2})](P_h\overline{v}_h)'_{j+1/2}, \\ T_b &:= \sum_{j=0}^{N-1} h_j \left[(bu)(x_{j+1/2}) - \frac{(bu)(x_j) + (bu)(x_{j+1})}{2}\right](P_h\overline{v}_h)'_{j+1/2}, \\ T_c &:= \sum_{j=0}^{N}\left[\frac{h_{j-1}+h_j}{2}(cu)(x_j) - \int_{x_{j-1/2}}^{x_{j+1/2}}(cu)(x)\,\mathrm{d}x\right]\overline{v}_j. \end{aligned} \tag{3.4}$$

We continue the proof by first considering the case $s \in \{1, 2\}$.
Estimate for T_a: Let the function w be defined by $w(\xi) := u(x_j + \xi h_j)$ for $\xi \in [0, 1]$. We have

$$u'(x_{j+1/2}) - (P_h u)'(x_{j+1/2}) := \frac{1}{h_j}\left[w'\left(\frac{1}{2}\right) - w(1) + w(0)\right]. \tag{3.5}$$

The functional

$$\lambda(g) := g'\left(\frac{1}{2}\right) - g(1) + g(0) \tag{3.6}$$

is bounded on $W^{2,1}(0,1)$ and vanishes for $g = 1,\ \xi$ and ξ^2. Thus, the Bramble–Hilbert Lemma gives the existence of a positive constant C such that

$$|\lambda(g)| \leqslant C\|g^{(r)}\|_{L^1(0,1)},$$

for $r \in \{2, 3\}$. The last estimate applied to $g := w$ and (3.5) yields

$$|u'(x_{j+1/2}) - (P_h u)'(x_{j+1/2})| \leqslant Ch_j^{s-1}\|u^{(1+s)}\|_{L^1(I_j)} \leqslant Ch_j^{s-1/2}\|u\|_{H^{(1+s)}(I_j)}, \tag{3.7}$$

and by summation, we obtain the bound

$$|T_a| \leqslant C\|a\|_\infty\left(\sum_{j=0}^{N-1} h_j^{2s}\|u\|^2_{H^{(1+s)}(I_j)}\right)^{1/2}\|P_h v_h\|_1. \tag{3.8}$$

Estimate for T_b: Let w be defined as before but with u replaced by bu. Then

$$\frac{(bu)(x_j) + (bu)(x_{j+1})}{2} - (bu)(x_{j+1/2}) = \frac{w(0) + w(1)}{2} - w\left(\frac{1}{2}\right).$$

The functional

$$\lambda(g) := \frac{g(0)+g(1)}{2} - g\left(\frac{1}{2}\right)$$

is bounded on $W^{2,1}(0,1)$ and vanishes for $g=1$ and ξ. Again, by the Bramble–Hilbert Lemma, the estimate

$$|\lambda(g)| \leqslant C\|g''\|_{L^1(0,1)}, \quad g \in W^{2,1}(0,1),$$

holds and we obtain the bound

$$|T_b| \leqslant C\|b\|_{2,\infty}\left(\sum_{j=0}^{N-1} h_j^4 \|u''\|^2_{L^2(I_j)}\right)^{1/2} \|P_h v_h\|_1, \tag{3.9}$$

where $\|\cdot\|_{2,\infty}$ is the usual norm in the Sobolev space $W^{2,\infty}(0,L)$.

Estimate for T_c: Recall that $x_{-1/2}=0,\ x_{N+1/2}=L,\ h_{-1}=h_N=0$. Thus, T_c may be written as the sum

$$T_c = (T_1+T_2)/2,$$

with

$$T_1 := \sum_{j=0}^{N-1}\left[\frac{h_j}{2}\left((cu)_j+(cu)_{j+1}\right) - \int_{x_j}^{x_{j+1}} cu\,\mathrm{d}x\right](\overline{v}_j+\overline{v}_{j+1}),$$

and

$$T_2 := \sum_{j=0}^{N-1}\left[\frac{h_j}{2}\left((cu)_{j+1}-(cu)_j\right) + \int_{x_j}^{x_{j+1/2}} cu\,\mathrm{d}x - \int_{x_{j+1/2}}^{x_{j+1}} cu\,\mathrm{d}x\right](\overline{v}_{j+1}-\overline{v}_j).$$

The sum in T_1 contains the errors of the trapezoidal rule that can be bounded with the aid of the Bramble–Hilbert Lemma by

$$\left|\frac{h_j}{2}((cu)_j+(cu)_{j+1}) - \int_{x_j}^{x_{j+1}} cu\,\mathrm{d}x\right| \leqslant Ch_j^2\|(cu)''\|_{L^1(I_j)}.$$

Since $P_h v_h$ is piecewise linear, the estimate

$$|T_1| \leqslant C\|c\|_{2,\infty}\left(\sum_{j=0}^{N-1} h_j^4 \|u\|^2_{H^2(I_j)}\right)^{1/2} \|P_h v_h\|_0 \tag{3.10}$$

follows. For T_2, we have only the first-order bound

$$\left|\frac{h_j}{2}((cu)_{j+1}-(cu)_j) + \int_{x_j}^{x_{j+1/2}} cu\,\mathrm{d}x - \int_{x_{j+1/2}}^{x_{j+1}} cu\,\mathrm{d}x\right| \leqslant Ch_j\|(cu)'\|_{L^1(I_j)},$$

for $u \in W^{1,1}(I_j)$. But the factor $(\overline{v}_{j+1}-\overline{v}_j)$ allows us to estimate T_2 with the same order as T_1 by

$$|T_2| \leqslant C\|c\|_{1,\infty}\left(\sum_{j=0}^{N-1} h_j^4 \|u\|^2_{H^1(I_j)}\right)^{1/2} \|P_h v_h\|_1. \tag{3.11}$$

The inequality (3.2) now follows from the bounds (3.8)–(3.11).

Now we come to the proof for $s \in (1/2, 1)$. For an interval $I \subset \mathbb{R}$, the norm in $H^{1+s}(I)$ is defined by

$$\|v\|_{H^{1+s}(I)} = \left(\|v\|^2_{H^1(I)} + |v'|^2_{H^s(I)}\right)^{1/2}, \tag{3.12}$$

where

$$|v|_{H^s(I)} = \left(\int_I \int_I \frac{|v(x) - v(y)|^2}{|x-y|^{1+2s}} \,\mathrm{d}x \,\mathrm{d}y\right)^{1/2}.$$

The functional λ from (3.6) is bounded on $H^{1+s}(0, 1)$ and by the Bramble–Hilbert Lemma (see e.g. in the form given in Braess, 1992, Lemma 6.2)

$$|\lambda(g)| \leqslant C|g'|_{H^s(0,1)}. \tag{3.13}$$

After applying this inequality to $g := w$, scaling to the interval I_j and summing with respect to j, we obtain the estimate (3.8) for T_a. Similarly, following the arguments which led to (3.9), (3.10) and (3.11), one derives for T_b and T_c the same bound as for T_a, thus proving the first inequality in (3.2). Then the second one follows from the obvious inequality

$$\sum_{j=0}^{N-1} \|u\|^2_{H^{1+s}(I_j)} \leqslant \|u\|^2_{H^{1+s}(0,L)}.$$

The proof for $s \in (1, 2)$ is similar, this time based on the bound

$$|\lambda(g)| \leqslant C|g''|_{H^{s-1}(0,1)},$$

in place of (3.13). Note that for $s \in (1, 2)$, the proof could be easily obtained for the interpolation norm in $H^s(0, L)$ from the already proved cases $s = 1$ and $s = 2$. □

REMARK 3.1 For the purposes of adaptive error control, the following error estimate, which is more detailed than (3.2), may be useful; it is obtained by collecting the quantities as they appear in the proof:

$$\|P_h(R_h u - u_h)\|_1 \leqslant C \left[\sum_{j=0}^{N-1} h_j^4 \left(\|a_{j+\frac{1}{2}} u'''\|^2_{L^2(I_j)} + \|(bu)''\|^2_{L^2(I_j)} + \|(cu)''\|^2_{L^2(I_j)} + \|(cu)'\|^2_{L^2(I_j)}\right)\right]^{1/2}. \tag{3.14}$$

REMARK 3.2 In (3.2), only the error of the P_h-projection of $R_h u - u_h$ is bounded in the H^1-norm. This is weaker than bounding the actual error $u - P_h u_h$ as in Roos & Linß (2001), where in the context of singularly perturbed problems an adequate ϵ-weighted norm is used for the derivative. In the FEM context, the second-order convergence of $P_h(R_h u - u_h)$ is the so-called supercloseness (see Wahlbin, 1995, p. 80) of the gradient of the (fully discrete) finite element method (FEM) approximation. It is known that a superclose approximation exhibits superconvergence in certain points.

REMARK 3.3 Supraconvergence takes place also for differential equations that are not given in divergence form, assuming the increased smoothness $a \in W^{2,\infty}(0, L)$ but keeping the regularity assumptions $b, c \in W^{2,\infty}(0, L)$ and $u \in H^3(0, L)$ as before.[1] For example, the scheme

$$-a\delta^{1/2}\delta^{1/2}u_h + b\delta u_h + cu_h = f_h, \quad \text{in } \mathbb{I}_h', \tag{3.15}$$

for the differential equation $-au'' + bu' + cu = f$ subject to, for simplicity, Dirichlet boundary conditions $u_0 = \gamma_0,\ u_N = \gamma_L$ is supraconvergent. We came to this observation through the suggestions of one of the referees.

In the following, we indicate a proof. We found it convenient to consider (3.15) as a perturbation of the scheme (1.4). Note first that it is sufficient to consider the case $a \equiv 1$. If this is not the case, divide (3.15) by the coefficient a leading to the approximation of a differential equation with $-u''$ as second-order term and obviously modified coefficients b and c. The right-hand side then has the form $a^{-1} f_h$ which is shown in Remark 3.4 to be a second-order perturbation of the integral average $(a^{-1} f)_h$. Assuming now $a \equiv 1$, we consider (3.15) as an approximation of the transformed differential equation

$$-u'' + (bu)' + (c - b')u = f.$$

From the relation

$$b_j(u_{j+1} - u_{j-1}) = (bu)_{j+1} - (bu)_{j-1} - [(b_{j+1} - b_j)u_{j+1} + (b_j - b_{j-1})u_{j-1}],$$

it is seen that the quantity in square brackets divided by $h_{j-1} + h_j$ is the approximation of $b'u$. Consequently, in the error analysis, an additional expression $\widetilde{T}_c$ appears in (3.4) which has the form

$$\widetilde{T}_c = \sum_{j=0}^{N}\left(\int_{x_{j-1/2}}^{x_{j+1/2}} b'u\,\mathrm{d}x - \frac{1}{2}\left[(b_{j+1} - b_j)u_{j+1} + (b_j - b_{j-1})u_{j-1}\right]\right)\overline{v}_j. \tag{3.16}$$

Taking into account that $v_0 = v_N = 0$, summing by parts in (3.16), we obtain

$$\widetilde{T}_c = \sum_{j=0}^{N-1} h_j F_j (P_h\overline{v}_h)'_{j+1/2},$$

where $F_0 = 0$ and

$$F_j := -\int_{x_{1/2}}^{x_{j+1/2}} b'u\,\mathrm{d}x + \frac{1}{2}\sum_{i=1}^{j}[(b_{i+1} - b_i)u_{i+1} + (b_i - b_{i-1})u_{i-1}], \quad j = 0, \ldots, N-1.$$

Note that the sum in the expression of F_j can be written as

$$\sum_{i=1}^{j}[(b_{i+1} - b_i)u_{i+1} + (b_i - b_{i-1})u_{i-1}] = (b_{j+1}u_{j+1} + b_ju_j) - (b_1u_1 + b_0u_0) - \sum_{i=1}^{j} b_i(u_{i+1} - u_{i-1}).$$

[1] The original proof by the authors needed $b \in W^{3,\infty}(0, L)$. We owe the proof under the present lower regularity assumption to one of the referees.

Then integration by parts allows us to write

$$F_j = F_0^{(1)} - F_j^{(1)} - F_j^{(2)},$$

where

$$F_j^{(1)} := (bu)_{j+1/2} - \frac{1}{2}(b_j u_j + b_{j+1} u_{j+1}), \quad j = 0, \dots, N-1,$$

and

$$F_j^{(2)} := \int_{x_{1/2}}^{x_{j+1/2}} bu' \,\mathrm{d}x - \frac{1}{2}\sum_{i=1}^{j} b_i (u_{i+1} - u_{i-1}), \quad j = 1, \dots, N-1.$$

We have already shown in (3.9) that

$$\left|\sum_{j=0}^{N-1} h_j F_j^{(1)} (P_h \overline{v}_h)'_{j+1/2}\right| \leqslant C\|b\|_{2,\infty} \left(\sum_{j=0}^{N-1} h_j^4 \|u''\|^2_{L^2(I_j)}\right)^{1/2} \|P_h v_h\|_1.$$

The corresponding sum with $F_j^{(1)}$ replaced by $F_0^{(1)}$ vanishes since $v_0 = v_N = 0$. We now estimate the part of $\widetilde{T}_c$ associated with $F_j^{(2)}$. By writing $u_{i+1} - u_{i-1} = (u_{i+1} - u_i) + (u_i - u_{i-1})$ and summing by parts, we obtain

$$\sum_{i=1}^{j} b_i (u_{i+1} - u_{i-1}) = b_j (u_{j+1} - u_j) + b_1 (u_1 - u_0) + \sum_{i=2}^{j} (u_i - u_{i-1})(b_{i-1} + b_i).$$

Thus,

$$F_j^{(2)} = \int_{x_j}^{x_{j+1/2}} bu' \,\mathrm{d}x - b_j \frac{u_{j+1} - u_j}{2} + \int_{x_{1/2}}^{x_1} bu' \,\mathrm{d}x - b_1 \frac{u_1 - u_0}{2}$$
$$+ \sum_{i=2}^{j} \left[\int_{x_{i-1}}^{x_i} bu' \,\mathrm{d}x - \frac{1}{2}(u_i - u_{i-1})(b_{i-1} + b_i)\right]. \tag{3.17}$$

By Taylor expansion of b and integration by parts, we derive for the first term in (3.17)

$$\int_{x_j}^{x_{j+1/2}} bu' \,\mathrm{d}x = \int_{x_j}^{x_{j+1/2}} [b_j + b'_j (x - x_j)] u' \,\mathrm{d}x + S$$
$$= b_j \int_{x_j}^{x_{j+1/2}} u' \,\mathrm{d}x + \frac{h_j^2}{8} b'_j u'_{j+1/2} - \frac{b'_j}{2} \int_{x_j}^{x_{j+1/2}} (x - x_j)^2 u'' \,\mathrm{d}x + S, \tag{3.18}$$

where S has the required order, as is also the case for the second and third quantity in (3.18). Also, by virtue of the Bramble–Hilbert Lemma,

$$\int_{x_j}^{x_{j+1/2}} u' \,\mathrm{d}x - \frac{u_{j+1} - u_j}{2}$$

is seen to be of second order. A similar argument applies to the difference of the third and fourth quantity in (3.17). Finally, for bounding the sum in (3.17), we first note that replacing $(u_i - u_{i-1})$ by $h_{i-1} u'_{i-1/2}$ and $(b_i + b_{i-1})$ by $2b_{i-1/2}$ introduces a second-order error only. We then end up with the composite midpoint rule that is also of second order.

REMARK 3.4 If the right-hand side f in (1.1) is in $H^2(0, L)$, its discretisation (1.7) as an integral average can be replaced by the pointwise restriction $R_h f$ of f to the grid. This follows from the observation that the corresponding perturbation in the right-hand side of (2.3) is of second order as seen from the estimate

$$|(f_h - R_h f, v_h)_h| = \left|\sum_{j=0}^{N}\left(\int_{x_{j-1/2}}^{x_{j+1/2}} f(x)\,\mathrm{d}x - \frac{h_{j-1}+h_j}{2} f(x_j)\right)\overline{v}_j\right|$$
$$\leqslant C\left(\sum_{j=0}^{N-1} h_j^4 \|f\|^2_{H^2(I_j)}\right)^{1/2} \|P_h v_h\|_1,$$

which can be proved in the same way as (3.10) and (3.11).

If $f \in W^{1,\infty}(0, L)$ and $g \in W^{2,\infty}(0, L)$, the quantity $(f_h R_h g - (fg)_h, v_h)_h$ arising from taking out the factor g from the integral can be bounded similarly. The proof of this fact is based on a Taylor expansion and an integration by parts:

$$\int_{x_{j-1/2}}^{x_{j+1/2}} gf\,\mathrm{d}x = \int_{x_{j-1/2}}^{x_{j+1/2}} (g(x_j) + g'(x_j)(x - x_j))f\,\mathrm{d}x + S_j$$
$$= g(x_j)\int_{x_{j-1/2}}^{x_{j+1/2}} f\,\mathrm{d}x + \frac{g'(x_j)}{8}[h_j^2 f_{j+1/2} - h_{j-1}^2 f_{j-1/2}]$$
$$- \frac{g'(x_j)}{2}\int_{x_{j-1/2}}^{x_{j+1/2}} (x - x_j)^2 f'\,\mathrm{d}x + S_j.$$

After multiplication by $\overline{v}_j$ and a summation, the contribution from the last two quantities has already the desired bound while this is also seen for the expression coming from the term in square brackets by a summation by parts.

REMARK 3.5 If pointwise evaluation of f is used in place of the integral mean value (1.7), the scheme is not second-order convergent if only $f \in H^1(0, L)$ is assumed. This happens to be already the case for equidistant grids and the norm of the error $e_h := P_h(R_h u - u_h)$ measured in $L^2(0, L)$ only. A proof of this fact can be based on the uniform boundness principle. Consider the problem

$$-u'' = f, \quad u(0) = u(L) = 0,$$

which is uniquely solvable in $H^3(0, L)$ for $f \in H^1(0, L)$. Assume that the scheme is second-order accurate for $f \in H^1(0, L)$. This implies that the sequence of maps

$$E_h\colon H^1(0, L) \ni f \to h^{-2} e_h \in L^2(0, L), \quad h \in H,$$

is pointwise bounded (here the symbol h is also used for the uniform step-size). Hence, it is also uniformly bounded. It is well known that the approximate solution u_h can be written in the form

$$u_h(x_j) = h\sum_{k=1}^{N-1} g(x_j, x_k) f(x_k), \quad j = 0, \ldots, N,$$

where g is the Green's function of the continuous problem. Thus, $e_h(x_j)$ is the quadrature error in approximating the integral

$$u(x_j) = \int_0^L g(x_j, y) f(y)\,\mathrm{d}y,$$

with the trapezoidal rule. It is known from the Euler–MacLaurin formula or also easily obtained by partial integration that

$$e_h(x_j) = \sum_{k=1}^{N-1} \int_{x_k}^{x_{k+1}} (x_{k+1/2} - y)(g(x_j, \cdot) f)'(y)\,\mathrm{d}y.$$

For given $h \in H$, choose $f = f_h$, with $f_h(0) = 0$ and f_h is defined by its derivative, where f_h' is the step function with values in $\{-1, 1\}$ and changing sign at each point x_k and $x_{k+1/2}$. It is not difficult to verify that $\| f_h\|_1$ is uniformly bounded while $\|E_h f_h\|_0 \to \infty$ $(h \in H)$, contradicting the uniform boundness of $\{E_h\}_H$.

REMARK 3.6 The sesquilinear form $a_h(\cdot, \cdot)$ in (2.1) is obtained from that in (2.4) by applying the midpoint rule to the first term and the trapezoidal rule to the remaining terms. The term corresponding to $(bv)'$ can also be obtained by first interpolating bv_h and then applying the midpoint rule, i.e. it has also the presentation

$$\sum_{j=0}^{N-1} h_j (P_h(bv_h))_{j+1/2}(P_h\overline{w}_h)'_{j+1/2}.$$

If, instead, the trapezoidal rule is applied to the first term $(av', w')_0$, the resulting finite difference scheme differs from (1.4) in that $a_{j+1/2}$ is replaced by $(a_j + a_{j+1})/2$. It is also superconvergent, assuming the additional regularity $a \in C^1[0, L]$. Note that the discrete inner product $(\cdot, \cdot)_h$ is not equal to the L^2 inner product $(\cdot, \cdot)_0$ when applied to linear splines. So the results of Theorem 3.1 are not a simple perturbation of the case without quadrature that could be analysed with the aid of Strang's Lemma. Such a situation would indeed arise by applying at least third-order accurate quadrature formulas to the continuous variational problem.

4. Numerical results

We present numerical results for the problem (see Carey & Humphrey, 1979)

$$-\left(\left(\frac{1}{\alpha} + \alpha(x - \overline{x})^2\right) u'\right)' = 2 + (2\alpha(x - \overline{x}))(\arctan(\alpha(x - \overline{x})) + \arctan(\alpha\overline{x})) \qquad (4.1)$$

subject to Dirichlet boundary conditions $u(0) = u(1) = 0$ which has the solution

$$u(x) = (1 - x)(\arctan(\alpha(x - \overline{x})) + \arctan(\alpha\overline{x})),$$

where $\overline{x}$ and α are parameters.

In our first example, we choose $\overline{x} = 0.36388$ and $\alpha = 1$. Figure 1 shows the numerical solution on 500 random meshes ($N - 1$ points placed in $(0, 1)$ at random), where N ranges from 500 to 3000.

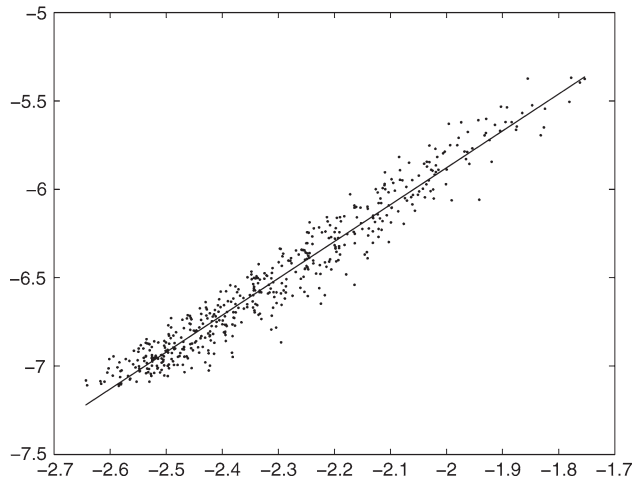

FIG. 1. $\log(\|e_h\|_1)$ versus $\log(h_{\max})$.

The logarithm of the norm of the error $e_h := P_h(R_h u - u_h)$ is plotted versus the logarithm of the maximum step-size. This kind of presentation has been also used in Manteuffel & White (1986). The straight line is the least-squares fit to the points and has the slope 2.085, which confirms the estimates given in Theorem 3.1.

In the second example, we take $\alpha = 100$ and $\alpha = 1000$. For large α, the solution has an interior layer in the neighbourhood of $x = \overline{x}$. We solve (4.1) with meshes that are adapted to the singular nature of the problem. Let us point out that our method is not especially designed for solving problems with layers for which schemes with some upwinding incorporated are in place (e.g. Kopteva & Stynes, 2001; Kopteva, 2001, and the references given there). But, imposing certain mesh restrictions, even central difference schemes have been used (see Kopteva, 1999) in this context. For our purpose, (4.1) provides a suitable test example that leads to strongly non-uniform meshes to illustrate the behaviour of our method with regard to that aspect.

Table 1 shows results for adaptive meshes. These meshes have a fixed number of $N + 1$ nodes and are initially uniform. The nodes are moved by numerically equidistributing the error terms in (3.14), i.e. equidistributing the quantities

$$\ell_j \approx h_j^5 C_j^{1/2}, \quad j = 0, \dots, N-1, \tag{4.2}$$

with

$$C_j := |au'''|^2_{j+\frac{1}{2}} + |(bu)''|^2_{j+\frac{1}{2}} + |(cu)''|^2_{j+\frac{1}{2}} + |(cu)'|^2_{j+\frac{1}{2}}. \tag{4.3}$$

Somewhat differently from (3.14), we have used in (4.2) the square root of the derivative terms which damps the influence of steep portions of the solution and turned out to give better results for not so large values of N. In Table 1, the rate of convergence is computed using the numerical solutions corresponding to meshes with $N + 1$ and $2(N + 1)$ points. We observe that asymptotically the rate of convergence is approximately 2. Equidistribution of the mesh based on the arc length leads to similar numerical results.

TABLE 1 *Equidistribution based on* (4.2)

	$\alpha = 100$				$\alpha = 10000$			
N	$h_{\max}$	$h_{\min}$	$\|e_h\|_1$	Rate	$h_{\max}$	$h_{\min}$	$\|e_h\|_1$	Rate
32	2.306×10^{-1}	2.408×10^{-3}	1.419×10^{-1}	1.98	2.997×10^{-1}	3.449×10^{-5}	3.301×10^{0}	2.06
64	1.524×10^{-1}	1.271×10^{-3}	3.592×10^{-2}	1.71	1.955×10^{-1}	1.822×10^{-5}	7.892×10^{-1}	1.93
128	1.023×10^{-1}	6.300×10^{-4}	1.101×10^{-2}	2.17	1.679×10^{-1}	9.133×10^{-6}	2.069×10^{-1}	1.99
256	6.675×10^{-2}	3.406×10^{-4}	2.442×10^{-3}	1.93	1.520×10^{-1}	4.538×10^{-6}	5.219×10^{-2}	1.99
512	4.428×10^{-2}	1.701×10^{-4}	6.410×10^{-4}	1.92	1.234×10^{-1}	2.236×10^{-6}	1.311×10^{-2}	1.96
1024	2.926×10^{-2}	8.502×10^{-5}	1.692×10^{-4}	1.92	9.168×10^{-2}	1.105×10^{-6}	3.371×10^{-3}	1.99
2048	1.926×10^{-2}	4.250×10^{-5}	4.485×10^{-5}	1.85	6.366×10^{-2}	5.518×10^{-7}	8.459×10^{-4}	1.92
4096	1.264×10^{-2}	2.125×10^{-5}	1.246×10^{-5}	1.93	3.898×10^{-2}	2.931×10^{-7}	2.233×10^{-4}	2.00
8192	8.284×10^{-3}	1.063×10^{-5}	3.276×10^{-6}	1.97	2.545×10^{-2}	1.471×10^{-7}	5.571×10^{-5}	1.99
16384	5.424×10^{-3}	5.313×10^{-6}	8.357×10^{-7}	—	1.634×10^{-2}	7.322×10^{-8}	1.400×10^{-5}	—

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