Channel Assignment and Weighted Coloring

Colin McDiarmid
Department of Statistics, University of Oxford, 1 South Parks Road, Oxford OX1 3TG, United Kingdom

Bruce Reed
Equipe Combinatoire, CNRS, Paris, France

In cellular telephone networks, sets of radio channels (colors) must be assigned to transmitters (vertices) while avoiding interference. Often, the transmitters are laid out like vertices of a triangular lattice in the plane. We investigated the corresponding weighted coloring problem of assigning sets of colors to vertices of the triangular lattice so that the sets of colors assigned to adjacent vertices are disjoint. We present a hardness result and an efficient algorithm yielding an approximate solution.

Keywords: radio channel assignment; cellular communication system; hexagonal cells; triangular lattice; weighted coloring; approximation algorithm

1. INTRODUCTION AND STATEMENT OF RESULTS

A basic problem concerning cellular telephone networks is to assign sets of radio channels (frequency bands) to transmitters so as to avoid unacceptable interference. The number w_v of bands demanded at transmitter v may vary between transmitters. We assume that the transmitters are located at vertices of a triangular lattice (with hexagonal cells) in the plane: this pattern is often used as it gives good coverage – see, for example, [10, 13]. We assume also that adjacent vertices must not be assigned the same channel, so as to avoid interference. There are more refined versions of this "channel assignment problem"; see, for example, [9, 17], in which we insist on a minimum separation between channels assigned to two transmitters depending on the proximity of the transmitters, but we consider only the most basic case here. For related work, see [1].

Let us recall some basic definitions. A (proper) *coloring* of a graph G is an assignment of a color to each vertex so that adjacent vertices receive distinct colors. Equivalently, it is a partition of the vertices into stable sets, where a set of vertices in G is *stable* (or independent) if no two are adjacent.

The channel assignment problem described above is a "weighted coloring" problem on the triangular lattice, where the weights correspond to the demands. A *weight vector* for a graph G is a nonzero vector w of nonnegative integers w_v indexed by the vertices v of G. Given a graph G and a weight vector w, a *weighted coloring* of the pair (G, w) is a family of stable sets with multiplicities such that each vertex v is in w_v of these sets. The least value of the total number of sets (counting multiplicities) for which there is such a coloring is the *weighted chromatic number* of (G, w). A weighted coloring may also be thought of as an assignment to each vertex v of a set of w_v distinct colors such that adjacent vertices receive disjoint sets of colors. The weighted chromatic number is then the least number of colors used in such a coloring. The *weighted coloring problem* is to find a weighted coloring using "few" colors.

There is a natural graph G_w associated with a pair (G, w) as above, obtained by replacing each vertex v by a complete graph on w_v vertices. Weighted colorings of the pair (G, w) correspond to usual vertex colorings of the graph G_w, and the weighted chromatic number of (G, w) is the chromatic number $\chi(G_w)$ of G_w.

We are interested here in weighted colorings of finite induced subgraphs of the triangular lattice graph, as this corresponds precisely to the basic channel assignment problem described above. This lattice graph may be described as follows. The vertices are all integer linear combinations $x\mathbf{p} + y\mathbf{q}$ of the two vectors $\mathbf{p} = (1, 0)$ and $\mathbf{q} = (\frac{1}{2}, \frac{\sqrt{3}}{2})$: thus, we may identify the vertices with the pairs (x, y) of integers. Two vertices are adjacent when the Euclidean distance between them is 1. Thus, each vertex (x, y) has the six neighbors $(x \pm 1, y), (x, y \pm 1), (x + 1, y - 1), (x - 1, y + 1)$ (see Fig. 1).

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Correspondence to: C. McDiarmid; e-mail: cmcd@stats.ox.ac.uk; B. Reed, email: reed@ecp6.jussieu.fr

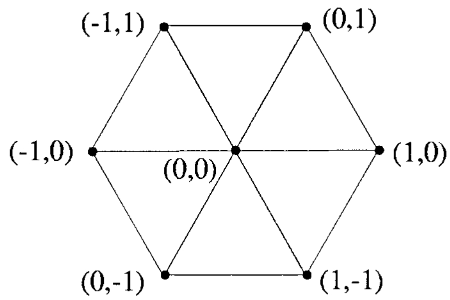

FIG. 1. The neighbors of $(0, 0)$.

We present two theorems, one a hardness result and one a result on the algorithmic approximation. Proofs of these results are given in the next section. First, however, we make two preliminary comments:

(a) If we fix a (finite) graph G and take as input a weight vector w for G, then an optimal weighted coloring can be found in polynomial time by using integer programming, since there are only a finite number of stable sets in G — see, for example, [8]. This comment is of theoretical rather than of practical interest.

(b) Recall that a graph G is *perfect* if for each induced subgraph H of G the chromatic number $\chi(H)$ equals the maximum number $\omega(H)$ of vertices in a complete subgraph of H. If a graph G is perfect, then so is the replicated graph G_w for any weight vector w, and, further, an optimal weighted coloring for (G, w) can be found in polynomial time by using the ellipsoid method – see [8]. If G is bipartite, for example, if it is a finite subgraph of the square or hexagonal lattice, then things are even easier – see Lemma 1 below. Of course, finite subgraphs of the triangular lattice graph need not be perfect–see the remarks at the end of this section.

Theorem 1. *It is NP-complete to determine, on input a set F of pairs of integers determining an induced subgraph G of the triangular lattice graph together with a corresponding weight vector w, if the graph G_w is 3-colorable.*

Given an input as above, it is, of course, easy to find the maximum size $\omega(G_w)$ of a complete subgraph of G_w in polynomial time, since each clique in G has at most three vertices.

Theorem 2. *There is a polynomial time combinatorial algorithm which, on input a set F of pairs of integers determining an induced subgraph G of the triangular lattice graph together with a corresponding weight vector w, finds a weighted coloring of (G, w) which uses at most $\frac{4\hat{\omega}+1}{3}$ colors, where $\hat{\omega} = \omega(G_w)$ (and which uses, at most, $3n$ distinct stable sets, where $n = |F|$ is the number of vertices in G).*

The algorithm is quite simple and practical. It has a distributed phase, in which it constructs color sets similar to the color sets of the 3-coloring of the triangular lattice graph and then a tidy-up phase which corresponds to coloring a forest. By using the algorithm, we can find quickly a weighted coloring for an induced subgraph of the triangular lattice such that the number of colors used is no more than about $4/3$ times the corresponding clique number of G_w and, hence, is no more than about $4/3$ times the optimal number. Further, by Theorem 1, we cannot guarantee to improve on the ratio $4/3$, assuming that $P \neq NP$.

However, perhaps we are being pessimistic. In typical radio channel assignment problems, the maximum number of channels demanded at a transmitter may be quite large. For example, the "Philadelphia problem" described in [2] involves a 21-vertex subgraph of the triangular lattice with demands ranging from 8 to 77 (although it also has constraints on the colors of vertices at distances up to 3, and so it is not a simple weighted coloring problem). Perhaps, we can improve on the ratio $4/3$ if there are large demands?

We note that the 9-cycle C_9 is an induced subgraph of the triangular lattice graph. Further, for any positive integer k, if we start with a C_9 and replicate each vertex k times, we obtain a graph with clique number $2k$ and chromatic number $\lceil \frac{9k}{4} \rceil$. Is this ratio $\frac{9}{8}$ of chromatic number to clique number asymptotically the worst (greatest) possible? This question has sparked off much further work [5, 6, 14], but is still not resolved.

2. PROOFS

Proof of Theorem 1. Suppose that we are given a planar graph G with each vertex degree at most 4. We show how to find, in polynomial time, an induced subgraph F of the triangular lattice and a set D of vertices of F, such that the graph F' obtained from F by duplicating each of the vertices in D has a 3-coloring if and only if G has. But it is NP-complete to determine if such a graph G has a 3-coloring [3, 4], and so this will complete the proof.

Let H denote the subgraph induced by the vertices at a graph distance at most 3 from a fixed vertex in the triangular lattice graph. Then, the infinite face of H is bounded by a regular hexagon. The six extreme points of this hexagon are the "contact points" of H. Observe that in any 3-coloring of H all the contact points must have the same color. Observe also that if P is an odd-length path in which every second internal vertex is doubled then in any 3-coloring of P the terminal vertices must have different colors.

The graph F and set D will be as follows. For each vertex v of G, there will be a copy H_v of the hexagon H, and all these subgraphs H_v will be suitably far apart. For each edge $e = \{u, v\}$ of G, there will be an odd-length path P_e between one of the contact points of H_u and one of the contact points of H_v. All these paths P_e will be completely disjoint, and for each internal vertex, its only neighbors in F will be the two on the path. The set D will contain every second internal vertex of each path P_e (starting from one end).

It is easily seen from the observations above that if we can construct such an induced subgraph F of the triangular lattice then it will do the trick, that is, G is 3-colorable if and only if the duplicated graph F' is. Apart from making the path lengths odd, this is a standard idea and we omit the details.

To ensure that all these path lengths are odd, we proceed as follows: We remark first that, for any two contact points of H, there are induced paths of both parities within H between these contact points. We consider the graph G' obtained from G by subdividing each edge exactly once. Thus, each edge e of G corresponds to two edges e' and e'' and a vertex v_e of G'. We construct H_x and P_f as above for each vertex x of G' and edge f of G', without insisting that the path lengths be odd. As remarked, standard techniques will do this. Now, for each edge e of G, we construct the path P_e from $P_{e'}$, $P_{e''}$ and a path of the appropriate parity in H_{v_e} between the appropriate two contact points of H_{v_e}. ■

Before giving the main part of the proof of Theorem 2, let us consider the weighted coloring of bipartite graphs.

Lemma 1. *Let G be a bipartite graph with n vertices and with a weight vector w. Then, we may obtain an optimal weighted coloring for (G, w) as follows.*

First, find a proper 2-coloring of G, partitioning the vertices into the two sets A and B, and determine $\hat{\omega} = \omega(G_w)$. To specify an optimal weighted coloring in terms of assigning sets of colors to the vertices, we use the colors $\{1, 2, \ldots, \hat{\omega}\}$, let $T_v = \{1, \ldots, w_v\}$ for $v \in A$, and let $T_v = \{\hat{\omega} - w_v + 1, \ldots, \hat{\omega}\}$ for $v \in B$. In $O(n \log n)$ arithmetic operations, we may describe this coloring in terms of stable sets with multiplicities, using at most n distinct stable sets.

Proof. Clearly, the sets T_v specify an optimal weighted coloring, since if $u \in A$ and $v \in B$ are adjacent, then $w_u \le \hat{\omega} - w_v$ and so the sets T_u and T_v are disjoint.

To describe this weighted coloring in terms of stable sets with multiplicities, list the distinct values in

$$\{w_v : v \in A\} \cup \{\hat{\omega} - w_v : v \in B\} \cup \{0, \hat{\omega}\}$$

in increasing order as $x_0, x_1, \ldots, x_k$. Note that k is at most the number n of vertices of G. For each $j = 1, \ldots, k$, let

$$S_j = \{v \in A : w_v \ge x_j\} \cup \{v \in B : w_v \ge \hat{\omega} - x_{j-1}\},$$

and

$$m_j = x_j - x_{j-1}.$$

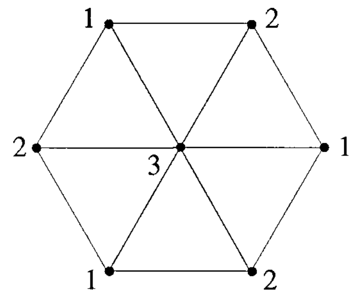

FIG. 2. The 3-coloring.

Then, each set S_j is stable, since $x_j + \hat{\omega} - x_{j-1} > \hat{\omega}$. Now, consider these stable sets S_j with the multiplicities m_j, which sum to $\hat{\omega}$. If $v \in A$ with $w_v = x_j$, then v is in the sets $S_1, \ldots, S_j$ and so v is covered $\sum_{i=1}^{j} m_i = x_j - x_0 = w_v$ times. If $v \in B$ with $w_v = \hat{\omega} - x_j$, then v is in the sets $S_{j+1}, \ldots, S_k$ and so v is covered $\sum_{i=j+1}^{k} m_i = x_k - x_j = w_v$ times. It is now clear that the sets S_j with the multiplicities m_j form an optimal weighted coloring of (G, w), and, indeed, they describe the "same" coloring as before. ■

Proof of Theorem 2. We first compute the 3-coloring f of G with values 1, 2, 3, where for vertex (x, y) we have $f(x, y) = x + 2y \pmod 3$ (see Fig. 2). Then, we compute $\hat{\omega} = \omega(G_w)$ and set $k = \lfloor \frac{\hat{\omega}+1}{3} \rfloor$. The algorithm now proceeds in two stages, which use disjoint sets of colors.

In the first stage, we use the $3k$ colors (i, j) for $i = 1, 2, 3$ and $j = 1, \ldots, k$. For each vertex v, we compute the value m_v, which is the maximum value of w_x over the neighbors x of v with $f(x) = f(v) + 1 \pmod 3$. Note that these are the right neighbor, the up-left neighbor, and the down-left neighbor if they are present. If v has no such neighbors, we set $m_v = 0$. We then compute the value $r_v = \min\{w_v - k, k - m_v\}$. We assign colors $(f(v), 1), \ldots, (f(v), \min\{k, w_v\})$ to vertex v, and if $r_v > 0$, then we assign the r_v further colors $(f(v) + 1, k - r_v + 1), \ldots, (f(v) + 1, k)$ (which will not appear on any neighbor of v). The colors used so far correspond to at most $2n$ distinct stable sets, as in Lemma 1 above.

Now, let U denote the set of vertices v of G still not fully colored after stage 1. Then, $v \in U$ if and only if $w_v > \max\{k, 2k - m_v\}$, and in this case, the

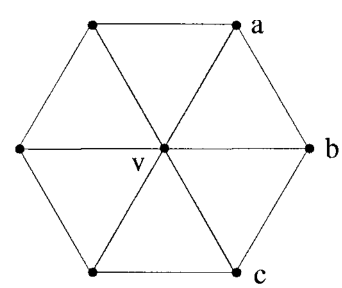

FIG. 3. The neighbors of v.

number of colors still to be assigned to v is $w'_v = w_v - \max\{k, 2k - m_v\}$. Let us use U also to denote the corresponding induced subgraph of G. In the second stage, we use new colors to complete the coloring of the vertices in U.

Note that $w'_v \leq w_v + m_v - 2k \leq \hat{\omega} - 2k$ for each vertex $v \in U$, and $w'_u + w'_v \leq \hat{\omega} - 2k$ for each adjacent pair of vertices $u, v \in U$. Also, the graph U has no triangles, since $w_v \geq k + 1$ for each $v \in U$. Hence, the replicated graph $U_{w'}$ has a clique number at most $\hat{\omega} - 2k$. We shall see that the graph U is acyclic and, thus, bipartite, and so by Lemma 1 above, we can quickly find a weighted coloring for (U, w') as required, using at most $\hat{\omega} - 2k$ colors and corresponding to at most n distinct stable sets. Then, putting the two stages together, we will have a weighted coloring for (G, w) using a total of at most

$$3k + \hat{\omega} - 2k = \hat{\omega} + k \leq (4\hat{\omega} + 1)/3$$

colors and corresponding to, at most, $3n$ distinct stable sets.

To complete the proof, we must show that the graph U is acyclic. Order the vertices in U so that, if in the embedding in the plane u is to the right of v, then u appears before v. Consider any vertex v in U. In the full triangular lattice, the vertex v has an up-right neighbor a, a right neighbor b, and a down-right neighbor c, where a and b are adjacent and b and c are adjacent (see Fig. 3). These are the only possible neighbors of v in U which occur earlier in the ordering: we shall see that at most one of them is in U. We have already seen that U has no triangles, so it remains only to show that we cannot have both a and c in U (and b not in U). Suppose then that a and c are in U, and let $s = \min\{w_a, w_c\}$, so that $s \geq k + 1$. Of course, $w_v + s \leq \hat{\omega}$, since, for example, v and a are adjacent. Further, if $m_v > 0$ and x is a neighbor of v at which m_v is attained, then v, x and either a or c form a triangle in G. Hence, $w_v + m_v + s \leq \hat{\omega}$ and so

$$1 \leq w'_v \leq w_v + m_v - 2k \leq \hat{\omega} - s - 2k \leq \hat{\omega} - 3k - 1 \leq 0,$$

a contradiction. ∎

For results related to Theorem 2 above, see [11, 12, 15, 16].

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