

# $L^p$ SPECTRAL THEORY OF HIGHER-ORDER ELLIPTIC DIFFERENTIAL OPERATORS

E. B. DAVIES

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1. Introduction

In this review we describe the  $L^p$  spectral theory of self-adjoint operators acting in either  $L^2(\mathbf{R}^N)$  or  $L^2(\Omega)$ , where  $\Omega$  is a region in  $\mathbf{R}^N$ , and satisfying the following list of conditions. We start with the expression

$$Hf(x) := \sum_{|\alpha| \leq m, |\beta| \leq m} (-1)^{|\alpha|} D^\alpha \{a_{\alpha, \beta}(x) D^\beta f(x)\}, \tag{1.1}$$

where  $a(x)$  is a bounded measurable complex self-adjoint matrix-valued function,  $\alpha, \beta$  are multi-indices, and  $|\alpha| := \alpha_1 + \dots + \alpha_N$ . We say that  $H$  is homogeneous of order  $2m$  if the only non-zero terms are those with  $|\alpha| = |\beta| = m$ . The domain of such an operator need not contain the space  $C_c^\infty$  of smooth functions with compact support in  $\mathbf{R}^N$  or in  $\Omega$ , so we define  $H$  using quadratic form techniques. Namely, we define the quadratic form  $Q$  on  $C_c^\infty$  by

$$Q(f) := \int_{\Omega} \sum_{|\alpha| \leq m, |\beta| \leq m} a_{\alpha, \beta}(x) D^\beta f(x) \overline{D^\alpha f(x)} dx. \tag{1.2}$$

This form is closable, and the domain of the closure is the Sobolev space  $W_0^{m,2}(\Omega)$  under the following conditions. Let  $S$  denote the set of multi-indices  $\alpha$  such that  $|\alpha| = m$ , and let  $\tilde{a}(x)$  denote the matrix obtained from  $a(x)$  by restricting the indices to  $S$ . We assume that there exist constant non-negative real symmetric matrices  $\{c_{i, \alpha, \beta}\}_{\alpha, \beta \in S}$  for  $i = 1, 2$ , such that  $c_1 \leq \tilde{a}(x) \leq c_2$  in the sense of matrices for all  $x \in \Omega$  and such that the operators

$$(-1)^m \sum_{|\alpha|=|\beta|=m} c_{i, \alpha, \beta} D^{\alpha+\beta} \tag{1.3}$$

are elliptic in the standard sense, that is,

$$\sum_{|\alpha|=|\beta|=m} c_{i, \alpha, \beta} \zeta^{\alpha+\beta} \geq b_i |\zeta|^{2m} \tag{1.4}$$

for some  $b_i > 0$  and all  $\zeta \in \mathbf{R}^N$ .

The term ‘elliptic operator’ below will always refer to the self-adjoint operator  $H$  associated as described in [48, Section 4.2] with the closure of a form  $Q$  satisfying all of the above conditions. This method of approach amounts to the choice of what are called (zero) Dirichlet boundary conditions (DBC). In classical terms these would be the conditions  $D^\alpha f(x) = 0$  for all  $x \in \partial\Omega$  and all  $|\alpha| \leq m-1$ , but since we base our calculations upon the quadratic form rather than the operator, we do not use this fact. It should be noted that the operator  $(\Delta^2)_{\text{DBC}}$  satisfies the boundary conditions  $f = \partial f / \partial n = 0$  on  $\partial\Omega$ , and has quite different spectrum from  $(\Delta_{\text{DBC}})^2$ .

If the coefficients of  $H$  are sufficiently smooth, then it can be written in the form

$$Hf(x) := \sum_{|\gamma|=0}^{2m} b_\gamma(x) D^\gamma f(x), \tag{1.5}$$

where

$$b_\gamma(x) := \sum_{\alpha+\beta=\gamma} a_{\alpha,\beta}(x) \tag{1.6}$$

provided  $|\gamma| = 2m$ . In this case the self-adjoint operator is the Friedrichs extension of its restriction to  $C_c^\infty$ . However, for operators with measurable coefficients, the theory of the two classes of operators proceeds separately.

Although there is by now a significant body of general theory for elliptic operators satisfying Dirichlet boundary conditions, we warn the reader that other boundary conditions cannot be treated by modifications of the same methods unless one makes much stronger conditions on the boundary. Even for  $H = -\Delta$ , the study of Neumann boundary conditions is far harder because of the lack of domain monotonicity of the eigenvalues. Maz'ya [127, Section 4.10.1] gives necessary and sufficient ‘geometrical’ conditions on the region for the Neumann Laplacian to have compact resolvent; here ‘geometrical’ means conditions involving bounds on capacities, etc., of all compact subsets. We refer to [63, 94, 100] and other papers cited there for examples in which essential spectrum and even absolutely continuous spectrum can occur specifically because of the choice of Neumann boundary conditions. We also mention [76, 62] for new results concerning the spectral theory of acoustic waveguides, for which the physically appropriate boundary conditions are the Neumann ones.

Our goal is to review results obtained within the last fifteen years on the spectral theory of such operators. We concentrate on those matters which do not naturally fall within the scope of the theory of pseudodifferential operators by avoiding regularity conditions on the coefficients and boundaries except to the extent that it has been discovered that these are really necessary for the validity of theorems of interest. It will be clear that many of the results which we describe are not in a final state. Nevertheless, much is known about the overall shape of the subject, and it seems desirable to make this available to non-experts. We focus on those matters which relate to  $L^p$  properties of the operators, and of the semigroups which those operators generate.

The second-order theory is so vast that it is already impossible to write a survey article of the type which we attempt here for higher-order operators. In order to avoid any misunderstanding, we state now that we shall mention second-order operators only to the extent that this throws some light on the higher-order theory. While some second-order results do extend to higher-order operators, in other cases the higher-order theory can be quite different, and there is at present no single clear guiding principle to distinguish between these.

Our interest in non-smooth coefficients and boundaries is not motivated by a love of pathologies. Although this review is of a pure-mathematical nature, there are important problems of applied mathematics which depend upon such matters. The study of the bulk properties of materials with periodic microscopic structure depends upon taking an asymptotic limit in which the periodic cell diameter vanishes, and the derivatives of the coefficients of the relevant PDEs become infinite. To control the limit, it is essential that one has estimates independent of bounds on those derivatives. The study of a random mixture of two materials involves considering operators whose coefficients have one of two values, but in a manner which varies randomly in

space; it is not helpful to regularize the coefficients so as to make them smooth. The study of stresses at inwardly pointing cracks in solid mechanics is clearly not going to be helped by smoothing out the crack [113]. The analysis of ‘Moffatt’ vortices of incompressible viscous fluids [134, 137] depends upon some rather interesting properties of the biharmonic equation at corners. Finally, a substantial part of the theory of non-linear PDEs depends upon recasting them as linear equations whose coefficients depend upon the solution and using the properties of the linear equations under the weakest possible assumptions on the coefficients [84, 138].

Finally, we mention that Chapter 3 of Kenig [106] contains a long list of open problems closely related to the subject matter of this review. The present article contains the solution of a small number of these, but much remains to be done!

With a very few exceptions, I have adopted a cut-off date of 1 September 1996 for entries in this review. I could have continued indefinitely adding new contributions, but the review would then never have been published!

Among the many colleagues who have helped me to learn about this complicated subject over the last five years, I should particularly like to thank Professors Maz’ya, Safarov, Solomyak and Vassiliev. I have also benefited from a large number of comments about errors and omissions in an early draft of this review, as a result of which it is certainly better than it was at first. The referee’s criticisms have been particularly valuable. I must, however, take responsibility for instances in which I have failed to refer to important work through ignorance or a failure of memory. Finally, I acknowledge support under EPSRC grant number GR/K00967.

## 2. Eigenvalues and eigenfunctions

2.1 *Spectral asymptotics.* If  $\Omega \subseteq \mathbf{R}^N$  is bounded and  $H$  is an elliptic operator of order  $2m$ , then  $H$  has discrete spectrum  $\{\lambda_n\}_{n=1}^\infty$ , where we write the eigenvalues in increasing order and repeat them according to multiplicity. If  $H$  has smooth coefficients and  $\partial\Omega$  is piecewise smooth, then the leading term in the asymptotic eigenvalue distribution is independent of the boundary conditions. Putting

$$N(\lambda) := \max\{n: \lambda_n \leq \lambda\} \quad (2.1.1)$$

and  $H = -(\Delta)_{\text{DBC}}$ , Weyl (1911–1913) obtained

$$N(\lambda) = \frac{|\Omega|}{4\pi} \lambda + o(\lambda) \quad \text{as } \lambda \rightarrow \infty \quad (2.1.2)$$

for bounded  $\Omega$  with piecewise smooth boundary in two dimensions (and the three-dimensional analogue), thus completing a partial proof of Rayleigh dating from 1903. For  $H = (\Delta^2)_{\text{DBC}}$  in two dimensions, Courant (1922) obtained

$$N(\lambda) = \frac{|\Omega|}{4\pi} \lambda^{1/2} + o(\lambda^{1/2}). \quad (2.1.3)$$

It should be mentioned that determining the spectral asymptotics of  $N(\lambda)$  as  $\lambda \rightarrow \infty$  is a much harder problem than that of the short-time asymptotics of  $\text{tr}[e^{-Ht}]$ . The leading terms of the two asymptotic expansions are often related to each other by Tauberian theorems. A two-term asymptotic expansion of  $N(\lambda)$  of the form of (2.1.4) below implies a corresponding two-term asymptotic expansion of  $\text{tr}[e^{-Ht}]$  by taking Laplace transforms, but the converse is false. Moreover, there are cases in which the

operator  $H$  has discrete spectrum even though the trace is infinite for all  $t > 0$ . The higher-order asymptotics of  $N(\lambda)$  must be investigated by an analysis of the periodic trajectories of the region or manifold, in conjunction with the wave equation, which yields much sharper information than that available from the heat equation.

There is a long history of improvements of the above results, providing sharper estimates of the remainder for more general elliptic operators [35, 42, 135, 147]. For a general elliptic operator with smooth coefficients acting in a region (or  $C^\infty$  manifold) with smooth boundary, two-term spectral asymptotics of the form

$$N(\lambda) = a\lambda^{N/2m} + b\lambda^{(N-1)/2m} + o(\lambda^{(N-1)/2m}) \tag{2.1.4}$$

were obtained in increasing generality by Duistermaat-Guillemin, Melrose, Ivrii and Vassiliev between 1975 and 1987, finally completing the proof of a conjecture of Weyl dating back to 1913. In this formula,

$$a = (2\pi)^{-N} \int_{\sigma(x, \xi) \leq 1} dx d\xi, \tag{2.1.5}$$

where the principal symbol  $\sigma$  of  $H$  is the function on the cotangent bundle given by

$$\sigma(x, \xi) := \sum_{|\alpha|+|\beta|=m} a_{\alpha, \beta}(x) \xi^\alpha \xi^\beta. \tag{2.1.6}$$

The coefficient  $b$  involves an integral of the spectral shift function over the boundary of the region or manifold, and the existence of such an expansion depends on the measure of the set of periodic and dead-end trajectories being zero with respect to a certain canonical measure on the co-sphere bundle [169; 155, Theorem 1.6.1]. This condition is satisfied generically for planar regions with smooth boundary. However, it has been verified in detail only for some very special regions in two dimensions [69, 155]. If the condition is not satisfied, then the second term of the asymptotic expansion has a different form, described in [154, 155].

Explicit comparisons of the one- and two-term spectral asymptotic formulae with the actual eigenvalues of a variety of more or less exactly solvable mechanical problems are given in [155, Chapter 6]. It is clear that in these examples, the two-term asymptotic formula is extremely accurate if  $N(\lambda) \leq 90$ . Nevertheless, an error estimate would be of value since it would imply bounds on the multiplicities of eigenvalues.

It is not surprising that if one allows the coefficients of  $H$  to be measurable rather than smooth, then such detailed results are not possible. However, one can prove the formula

$$N(\lambda) = a\lambda^{N/2m} + o(\lambda^{N/2m}) \tag{2.1.7}$$

in great generality, even when the highest-order coefficients lie in only certain  $L^p$  spaces. We content ourselves with the statement that this formula holds when  $\Omega$  is any bounded region in  $\mathbf{R}^N$  and the operator is uniformly elliptic with bounded measurable coefficients, and subject to DBC as usual. The proof involves approximating the operator in a sufficiently controlled manner by a sequence of operators with bounded smooth coefficients [29, 30]. A simpler exposition of the second-order case is given in [32].

For generalized (that is, higher-order) Schrödinger operators of the form

$$Hf(x) := (-\Delta)^m f(x) + V(x)f(x) \tag{2.1.8}$$

with non-negative potential  $V$ , Maz'ya [127, Theorem 12.5.5] has necessary and

sufficient conditions on the potential for the resolvent to be compact. It is clear that the leading term of the spectral asymptotics must involve the whole symbol rather than just the principal term. If the potential increases in a suitably regular manner at infinity, then one has

$$\begin{aligned} N(\lambda) &\sim (2\pi)^{-N} \int_{a(x, \xi) < \lambda} dx d\xi \\ &= (2\pi)^{-N} v_N \int_{\mathbf{R}^N} (\lambda - V(x))_+^{N/2m} dx \end{aligned} \quad (2.1.9)$$

as  $\lambda \rightarrow \infty$ , where  $a(x, \xi) := |\xi|^{2m} + V(x)$ , and  $v_N$  is the volume of the unit ball in  $\mathbf{R}^N$ . See [153, 146] for details.

There is also extensive work on the negative eigenvalues of generalized Schrödinger operators for which the potential is negative and converges to zero at infinity. Much of this concentrates on inequalities satisfied by  $N(\lambda)$  for all  $\lambda < 0$ , and the results are necessarily simpler if  $N > 2m$  [31, 33]. We finally refer to [140] for Lieb–Thirring type inequalities satisfied by the negative eigenvalues of such generalized Schrödinger operators.

Finally, there are cases of degenerate or irregular elliptic operators for which even this formula fails, possibly because the integral is infinite, and for which other formulae have been developed [119, 120, 31, 33, 28]. As a very special case of this, the spectral asymptotics of the Laplacian in horn-shaped regions such as

$$\{(x, y) : 0 < x < \infty, y \in \mathbf{R}^{N-1}, |y| < f(x)\}, \quad (2.1.10)$$

where  $f$  is positive and slowly decreasing with  $\lim_{x \rightarrow \infty} f(x) = 0$ , depends very sensitively upon the shape of the horn and on the boundary conditions [100, 119, 120, 27, 28, 63].

*2.2 The isoperimetric problem.* In 1877 Rayleigh made two conjectures, the first of which was solved by Faber [79] in 1923; this was the matter of proving that the plane region of unit area for which the Laplacian subject to Dirichlet boundary conditions has minimum principal eigenvalue is a circular disc, and is known as the isoperimetric problem for a membrane. The higher-dimensional analogue was solved by Krahn [115, 116] in 1925/26. The second conjecture was that among all clamped plates with a given area, the circular disc again has the minimal principal eigenvalue  $\lambda_1(\Omega)$ . The clamped plate may be identified mathematically with the biharmonic operator  $\Delta^2$  subject to Dirichlet boundary conditions. Rayleigh's second problem was solved by Szegő in 1950 subject to the plausible (but eventually disproved) hypothesis that the ground state eigenfunction of every bounded region must be positive [151, 166]. Nadirashvili [139] recently solved the isoperimetric problem in two dimensions, extending earlier partial results of Talenti. Ashbaugh and Benguria solved the corresponding problem in three dimensions by a related method which depends heavily on the properties of Bessel functions [9]. In a later paper, Ashbaugh and Laugesen investigated the same problem in all dimensions [11]. They proved Rayleigh's conjecture up to a multiplicative factor  $d_n$ , where  $d_4 \doteq 0.95$  and  $d_n \rightarrow 1$  as  $n \rightarrow \infty$ . These two papers and the recent review [10] contain a host of other results and open questions on this and the related buckling problem.

2.3 *The Rellich inequality.* The simplest form of the Rellich inequality is

$$\frac{9}{16} \int_0^\infty \frac{|f|^2}{x^4} dx \leq \int_0^\infty |f''|^2 dx, \tag{2.3.1}$$

valid for all  $f \in C_c^\infty(0, \infty)$ . Owen [143] deduced from this that for any region  $\Omega$  in  $\mathbf{R}^N$  with Lipschitz boundary, there exists  $c > 0$  such that

$$c \int_\Omega \frac{|f|^2}{d(x)^4} dx \leq \int_\Omega |\Delta f|^2 dx \tag{2.3.2}$$

for all  $f \in C_c^\infty(\Omega)$ , where

$$d(x) := \inf\{|x - y| : y \in \partial\Omega\}. \tag{2.3.3}$$

The same holds for many regions with self-similar fractal boundaries [143]. In particular, Owen obtained the sharp value  $c = 9/16$  for any convex set  $\Omega$ . There are analogous inequalities for  $(-\Delta)^m$  for all  $m \geq 2$ . One can use (2.3.2) to obtain a lower bound on the first eigenvalue of  $\Delta^2$  in terms of the inradius of the region, but better results follow by combining the quadratic form inequality

$$0 \leq (\Delta_{\text{DBC}})^2 \leq (\Delta^2)_{\text{DBC}} \tag{2.3.4}$$

with known lower bounds on the first eigenvalue of the Laplacian [16]. However, (2.3.2) cannot be proved in this manner.

There are many papers on the Rellich inequality, but mostly in an  $L^2$  setting. One approach is based upon the boundedness of operators of the type  $f(Q)g(P)$ , where

$$f(Q)g(P)\phi(x) := \frac{1}{(2\pi)^N} \int_{\mathbf{R}^N} e^{ix \cdot \xi} f(x)g(\xi)\hat{\phi}(\xi) d\xi. \tag{2.3.5}$$

If  $N > 2m$ , then by suitable choices of  $f, g$  in the above formula, one may prove that there exists  $c$  such that

$$\int_{\mathbf{R}^N} |V||f|^2 dx \leq c \int_{\mathbf{R}^N} \{(-\Delta)^m f\}\bar{f} dx \tag{2.3.6}$$

for all  $f \in W^{m,2}$ , provided  $V \in L_{\text{weak}}^{N/2m}$  [161, 33]. There are several ways of characterizing the non-negative measures  $\mu$  for which

$$\int_{\mathbf{R}^N} |f|^2 d\mu \leq c \int_{\mathbf{R}^N} \{(-\Delta)^m f\}\bar{f} dx \tag{2.3.7}$$

holds for some constant  $c$  and all  $f \in W^{m,2}$  [127, Theorems 2.5.2, 12.5.1; 132].

Sharp constants for such Rellich type inequalities in  $L^p$  were obtained in [109] for Euclidean space using rearrangement inequalities, and in [59] by a method applicable to Riemannian manifolds. In particular, the sharp constant  $c$  for

$$\int_{\mathbf{R}^N} \frac{|u(x)|^p}{|x|^{\beta+2mp}} dx \leq c \int_{\mathbf{R}^N} \frac{|\Delta^m u(x)|^p}{|x|^\beta} dx \tag{2.3.8}$$

is known for all  $N, \beta, m, p$ . Such inequalities play an important role in determining the sharp relative bounds of various classes of potentials with respect to powers of the Laplacian in  $L^p$ ; see [109] for details.

2.4. *Numerical studies of the rectangular plate.* The numerical study of eigenvalues of the biharmonic operator in various simple plane regions goes back to

Weinstein (1937) but has been repeatedly re-investigated as computing facilities have improved. The only regions for which the eigenvalues can be computed in closed form are the circular disc and the annuli of various internal and external radii; see Fichera [80] for tabulations. In particular, the eigenvalues of rectangles cannot be computed by separation of variables.

We look in more detail at one particular problem, that of computing the minimum eigenvalue  $\lambda_1$  and the corresponding eigenfunction  $f_1$  of the square or rectangular clamped plates  $\Omega := (0, h) \times (0, 1)$ , where  $0 < h < \infty$ . This may be formulated precisely as finding solutions  $f: \Omega \rightarrow \mathbf{R}$  and  $\lambda \in \mathbf{R}$  of  $\Delta^2 f = \lambda f$  where  $f$  satisfies the ‘Dirichlet’ boundary conditions  $f = \partial f / \partial n = 0$  on  $\partial\Omega$ .

Early results for the square clamped plate are reviewed by Fichera [80, pages 74–79], who gives bounds ranging from

$$1294.93307 \leq \lambda_1 \leq 1294.93502 \quad (2.4.1)$$

to

$$4.446 \times 10^5 \leq \lambda_{45} \leq 5.233 \times 10^5. \quad (2.4.2)$$

The development of interval arithmetic programming languages such as Pascal-XSC by Kulisch and his group is transforming the status of numerical analysis. In particular, the final word on the eigenvalue problem for the square plate is due to Wieners [173], who computed the first 100 eigenvalues. He obtained rigorous enclosures ranging from

$$\lambda_1 = 1294.9339_4^9 \quad (2.4.3)$$

to

$$\lambda_{100} = 2.3_{65}^{93} \times 10^6. \quad (2.4.4)$$

His proof is based upon a boundary homotopy method, that is, the behaviour of the quadratic form

$$Q_t(f) := \|\Delta f\|_{L^2(\Omega)}^2 + t \left\| \frac{\partial f}{\partial n} \right\|_{L^2(\partial\Omega)}^2 \quad (2.4.5)$$

for  $0 \leq t < \infty$ , where  $\Omega := (0, 1)^2$ . The form was restricted to a finite element subspace, and the errors involved were controlled using interval arithmetic.

The clamped rectangular plate  $(0, h) \times (0, 1)$ , has been studied independently in two complementary papers, by Behnke [23] and Owen [143]. Behnke used interval analysis to obtain rigorous enclosures of the eigenvalues for the first ten eigenvalues for  $0 \leq h \leq 5$  and for a somewhat more general class of constant-coefficient fourth-order elliptic operators than the biharmonic operator. He observed the phenomenon of ‘curve veering’: the eigenvalue curves almost cross, but on closer inspection may be seen to change direction just before the crossings occur. A convincing explanation of this phenomenon remains to be given. His method is described in the next section and in [24], where a similar problem with different boundary conditions is described in detail.

Owen [143] obtained rigorous but less precise bounds on various spectral properties uniformly over the entire range  $1 \leq h < \infty$ . Among his results were the definitions of two explicit functions  $\mu_1(h)$  and  $\nu_1(h)$  such that the smallest eigenvalue  $\lambda_1(h)$  satisfies

$$\mu_1(h) \leq \lambda_1(h) \leq \nu_1(h) \leq 1.0072\mu_1(h) \quad (2.4.6)$$

for all such  $h$ , which allowed him to prove the asymptotic formula

$$\lambda_1(h) = c^4 + bh^{-2} + O(h^{-3}), \quad (2.4.7)$$

where  $c \doteq 4.73004$  is the first positive solution of the equation  $\cos(c) \cosh(c) = 1$ , and  $b$  is also given explicitly.

2.5 *The ground state.* One may ask whether the ground state eigenfunction  $f_1$  of a higher-order elliptic operator in a bounded region subject to DBC is positive, and whether it has multiplicity one. There seems to be no general principle for dealing with this question. Even for the biharmonic operator  $H := \Delta^2$  in a bounded plane region with smooth boundary, the question is not simple. For a ball, the answer is positive by a direct computation. However, for the annulus  $\{x \in \mathbf{R}^2: \varepsilon < |x| < 1\}$ , the ground state is degenerate, and the ground state eigenfunctions are non-positive if  $\varepsilon < (762.36)^{-1} = 0.00131$ . The proof depends upon separating variables in polar coordinates, and then using delicate properties of Bessel functions, but for  $\varepsilon = 0$ , the punctured disc, a simpler approach was described in Coffman and Duffin [37], who also gave references to the earlier literature.

Bauer and Reiss [22] were the first to report that for the square clamped plate,  $f_1$  has nodal lines near the corners; they drew attention to the problems this causes for Szegő's approach to the isoperimetric problem. Their calculation was, however, not for the biharmonic operator but for a discretization. Their conclusion was questioned at the time because the effect was so small and the result so surprising, but it has since been proved to have been entirely correct.

Wiener [174] used interval arithmetic to prove that  $f_1$  becomes negative near the corner  $(0, 0)$ . Specifically, he proved that on the diagonal,  $f$  vanishes near  $x = y = 0.03$ , and that at  $x = y = 0.02$ , it has a value  $\sim -10^{-6}$  subject to the normalization  $\|u\|_{W_0^{2,2}(\Omega)} = 1$ , or  $u(\frac{1}{2}, \frac{1}{2}) \sim 0.07$ . His method involves approximating  $f_1$  in the Sobolev space  $W^{2,2}$  with controlled errors using a finite element method, and then using the explicit Sobolev embedding constant of  $W^{2,2}$  into  $C(\bar{\Omega})$  to prove that  $f_1$  is indeed negative where the approximate eigenfunction is most negative.

Finally, Owen [143] has proved that for the clamped rectangular plate  $(0, h) \times (0, 1)$ , the negative part of the eigenfunction is small in the sense that

$$0 < \frac{\|f_1^-\|_2}{\|f_1\|_2} \leq 0.0484 \tag{2.5.1}$$

uniformly for all  $1 \leq h < \infty$ , and that the ratio is of order  $h^{-1/2}$  as  $h \rightarrow \infty$ .

This question may also be approached in a more theoretical manner, yielding information which is stronger in some respects and weaker in others. Suppose that a two-dimensional region  $\Omega$  has a right-angled corner, and consider the behaviour of the ground state eigenfunction  $f_1$  at this corner. Coffman [36, Theorem 9.1] used the Mellin transform to prove that  $f_1$  changes sign infinitely often as one approaches the corner. He also showed that  $f_1$  can be expanded as a linear combination of standard functions each depending upon a complex solution  $p$  of

$$1 + p + \cos(\pi p/2) = 0. \tag{2.5.2}$$

Numerical calculations show that this equation has solutions

$$-1, \quad 1.740 \pm 1.119i, \quad 5.845 \pm 1.682i, \quad 9.886 \pm 1.970i, \tag{2.5.3}$$

etc., the first of which is not relevant to this problem because the corresponding eigenfunction does not lie in  $W_0^{2,2}(\Omega)$ . Putting  $p := 1.740 + 1.119i$ , one would expect  $f_1$  to be of the form

$$f_1(x, x) \sim \text{Re}\{cx^{p+2}\} \tag{2.5.4}$$

near the origin unless the first coefficient in its expansion vanishes.

Kozlov, Kondrat'ev and Maz'ya [112] obtained similar results for eigenfunctions of a class of constant-coefficient higher-order elliptic operators in higher space dimensions at conical vertices with sufficiently small angles. They also proved by a perturbation theory argument that there exists a convex plane region with smooth boundary for which the ground state is of variable sign. Coffman and these three authors used a well-established asymptotic expansion of the eigenfunction around the corner, and noted that the leading power of  $r$  is complex. The difficult parts of their papers consisted of proving that the eigenfunction cannot vanish to infinite order at the corner, since this would render invalid any conclusions based upon use of the asymptotic expansion.

Both [36] and [112] yield the fact that at a two-dimensional vertex with a total internal angle of  $\theta$ , the indicial equation (2.5.2) for  $p$  above should be replaced by

$$p + 1 + \frac{\sin((p + 1)\theta)}{\sin(\theta)} = 0. \tag{2.5.5}$$

This has a positive real root only if the total internal angle is larger than  $0.8128\pi \doteq 146.30^\circ$ . As  $\theta \rightarrow 0$ , the real and imaginary parts of the solutions increase indefinitely. The significance of the indicial equation (2.5.5) and of the angle  $146.30^\circ$  was, however, appreciated in fluid mechanics as long ago as 1949 [64]. A famous article of Moffatt [137] explains how the solutions of the biharmonic equation  $\Delta^2 f = 0$  describe the behaviour of an incompressible viscous fluid with Reynolds number zero at a sharp corner, and describes in considerable detail the connection between the complex solutions of (2.5.5) and ‘Moffatt’ eddies at corners, which have been experimentally observed. Meleshko [134] has recently compared various theoretical and numerical approaches to this problem in the fluids literature.

An analysis of the bottom eigenvalue  $\lambda_1$  and eigenfunction  $f_1$  of the unit square has also been given in [34], where Brown and Davies independently adopted the same approach as Behnke [23], but without using interval arithmetic. That is, both [34] and [23] considered an approximate eigenfunction on  $(0, 1)^2$  of the form

$$f_n(x, y) = \varphi(x)\varphi(y)P_n(x, y), \tag{2.5.6}$$

where

$$\varphi(x) = x^2(1 - x)^2, \tag{2.5.7}$$

$$P_n(x, y) = \sum_{r=0}^n \sum_{s=0}^n \beta_{rs} D_r^\alpha(x) D_s^\alpha(y), \tag{2.5.8}$$

$$D_n^\alpha(t) = C_n^\alpha(2t - 1), \tag{2.5.9}$$

and where the Gegenbauer polynomials  $C_n^\alpha(x)$  are defined on  $(-1, 1)$  for  $\alpha = 9/2$  as in [1, page 773]. Calculations confirm that  $f_1$  is slightly negative near enough to the origin, the numerical data being similar to those obtained by Wieners [174].

*2.6 Stability properties.* Let  $\Omega$  be a bounded region in  $\mathbf{R}^N$ , and let  $H_n$  be a sequence of elliptic operators all acting in  $L^2(\Omega)$  and satisfying a uniform ellipticity condition. More precisely, we assume that

$$c_1 \leq \tilde{a}_n(x) \leq c_2 \tag{2.6.1}$$

for all integers  $n$  and all  $x \in \Omega$ , where  $\tilde{a}(x)$  was defined in Section 1 and  $c_1, c_2$  are matrices satisfying (1.4). We discuss conditions under which the operators  $H_n$  converge in the norm resolvent sense to a limit operator  $H$  of the same form.

If the highest-order coefficients of  $H_n$  and  $H$  coincide, then the difference is a lower-order perturbation, and the convergence can often be controlled by standard theorems about relatively bounded operator or quadratic form perturbations. From now on we assume that  $H_n$  and  $H$  are all homogeneous of order  $2m$ , so that only top-order coefficients are involved, and that  $a_n(x) \rightarrow a(x)$  pointwise as  $n \rightarrow \infty$ .

Sophisticated estimates yielding comparisons between the spectral asymptotics of  $H_n$  and  $H$  are given in [29, 30, 32]. The approach of Barbatis [18, 19, 20] concentrates on the comparison of small eigenvalues and the corresponding resolvent operators. It depends upon a fundamental theorem of Meyers [136] to the effect that if  $2m = 2$ , then there exists  $p > 2$  such that  $(H + 1)^{-1} L^p \subseteq W^{1,p}$ , and a higher-order analogue of this result in [15, 20]. Barbatis [20] also obtained estimates of the type

$$\|(H_n + 1)^{-1} - (H + 1)^{-1}\|_p \leq c \|a_n - a\|_q \tag{2.6.2}$$

for certain values of  $p, q$ , where the norms on the left-hand side are Schatten trace ideal norms, and those on the right-hand side are  $L^q$  norms. These estimates imply norm resolvent convergence if the coefficients converge pointwise—subject to the condition (2.6.1).

There are, however, important cases in which one has operator convergence even when the coefficients do not converge pointwise anywhere. If the operator  $H_\gamma$  has coefficients which are periodic with period  $\gamma$ , then there often exists an operator  $H$  with constant coefficients obtained as the  $\gamma \rightarrow 0$  limit of  $H_\gamma$ . This subject is studied systematically in [145], and the one-dimensional case is explored in more detail in [51]. Suffice it to say that the coefficients of the limit operator are not simply the averages of the coefficients of the periodic approximants, and that the procedure for determining them is both complicated and dimension-dependent.

*2.7 The ideal column.* Lest it be thought that higher-order one-dimensional equations are of little mathematical interest, I should mention the problem of the ideal column, ‘solved’ incorrectly by Lagrange and others before its eventual resolution in [45]. If  $a$  is a positive continuous function on  $[0, 1]$ , then the buckling load for a column of variable cross-sectional area  $a$  and length 1 is, after normalization, obtained by finding the smallest eigenvalue of the differential equation

$$(a(x)^2 f''(x))'' + \lambda f''(x) = 0, \tag{2.7.1}$$

subject to various possible boundary conditions at  $x = 0, 1$ . The eigenvalue may, alternatively, be defined by

$$\lambda(a) = \inf_f \left\{ \frac{\int_0^1 a(x)^2 |f''|^2 dx}{\int_0^1 |f'|^2 dx} \right\}, \tag{2.7.2}$$

where the class of  $f$  involved depends upon the boundary conditions. The ideal, or strongest, column is defined by that function  $a$  for which  $\lambda(a)$  takes the largest possible value, subject to the column having a fixed amount of material:

$$\int_0^1 a(x) dx = 1. \tag{2.7.3}$$

The history of this problem and its final, surprisingly complicated, solution are described in [44, 45].

### 3. Solving elliptic equations

If  $\Omega$  is a region in  $\mathbf{R}^N$ , then the solution of elliptic equations of the form  $Hf = g$ , subject to various boundary conditions, can be approached from several different points of view. One can seek to prove  $L^p$  properties of  $H^{-1}$  by operator-theoretic methods, incorporating the boundary conditions into the definition of the domain of  $H$ . One can try to find properties of the Green function of the operator by a variety of methods. Finally, one can use the techniques of harmonic analysis to study the Dirichlet problem for the most general possible boundary values. The literature on these problems is enormous, particularly for second-order operators, and we can do no more than provide some indication of the type of results possible, with sources for further reading.

The Green function of  $H$  is defined to be the integral kernel of the bounded or unbounded operator  $H^{-1}$ , assuming that  $H \geq 0$  and that 0 is not an eigenvalue of  $H$ .

3.1 *One-dimensional results.* Let  $a(x)$  be a positive continuous, or even measurable, function, and put

$$Hf := \frac{d^2}{dx^2} \left( a(x) \frac{d^2 f}{dx^2} \right) \quad (3.1.1)$$

on a bounded interval  $(\alpha, \beta)$  subject to DBC. The Green function of  $H$  may be written explicitly as a combination of the four independent solutions of the homogeneous equation. It may then be seen that the Green function is positive on  $(\alpha, \beta)^2$  for any choice of the coefficient function  $a(x)$ . Owen [143] has proved that the operator  $Hf = f'''' - \alpha f''$  subject to DBC on a bounded interval has a positive Green function for all positive constants  $\alpha$ .

The book by Kamke [103] develops the notion of Wronskian for such operators, and contains the solution of a variety of exactly soluble problems in one dimension. Information about the Wronskian and computation of the  $M$ -matrix for fourth-order ODEs may be found in [26]. A complete spectral and scattering analysis of constant-coefficient ordinary differential operators on the half line subject to arbitrary self-adjoint boundary conditions has been given by Holst [98].

On the numerical side, there are a variety of packages for solving one-dimensional eigenvalue and initial value problems, and SLEUTH is particularly suitable for regular fourth-order operators. The computation of very high eigenvalues has been aided by the development of an oscillation theory for fourth-order ODEs in [86], which starts by transforming the equation into a first-order system and using a variant of the Morse index theorem, but also develops further ideas relevant to the actual computational problem.

The forthcoming book of Kozlov and Maz'ya [114] contains an account of the theory of ordinary differential equations with constant and non-constant operator-valued coefficients. This far-reaching generalization of the classical theory can be applied to PDEs by choosing one particular space variable to be considered separately from the others.

3.2 *Constant-coefficient operators.* The Green function of a constant-coefficient elliptic operator  $H \geq 0$  of homogeneous order  $2m$  acting in  $L^2(\mathbf{R}^N)$  is a locally  $L^1$

function of  $(x - y)$  for all  $m, N$ . If  $N > 2m$ , then it is of the form  $G(x - y)$ , where  $G$  is a homogeneous distribution of order  $2m - N$  which is smooth away from the origin [99, Theorem 7.1.20]. If  $2m = 2$ , then it must be a positive function, but for higher-order operators it may or may not be positive. It is not obvious that this is a matter of great importance, but Maz'ya and Nazarov [128] have proved that non-positivity of the Green function in  $L^2(\mathbf{R}^N)$  implies the existence of bounded regions  $\Omega$  for which the solutions of the equation  $Hf = g$  in  $L^2(\Omega)$  subject to DBC may diverge at corner singularities, and for which there may not exist an  $L^p$  theory for large enough  $p$ . There is currently no useful characterization of the symbols of those operators for which the Green function is not positive, but a variety of examples are given in [128] and [57]. In particular, if  $2m \geq 4$ , then the operator

$$(-\Delta_r)^m + (-\Delta_s)^m, \tag{3.2.1}$$

where  $-\Delta_r$  denotes the Laplacian acting in  $L^2(\mathbf{R}^r)$ , has non-positive Green function if  $r \geq 1$  and  $s = 2m + 3$ , so that  $N \geq 2m + 4$  [57]. An example with non-convex symbol may be constructed in dimension  $N = 2m + 3$  [57].

*3.3 Green functions of bounded regions.* If  $\Omega$  is a bounded region in  $\mathbf{R}^N$ , then an elliptic operator  $H$  in  $L^2(\Omega)$  satisfying DBC has compact inverse operator  $H^{-1}$ . If this operator has a positive integral kernel  $G$ , then the bottom eigenvalue of  $H$  is of multiplicity 1, and the corresponding eigenfunction is positive.

The Green function  $G(x, y)$  of the polyharmonic operator  $(-\Delta)^m$  subject to DBC in the unit ball  $B$  of  $\mathbf{R}^N$  is known explicitly (Boggio, 1905), and is positive. An alternative expression for  $G(x, y)$ , which is also visibly positive, may be found in [91]. The formula and explicit estimates of the rate of vanishing of  $G(x, y)$  and its partial derivatives as  $x \rightarrow \partial B$  are given by Grunau and Sweers [90]; they also obtain similar results for the polyharmonic operator subject to a sufficiently small lower-order perturbation. Typical of their results is the uniform estimate in terms of  $d(x) := 1 - |x|$  for  $N < 2m$ , namely

$$G(x, y) \sim d(x)^{m-N/2} d(y)^{m-N/2} \min \left\{ 1, \frac{d(x)^{N/2} d(y)^{N/2}}{|x-y|^N} \right\}.$$

In 1909, Hadamard noted that Boggio's conjecture, that the biharmonic Green function of any bounded region is positive, is false for the annulus

$$\{x \in \mathbf{R}^2: \varepsilon < |x| < 1\} \tag{3.3.1}$$

if  $\varepsilon > 0$  is small enough. An explicit but complicated expression for the Green function for the annulus has just been obtained by Engliš and Peetre [75]. Their formula enables them to show that the Green function is non-positive for all  $0 < \varepsilon < 1$ . The corresponding conjecture for convex regions was proved false for the infinite strip by Duffin [66] long before it was known that the ground state of the unit square is of variable sign. An interesting historical discussion of this problem was given by Hedenmalm [92], who also obtained explicit Green functions for certain weighted Laplace operators in two dimensions.

The ground state results of Subsection 2.5 imply that the Green function is non-positive for any  $n$ -sided polygon with  $n \leq 10$ . Presumably it is positive for the regular 11-gon.

**3.4 The Poisson problem.** If  $\Omega$  is a bounded region in  $\mathbf{R}^N$  and  $H$  is an elliptic operator of order  $2m$  acting in  $L^2(\Omega)$  subject to Dirichlet boundary conditions, then the Poisson problem concerns the solution of  $Hf = g$  for a given function  $g$  on  $\Omega$ . If  $H$  is second order and  $g$  is bounded, then  $f$  is also bounded, but whether the same is true for higher-order elliptic operators depends upon the regularity of the coefficients, the space dimension  $N$  and the order  $2m$ .

If one assumes minimal regularity properties of the highest-order coefficients, for example uniform or Hölder continuity, then the situation for higher-order elliptic operators in  $L^p$  is well understood. Unfortunately, the formulation of the results in [3, 4, 83, 138] involves so many definitions that we must leave the interested reader to consult those works. Henceforth we consider only elliptic operators of order  $2m \geq 4$  with bounded measurable coefficients.

Consider first the case  $N < 2m$ , and suppose that  $\Omega$  is bounded but with no particular boundary regularity properties. The  $L^2$  domain of the operator is contained in  $W_0^{m,2}(\Omega)$ , and a standard Sobolev embedding theorem [2, Theorem 5.4, part III] states that this is contained in a space  $C_0^\alpha(\bar{\Omega})$  of Hölder continuous functions on  $\bar{\Omega}$  which vanish on the boundary  $\partial\Omega$ . Therefore  $g \in L^2$  implies  $f \in C^\alpha(\bar{\Omega})$ . It also follows from results in [52] that  $H^{-1}$  is a bounded operator on  $L^p$  for all  $p \in [1, \infty]$ , and that it has a continuous integral kernel which vanishes on the boundary of  $\Omega$ . Therefore  $g \in L^p$  for any  $1 \leq p \leq \infty$  implies  $f \in L^p$ , the solution  $f$  being consistent if  $g$  lies in two different  $L^p$  spaces. The results for  $N = 2m$  are very similar, but the methods of proof are harder [14, 15, 74].

If  $N > 2m$ , then the behaviour of solutions of the Poisson problem depends on the value of  $p$  and on the boundary properties of  $\Omega$ . If  $q_c \leq p \leq p_c$ , where

$$q_c := 2N/(N+2m), \quad p_c := 2N/(N-2m), \quad (3.4.1)$$

then it follows from [52] as above that one has  $L^p$  elliptic regularity in the sense that  $g \in L^p$  implies  $f \in L^p$ , but the situation for  $p$  outside this range is not completely understood even if  $H$  has constant coefficients, which we assume for the rest of this paragraph. If  $\partial\Omega$  is smooth, then  $g \in C^\infty(\bar{\Omega})$  implies  $f \in C^\infty(\bar{\Omega})$  by standard elliptic regularity theorems. If  $m = 1$  and the boundary of  $\Omega$  is regular in the sense that it is determined by a differentiable function such that the modulus of continuity  $\omega(t)$  of its gradient satisfies  $\omega(t)/t \in L^p(0, 1)$ , then  $g \in L^p(\Omega)$  implies  $f \in W^{2,p}(\Omega)$  [131]; there is also an improvement and extension of this result to higher-order operators. However, if  $2m \geq 4$  and  $\partial\Omega$  is piecewise smooth, then  $f$  may be unbounded at a corner of  $\partial\Omega$  even if  $g \in C^\infty(\bar{\Omega})$ . Our present understanding of the subject depends upon the use of the methods of Subsection 3.6, and upon the construction of particular examples in space dimensions  $N \geq 2m + 3$  [128, 57].

**3.5 The Dirichlet problem.** In this section we consider only the polyharmonic equation  $\Delta^m f = 0$  in a bounded region  $\Omega \subset \mathbf{R}^N$  with a Lipschitz boundary. The  $L^p$  Dirichlet problem  $DP_p$  is to prove the existence of a solution to this equation with prescribed boundary values in  $L^p$ . Somewhat more precisely, we require that almost everywhere boundary values of  $D^\alpha f$  equal a specified function  $g_\alpha \in L^p(\partial\Omega)$  for all  $|\alpha| \leq m-1$ . The boundary functions  $g_\alpha$  are required to satisfy certain consistency conditions which we do not spell out. The second-order theory has been developed to an extraordinary level of sophistication, and we can do no better than to refer to [106] for an account.

If  $2m \geq 4$ , the results for this problem are both complicated and incomplete. For any  $N, m$ , and any  $\Omega$  with a Lipschitz boundary, there exists  $\varepsilon > 0$  for which  $DP_p$  is soluble for all  $2 - \varepsilon < p < 2 + \varepsilon$  [47, 170], but for any  $p < 2$ ,  $m \geq 1$  and  $N \geq 2$ , there exists a region  $\Omega$  for which  $DP_p$  is insoluble. For  $N = 2, 3$ ,  $DP_p$  is soluble for all  $m \geq 1$  and  $p > 2$ , but for  $N \geq 4$ , the situation is quite different even for the biharmonic operator [148, 149]; in particular, a certain weak maximum principle holds for the biharmonic equation in bounded domains in  $\mathbf{R}^N$  with Lipschitz boundaries when  $N = 2, 3$ , but not for  $N \geq 4$ . The most complete current results, for general higher-order constant-coefficient elliptic operators, may be found in [150]. Counterexamples to some of the conjectures are obtained by considering regions with conical points, as described in the next subsection; see also [113].

**3.6 Conical points.** Many of the crucial examples in the subject depend upon the asymptotic expansion of solutions of elliptic equations at a conical point, a subject initiated by Kondrat'ev, Maz'ya, Plamenevskii and others in the 1960s; see [107, 130]. Let  $\Omega$  be a region in the unit sphere  $S^{N-1}$  of  $\mathbf{R}^N$  which has a non-empty smooth boundary, and define its associated conical region with vertex at the origin by

$$C_{\Omega,r} := \{x: 0 < |x| < r, x/|x| \in \Omega\}. \tag{3.6.1}$$

Let  $H$  be a constant-coefficient elliptic operator of order  $2m$  acting in  $L^2(C_{\Omega,r})$ . When we consider solutions of equations such as  $Hf = \mu f$  in  $C_{\Omega,r}$  below, we shall suppose that  $f \in W^{m,2}(C_{\Omega,r})$  and that  $f$  satisfies any of a range of appropriate boundary conditions on

$$\{x: 0 \leq |x| < r, x/|x| \in \partial\Omega\} \cup \{x: |x| = r, x/|x| \in \Omega\}. \tag{3.6.2}$$

It is known that any solution has an asymptotic expansion, the terms of which are functions of the form

$$f(r\omega) \sim r^\lambda \sum_{s=0}^n \frac{(\log r)^s}{s!} u_{n-s}(\omega) \tag{3.6.3}$$

near the origin [107, 130, 129], where  $\text{Re } \lambda > m - N/2$ . The procedure for determining the possible values of  $\lambda$  and  $n$  is specified in these sources. It is possible, in principle, that a solution has a ‘strong zero’ at the origin, by which one means a solution which vanishes to all orders at the origin, so that all terms in the asymptotic expansion vanish.

There are several conditions on  $H$  which ensure that strong zeros do not exist. In particular, in [110] conditions on the coefficients  $a_r$  of an operator on  $(0, \infty)^2$  of the form

$$H := \prod_{r=1}^{2m} \left( \frac{\partial}{\partial x} - a_r \frac{\partial}{\partial y} \right) \tag{3.6.4}$$

are given which ensure that strong zeros do not exist, but in [111] an operator of this form which does possess strong zeros is presented. In [112] it is shown that  $(-\Delta)^2$  does not have strong zeros in any sector

$$\{z \in \mathbf{C}: 0 < \arg(z) < \phi\} \tag{3.6.5}$$

for  $0 < \phi \leq 2\pi$ . Also obtained in [112] are related results concerning the non-existence of positive solutions for a class of constant-coefficient elliptic operators in higher dimensions with respect to sufficiently sharp conical regions.

There is a particularly complete theory for the Dirichlet problem  $Hf = 0$  when  $H$  is a constant-coefficient homogeneous elliptic differential operator of order  $2m$  acting in a conical region  $C_{\Omega, \infty}$ , surveyed and extended in [113]. For any  $\lambda \in \mathbf{C}$ , one may look for solutions of the (exact rather than asymptotic) form

$$f(r\omega) := r^\lambda \sum_{s=0}^n \frac{(\log r)^s}{s!} u_{n-s}(\omega), \quad (3.6.6)$$

where  $0 < r < \infty$  and  $|\omega| = 1$ . The particular case  $f(r\omega) := r^\lambda u(\omega)$  was studied in detail for the biharmonic operator by Moffatt in 1963 [137]. The equation (3.6.6) leads to auxiliary equations for the sequence of functions  $u_r: \Omega \rightarrow \mathbf{C}$  which depends in a non-linear manner on  $\lambda$ , and which is called an operator pencil. The values of  $\lambda$  for which the equation has non-zero solutions are called eigenvalues, and one need consider only the case  $\operatorname{Re} \lambda > m - N/2$  if one imposes the usual condition that the function should lie in  $W_{\text{loc}}^{m,2}$  at the origin. A solution corresponding to a complex value of  $\lambda$  leads to a function  $f$  which oscillates in sign infinitely often as  $|x| \rightarrow 0$ . If  $N \leq 2m$  or there are no eigenvalues in the strip  $\{\lambda: m - N/2 < \lambda \leq 0\}$ , then it follows that the origin is a regular point of the operator [113]. On the other hand, a solution with negative  $\lambda$  leads to a solution which diverges as  $|x| \rightarrow 0$ , and which fails to lie in  $L_{\text{loc}}^p$  for large enough  $p$ . We have already mentioned such possibilities in Subsection 3.5, and leave the reader to consult the original papers for results concerning the possible location of such eigenvalues in the complex plane.

#### 4. The semigroup $e^{-Ht}$

In the remainder of this review, when we write about a ‘semigroup’  $e^{-Ht}$ , we shall assume that  $0 \leq t \in \mathbf{R}$  unless otherwise stated, and that the operators are bounded in norm on the Banach space being considered, uniformly for  $t$  in any bounded interval. The  $L^p$  behaviour of the semigroup  $e^{-Ht}$ , where  $H$  is an elliptic operator, is a subject in its own right. We assume that  $H$  is elliptic of homogeneous order  $2m$ , and that it acts in  $\mathbf{R}^N$  or in a subregion  $\Omega$  of  $\mathbf{R}^N$  subject to Dirichlet boundary conditions. Analogous results hold in the non-homogeneous case.

If  $2m = 2$  and  $H$  has real coefficients, then the quadratic form of the operator, defined in the Introduction, has domain  $W_0^{1,2}(\Omega)$ , and has the properties

$$Q(|f|) \leq Q(f), \quad Q(0 \vee (1 \wedge f)) \leq Q(f), \quad (4.1)$$

characteristic of a Dirichlet form. These properties imply, by a purely functional analytic argument, that the semigroup  $e^{-Ht}$  extends to a contraction semigroup on  $L^p$  for all  $1 \leq p \leq \infty$  [50, Theorem 1.3.5; 82; 125]. Moreover, there is a Markov process whose sample paths are functions from  $[0, \infty)$  to  $\Omega$  from which the semigroup can be reconstructed. This, and the fact that the semigroup is positivity-preserving, profoundly affects the form of the whole theory.

**4.1 Elliptic systems.** There is an enormous difference between the second-order theory when the coefficient matrix  $a_{\alpha,\beta}(x)$  is real-valued and all other cases. Most of the  $L^p$  problems which we shall encounter for the case  $2m \geq 4$  also occur for  $2m = 2$  if one considers second-order elliptic systems—by which we mean operators acting

on spaces of vector-valued functions. Important and closely related examples of such operators were discovered independently and simultaneously by de Giorgi [85] and Maz'ya [126]; see also [83, page 54]. Let  $H$  act in  $L^2(\mathbf{R}^N, \mathbf{C}^N)$  according to the formula

$$H_2 f(x) := -\Delta f(x) + B^* B f(x), \tag{4.1.1}$$

where  $N \geq 3$  and  $B$  takes  $\mathbf{C}^N$ -valued functions to complex-valued functions according to the formula

$$B f(x) := \sum_{\alpha, i=1}^N \left\{ \mu \delta_{i, \alpha} + \nu \frac{x_i x_\alpha}{|x|^2} \right\} \frac{\partial f_i}{\partial x_\alpha} \tag{4.1.2}$$

for certain real constants  $\mu$  and  $\nu$ . We may also write  $H_2$  in the form

$$(H_2 f)_i = - \sum_{j, \alpha, \beta} \frac{\partial}{\partial x_\alpha} \left\{ a_{\alpha, \beta}^{i, j}(x) \frac{\partial f_j}{\partial x_\beta} \right\}. \tag{4.1.3}$$

We note that the coefficients  $a_{\alpha, \beta}^{i, j}(x)$  are smooth and bounded for  $x \neq 0$ .

It may be shown by a direct computation that there exists a function  $f \in W_{loc}^{1,2}$  such that  $H_2 f = 0$  but with  $f$  unbounded at the origin. By choosing  $\mu$  and  $\nu$  carefully, one can prove [57] that this system fails to have any  $L^p$  elliptic regularity properties for any chosen value of  $p$  outside the interval  $[q_e, p_e]$  where  $p_e, q_e$  are as in (3.4.1).

The same conclusion can be obtained for  $2m = 4$  and  $N \geq 5$  by considering the (scalar) operator  $H_4 := \nabla \cdot H_2 \nabla$ . Higher-order analogues can then be obtained for all  $N > 2m \geq 4$  by pre- and post-multiplying one of the above by (the same) power of  $\Delta$  [57]. These examples underlie our comments in Subsection 4.2.

Similar results have been proved for second-order elliptic operators with complex-valued coefficients [14]. If  $N = 1, 2$ , then Gaussian heat kernel bounds have been proved in [14] for operators with bounded measurable coefficients. It is known [13] that there exists a counterexample to the existence of heat kernel bounds for measurable coefficients if  $N \geq 5$ , but such a counterexample has not yet been found for  $N = 3, 4$ . See also Subsection 6.2.

**4.2 Extensions to  $L^p$ .** If  $N < 2m$ , then the semigroup may be extended to  $L^p$  for all  $1 \leq p < \infty$  even if the coefficients are measurable. If the coefficients are in  $C_b^\infty$ , then the same holds without restriction on the order [108, 152]. It is interesting that for  $2m \geq 4$ , the proofs of these facts depend upon first obtaining appropriate kernel bounds, for example those discussed in the next section, whereas for  $2m = 2$  the  $L^p$  property is used to prove heat kernel bounds. The extension of the results to the case  $N = 2m$  may be found in [14, 15, 74].

If  $N > 2m \geq 4$  and  $p$  lies outside the interval  $[q_e, p_e]$ , then there exists an operator or system  $H$  for which the coefficients are smooth away from a single point of  $\mathbf{R}^N$ , but the semigroup may not be extended to  $L^p$  for any  $t > 0$ . For odd  $m$ , the example constructed in [57] is a system, but for even  $m$ , it is an operator; we do not expect that this is an important distinction. For every individual operator, one may extend the semigroup to  $L^p$  for all  $p$  in an interval properly containing  $[q_e, p_e]$  by the use of a generalized Meyers theorem [15].

It is a long established fact that the semigroup  $e^{-Ht}$  cannot be a contraction semigroup on  $L^1$  if  $2m \geq 4$ . Langer and Maz'ya [118] have shown for a very general class of non-self-adjoint differential operators  $H$  that such a semigroup cannot be a contraction semigroup on  $L^p$  for any  $p \neq 2$  if  $2m \geq 4$ . However, even the issue of uniform boundedness of  $\|e^{-Ht}\|$  in  $L^p$  can cause trouble (if one regards the second-

order case as the standard). If one puts  $H := (\Delta + c)^2$  in  $L^2(\mathbf{R}^N)$  with  $c > 0$ , then  $H \geq 0$  in the  $L^2$  spectral sense. The semigroup  $e^{-Ht}$  is uniformly bounded on  $L^1$  if  $N = 1$ , but the  $L^1$  norms diverge as  $t \rightarrow \infty$  if  $N = 3$ , and probably also for all  $N \geq 2$  [55].

An examination of the proof shows that the above phenomenon is a consequence of the operator not being homogeneous. However, there exists a homogeneous elliptic operator with smooth and uniformly bounded coefficients of any order  $2m \geq 4$  for which the semigroup is not uniformly bounded in norm on  $L^p$ , provided  $p > p_c$  and  $N > 2m$  [57].

**4.3 Generalized Schrödinger operators.** We next turn to the study of generalized Schrödinger operators of the form  $H := H_0 + V$  acting in  $L^p(\mathbf{R}^N)$ , where  $H_0$  is a non-negative homogeneous elliptic operator with constant coefficients, and  $V$  is a non-negative locally  $L^1$  potential. Since  $H$ , defined as a quadratic form sum, satisfies  $H \geq 0$ , it generates a contraction semigroup  $e^{-Ht}$  on  $L^2$ . It is a consequence of the theory of Dirichlet forms that for  $2m = 2$ , this ‘Schrödinger’ semigroup can be extended to a contraction semigroup on  $L^p$  for all  $p \in [1, \infty)$  [50, Theorem 1.3.5]. The integral kernel of  $e^{-Ht}$  is given by the famous Feynman–Kac formula, and the whole theory is profoundly enriched by its connections with Brownian motion [162, 163]. The situation for  $2m > 2$  is much more complicated, and depends upon the dimension and order.

If  $N < 2m$ , then the Schrödinger semigroup is strongly continuous and uniformly bounded on  $L^p$  for all  $1 \leq p < \infty$ . If  $N \geq 2m + 3$ , then there exist  $H_0$ ,  $p < \infty$  and a non-negative smooth potential  $V$  such that the Schrödinger semigroup  $e^{-Ht}$  cannot be extended from  $L^2$  to a strongly continuous semigroup on  $L^p$  [57]. The reason is that if for some  $H_0$  and  $p$  such an extension is possible for all smooth  $V \geq 0$ , then it follows that  $H_0$  has a certain  $L^p$  elliptic regularity property in every subregion  $\Omega$  of  $\mathbf{R}^N$ , subject to DBC; see [57]. It is possible to construct  $H_0$  for which this is false by first proving that the  $\mathbf{R}^N$  Green function of such an operator can be negative; see Subsection 3.2 and [128].

For  $H_0 := (-\Delta)^m$ , the above method cannot be applied to resolve the problem concerning Schrödinger operators, because the  $\mathbf{R}^N$  Green function is clearly positive. The logical argument in [57] flows in only one direction, so no conclusions can be drawn.

If one allows non-negative time-dependent potentials, then there is an example even in one space dimension for which the semigroup is not uniformly bounded in time on  $L^1$  [55]. It is probably possible to find an example for which the semigroup is not extendible to  $L^1$  at all.

**4.4 Kato class potentials.** The so-called ‘Kato class’ potentials have been studied in great detail in a number of papers since the seminal review of [163]. Various generalizations have been introduced, but all restricted to the case in which the unperturbed operator is of second order. Here we consider the corresponding class of potentials  $V$  when  $H_0$  is elliptic of homogeneous order  $2m \geq 4$ . If  $0 < \alpha < N$ , we say that  $V \in K_\alpha$  if

$$\limsup_{\delta \rightarrow 0} \sup_{x \in \mathbf{R}^N} \int_{|x-y| < \delta} |x-y|^{\alpha-N} |V(y)| dy = 0, \quad (4.4.1)$$

so that the usual Kato class corresponds to the choice  $\alpha = 2$ , and for our purposes we need to put  $\alpha := 2m$  and assume that  $N > 2m$ ; the cases for which  $N \leq 2m$  can also be treated in a modified way.

There are two results in [60] which we mention here. The first is an extension of a result of Grinshpun [88] to the higher-order case. The basic assumption, for the rest of this section including (4.4.4), is that the semigroup  $e^{-H_0 t}$  has a heat kernel  $K$  whose absolute value has a Gaussian upper bound. This implies that the semigroup extends to a uniformly bounded semigroup on  $L^p$  for all  $1 \leq p \leq \infty$ . In this case, one can prove that  $V \in K_{2m}$  if

$$\int_{\mathbf{R}^N} |V(x)| |f(x)|^2 dx \leq \varepsilon Q_0(f) + \beta(\varepsilon) \|f\|_2^2 \tag{4.4.2}$$

for all  $f \in C_c^\infty$  and  $\varepsilon > 0$ , where  $\beta$  is a positive decreasing function on  $(0, \infty)$  such that

$$\int_0^\delta \log[\beta(\varepsilon)] d\varepsilon < \infty \tag{4.4.3}$$

for some  $\delta > 0$ . Moreover, under the same assumption, the semigroup  $e^{-Ht}$  extends from  $L^2$  to a holomorphic semigroup on  $L^p$  for all  $1 \leq p < \infty$ .

The second result from [38, 60] concerns the regularizing properties of  $e^{-Ht}$  when  $H := H_0 + V$ , if  $H_0$  is elliptic of homogeneous order  $2m \geq 4$  and  $V$  is a Kato potential; unlike the second-order case, the condition is on  $V$  and not just on its negative part. Under these assumptions, if  $1 \leq p \leq q \leq \infty$ , then  $e^{-Ht}$  is bounded from  $L^p$  to  $L^q$  for all  $t > 0$ , and its norm is bounded by

$$\|e^{-Ht}\|_{q,p} \leq ct^{-N(p^{-1}-q^{-1})/2m} e^{\alpha t} \tag{4.4.4}$$

for some positive constants  $c, \alpha$ . Coulhon and Saloff-Coste [39, 41] have even shown that if  $e^{-Ht}$  is an abstract non-self-adjoint holomorphic semigroup on  $L^p$ , then bounds of the type of (4.4.4) hold if and only if a high enough power of  $H$  satisfies a Galgiardo–Nirenberg type of inequality.

### 5. Heat kernel bounds

If  $H$  is a non-negative self-adjoint operator acting in  $L^2(M, dx)$  for some locally compact space  $M$ , its heat kernel  $K(t, x, y)$  is defined to be a function such that

$$e^{-Ht}f(x) = \int_M K(t, x, y)f(y) dy \tag{5.1}$$

for all  $f \in L^2$ . We require that the above integral is absolutely convergent for all  $f \in L^2$ , all  $x \in M$  and  $t > 0$ . Such a kernel need not exist, and indeed its existence and properties are matters of major importance.

There exists a non-negative potential  $V$  which is regular with respect to  $-\Delta$  in the sense of [171, 142] such that although the heat kernel of  $H := -\Delta + V$  satisfies

$$0 \leq K(t, x, y) \leq \frac{1}{(4\pi t)^{N/2}} \exp\left[-\frac{|x-y|^2}{4t}\right] \tag{5.2}$$

for all  $t > 0$  and  $x, y \in \mathbf{R}^N$ , one has

$$\text{Dom}(H) \cap C_0(\mathbf{R}^N) = \{0\}. \tag{5.3}$$

This implies that if  $f \in L^2$  and  $e^{-Ht}f \in C_0(\mathbf{R}^N)$ , then  $f = 0$ . Hence  $e^{-Ht}$  cannot have the Feller property, and its kernel  $K$  cannot be a continuous function of  $x, y$  for any  $t > 0$ . On the other hand, if  $V$  lies in the Kato class, then continuity and strict positivity of the heat kernel is known, for second-order operators [163].

In spite of the above example, all of the heat kernels introduced below are continuous if they exist at all, and we shall assume this to be part of the definition of a heat kernel. There are (at least) three reasons for being interested in heat kernel bounds. The first is because the evolution equation

$$f'(t) = -Hf(t) \tag{5.4}$$

may be of intrinsic physical interest. While this is abundantly clear when  $H$  is second order, it does not seem to apply to higher-order operators, for which the wave equation is usually the equation of significance.

Secondly, it is of interest to find out to what extent results proved for the heat equation when  $H$  is second order really depend upon the existence of the probabilistic interpretation available in that case. Sometimes proofs which are not apparently probabilistic turn out to have an obscure lemma whose only proof depends upon positivity preservation.

Thirdly, when one is proving results in operator theory, one often needs to show that certain operators are bounded or even compact in  $L^p$ , or Hilbert–Schmidt in  $L^2$ . One of the ways of proving this is to obtain bounds on the integral kernels of these operators. Heat kernel bounds turn out to be an ideal starting point for this kind of result, since one can use them to obtain bounds on various other operators by integration.

*5.1 General properties of heat kernels.* Most of the results below are specific to uniformly elliptic operators acting in  $L^2(\mathbf{R}^N)$ , but the following conclusions can be derived by an extremely general analysis [56]. Suppose that  $K$  is the heat kernel of an abstract non-negative self-adjoint operator  $H$  acting in  $L^2(M, dx)$ , where  $dx$  is a regular Borel measure on a locally compact metric space  $M$ . The function  $t \rightarrow K(t, x, x)$  is always decreasing, log-convex and an analytic function of  $t > 0$ . Bounds on the time derivatives of  $K$  can be obtained by methods of analytic function theory. In particular,

$$\left| \frac{\partial^n}{\partial t^n} K(t, x, y) \right| \leq \frac{n!}{(t-s)^n} K(s, x, x)^{1/2} K(s, y, y)^{1/2} \tag{5.1.1}$$

for all  $n > 0$ ,  $0 < s < t$  and  $x, y \in M$ . In the second-order case, it is also shown in [56] that the long-time behaviour of  $K(t, x, y)$  is independent of  $x$  and  $y$  as far as the order of magnitude is concerned, without any assumptions of bounded geometry or volume doubling conditions.

*5.2 Gaussian upper bounds.* If  $H$  is an elliptic operator acting in  $L^2(\mathbf{R}^N)$ , then it possesses a heat kernel in the sense of the start of this section, and one may obtain a ‘Gaussian’ heat kernel bound when the coefficients are measurable, provided that the dimension  $N$  is lower than the order  $2m$  of the operator [52]. The bound is of the form

$$|K(t, x, y)| \leq c_1 t^{-N/2m} e^{\gamma(t, x, y)}, \tag{5.2.1}$$

where

$$\gamma(t, x, y) := -c_2 d(x, y)^{2m/(2m-1)} t^{-1/(2m-1)} + c_3 t, \tag{5.2.2}$$

and where  $c_i > 0$  and  $d(x, y)$  is some measure of the distance between  $x$  and  $y$ . The proof of this type of upper bound extends to elliptic systems, provided that  $K(t, x, y)$  is interpreted as a matrix-valued function.

The dimensional restriction can be weakened to  $N \leq 2m$  for operators with measurable coefficients, by the method of [14, 15, 74]. However, if one assumes

Hölder continuous or more regular coefficients, then the dimensional restriction becomes entirely unnecessary [108, 152, 13, 14]. Auscher [12] has extended this result to the case of uniformly continuous coefficients by the use of Morrey spaces, but bounded continuous coefficients do not suffice. No such theorem can be proved for  $N > 2m \geq 4$  if the operator has measurable coefficients, because of the examples described in Subsection 4.1. Hieber [96] has proved that for a class of non-self-adjoint holomorphic semigroups acting on  $L^p(\Omega)$ ,  $\Omega \subseteq \mathbf{R}^N$ , the existence of a Gaussian upper bound of the form (5.2.1), (5.2.2) is equivalent to the existence of certain pointwise upper bounds on a high enough power of the resolvent. We continue below to concentrate on the case of measurable coefficients.

The history of the gradual sharpening of such estimates is itself interesting. One finds that  $-c_3$  can be taken as close as one likes to the bottom of the  $L^2$  spectrum, at the cost of increasing the constant  $c_1$ . For constant-coefficient homogeneous operators, one can put  $c_3 = 0$  by a scaling argument. The situation concerning  $c_2$  is much more complicated.

*5.3 Sharp constants.* The obvious choice of  $d(x, y)$  is the Euclidean distance between  $x$  and  $y$ . However, even for the Laplacian this is not always the best choice. If  $H$  is second order and acts in  $L^2(\Omega)$ , where  $\Omega$  is non-convex, then it is better to take  $d$  to be the infimum of the length of curves from  $x$  to  $y$  which stay entirely inside  $\Omega$ . The possibility of doing this was demonstrated in [49] for the Laplacian, with the obvious generalization for other second-order elliptic operators [50, Theorem 3.2.7]. Of course, the choice of  $c_2$  is also important, and the same papers showed that one could take the ‘standard’ choice  $c_2 = 1/4$  for second-order operators.

The situation for higher-order operators is more complex. The sharp constant  $c_2$  was obtained in [21], which used the Euclidean distance; by ‘sharp’ here and below we mean the supremum of all constants for which such a bound exists. The formula is

$$c_2 = (2m - 1)(2m)^{-2m/(2m-1)} \sin\left(\frac{\pi}{4m-2}\right). \tag{5.3.1}$$

Much earlier, Tintarev [167] introduced the Finsler metric associated with the highest-order coefficients to obtain a sharp short-time asymptotic formula for the heat kernel associated with a variable-coefficient elliptic operator, provided that  $x, y$  are sufficiently close.

The Finsler metric is defined by

$$d(x, y) := \sup \{ \xi(x) - \xi(y) : \xi \in \mathcal{E} \}, \tag{5.3.2}$$

where  $\mathcal{E}$  is the class of functions on  $\mathbf{R}^N$  satisfying some smoothness condition together with the crucial inequality

$$\sum_{|\alpha|=|\beta|=m} a_{\alpha,\beta}(x) (D\xi)^\alpha (D\xi)^\beta \leq 1 \tag{5.3.3}$$

for all  $x \in \mathbf{R}^N$ , where

$$(D\xi)^\alpha := \prod_{r=1}^N \left( \frac{\partial \xi}{\partial x_r} \right)^{\alpha_r}. \tag{5.3.4}$$

The Finsler metric reduces to the Riemannian metric associated with the operator if  $2m = 2$ .

Tintarev’s paper does not make any reference to the importance of the notions of convexity and strong convexity for elliptic symbols. Evgrafov and Postnikov [77, 78]

had already shown that strong convexity is essential for the proof of such asymptotic formulae, even for constant-coefficient elliptic operators (the issue does not arise for second-order operators, because all symbols have these properties).

If  $\sigma(\xi)$  is a polynomial symbol of homogeneous order  $2m$ , it is said to be strongly convex if it is of the form

$$\sigma(\xi) := \sum_{\gamma=m} \frac{(2m)!}{\gamma!} u_{\gamma} \xi^{\gamma}, \quad (5.3.5)$$

where  $\{u_{\alpha+\beta}\}_{\alpha,\beta}$  is a positive self-adjoint matrix. The example

$$\sigma(\xi) := \xi_1^8 + 2\mu \xi_1^4 \xi_2^4 + \xi_2^8 \quad (5.3.6)$$

is elliptic if and only if  $\mu > -1$ , convex if and only if  $0 \leq \mu \leq 7$ , but not strongly convex for any  $\mu$ .

Barbatis [17] has recently obtained pointwise upper bounds on the heat kernel for variable-coefficient higher-order elliptic operators with sufficiently differentiable strongly convex elliptic symbols in terms of the Finsler metric for the sharp value of the constant  $c_2$ . For variable-coefficient operators, the definition of strong convexity involves imposing the obvious modification of (5.3.5) at each point  $x \in \mathbf{R}^N$ .

**5.4 Complex time bounds.** We discuss the extension of the semigroup  $e^{-Ht}$  acting on  $L^p$  from real to complex and possibly even imaginary times, starting with the second-order case.

An application of the Riesz–Thorin convexity theorem yields a proof that a symmetric Markov semigroup (for example, the semigroup associated with a homogeneous second-order elliptic operator) can be extended to a holomorphic semigroup in  $L^p$ , but one cannot obtain the sharp angle of holomorphicity by this method. The best possible result in this abstract context may be found in [122, 123]; an example showing that the angle obtained in [122, 123] cannot be improved was given in [172].

Now suppose that  $H := H_0 - V$  acting in  $L^2(\mathbf{R}^N)$ , where  $H_0$  is a homogeneous second-order elliptic operator, and  $V$  is a non-negative potential with quadratic form relative bound less than 1. The fact that  $e^{-Ht}$  may be extended to a holomorphic semigroup acting on  $L^p$  for all  $p$  in a certain interval around  $p = 2$  was proved in [25, 122]. In particular, if the relative bound is zero, then the result holds for all  $1 < p < \infty$ . Semenov [160] extended the size of the interval, and proved that his result is optimal if one is provided with only the relative bound of  $V$ . Schreieck and Voigt [157] proved in this context that the  $L^p$  spectrum of  $H$  is independent of  $p$ .

It is fairly clear that such  $L^p$  results follow immediately for all  $1 \leq p \leq \infty$  if it is possible to prove suitable complex-time heat kernel bounds. It is rather surprising that one can obtain complex-time bounds for all  $t$  such that  $\operatorname{Re}(t) > 0$  rather easily from the real-time bounds, even for higher-order elliptic operators. If  $H = H^* \geq 0$  acts in  $L^2(\mathbf{R}^N)$  and

$$|K(t, x, y)| \leq ct^{-N\beta/\alpha} \exp[-a|x-y|^\alpha t^{-\beta}] \quad (5.4.1)$$

for all  $x, y$  and all  $t > 0$ , where all the constants are positive, then one also has

$$|K(t, x, y)| \leq c(r \cos(\theta))^{-N\beta/\alpha} \exp[-a|x-y|^\alpha r^{-\beta} \cos(\theta)] \quad (5.4.2)$$

for all  $x, y$  and all  $t = re^{i\theta}$ , where  $r > 0$  and  $|\theta| < \pi/2$ ; in these inequalities,  $a, c$  may be different but  $\alpha, \beta$  are unchanged [52; 141; 50, page 103]. The proof involves an

application of the Phragmén–Lindelöf theorem. This estimate fails for purely imaginary  $t$ , the reason being that heat kernel bounds for  $t > 0$  provide almost no information about the nature of the spectrum of  $H$ , except near the bottom.

It turns out that the complex-time heat kernel bounds are of great importance for one approach to  $L^p$  spectral theory described in the next section. If  $H$  acts in  $L^2(\mathbf{R}^N)$ , then it follows immediately from the estimate (5.4.2) that one has the explicit bound

$$\|e^{-Ht}\|_{p,p} \leq c \cos(\theta)^{-N(1+\beta)/\alpha} \tag{5.4.3}$$

for all  $1 \leq p < \infty$  and  $\operatorname{Re}(t) > 0$ .

Even for second-order elliptic operators, the investigation of the imaginary-time heat kernel (that is, the fundamental solution of the Schrödinger equation) is a very hard problem. It is shown in [46, 104] that if the second-order coefficients converge to those of  $-\Delta$  sufficiently rapidly at infinity, and no point in the phase space  $T^*(\mathbf{R}^N)$  is trapped forwards or backwards by the bicharacteristic flow, then the fundamental solution of the Schrödinger equation is smooth for all  $t \neq 0$  and  $x, y \in \mathbf{R}^N$ . The method of [104] involves the use of a direct formula expressing the fundamental solution of the Schrödinger equation as a direct integral of the fundamental solution of the wave equation, and then using known properties of the latter. Higher-order versions of the above theorems are not known, and effective long-time bounds on the fundamental solution of the Schrödinger equation are also not known.

*5.5 Pointwise lower bounds.* If  $H$  is second-order elliptic acting in  $L^2(\mathbf{R}^N)$ , then there are Gaussian lower bounds on the heat kernel which complement the upper bounds, provided that the coefficients of the operator are real; see [168, 50, 152] for the extensive literature on this problem, and its generalization to Riemannian manifolds  $M$  under suitable geometrical conditions on the latter. If  $M$  is non-compact, then a principal object of interest is the long-time asymptotics of the heat equation. If the spectral projection  $E_\lambda$  defined for all  $\lambda \geq 0$  has an integral kernel  $E(\lambda, x, y)$ , then

$$K(t, x, y) = \int_0^\infty e^{-\lambda t} E(d\lambda, x, y). \tag{5.5.1}$$

Tauberian theorems relating the long-time asymptotics of  $K$  with the low-energy asymptotics of  $E$  are given in [89].

If  $H = -\Delta$ , then it is entirely possible that  $\sup_x K(t, x, x)$ ,  $\inf_x K(t, x, x)$  and  $K(t, x, x)$  itself may all have different long-time asymptotic behaviour [56], even for rotationally invariant metrics on  $M := S^{N-1} \times \mathbf{R}$ . The examples involve the two ends of the manifold having different asymptotic dimensions, and are closely related to the existence of certain positive harmonic functions on the manifolds in question; see [121] for a deep analysis of this issue. The situation is, however, much simpler for manifolds with non-negative Ricci curvature, and for the manifolds with the volume-doubling property

$$\operatorname{vol}(x, 2r) \leq c \operatorname{vol}(x, r) \tag{5.5.2}$$

for some  $c > 0$  and all  $x \in M$ ,  $r > 0$ ; see [87, 152, 156, 168] for references to some of the enormous recent literature on this problem. We draw particular attention to [40], where diagonal lower bounds on  $K(t, x, x)$  are obtained without recourse to the use of the parabolic Harnack inequality, which is an intrinsically second-order property.

It is a classical result that if  $H$  is an elliptic operator of order  $2m$ , then  $K(t, x, y)$  cannot be non-negative for all  $t > 0$  and  $x, y \in M$  unless  $m = 1$ . The reason for this is

that non-negative heat kernels are associated with Dirichlet forms, and it is intuitively clear that the integrals defining Dirichlet forms can involve only first-order derivatives. However, the heat kernel is the kernel of a non-negative self-adjoint operator, so it is non-negative on the diagonal  $x = y$  for any  $m$ . The author [58] has proved that if  $H$  is any (uniformly) elliptic operator of order  $2m$  acting in  $L^2(\mathbf{R}^N)$  for which there exist Gaussian upper bounds on the heat kernel, then there exists a constant  $c > 0$  such that

$$K(t, x, x) \geq ct^{-N/2m} \quad (5.5.3)$$

for all  $x \in \mathbf{R}^N$  and  $t \in (0, 1)$ . If  $H$  is homogeneous, then the same holds for all  $t \in (0, \infty)$ . There appear to be no other results of this type in the literature for higher-order operators, apart from the abstract functional analytic bounds of [40, 56].

## 6. $L^p$ spectral theory

**6.1 Functional calculus.** In this section we ask under what circumstances the  $L^p$  spectrum of an elliptic operator  $H$  is independent of  $p \in [1, \infty)$ . The validity of many of the results of this section for  $p = 1$  is in strong contrast to what is provable using the holomorphic functional calculus, described in the next section. Before asking about the  $L^p$  spectrum of  $H$ , one has to agree upon a proper definition of  $H_p$  acting in  $L^p$ , and this is not straightforward since even in  $L^2$  the domain of  $H$  is not easy to specify. We adopt the view that  $-H_p$  should be defined as the infinitesimal generator of the semigroup  $e^{-Ht}$ , assuming that this semigroup can be extended consistently from  $L^2$  to a strongly continuous semigroup on  $L^p$ . It then follows from the formula

$$(H - \lambda)^{-1}f = \int_0^\infty e^{-Ht + \lambda t} f dt, \quad (6.1.1)$$

valid for all  $f \in L^2 \cap L^p$ , that the resolvent operators are consistent provided that  $\operatorname{Re}(\lambda)$  is sufficiently negative.

The seminal paper in the study of  $L^p$  spectral independence was that of Hempel and Voigt [95], who proved this fact for  $1 \leq p < \infty$  when  $H := -\Delta + V$  is a Schrödinger operator acting in  $L^2(\mathbf{R}^N)$ . They made the minimal assumption that  $V_+$  lies in  $L^1_{\text{loc}}$  but that  $V_-$  lies in the Kato class. However, it is not always the case that the  $L^p$  spectrum of an elliptic operator  $H$  is independent of  $p$ , even if we restrict attention to the Laplace–Beltrami operator of a complete Riemannian manifold of bounded geometry. It was shown in [124, 61] that for  $p \neq 2$ , the  $L^p$  spectrum of hyperbolic space  $H^{n+1}$  is the complex region on or inside the parabola

$$t \in \mathbf{R} \rightarrow n^2 p^{-1}(1 - p^{-1}) + t^2 + itn(1 - 2/p). \quad (6.1.2)$$

Other simple examples of operators for which the  $L^p$  spectrum depends upon  $p$  may be found in [8]. The best result in the reverse direction for Laplace–Beltrami operators is that of Sturm, who proved that if large balls of the manifold have uniformly subexponential volume growth, then one has  $p$ -independence of the spectrum [165]. The method which we describe here can be extended to complete Riemannian manifolds for which the volumes of large balls are polynomially bounded, but we confine the discussion to the case of Euclidean space.

The above question turns out to be intimately related to the construction of a functional calculus. By this we mean a systematic way of associating a bounded operator to the expression  $f(H)$  when  $f$  is a function defined on the spectrum of  $H$ .

The paradigm of such a calculus is the one for self-adjoint operators on Hilbert space, but we are interested in analogues on  $L^p$  spaces. Unfortunately, it is easy to construct functional calculi using abstract methods which turn out to have very few applications; we do not give examples, in order to save embarrassment. The real task is to define a functional calculus which has hypotheses which are true for operators of the type in which we are interested, and which encode some substantial special information about those operators so that the calculus is not too flabby. We describe two approaches which meet these criteria, at least to a certain extent.

**6.2 Helffer–Sjöstrand calculus.** This calculus depends upon a formula for  $f(H)$  which appeared in a paper of Helffer and Sjöstrand for self-adjoint operators [93], and which was subsequently extended to abstract Banach spaces for application to  $L^p$  in [53, 54, 101, 102]. The starting point is an unbounded operator  $H$  with real spectrum acting in a Banach space  $\mathcal{B}$ . It is assumed that the resolvent operators satisfy the norm bounds

$$\|(z - H)^{-1}\| \leq c |\operatorname{Im} z|^{-1} \left( \frac{\langle z \rangle}{|\operatorname{Im} z|} \right)^\alpha \tag{6.2.1}$$

for some  $\alpha \geq 0$  and all  $z \notin \mathbf{R}$ , where  $\langle z \rangle := (1 + |z|^2)^{1/2}$ . For self-adjoint operators, this bound is well-known with  $\alpha = 0$  and  $c = 1$ , but its validity for a substantial range of elliptic operators in  $L^p$  is a highly non-trivial technical input [101, 102, 144].

It seems possible that the methods described below can be extended to operators for which the norms of the resolvents diverge at a subexponential rate as  $\operatorname{Im}(z) \rightarrow 0$ . This would enable one to obtain the full version of Sturm’s result mentioned above [165].

We define  $S^\beta$  for any  $\beta \in \mathbf{R}$  to be the space of smooth functions  $f: \mathbf{R} \rightarrow \mathbf{C}$  such that

$$f^{(r)}(x) := \frac{d^r f}{dx^r} = O(\langle x \rangle^{\beta-r}) \tag{6.2.2}$$

as  $|x| \rightarrow \infty$  for all  $r \geq 0$ , and then put  $\mathcal{A} := \bigcup_{\beta < 0} S^\beta \subseteq C_0(\mathbf{R}^N)$ .

We use a version of Hörmander’s concept of almost analytic extensions. Given  $f \in \mathcal{A}$  and  $n \geq 0$ , we define an almost analytic extension of  $f$  to  $\mathbf{C}$  by the formula

$$\tilde{f}(x, y) := \left( \sum_{r=0}^n f^{(r)}(x) (iy)^r / r! \right) \sigma(x, y), \tag{6.2.3}$$

where

$$\sigma(x, y) := \tau(y/\langle x \rangle) \tag{6.2.4}$$

and  $\tau$  is a non-negative  $C^\infty$  function such that  $\tau(s) = 1$  if  $|s| \leq 1$  and  $\tau(s) = 0$  if  $|s| \geq 2$ .

The Helffer–Sjöstrand functional calculus is defined under the above hypotheses by the operator norm convergent formula

$$f(H) := -\frac{1}{\pi} \int_{\mathbf{C}} \frac{\partial \tilde{f}}{\partial \bar{z}} (z - H)^{-1} dx dy. \tag{6.2.5}$$

It is remarkable that the operator  $f(H)$  is bounded and independent of the particular choice of the cut-off function  $\sigma$  and of  $n$ , provided that  $n$  is large enough. Moreover, the map  $f \rightarrow f(H)$  is an algebra homomorphism which relates to the operator  $H$  in an appropriate way. In particular,  $f(H) = 0$  whenever the support of  $f$  is disjoint from the spectrum of  $H$ .

The exact class of functions  $f$  for which  $f(H)$  can be defined under the above assumptions on its resolvent is not known. We mention, however, that Dynkin [70] has obtained a precise relationship between the local regularity properties of a function  $f: \mathbf{R} \rightarrow \mathbf{C}$  and the rate of convergence of an almost analytic extension as  $\text{Im}(z) \rightarrow 0$ .

A great merit of the above formula is that it provides a direct passage from resolvents, which are relatively easy to control in particular situations, to the much more abstract operators  $f(H)$ , which are often of great importance in carrying through technical arguments. In multibody scattering theory, one may estimate  $f(H_1) - f(H_2)$  provided that one has strong enough information about the difference of the resolvents. We refer to the cited sources for further information about the many applications of the Helffer–Sjöstrand formula.

**6.3 Spectral independence.** We describe the proof of  $L^p$  spectral independence given in [52, 53, 54] using the above functional calculus; other approaches, mostly for second-order operators, may be found in [8, 95, 108, 159, 160, 165]. The starting point is the complex-time Gaussian heat kernel bound for higher-order elliptic operators of Subsection 5.4. If the volumes of large balls in the space have a uniform polynomial upper bound, then one can deduce  $L^p$  norm bounds on  $e^{-Ht}$  for all  $\text{Re}(t) > 0$  and all  $1 \leq p < \infty$ . By integration with respect to  $t$ , one finds that the  $L^p$  resolvent operators satisfy the hypothesis of the Helffer–Sjöstrand calculus. The method of definition guarantees that the operator  $f(H)$  on  $L^p$  is consistent between different values of  $p$ , and the properties of the functional calculus imply immediately that the  $L^p$  spectrum is independent of  $p$ .

If  $N > 2m$ , then one still has Gaussian off-diagonal decay, in the sense that one can obtain an appropriate upper bound on the  $L^2$  operator norm

$$\|\chi_E e^{-Ht} \chi_F\|$$

in terms of the distance between the two compact subsets  $E$  and  $F$  of  $\mathbf{R}^N$ . This is enough to prove that the  $L^p$  spectrum of  $H$  is independent of  $p$  for  $q_c \leq p \leq p_c$  [52]. The same method can be applied to  $L^p(\mathbf{R}^N, (1 + |x|^2)^\gamma dx)$  for any value of the constant  $\gamma$ , and to a variety of other Banach function spaces [54]. An abstract criterion for the equality of the spectrum of an operator  $H$  acting simultaneously in two abstract Banach spaces is given in [52].

## 7. Non-self-adjoint operators

Everything written in preceding sections has to be re-examined if one wishes to allow the elliptic operators under consideration to be non-self-adjoint. If one assumes that the highest-order coefficients of  $H$  are uniformly continuous, then there is a large literature on elliptic regularity problems [3, 4, 138, 83],  $L^p$  properties of the semigroup  $e^{-Ht}$  and the  $L^p$  functional calculus [7, 5, 67, 68]. Some aspects of the self-adjoint theory need only small modifications, but there are new problems which do not arise in the self-adjoint case. In the next subsection we discuss the second-order case, and then go on to the construction of the holomorphic functional calculus of McIntosh.

**7.1 Second-order operators.** We consider operators  $H$  acting in  $L^2(\mathbf{R}^N)$  of the form

$$Hf(x) := - \sum_{i,j=1}^N \frac{\partial}{\partial x_i} \left\{ a_{i,j}(x) \frac{\partial f}{\partial x_j} \right\}, \quad (7.1.1)$$

where the coefficients are assumed to be bounded and measurable. Lower-order derivatives can be accommodated easily if the lower-order coefficients are bounded, and this condition can surely be weakened without difficulty.

We assume that the coefficients satisfy

$$\operatorname{Re} \left( \sum_{i,j=1}^N a_{i,j}(x) \xi_j \bar{\xi}_i \right) \geq c|\xi|^2 \tag{7.1.2}$$

for some  $c > 0$  and all  $\xi \in \mathbf{C}^N$ . This forces the quadratic form

$$Q(f) := \int_{\mathbf{R}^N} \sum_{i,j=1}^N a_{i,j}(x) \frac{\partial f}{\partial x_j} \frac{\partial \bar{f}}{\partial x_i} dx \tag{7.1.3}$$

defined on  $W^{1,2}(\mathbf{R}^N)$  to be sectorial, and the associated operator  $H$  to be maximal accretive [105, 125]. Every such operator has a sectorial angle  $\alpha(H)$  defined as the supremum of the angles  $\theta$  such that  $e^{\pm i\theta}H$  still lies in the stated class. If  $H$  is self-adjoint, then the sectorial angle is  $\pi/2$ , but in general it is smaller. Finally,  $e^{-Ht}$  is a contraction semigroup on  $L^2$  for  $t \geq 0$ .

It was shown in [14] that if  $N = 1, 2$ , then the semigroup  $e^{-Ht}$  has a continuous heat kernel satisfying Gaussian bounds. These bounds can be extended to complex times in the sector

$$\{t: |\arg(t)| < \alpha(H)\}. \tag{7.1.4}$$

For  $N \geq 3$ , the same was proved subject to Hölder continuity of the coefficients in [14] and subject to uniform continuity of the coefficients in [12]. The most complete analysis of the relationship between assumptions on the coefficients and properties of the heat kernel has recently been given by Elst and Robinson [71, 72, 73], who worked in a Lie group context.

Under the above assumptions, it may be shown that the spectrum of  $H$  is contained in the sector

$$\left\{ z: |\arg(z)| < \frac{\pi}{2} - \alpha(H) \right\}, \tag{7.1.5}$$

and that for any  $\gamma > \frac{\pi}{2} - \alpha(H)$ ,

$$\|(z - H)^{-1}\| \leq c|z|^{-1} \tag{7.1.6}$$

whenever  $|\arg(z)| \geq \gamma$ . This is the jumping-off point for the theory of the next subsection.

*7.2  $L^p$  multiplier theory.* If  $H := -\Delta$ , then  $e^{-Ht}$  is a contraction on  $L^p(\mathbf{R}^N)$  for all  $t \geq 0$  and all  $1 \leq p \leq \infty$ . However, the situation for the operators  $H^{it}$  where  $t \in \mathbf{R}$  is quite different. The operators are given on  $L^2$  by the formula

$$H^{it}f(x) = \frac{1}{(2\pi)^N} \int_{\mathbf{R}^N} e^{ix \cdot \xi} |\xi|^{2it} \hat{f}(\xi) d\xi, \tag{7.2.1}$$

and yield a one-parameter unitary group. It is a consequence of standard multiplier theorems [164, page 96] that the operators extend consistently to bounded operators

on  $L^p$  for  $1 < p < \infty$ . However, the multiplier  $\zeta \rightarrow |\zeta|^{2it}$  is not continuous at  $\zeta = 0$  if  $t \neq 0$ , so the same function cannot be an  $L^1$  multiplier, or, equivalently, the operators do not extend to bounded operators on  $L^1$ .

The above results are not restricted to constant-coefficient operators and do not depend upon the self-adjointness of  $H$ . The natural context is that of sectorial operators in Banach spaces, to which we now turn.

*7.3  $H^\infty$  functional calculus.* We give a short and slightly over-simplified account of McIntosh's theory of the  $H^\infty$  functional calculus, following particularly [133, 5, 7, 43]. Let  $H$  be an abstract unbounded operator with dense domain in a Banach space  $\mathcal{B}$  and with spectrum contained in the sector

$$S_\beta := \{z: |\arg(z)| \leq \beta\} \cup \{0\} \quad (7.3.1)$$

for some  $0 < \beta < \pi$ . If  $\gamma > \beta$ , put

$$S_\gamma^0 := \{z \neq 0: |\arg(z)| < \gamma\}, \quad (7.3.2)$$

and let  $H_\gamma^\infty$  be the space of bounded holomorphic functions on  $S_\gamma^0$ , with the supremum norm  $\| \cdot \|_\infty$ . One may then define a possibly unbounded operator  $f(H)$  for all  $f \in H_\gamma^\infty$  by the formula

$$f(H)\phi := (2\pi i)^{-1} \int_\sigma f(z)(H-z)^{-1}\phi dz, \quad (7.3.3)$$

where  $\sigma$  is the infinite clockwise oriented path  $\{re^{\pm i\theta}: 0 \leq r < \infty\}$  and  $\beta < \theta < \gamma$ . The convergence of this integral at  $r = 0, \pm\infty$  is obviously problematical, so a regularization procedure is necessary. An application of Cauchy's formula establishes that the operator obtained is independent of the choice of  $\theta$  within the stated limits. We say that  $H$  has a bounded  $H^\infty$  functional calculus if for some  $\gamma > \beta$  one has

$$\|f(H)\| \leq c_\gamma \|f\|_\infty \quad (7.3.4)$$

for every  $f \in H_\gamma^\infty$ . It is elementary that if  $H$  has a bounded  $H^\infty$  functional calculus, then  $H^{it}$  is a bounded operator for all  $t \in \mathbf{R}$ , and it is known that the converse is also true in a Hilbert space, but not in greater generality [43].

There is a simple abstract proof that every maximal accretive operator in a Hilbert space has a bounded  $H^\infty$  functional calculus [5], but the situation for  $L^p$  requires much deeper study. The existence of such a calculus is known for several different classes of non-self-adjoint elliptic operators [133, 5, 67, 97]. The proofs depend upon  $L^p$  properties of the semigroup, off-diagonal decay properties of the heat kernel, interpolation-type theorems and techniques of harmonic analysis, which imply the eventual conclusion provided  $1 < p < \infty$ .

*7.4 Applications of the  $H^\infty$  functional calculus.* One of the reasons for the importance of these results depends upon their close connections with questions in harmonic analysis and interpolation theory. Another concerns their application to the solution of non-linear problems such as the Navier–Stokes equation, for which the solution does not have very much regularity with respect to the time variable. We do not consider the non-linear application directly, but describe the relevant class of linear differential equations briefly.

Consider the evolution equation

$$\frac{du}{dt} + Au(t) = f(t) \tag{7.4.1}$$

in the Banach space  $\mathcal{B}$ , where  $f \in L^p((0, T), \mathcal{B})$  for some  $1 < p < \infty$ . It turns out to be useful to reformulate the equation in the form

$$(B + A)u = f, \tag{7.4.2}$$

interpreted as acting in  $L^p((0, T), \mathcal{B})$ , where  $B := d/dt$  commutes with  $A$ . One may hope to solve this equation by using the formula

$$(B + A)^{-1} = \frac{1}{2i} \int_{c-i\infty}^{c+i\infty} \frac{A^{-z} B^{z-1}}{\sin(\pi z)} dz, \tag{7.4.3}$$

provided that  $0 < c < 1$  and the complex powers  $A^z$  and  $B^z$  are sufficiently well-behaved. It turns out that the theory depends upon  $\mathcal{B}$  being a  $\zeta$ -convex space, in the sense that the Hilbert transform is bounded on  $L^p(\mathbf{R}, \mathcal{B})$  for  $1 < p < \infty$ . See [6, 65] for more details concerning the proper interpretation of the above formula and its applications. We conclude by mentioning that other aspects of the theory of the  $H^\infty$  functional calculus also depend upon assumptions concerning the geometry of the Banach spaces involved [117].

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Department of Mathematics  
King's College London  
Strand  
London WC2R 2LS