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Subclass of Starlike Functions with Respect to Symmetric Conjugate Points

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Abstract

This paper consider $S^*_{sc}(A,B)$ as a class of functions f which are analytic in an open unit disc $\mathcal{D} = \{z : |z| < 1\}$ and satisfying the condition $\frac{2zf'(z)}{f(z)-\overline{f(-\bar{z})}} \prec \frac{1+Az}{1+Bz}$, $-1 \leq B < A \leq 1$, $z \in \mathcal{D}$. We obtain some properties of functions $f \in S^*_{sc}(A,B)$ such as coefficient estimates, distortion theorem, growth result and integral operator.

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1 Introduction

Let $\mathcal{U}$ be the class of functions which are analytic in the open unit disc $\mathcal{D} = \{z : |z| < 1\}$ given by

$$w(z) = \sum_{k=1}^{\infty} b_k z^k$$

and satisfying the conditions

$$w(0) = 0,\ |w(z)| < 1,\ z \in \mathcal{D}.$$

Let $\mathcal{S}$ denote the class of functions f which are analytic and univalent in $\mathcal{D}$ of the form

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n,\ z \in \mathcal{D}. \tag{1}$$

Also, let $\mathcal{S}_s^*$ be the subclass of $\mathcal{S}$ consisting of functions given by (1) satisfying

$$Re\ \left\{\frac{zf'(z)}{f(z) - f(-z)}\right\} > 0, \quad z \in \mathcal{D}.$$

These functions are called starlike with respect to symmetric points and were introduced by Sakaguchi in 1959. El-Ashwah and Thomas in [2], introduced two other classes namely the class $\mathcal{S}_c^*$ consisting of functions starlike with respect to conjugate points and $\mathcal{S}_{sc}^*$ consisting of functions starlike with respect to symmetric conjugate points.

Further, let $f, g \in \mathcal{U}$. Then we say that f is subordinate to g, and we write $f \prec g$, if there exists a function $w \in \mathcal{U}$ such that $f(z) = g(w(z))$ for all $z \in \mathcal{D}$. Specially, if g is univalent in $\mathcal{D}$, then $f \prec g$ if and only if $f(0) = g(0)$ and $f(\mathcal{D}) \subseteq g(\mathcal{D})$.

In terms of subordination, Goel and Mehrok in 1982 introduced a subclass of $\mathcal{S}_s^*$ denoted by $\mathcal{S}_s^*(A, B)$. Let $\mathcal{S}_s^*(A, B)$ denote the class of functions of the form (1) and satisfying the condition

$$\frac{2zf'(z)}{f(z) - f(-z)} \prec \frac{1 + Az}{1 + Bz}, \quad -1 \leq B < A \leq 1,\ z \in \mathcal{D}.$$

In this paper, let consider $\mathcal{S}_{sc}^*(A, B)$ be the class of functions of the form (1) and satisfying the condition

$$\frac{2zf'(z)}{f(z) - \overline{f(-\bar{z})}} \prec \frac{1 + Az}{1 + Bz}, -1 \leq B < A \leq 1,\ z \in \mathcal{D}.$$

Obviously $\mathcal{S}_{sc}^*(A, B)$ is a subclass of the class $\mathcal{S}_{sc}^* = \mathcal{S}_{sc}^*(1, -1)$.

By definition of subordination, it follows that $f \in \mathcal{S}_{sc}^*(A, B)$ if and only if

$$\frac{2zf'(z)}{f(z) - \overline{f(-\bar{z})}} = \frac{1 + Aw(z)}{1 + Bw(z)} = P(z), \quad w \in \mathcal{U} \tag{2}$$

where

$$P(z) = 1 + \sum_{n=1}^{\infty} p_n z^n. \tag{3}$$

We study the class $\mathcal{S}^*_{sc}(A, B)$ and obtain coefficient estimates, distortion theorem, growth result and integral operator.

2 Preliminary Result

We need the following preliminary lemmas, required for proving our result.

Lemma 2.1 *([3]) If $P(z)$ is given by (3) then*

$$|p_n| \leq (A - B). \tag{4}$$

Lemma 2.2 *([3]) Let $N(z)$ be analytic and $M(z)$ starlike in D and $N(0) = M(0) = 0$. Then*

$$\frac{\left|\left(\frac{N'(z)}{M'(z)} - 1\right)\right|}{\left|\left(A - B\frac{N'(z)}{M'(z)}\right)\right|} < 1$$

implies

$$\frac{\left|\left(\frac{N(z)}{M(z)} - 1\right)\right|}{\left|\left(A - B\frac{N(z)}{M(z)}\right)\right|} < 1, \quad z \in \mathcal{D}.$$

3 Main Result

We give the coefficient inequalities for the class $S^*_{sc}(A, B)$.

Theorem 3.1 *Let $f \in S^*_{sc}(A, B)$, then for $n \geq 1$,*

$$|a_{2n}| \leq \frac{(A - B)}{n!2^n} \prod_{j=1}^{n-1} (A - B + 2j), \tag{5}$$

and

$$|a_{2n+1}| \leq \frac{(A - B)}{n!2^n} \prod_{j=1}^{n-1} (A - B + 2j). \tag{6}$$

Proof.

For (2) and (3), we have

$$z + 2a_2z^2 + 3a_3z^3 + ... + 2na_{2n}z^{2n} + (2n+1)a_{2n+1}z^{2n+1} + ...$$

$$\begin{aligned} = \quad & (z + a_3z^3 + a_5z^5 + ... + a_{2n-1}z^{2n-1} + a_{2n+1}z^{2n+1} + ...) \\ & \bullet (1 + p_1z + p_2z^2 + ... + p_{2n}z^{2n} + p_{2n+1}z^{2n+1} + ...) \end{aligned}$$

Equating the coefficients of like powers of z, we have

$$2a_2 = p_1, \quad 2a_3 = p_2 \tag{7}$$

$$4a_4 = p_3 + a_3p_1, \quad 4a_5 = p_4 + a_3p_2 \tag{8}$$

$$(2n)a_{2n} = p_{2n-1} + a_3p_{2n-3} + a_3p_{2n-5} + ... + a_{2n-1}p_1 \tag{9}$$

$$(2n)a_{2n+1} = p_{2n} + a_3p_{2n-2} + a_5p_{2n-4} + ... + a_{2n-1}p_2. \tag{10}$$

Easily using Lemma 2.1 and (7), we get

$$|a_2| \leq \frac{(A-B)}{2}, \quad |a_3| \leq \frac{(A-B)}{2}. \tag{11}$$

Again by applying (11) and followed by Lemma 2.1, we get from (8)

$$|a_4| \leq \frac{(A-B)(A-B+2)}{2!2^2}, \quad |a_5| \leq \frac{(A-B)(A-B+2)}{2!2^2}.$$

It follows that (5) and (6) hold for n=1,2. We now prove (5) using induction. Equation (9) in conjuction with Lemma 2.1 yield

$$|a_{2n}| \leq \frac{(A-B)}{2n} \left[1 + \sum_{k=1}^{n-1} |a_{2k+1}|.\right] \tag{12}$$

We assume that (5) holds for k=3,4,...,(n-1). Then from (12), we obtain

$$|a_{2n}| \leq \frac{A-B}{2n} \left[1 + \sum_{k=1}^{n-1} \frac{A-B}{k!2^k} \prod_{j=1}^{k-1} (A-B+2j)\right]. \tag{13}$$

In order to complete the proof, it is sufficient to show that

$$\frac{A-B}{2m}\left[1+\sum_{k=1}^{m-1}\frac{A-B}{k!2^k}\prod_{j=1}^{k-1}(A-B+2j)\right]$$

$$=\frac{A-B}{m!2^m}\prod_{j=1}^{m-1}(A-B+2j), \quad (m=3,4,...,n). \tag{14}$$

(14) is valid for $m=3$.

Let us suppose that (14) is true for all $m,\ 3<m\leq(n-1)$. Then from (13)

$$\frac{A-B}{2n}\left[1+\sum_{k=1}^{n-1}\frac{A-B}{k!2^k}\prod_{j=1}^{k-1}(A-B+2j)\right]$$

$$\begin{aligned}
&= \left(\frac{n-1}{n}\right)\left(\frac{A-B}{2(n-1)}\left(1+\sum_{k=1}^{n-2}\frac{A-B}{k!2^k}\prod_{j=1}^{k-1}(A-B+2j)\right)\right)\\
&\quad+\frac{A-B}{2n}\ \frac{A-B}{(n-1)!2^{n-1}}\prod_{j=1}^{n-2}(A-B+2j)\\
&= \frac{n-1}{n}\ \frac{A-B}{(n-1)!2^{n-1}}\prod_{j=1}^{n-2}(A-B+2j)\\
&\quad+\frac{A-B}{2n}\ \frac{A-B}{(n-1)!2^{n-1}}\prod_{j=1}^{n-2}(A-B+2j)\\
&= \frac{A-B}{(n-1)!2^{n-1}}\prod_{j=1}^{n-2}(A-B+2j)\frac{(A-B+2(n-1))}{2n}\\
&= \frac{A-B}{n!2^n}\prod_{j=1}^{n-1}(A-B+2j)
\end{aligned}$$

Thus, (14) holds for $m=n$ and hence (5) follows. Similarly, we can prove (6).

Next, we give distortion bound, growth result and preserving integral operator for the class $S^*_{sc}(A,B)$.

Theorem 3.2 *Let $f\in S^*_{sc}(A,B)$, then for $|z|=r,\ 0<r<1$,*

$$\frac{1-Ar}{(1-Br)(1+r^2)}\leq|f'(z)|\leq\frac{1+Ar}{(1+Br)(1-r^2)} \tag{15}$$

and

$$\frac{1}{1+B^2}\left((A-B)ln\left(\frac{1-Br}{\sqrt{1+r^2}}\right)+(1+AB)\tan^{-1}r\right)\leq|f(z)|$$

$$\frac{1}{1-B^2}\left((A-B)ln\left(\frac{1+Br}{\sqrt{1-r^2}}\right)+(1-AB)ln\left(\frac{1+r}{1-r}\right)^{\frac{1}{2}}\right). \tag{16}$$

The bounds are sharp.

Proof.

Put $h(z)=\frac{f(z)-\overline{f(-\bar{z})}}{2}$. Then from (2), we obtain

$$|zf'(z)|=|h(z)|\left|\frac{1+Aw(z)}{1+Bw(z)}\right|. \tag{17}$$

Since h is odd starlike, it follows that (see [1])

$$\frac{r}{(1+r^2)}\leq|h(z)|\leq\frac{r}{(1-r^2)}. \tag{18}$$

Furthermore, for $w\in\mathcal{U}$, it can also be easily established that

$$\frac{1-Ar}{1-Br}\leq\left|\frac{1+Aw(z)}{1+Bw(z)}\right|\leq\frac{1+Ar}{1+Br}. \tag{19}$$

Applying results (18) and (19) in (17) we obtain (15). Next, set $|z|=r$, and upon elementary integration of (15) will give the results in (16). The extremal functions corresponding to the left and right sides of (15) and (16) are, respectively

$$f(z)=\int_0^z\frac{(1-At)}{(1-Bt)(1+t^2)}\,dt$$

and

$$f(z)=\int_0^z\frac{(1+At)}{(1+Bt)(1-t^2)}\,dt.$$

Theorem 3.3 *If* $f\in S^*_{sc}(A,B)$ *then* $F\in S^*_{sc}(A,B)$, *where*

$$F(z)=\frac{2}{z}\int_0^z f(t)\,dt.$$

Proof.

With the given F above, consider

$$\frac{2zF'(z)}{F(z)-\overline{F(-\bar{z})}} = \frac{zf(z)-\int_0^z f(t)dt}{\frac{1}{2}\left[\int_0^z f(t)dt-\int_0^z \overline{f(-\bar{t})}dt\right]}.$$

Suppose, we let $N(z)$ and $M(z)$ be the numerator and denominator functions respectively. It can be shown that

$$M(z)=\frac{1}{2}\left[\int_0^z f(t)dt-\int_0^z \overline{f(-\bar{t})}dt\right]$$

is starlike. Furthermore,

$$\frac{N'(z)}{M'(z)}=\frac{2zf'(z)}{f(z)-\overline{f(-\bar{t})}} \quad with \quad f\in S^*_{sc}(A,B).$$

Thus

$$\frac{N'(z)}{M'(z)}=\frac{1+Aw(z)}{1+Bw(z)}, \quad w\in\mathcal{U}.$$

This implies that

$$\frac{\left|\left(\frac{N'(z)}{M'(z)}-1\right)\right|}{\left|\left(A-B\frac{N'(z)}{M'(z)}\right)\right|}<1.$$

Hence, by Lemma 2.2, we have

$$\frac{\left|\left(\frac{N(z)}{M(z)}-1\right)\right|}{\left|\left(A-B\frac{N(z)}{M(z)}\right)\right|}<1, \quad z\in\mathcal{D}$$

or equivalently,

$$\frac{N(z)}{M(z)}=\frac{1+Aw_1(z)}{1+Bw_1(z)}, \quad w_1\in\mathcal{U}.$$

Thus $F\in S^*_{sc}(A,B)$.

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