Generalized Fourier Transforms, Their Nonlinearization and the Imaging of the Brain

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Introduction

Among the most important applications of the Fourier transform are the solution of the Cauchy problem of linear evolution PDEs, as well as the solution of certain inverse problems such as the one appearing in computerized tomography (the inversion of the Radon transform).

The first goal of this article is to show that Fourier transforms (FT) can be both *nonlinearized* and *generalized*. Nonlinear FTs can be used for the solution of the Cauchy problem of certain nonlinear evolution PDEs such as the Korteweg-deVries equation. Generalized FTs can be used for the analytic inversion of certain integrals, such as the one arising in single particle emission computerized tomography (the inversion of the so-called attenuated Radon transform), as well as integrals characterizing the Dirichlet to Neumann map for a large class of linear and nonlinear PDEs.

The second goal of this article is to establish that certain *abstract* integral representations, sometimes called the Ehrenpreis-Palamodov representations, which have been shown to represent the general solution of linear PDEs in convex domains, can be (a) *made effective* and (b) *nonlinearized*. As illustrations of (a) and (b) we will solve initial-boundary value problems on the half-line for linear and certain nonlinear evolution PDEs, respectively.

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The third goal of this article is to emphasize the emergence of a *mathematical unification*. This encompasses:

(i) The derivation of the classical transforms and their application to the solution of linear PDEs.
(ii) The use of several specialized techniques for the solution of linear PDEs, such as the method of images, and the so-called Wiener-Hopf technique.
(iii) The Green's function approach.
(iv) The solution of certain inverse problems.
(v) The Ehrenpreis-Palamodov integral representations.
(vi) The inverse scattering method for solving the Cauchy problem of integrable nonlinear PDEs.
(vii) The new method for analyzing boundary value problems for linear and certain nonlinear PDEs introduced by one of the authors.
(viii) The analytic inversion of integrals characterizing the unknown boundary values of a large class of linear and nonlinear boundary value problems in terms of the given boundary conditions (generalized Dirichlet to Neumann maps).

The nonlinearization and the generalization of FTs are based on a novel approach to the spectral analysis of a *single* eigenvalue equation. The effectuation and nonlinearization of the Ehrenpreis-Palamodov representation is based on the *simultaneous* spectral analysis of a compatible *pair* of eigenvalue equations (Lax pair), as well as on the analysis of an equation relating all boundary values of a given boundary value problem. The systematic investigation of this equation, which we have called the *global relation*, reduces the problem of determining the unknown boundary values (the construction of the generalized Dirichlet to Neumann map), to the problem of inverting certain integrals. Such integrals can be analytically inverted using generalized FTs.

The relevant spectral analysis is based on two fundamental mathematical techniques of complex analysis, namely the Riemann-Hilbert [1] and the ∂-bar [2] problems.

A schematic summary of the above discussion is given below:

I. Fourier transforms can be

nonlinearized	and	generalized
↓		↓
Cauchy problem for certain nonlinear PDEs		analytic inversion of certain integrals (single particle emission computerized tomography)

II. The Ehrenpreis-Palamodov integral representations can be

made effective	and	nonlinearized
↓		↓
boundary value problems for linear PDEs		boundary value problems for nonlinear PDEs

III. Mathematical unification

Parts I and II will be discussed in the following sections. Part III and further generalizations will be discussed in the final section.

The Cauchy Problem for Linear Evolution PDEs on the Line

Let $\omega(k)$ be a real polynomial of degree n. Let $q(x,t)$ satisfy the Cauchy problem for the linear PDE[1]

$$\text{(1a)} \quad \partial_t q + i\omega(-i\partial_x)q = 0, \quad -\infty < x < \infty, \ t > 0,$$

$$\text{(1b)} \quad q(x,0) = q_0(x), \quad -\infty < x < \infty,$$

where $q_0(x) \in S(\mathbb{R})$.[2] The solution of this problem involves the construction of two maps: The *direct map* is defined by

$$\text{(2)} \quad q_0(x) \to \hat{q}_0(k) = \int_{-\infty}^{\infty} e^{-ikx} q_0(x)dx, \quad k \in \mathbb{R},$$

while the *inverse map* is defined by

$$\text{(3)} \quad e^{-i\omega(k)t}\hat{q}_0(k) \to q(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx}e^{-i\omega(k)t}\hat{q}_0(k)dk, \quad -\infty < x < \infty, \quad t > 0.$$

The solution $q(x,t)$ defined by equation (3) has several remarkable features: (i) It is general; i.e., it is valid for any dispersive PDE with symbol $\omega(k)$. (ii) It is "spectrally decomposable"; namely, the spectral function $\hat{q}_0(k)$ is separated from the (x,t) dependence, and the latter dependence appears in an exponential form. (iii) It is both elegant and useful. For example, it is straightforward to compute the large t asymptotics. (iv) It is straightforward to make it rigorous. In this respect we note that an applied mathematician starts with equation (1) and then by applying the FT derives equation (3); at this stage the problem is considered solved. In contrast, for an analyst the derivation of equation (3) is just the *starting point* of solving the Cauchy problem (1). Indeed, an analyst *defines* $\hat{q}_0(k)$ in terms of the given function $q_0(x)$ by equation (2) and also *defines* $q(x,t)$ in terms of $\hat{q}_0(k)$ by equation (3); then he/she proves that this function $q(x,t)$ satisfies the PDE (1a) and also the initial condition (1b).

An Alternative Type of Separability

The solution expressed by equation (3) is based on the separability of equation (1a). Indeed, separation of variables gives rise to the eigenvalue problem

$$k^n\psi(x,k) + i\omega\left(-i\frac{d}{dx}\right)\psi(x,k) = \delta(x), \quad -\infty < x < \infty, \quad k \in \mathbb{C}, \quad \psi(x,k) \to 0 \quad \text{as } |x| \to \infty.$$

The spectral analysis of this problem gives rise to the classical FT, which is the appropriate integral transform for the associated Cauchy problem.

We claim that there exists an alternative, perhaps deeper, kind of separability: Equation (1a) is the *compatibility* condition of the following equations satisfied by the scalar function $\mu(x,t,k)$:

$$\text{(4a)} \quad \frac{\partial\mu}{\partial x} - ik\mu = q(x,t),$$

$$\text{(4b)} \quad \frac{\partial\mu}{\partial t} + i\omega(k)\mu = \sum_{j=0}^{n-1} c_j(k)(-i\partial_x)^j q(x,t), \quad k \in \mathbb{C},$$

where the polynomials $\{c_j(k)\}_0^{n-1}$ are determined by

$$\text{(5)} \quad \frac{\omega(l)-\omega(k)}{l-k} = -\sum_{j=0}^{n-1} c_j(k) l^j.$$

Equations (4), which are *two* equations for the *single* function μ, are compatible provided that q satisfies the PDE (1a). Indeed, if (1a) is valid, then

[1] *For simplicity we assume that $\omega(k)$ is real; i.e., we study only dispersive PDEs.*

[2] *For simplicity we assume throughout this paper, except for the section on linear evolution PDEs on the half-line, that the initial conditions $q_0(x)$ or $q_0(x_1,x_2)$ belong in the Schwartz space denoted by $S(\mathbb{R})$ or $S(\mathbb{R}^2)$. It is of course possible to derive similar results on less restrictive function spaces.*

equation (4a) is compatible with the equation $(\partial_t + i\omega(-i\partial_x))\mu = 0$. Replacing in the latter equation the x-derivatives of μ using equation (4a) turns it into equation (4b).

Our claim regarding the importance of the formulation (4) is based on the following considerations:

(i) There exist certain nonlinear PDEs, called *integrable*, which are amenable to exact analysis. Since these equations are nonlinear, they are *not* separable, but they *do* admit a formulation which is a natural generalization of equations (4). The simplest example of an integrable nonlinear PDE is the celebrated nonlinear Schrödinger equation (NLS)

$$iq_t + q_{xx} - 2|q|^2 q = 0. \tag{6}$$

It can be verified [3] that this equation is the compatibility condition of the following pair of eigenvalue equations for the 2×2 matrix-valued function $\Psi(x, t, k)$:

$$\frac{\partial \Psi}{\partial x} + ik\sigma_3 \Psi = Q\Psi, \qquad k \in \mathbb{C}, \tag{7a}$$

$$\frac{\partial \Psi}{\partial t} + 2ik^2\sigma_3\Psi = \left(2kQ - i\frac{\partial Q}{\partial x}\sigma_3 - i|q|^2\sigma_3\right)\Psi, \tag{7b}$$

where σ_3 is the third Pauli matrix and Q is defined in terms of q,

$$\sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad Q(x,t) = \begin{bmatrix} 0 & q(x,t) \\ \bar{q}(x,t) & 0 \end{bmatrix}. \tag{8}$$

The compatible pair of linear eigenvalue equations associated with a given integrable PDE is called a Lax pair [4]. We note that for the linearized version of the NLS, $\omega(k) = k^2$; thus equations (4) become

$$\frac{\partial \mu}{\partial x} - ik\mu = q, \tag{9a}$$

$$\frac{\partial \mu}{\partial t} + ik^2\mu = i\frac{\partial q}{\partial x} - kq. \tag{9b}$$

Equations (7) are a non-Abelian version of equations (9).

(ii) Although the formulation (4) provides *no* advantage for the solution of the Cauchy problem for equation (1a), it *does* provide a novel approach for the solution of boundary value problems for equation (1a); see below.

The solution of the Cauchy problem on the infinite line for the NLS equation is based on the analysis of the Lax pair (7). Thus, for pedagogical reasons we conclude this section by using equations (4) to rederive the solution (3).

We first *assume* that the solution of the Cauchy problem on the infinite line exists and has sufficient smoothness and decay. In order to construct the direct map, we treat equation (4a) as an ODE for the unknown function $\mu(x, t, k)$ in terms of $q(x, t)$, where $t > 0$ is fixed and k is an arbitrary *complex* parameter. By integrating with respect to x from either $-\infty$ or $+\infty$, it follows that $\mu = \mu^+$, $k \in \mathbb{C}^+$ and $\mu = \mu^-$, $k \in \mathbb{C}^-$, where $\mathbb{C}^\pm = \{k \in \mathbb{C}, \operatorname{Im} k \gtreqless 0\}$, and

$$\begin{aligned} \mu^+(x,t,k) &= \int_{-\infty}^{x} e^{ik(x-\xi)} q(\xi, t) d\xi, \\ \mu^-(x,t,k) &= \int_{\infty}^{x} e^{ik(x-\xi)} q(\xi, t) d\xi. \end{aligned} \tag{10}$$

Equations (10) express $\mu(x, t, k)$ in terms of $q(x, t)$. In order to construct the inverse map, we ask the following nontrivial question: Does there exist an alternative representation for μ which expresses μ, not in terms of $q(x, t)$, but in terms of "some spectral function"? We can answer this question utilizing the fact that the function μ is defined for all complex values of k. For k real, both functions μ^+ and μ^- are well defined, and since they both satisfy the same ODE (4a), it follows that for $k \in \mathbb{R}$ these functions are simply related:

$$\mu^+(x,t,k) - \mu^-(x,t,k) = e^{ikx}\hat{q}(k,t), \quad k \in \mathbb{R}, \tag{11}$$

where $\hat{q}(k, t)$ is the FT of $q(x, t)$. We note that $\hat{q}(k, t)$ can be thought of as a "scattering function",

$$\hat{q}(k,t) = \lim_{x\to\infty} e^{-ikx}\mu^+(x,t,k). \tag{12}$$

The functions μ^+ and μ^- defined by equations (10) are analytic in k for $\operatorname{Im} k > 0$ and $\operatorname{Im} k < 0$ respectively. Furthermore, equations (10) imply that $\mu = O(1/k)$ as $k \to \infty$. These facts together with the "jump condition" (11) define a scalar Riemann-Hilbert (RH) problem for the sectionally analytic function $\mu(x, t, k)$. The unique solution of this problem is

$$\mu(x,t,k) = \frac{1}{2\pi i}\int_{-\infty}^{\infty} \frac{e^{ilx}\hat{q}(l,t)}{l-k} dl, \ k \in \mathbb{C}, \ \operatorname{Im} k \neq 0. \tag{13}$$

Equations (10) and (13) express μ in terms of $q(x, t)$ and $\hat{q}(k, t)$ respectively; hence q and $\hat{q}$ are related. Indeed, substituting (13) into equation (4a), we find the classical inverse FT formula.

The above analysis of equation (4a) will be referred to as the *spectral analysis*. In order to derive equation (3), all that remains is to show that $\hat{q}_t + i\omega(k)\hat{q} = 0$. This equation is a direct consequence of equation (4b). Indeed, it follows by substituting equation (12) into equation (4b) and using the fact that $\{\partial_x^j q(x,t)\}_0^{n-1}$ vanishes as $x \to \infty$. In summary, the spectral analysis of equation (4a) yields the FT, while equation (4b) yields the time evolution of the Fourier transform. Although we have constructed $q(x, t)$ under the assumption of existence, we can rigorously justify a posteriori this formula *without* this assumption. Indeed, the above construction motivates the *definitions* of both the direct and the inverse maps (2) and (3) respectively. In order to prove that

$q(x,0) = q_0(x)$ we must derive the inverse FT of $q_0(x)$ in terms of $\hat{q}_0(k)$. This can be achieved by performing the spectral analysis of equation (4a) evaluated at $t = 0$. In this case, instead of $q(x,t)$ we have the *known* function $q_0(x)$; thus every step of the spectral analysis can be rigorously justified.

The Cauchy Problem for the NLS on the Line

The analysis is conceptually similar to the analysis of equations (4) presented in the previous section. Indeed, we first assume that $q(x,t)$ exists and perform the spectral analysis of equation (7a). This yields a "nonlinear" or, more precisely, a non-Abelian FT. Furthermore, the time evolution of this nonlinear FT, which turns out to be linear, is a direct consequence of equation (7b). Having obtained the correct formula for $q(x,t)$, we can rigorously justify this formula following steps conceptually similar to those of the analogous treatment of equations (4). In what follows we present the nonlinear versions of equations (2) and (3) [5].

Direct Map $q_0(x) \mapsto \{a(k), b(k)\}$

$$a(k) = 1 - \int_{-\infty}^{\infty} \bar{q}_0(x)\phi_1(x,k)\,dx, \qquad b(k) = -\int_{-\infty}^{\infty} e^{2ikx} q_0(x)\phi_2(x,k)\,dx, \tag{14}$$

where the vector $(\phi_1(x,k), \phi_2(x,k))$ is defined in terms of $q_0(x)$ through the solution of the linear Volterra integral equation

$$\phi_1(x,k) = \int_{\infty}^{x} e^{-2ik(x-x')} q_0(x')\phi_2(x',k)\,dx', \qquad \phi_2(x,k) = 1 + \int_{\infty}^{x} \bar{q}_0(x')\phi_1(x',k)\,dx' \tag{15}$$

for $-\infty < x < \infty$ and $\operatorname{Im} k \geq 0$.

Inverse Map $\{a(k), e^{-4ik^2 t} b(k)\} \mapsto q(x,t)$

$$q(x,t) = -\frac{1}{\pi}\int_{-\infty}^{\infty} \Big[e^{-i(4k^2 t + 2kx)}\gamma(k) M_{11}^{+}(x,t,k) + |\gamma(k)|^2 M_{12}^{+}(x,t,k)\Big]dk, \tag{16}$$

where $\gamma(k) = b(k)/\bar{a}(k)$, and $M(x,t,k)$ is defined in terms of $\gamma(k)$ through the solution of the following matrix RH problem:

(i) M is analytic in $k \in \mathbb{C} \setminus \mathbb{R}$.
(ii) $M = \operatorname{diag}(1,1) + O(1/k)$, $k \to \infty$.
(iii) Let $M = M^+$ for $\operatorname{Im} k \geq 0$, $M = M^-$ for $\operatorname{Im} k \leq 0$; then M satisfies the "jump condition":

$$M^-(x,t,k) = M^+(x,t,k)\begin{bmatrix} 1 & -e^{-i(4k^2 t + 2kx)}\gamma(k) \\ e^{i(4k^2 t + 2kx)}\bar{\gamma}(k) & 1 - |\gamma(k)|^2 \end{bmatrix}, \quad k \in \mathbb{R}. \tag{17}$$

Suppose that $q_0(x) \in S(\mathbb{R})$. The rigorous justification of the solution of the Cauchy problem of the NLS equation (6) involves the following steps. The vector (ϕ_1, ϕ_2) is defined in terms of $q_0(x)$ through the solution of the linear Volterra integral equation (15); thus the direct map is well defined. The inverse map is also well defined provided that the RH problem which defines M in terms of $a(k)$ and $b(k)$ has a unique global solution. This is indeed the case: If the coefficients of the "jump matrix" appearing in equation (17) are in $H^1(\mathbb{R})$, then the RH problem is equivalent to a Fredholm integral equation of index zero. Using the fact that the (Hermitian) symmetric part of the jump matrix is positive definite, one can show that the Fredholm integral equation has a unique global solution. After we have established that the direct and the inverse maps are well defined, it remains to show that (a) $q(x,t)$ satisfies the NLS equation and (b) $q(x,0) = q_0(x)$. For (a) one uses ideas of the so-called dressing method introduced by Zakharov and Shabat; namely, one shows that if M satisfies an RH problem with the jump condition (17) and $q(x,t)$ is defined in terms of M by equation (16), then M satisfies the Lax pair (7). Hence q satisfies the NLS. For (b) one evaluates equation (17) at $t = 0$ and then uses the fact that if $\mathbf{S}$ denotes the direct map (14) and $\mathbf{Q}$ denotes the inverse map (16) evaluated at $t = 0$, then $\mathbf{Q}^{-1} = \mathbf{S}$.

If $\bar{q}$ in the definition of Q (equation (8)) is replaced by $-\bar{q}$, then the resulting RH problem is singular; namely, $a(k)$ can have zeros. These zeros are important because they give rise to *solitons*. A singular RH problem can be mapped to a regular RH problem supplemented with a set of algebraic equations.

Equations (14) and (16) provide a nonlinear analog of the direct and inverse Fourier transform formulae (2) and (3) respectively (with $\omega(k) = k^2$). Indeed, if $q_0(x)$ is small, then equations (15) imply $\phi_1 \sim 0$ and $\phi_2 \sim 1$; thus $a(k) \sim 1$ and $b(k)$ tends to the FT of $q_0(x)$. Furthermore, if $a \sim 1$ and b is small, then the RH problem defining M implies that $M \sim \operatorname{diag}(1,1)$; thus equation (16) implies that $q(x,t)$ tends to the formula (3) (with k replaced by $2k$ and $\omega(k) = k^2$).

Fourier Transform on the Plane

We saw above that the Fourier transform on the line can be derived through the spectral analysis of the differential operator d/dx. In this section we show that the Fourier transform and inverse Fourier transform on the plane can be derived as the direct and inverse maps in the spectral analysis of the ∂-bar operator.

Consider the following spectral problem:

$$(18)\qquad \frac{\partial\mu}{\partial\bar{x}} - \frac{ik}{2}\mu = q(x_1, x_2),$$

where $\partial/\partial\bar{x} = (1/2)(\partial/\partial x_1 + i\partial/\partial x_2)$ is the ∂-bar operator for the complex variable $x = x_1 + ix_2$, $k = k_1 + ik_2$, and $q \in S(\mathbb{R}^2)$. Using the inversion of the ∂-bar operator, it follows that the solution of (18) which vanishes at ∞ is given by

$$(19)\quad \mu(x,k) = \frac{1}{\pi}\int_{\mathbb{R}^2} \frac{e^{i[k(\bar{x}-\bar{x}')+\bar{k}(x-x')]/2} q(x_1', x_2')}{x - x'}\, dx_1' dx_2' \qquad (x' = x_1' + ix_2').$$

This equation expresses $\mu(x,k)$ in terms of $q(x_1, x_2)$. Using the fact that μ is defined for all complex k, one can find an alternative representation of μ. Indeed (19) implies

$$(20)\qquad \lim_{k\to\infty} \mu(x,k) = 0$$

and

$$(21)\qquad \frac{\partial\mu}{\partial\bar{k}} = e^{i(k_1x_1+k_2x_2)}\hat{q}(k_1, k_2),$$

where

$$(22)\quad \hat{q}(k_1, k_2) = \frac{i}{2\pi}\int_{\mathbb{R}^2} e^{-i(k_1x_1'+k_2x_2')} q(x_1', x_2')\, dx_1' dx_2'$$

is the Fourier transform of q.

The solution of (21) satisfying the boundary condition (20) is given by

$$(23)\quad \mu(x,k) = \frac{1}{\pi}\int_{\mathbb{R}^2} \frac{e^{i(k_1'x_1+k_2'x_2)}\hat{q}(k_1', k_2')}{k - k'} dk_1' dk_2' \qquad (k' = k_1' + ik_2').$$

Substituting (23) into (18) yields the Fourier inversion formula

$$(24)\quad q(x_1, x_2) = \frac{-i}{2\pi}\int_{\mathbb{R}^2} e^{i(k_1x_1+k_2x_2)}\hat{q}(k_1, k_2)\, dk_1 dk_2.$$

The Cauchy Problem for DSII on the Plane

The Fourier transform on the plane can be *nonlinearized* and the resulting transform can be used to solve certain nonlinear evolution equations in two space dimensions. The starting point of the nonlinearization is the following non-Abelian analog of (18):

$$(25)\qquad \frac{\partial\mu}{\partial\bar{x}} - \frac{ik}{4}\mu = Q\bar{\mu},$$

where μ is a 2×2 matrix and

$$(26)\qquad Q = \begin{bmatrix} 0 & q_{12}(x_1, x_2) \\ q_{21}(x_1, x_2) & 0 \end{bmatrix}$$

is an off-diagonal 2×2 matrix whose components belong to $S(\mathbb{R}^2)$. We look for solutions of (25) of the form

$$(27)\qquad \mu = e^{i(k_1x_1+k_2x_2)/2}\psi = e^{i(k\bar{x}+\bar{k}x)/4}\psi$$

and

$$(28)\qquad \lim_{x\to\infty}\psi = I,$$

where I denotes the 2×2 identity matrix.

Equation (25) implies that ψ satisfies the equation

$$(29)\qquad \frac{\partial\psi}{\partial\bar{x}} = e^{-i(k\bar{x}+\bar{k}x)/2}Q\bar{\psi}.$$

The solution of (29) satisfying (28) is defined by the integral equation

$$(30)\qquad \psi(x_1, x_2, k_1, k_2) = I + \frac{1}{\pi}\int_{\mathbb{R}^2} \frac{e^{-i(k\bar{x}'+\bar{k}x')/2}}{x - x'} Q(x_1', x_2')\overline{\psi(x_1', x_2', k_1, k_2)}\, dx'.$$

This equation implies

$$(31)\qquad \lim_{k\to\infty}\psi_d = I$$

and

$$(32)\quad \lim_{k\to\infty}\frac{-ik}{2}e^{i(k\bar{x}+\bar{k}x)/2}\psi_o(x_1, x_2, k_1, k_2) = Q(x_1, x_2),$$

where ψ_d (resp. ψ_o) denotes the diagonal (resp. off-diagonal) part of ψ.

Let $\nu = \bar{\psi}_d + e^{i(k\bar{x}+\bar{k}x)/2}\psi_o$. It follows from (31) and (32) that

$$(33)\qquad \lim_{k\to\infty}\nu = I$$

and

$$(34)\qquad \lim_{k\to\infty}\frac{-ik}{2}\nu_o(x_1, x_2, k_1, k_2) = Q(x_1, x_2).$$

Furthermore, equation (30) implies that ν satisfies

$$(35)\qquad \frac{\partial\nu}{\partial\bar{k}} = e^{i(k\bar{x}+\bar{k}x)/2}\bar{\nu}\hat{Q},$$

where $\hat{Q}$ is the 2×2 off-diagonal matrix defined by

$$(36)\quad \hat{Q}(k_1, k_2) = \frac{i}{2\pi}\int_{\mathbb{R}^2} e^{-i(k_1x_1+k_2x_2)}Q(x_1, x_2)\overline{\psi_d(x_1, x_2, k_1, k_2)}\, dx_1 dx_2$$

(and ψ_d is defined in terms of Q by equation (30)).

Given $\hat{Q}$, the solution of (35) with the boundary condition (33) is defined by the integral equation

(37)
$$\nu(x_1,x_2,k_1,k_2) = I + \frac{1}{\pi}\int_{\mathbb{R}^2} \frac{e^{i(k'\bar{x}+\bar{k}'x)/2}}{k-k'}\overline{\nu(x_1,x_2,k_1',k_2')}\,\hat{Q}(k_1',k_2')\,dk_1'dk_2'.$$

The asymptotic relation (34) and equation (37) imply

(38)
$$Q(x_1,x_2) = \frac{-i}{2\pi}\int_{\mathbb{R}^2} e^{2i(k_1x_1+k_2x_2)}\overline{\nu_d(x_1,x_2,k_1,k_2)}\,\hat{Q}(k_1,k_2)\,dk_1dk_2$$

(where ν_d is defined in terms of $\hat{Q}$ by equation (37)).

The *direct map* $Q \mapsto \hat{Q}$ defined by equation (36) and the *inverse map* $\hat{Q} \mapsto Q$ defined by equation (38) provide a nonlinear generalization of the two-dimensional FT. Indeed, if $Q(x_1,x_2)$ is small, then equation (30) implies that $\psi \sim \mathrm{diag}\,(1,1)$ and equation (36) implies that $\hat{Q}(k_1,k_2)$ tends to the FT of $Q(x_1,x_2)$. Furthermore, if $\hat{Q}$ is small, equation (37) implies that $\nu \sim \mathrm{diag}(1,1)$; thus equation (38) implies that $Q(x_1,x_2)$ tends to the inverse FT of $\hat{Q}(k_1,k_2)$.

If we now let $\hat{Q}$ evolve according to the differential equation

(39)
$$\frac{d\hat{Q}}{dt} = i(k_1^2 - k_2^2)\hat{Q}$$

and denote the function obtained from $\hat{Q}(k_1,k_2,t)$ through the inverse map by $Q(x_1,x_2,t)$, then Q satisfies the partial differential equation

(40)
$$\frac{\partial Q}{\partial t} = i\Big[\frac{\partial^2 Q}{\partial x_2^2} - \frac{\partial^2 Q}{\partial x_1^2}\Big] - 8i\begin{bmatrix}\phi & 0\\ 0 & \bar{\phi}\end{bmatrix}Q,$$

where ϕ is defined by

(41)
$$\frac{\partial^2\phi}{\partial x_1^2} + \frac{\partial^2\phi}{\partial x_2^2} = \Big[\frac{\partial^2}{\partial x_2^2} - \frac{\partial^2}{\partial x_1^2}\Big](q_{12}\bar{q}_{21}), \quad \lim_{x\to\infty}\phi = 0.$$

In the particular case where $q_{12} = q = \pm q_{21}$, equations (40) and (41) reduce to the Davey-Stewartson-II equations:

(42)
$$\frac{\partial q}{\partial t} = i\Big(\frac{\partial^2}{\partial x_2^2} - \frac{\partial^2}{\partial x_1^2}\Big)q - 8i\phi q,$$
$$\frac{\partial^2\phi}{\partial x_1^2} + \frac{\partial^2\phi}{\partial x_2^2} = \pm\Big(\frac{\partial^2}{\partial x_2^2} - \frac{\partial^2}{\partial x_1^2}\Big)|q|^2.$$

The rigorous justification of the solution of the Davey-Stewartson-II equations by the nonlinear Fourier transform associated with the spectral problem (25) depends on the solvability of the integral equations (30) and (37) [6-8]. These integral equations are Fredholm equations with index zero on the space of bounded continuous functions on $\mathbb{R}^2$. They are uniquely solvable if Q and $\hat{Q}$ are small. In the case where $q_{12} = q_{21}$ (resp. $\hat{q}_{12} = \hat{q}_{21}$), any solution of the homogeneous equation is a *generalized analytic* function that vanishes at ∞, and the only such solution is the zero solution by the *generalized Liouville* theorem. Therefore the integral equations are uniquely solvable without assuming Q or $\hat{Q}$ to be small. This means that when the sign in the equation for ϕ is positive, the Cauchy problem of (42) is uniquely solvable for any initial data in $S(\mathbb{R}^2)$.

The long-time decay of the Davey-Stewartson-II equations was studied in [8] using the formula (38). We note that so far such results cannot be obtained by classical PDE techniques.

Generalized Fourier Transforms and the Imaging of the Brain

One of the authors and I. M. Gel′fand emphasized in [9] that the approach illustrated in the spectral analysis of equations (4a) and (18) provides a novel approach for deriving linear transforms. As an application of this approach, one of the authors and R. Novikov rederived the celebrated Radon transform by performing the spectral analysis of the following eigenvalue equation for the scalar function $\mu(x_1,x_2,k)$:

(43)
$$\frac{1}{2}(k+\frac{1}{k})\frac{\partial\mu}{\partial x_1} + \frac{1}{2i}(k-\frac{1}{k})\frac{\partial\mu}{\partial x_2} = f(x_1,x_2), \quad -\infty < x_1,x_2 < \infty, \quad k\in\mathbb{C},$$

where $f \in S(\mathbb{R}^2)$. Although the Radon transform can be derived in a simpler way by using the two-dimensional FT, the advantage of the derivation of [10] was demonstrated recently by R. Novikov, who showed that a similar analysis applied to a slight generalization of equation (43) yields the inversion of the so-called attenuated Radon transform [11]. Implementing this inversion was one of the most important open problems in the field of medical imaging. We recall that the Radon transform provides the mathematical basis of computerized tomography (CT). Similarly, the attenuated Radon transform provides the mathematical basis of a new imaging technique of great significance, namely, of the so-called Single Photon Emission Computerized Tomography (SPECT). Before discussing the mathematics of CT and of SPECT, we first present a brief introduction of these remarkable imaging techniques.

CT

In brain imaging, computerized tomography is the computer-aided reconstruction of a mathematical function that represents the x-ray attenuation coefficient of the brain tissue (and is therefore related to its density). Let $f(x_1,x_2)$ denote the x-ray attenuation coefficient at the point (x_1,x_2). This means that x-rays transversing a small distance $\Delta\tau$ at (x_1,x_2) suffer a relative intensity loss $\Delta I/I = -f\Delta\tau$. Taking the limit and solving the resulting ODE, we find $I_1/I_0 = \exp[-\int_L f d\tau]$, where L denotes the part of the line that transverses the

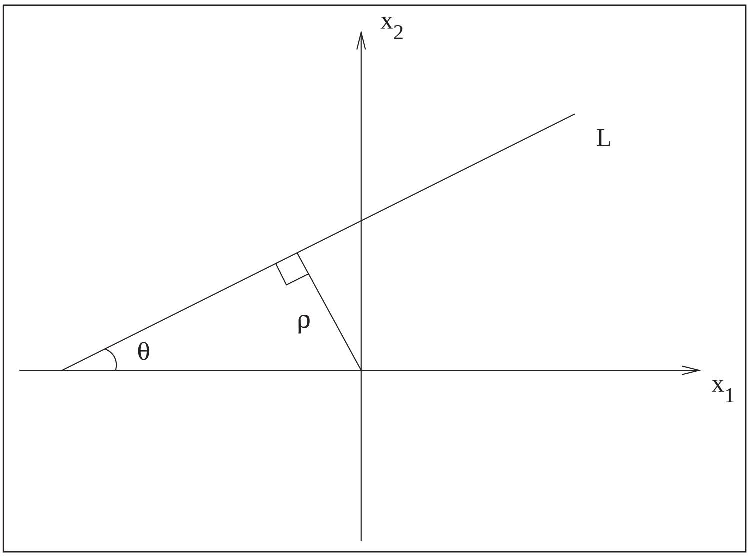

Figure 1. Local coordinates for the mathematical formulation of PET and SPECT.

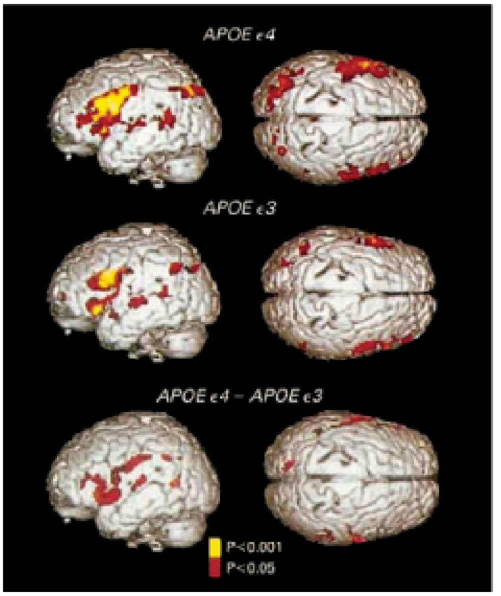

Figure 2. Images showing the difference in brain activation during a memory test between two groups of subjects. One group carries the apolipoprotein Eε4 allele, which is a known risk factor for Alzheimer's disease. The increase in activation may reflect a need to compensate for minor defects in memory, even though both groups appear normal. ***(Image reproduced with permission from: Bookheimer, S. Y., Strojwas, B. S., Cohen, M. S., et al., 2000. "Patterns of Brain Activation in People at Risk for Alzheimer's Disease",*** **New England Journal of Medicine** ***343 (7), 450–456. Copyright ©2000 Massachusetts Medical Society.)***

tissue. Since I_1/I_0 is known from the measurements, the basic mathematical problem of CT is to reconstruct a function from the knowledge of its line integrals. The line integral of a function is called its Radon transform. In order to define this transform, we introduce local coordinates: Let the line L make an angle θ with the positive x_1-axis. A point (x_1, x_2) on this line can be specified by the variables (τ, ρ), where ρ is the distance from the origin and τ is a parameter along the line; see Figure 1.

The variables (x_1, x_2) and (τ, ρ), for fixed θ, are related by the equations

$$\begin{aligned} x_1 &= \tau\cos\theta - \rho\sin\theta, \\ x_2 &= \tau\sin\theta + \rho\cos\theta. \end{aligned} \tag{44}$$

We will denote a function $f(x_1, x_2)$ written in the local coordinates (τ, ρ, θ) by $F(\tau, \rho, \theta)$, i.e.,

$$F(\tau,\rho,\theta) = f(\tau\cos\theta - \rho\sin\theta, \tau\sin\theta + \rho\cos\theta). \tag{45}$$

Thus the Radon transform of the function $f(x_1, x_2)$, which we will denote by $\hat{f}(\rho, \theta)$, is defined by

$$\hat{f}(\rho,\theta) = \int_{-\infty}^{\infty} F(\tau,\rho,\theta)d\tau. \tag{46}$$

In summary, the basic mathematical problem in CT is to reconstruct a function $f(x_1, x_2)$ from the knowledge of its Radon transform $\hat{f}(\rho, \theta)$.

The advent of CT made direct images of the brain tissue possible for the first time. Furthermore, the subsequent development of Magnetic Resonance Imaging (MRI) allowed striking discrimination between grey and white matter. This has had a tremendous impact on the entire field of medical imaging. Although the first applications of CT and MRI were in brain imaging, later these techniques were applied to many other areas of medicine. Indeed, it is impossible to think of medicine today without CT and MRI. However, in spite of their enormous impact, these techniques are capable of imaging only *structures* as opposed to *functional* characteristics.

PET and SPECT

The study of functional characteristics became possible only in the late 1980s with the development of functional MRI, of Positron Emission Tomography (PET), and of SPECT. Using these new techniques it is now possible to observe neural activity in living humans with ever-increasing precision. For example, we now know that understanding of the context of language is associated mainly with the left prefrontal part of the brain, while understanding of the emotional tone of speech is associated mainly with the right. Another example, perhaps of interest to mathematicians, is that during the performance of mathematical calculations there exists bilateral activation of the frontal cortex and of the posterial parietal cortex. There exist a vast number of clinical applications from epilepsy and migraine to differential diagnosis of schizophrenia and of Alzheimer's disease. Furthermore, just as with CT and MRI, the aforementioned new techniques are now used beyond neuroscience in a wide range of medical areas, which include pharmacology, oncology, and cardiology.

In PET the patient is injected with a dose of fluorodeoxyglucose (FDG), which is a normal molecule of glucose attached to an atom of radioactive fluorine. The more active cells absorb more FDG. The fluorine atom in FDG suffers a radioactive decay, emitting a positron which, when colliding with an electron, liberates energy in the form of *two* beams of gamma rays, which are picked up by the PET scanner simultaneously. In SPECT the situation is similar, but instead of FDG one uses Xenon-133, which emits a *single* photon.

Let $f, g, L(x)$ denote the x-ray attenuation coefficient, the distribution of the radioactive material, and the part of the ray from the tissue to the detector. Then in SPECT the following integral I is known from the measurements:

$$I = \int_L e^{-\int_{L(x)} f ds} g d\tau.$$

This integral is called the attenuated (with respect to f) Radon transform of g and will be denoted by $\hat{g}_f$. Writing f and g in local coordinates, we find

$$\hat{g}_f(\rho, \theta) = \int_{-\infty}^{\infty} e^{-\int_\tau^\infty F(s,\rho,\theta)ds} G(\tau, \rho, \theta) d\tau. \quad (47)$$

Thus the basic mathematical problem of SPECT is to reconstruct the function $g(x_1, x_2)$ from the knowledge of its attenuated Radon transform $\hat{g}_f$ and of the associated x-ray attenuation coefficient $f(x_1, x_2)$. In PET, because the two rays are picked up simultaneously, the basic mathematical problem reduces to the inversion of the classical Radon transform.

The Mathematics of PET and SPECT

It has recently been shown [12] that by scrutinizing the analysis of [10], it is possible using the basic result of [10] to derive the attenuated Radon transform almost immediately.[3] In this respect, we first review the main steps in the spectral analysis of equation (43): The left-hand side of this equation motivates the introduction of the complex variable z:

$$z = \frac{1}{2i}(k - \frac{1}{k})x_1 - \frac{1}{2}(k + \frac{1}{k})x_2. \quad (48)$$

Equation (43), after changing variables from (x_1, x_2) to $(z, \bar{z})$, becomes

$$\frac{1}{2i}\left(\frac{1}{|k|^2} - |k|^2\right)\frac{\partial \mu}{\partial \bar{z}} = f, \quad |k| \neq 1. \quad (49)$$

[3] *A numerical implementation of the inverse attenuated Radon transform is presented in [12].*

Using the inverse ∂-bar formula, one finds that the unique solution of this equation satisfying the boundary condition $\mu = O(1/z)$ as $z \to \infty$ is

(50)
$$\mu(x_1, x_2, k) = \frac{1}{2\pi i} sgn\left(\frac{1}{|k|^2} - |k|^2\right) \times \int_{\mathbb{R}^2} \frac{f(x_1', x_2')dx_1'dx_2'}{z' - z}, \quad |k| \neq 1.$$

This equation provides the solution of the direct problem; i.e., it expresses μ in terms of $f(x_1, x_2)$. In order to solve the inverse problem, i.e., in order to find an alternative representation of μ (in terms of an appropriate spectral function), we note that μ is an analytic function of k in the entire complex k-plane (including infinity) except for the unit circle. Thus, in order to reconstruct μ, it is sufficient to compute the "jump" $\mu^+ - \mu^-$, where μ^+ and μ^- denote the limits of μ as k approaches the unit circle from inside and outside the unit disk. A simple computation yields

(51)
$$\mu^\pm = \lim_{\varepsilon \to 0} \mu(x_1, x_2, (1 \mp \varepsilon)e^{i\theta}) = \mp P^\mp \hat{f}(\rho, \theta) - \int_\tau^\infty F(s, \rho, \theta) ds,$$

where $P^\mp$ denote the usual projectors in the variable ρ, i.e.,

$$(P^\mp f)(\rho) = \mp \frac{f}{2} + \frac{1}{2i\pi} Hf,$$

and H denotes the Hilbert transform.

Equations (51) imply $\mu^+ - \mu^- = -H\hat{f}(\rho, \theta)/i\pi$, where H acts on the first variable. Substituting this expression in the equation

(52)
$$\mu = \frac{1}{2i\pi} \int_0^{2\pi} \frac{(\mu^+ - \mu^-)(e^{i\theta'}) i e^{i\theta'} d\theta'}{e^{i\theta'} - k}, \quad |k| \neq 1,$$

we find μ in terms of $\hat{f}$:

$$\mu(x_1, x_2, k) = -\frac{1}{2i\pi^2} \int_0^{2\pi} \frac{e^{i\theta'}[H\hat{f}(x_2 \cos\theta' - x_1 \sin\theta', \theta')] d\theta'}{e^{i\theta'} - k}.$$

Substituting this expression in equation (43), we find the inverse Radon transform formula

(53)
$$f(x_1, x_2) = -\frac{i}{4\pi^2}\left(\partial_{x_1} - i\partial_{x_2}\right) \times \int_0^{2\pi} e^{i\theta}[H\hat{f}(x_2 \cos\theta - x_1 \sin\theta, \theta)] d\theta.$$

We now present the derivation of the inverse attenuated Radon transform. Instead of starting with equation (49), we start with the equation

$$\nu(k)\frac{\partial\mu}{\partial\bar{z}} + f(x_1,x_2)\mu = g(x_1,x_2),$$
(54)
$$\nu(k) := \frac{1}{2i}\left(\frac{1}{|k|^2} - |k|^2\right),$$

where $f \in S(\mathbb{R}^2)$, $g \in S(\mathbb{R}^2)$, $k \in \mathbb{C}$, and $|k| \neq 1$. Equation (54) implies

(55) $$\mu \exp\left[\partial_{\bar{z}}^{-1}\frac{f}{\nu}\right] = \partial_{\bar{z}}^{-1}\left(\frac{g}{\nu}\exp\left[\partial_{\bar{z}}^{-1}\frac{f}{\nu}\right]\right).$$

This equation provides the solution of the direct problem; i.e., it expresses μ in terms of f and g. Since μ is an analytic function of k in the entire complex k-plane except for the unit circle, it follows that μ satisfies the alternative representation (52). Thus, in order to express μ in terms of an appropriate spectral function, all that remains is to compute $\mu^+ - \mu^-$. But this can be easily achieved using equation (51). Indeed, equation (51) can be rewritten in the form

$$\lim_{k\to k^\pm}\left\{\partial_{\bar{z}}^{-1}\left(\frac{f(x_1,x_2)}{\nu(k)}\right)\right\} = \mp P^{\mp}\hat{f}(\rho,\theta) - \int_\tau^\infty F(s,\rho,\theta)ds.$$

Using this equation, equation (55) can be used to compute $\mu^\pm$, and then equation (52) provides an alternative representation of μ in terms of $\hat{g}_f(\rho,\theta)$ and of F. Substituting this representation in equation (54), we find the inverse attenuated Radon transform

$$g(x_1,x_2) = -\frac{1}{4\pi}(\partial_{x_1} - i\partial_{x_2})\int_0^{2\pi} J(x_1,x_2,\theta)e^{i\theta}d\theta,$$

where

$$J(x_1,x_2,\theta) = -e^{\int_\tau^\infty F(s,\rho,\theta)ds}$$
$$\times\left(e^{P^-\hat{f}(\rho,\theta)}P^-e^{-P^-\hat{f}(\rho,\theta)} + e^{-P^+\hat{f}(\rho,\theta)}P^+e^{P^+\hat{f}(\rho,\theta)}\right)\hat{g}_f(\rho,\theta).$$

Linear Evolution PDEs on the Half-Line

After solving the Cauchy problem for integrable evolution PDEs on the line and on the plane, the most important open problem in the analysis of integrable PDEs became the solution of boundary value problems, such as the solution of the NLS on the half-line. This problem remained essentially open for almost thirty years until the recent emergence of a unified transform method introduced by one of the authors. The difficulty of this type of problem can be illustrated by considering the Korteweg-deVries equation on the half-line, $0 < x < \infty$. This equation, which is another well-known integrable PDE, is

(56) $$q_t + q_x + q_{xxx} + 6qq_x = 0.$$

It was noted above that the solution of the Cauchy problem on the infinite line for an integrable evolution PDE can be constructed by using a nonlinear FT. Thus, a natural strategy for solving equation (56) is to solve the linearized version of equation (56) by the appropriate x-transform and then to nonlinearize this transform. However, this strategy fails: Let us consider an initial-boundary value problem for the equation

(57) $$q_t + q_x + q_{xxx} = 0, \quad 0 < x < \infty, \quad 0 < t < T,$$

where T is a positive fixed constant and q decays for large x. Although it is *not* immediately obvious how many boundary conditions must be prescribed at $x = 0$, it turns out that this problem is well-posed with *one* boundary condition. Thus, we consider equation (57) with $q(x,0) = q_0(x) \in H^1(\mathbb{R}^+)$ and $q(0,t) = g_0(t) \in H^1(0,T)$. In order to derive the appropriate x-transform we analyze the eigenvalue equation obtained by separation of variables. It turns out that the associated ordinary differential operator is *not* self-adjoint, and furthermore it does *not* admit a self-adjoint extension. This implies that there does *not* exist an appropriate x-transform for third-order evolution PDEs defined in the half-line (i.e., there do *not* exist proper analogs of the sine and of the cosine transforms which are the appropriate transforms for the Dirichlet and the Neumann problems respectively of second-order evolution PDEs on the half-line). Of course, one can use the appropriate t-transform, which is actually the Laplace transform. However, since in this transform $0 < t < \infty$, the Laplace transform is problematic if the growth of $g_0(t)$ as $t \to \infty$ is faster than linearly exponential functions. Furthermore, as was mentioned earlier, the solution of the Cauchy problem of the associated nonlinear PDE (56) was based on the nonlinearization of the x-transform and not of the t-transform.

The above failure of the classical transforms should be contrasted with the existence of the abstract Ehrenpreis-Palamodov integral representation [13]. For equation (1a) this fundamental representation implies that if a well-posed problem is defined in a convex domain,[4] then there exists a measure $\rho(k)$ such that the solution $q(x,t)$ is given by

(58) $$q(x,t) = \int_L e^{ikx - i\omega(k)t}d\rho(k).$$

The limitation of this result is that it does *not* provide an algorithm for constructing L and $\rho(k)$ in terms of the differential operator and the given initial and boundary conditions. However, it does imply that there must exist a representation for the solution of (57) which can be expressed in the form (58). This is indeed the case. Actually, the following general result has been proved by

[4] *In the abstract formulation it is required that the domain be bounded and smooth, so this result does not apply immediately to our domain.*

the authors: Consider the linear evolution PDE (1a) in the domain $0 < x < \infty$, $0 < t < T$, with the initial condition $q(x,0) = q_0(x) \in H^{\tilde{n}/2}(\mathbb{R}^+)$, and the N boundary conditions $\partial_x^l q(0,t) = g_l(t) \in H^{\frac{1}{2}+\frac{2\tilde{n}-2l-1}{2n}}(0,T)$, $l = 0, 1, \ldots, N-1$, where

$$N = \begin{cases} \frac{n}{2} & \text{if } n \text{ is even,} \\ \frac{n+1}{2} & \text{if } n \text{ is odd and } \alpha_n > 0, \\ \frac{n-1}{2} & \text{if } n \text{ is odd and } \alpha_n < 0, \end{cases} \quad \tilde{n} = \begin{cases} \frac{n}{2} & \text{if } n \text{ is even,} \\ \frac{n+1}{2} & \text{if } n \text{ is odd,} \end{cases}$$

and $\alpha_n \neq 0$ is the coefficient of k^n in $\omega(k)$. Assume that the functions $q_0(x)$ and $g_l(t)$ are compatible at $x = t = 0$; i.e., $g_l(0) = \partial_x^l q_0(0)$, $0 \le l \le N-1$. Then the unique solution of (1) is given by the Ehrenpreis-Palamodov representation

$$(59) \qquad q(x,t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{ikx-i\omega(k)t}\hat{q}_0(k)\,dk + \frac{1}{2\pi}\int_{\partial D_+} e^{ikx-i\omega(k)t}\tilde{g}(k)\,dk,$$

where ∂D_+ is the boundary of the domain $D_+ = \{k \in \mathbb{C} : \operatorname{Im}\omega(k) > 0,\ \operatorname{Im} k > 0\}$ oriented so that D_+ is to the left of the increasing direction, $\hat{q}_0(k)$ is the Fourier transform of $q_0(x)$, and $\tilde{g}(k)$ can be written explicitly in terms of $\{\tilde{g}_l(k)\}_0^{N-1}$ which are the t-transforms of $\{g_l(t)\}_0^{N-1}$,

$$(60) \qquad \tilde{g}_l(k) = \int_0^T e^{i\omega(k)t} g_l(t)\,dt, \qquad k \in \mathbb{C}, \quad 0 \le l \le N-1,$$

and in terms of $\hat{q}_0(\nu_i(k))$, $i = 1, \ldots, n-1$, where $\nu_i(k)$ are determined by solving $\omega(\nu_i(k)) - \omega(k) = 0$.

For equation (57), $\omega(k) = k - k^3$, ∂D_+ consists of part of the real k-axis and of parts of the curve $k_I^2 - 3k_R^2 + 1 = 0$ $(k = k_R + ik_I)$ (see Figure 3), and $\tilde{g}(k)$ is given by the following expression:

$$(61) \qquad \tilde{g}(k) = \frac{\nu_1 - k}{\nu_2 - \nu_1}\hat{q}_0(\nu_2) + \frac{k - \nu_2}{\nu_2 - \nu_1}\hat{q}_0(\nu_1) + (1 - 3k^2)\tilde{g}_0(k),$$

where $\tilde{g}_0(k)$ is the t-transform of $g_0(t)$ (see equation (60) with $l = 0$ and $\omega = k - k^3$), and ν_1, ν_2 are the nontrivial solutions of the equation $\nu - \nu^3 = k - k^3$; i.e., they are the two solutions of the quadratic equation $\nu^2 + \nu k + k^2 - 1 = 0$.

This result is obtained by implementing a general approach initiated by one of the authors [14] to the initial-boundary value problem of (57), formulated earlier. In this approach the PDE, the domain, and the given initial and boundary conditions are treated as follows:

(i) The PDE is rewritten in the form of a Lax pair. For equation (1a) the associated Lax pair is given by equations (4).

(ii) For a given domain the *simultaneous* spectral analysis of the Lax pair gives rise to two maps, a direct and an inverse map. The direct map constructs, from the *initial* and *boundary values*, appropriate x and t transforms, which we call *spectral functions*. The inverse map constructs, from the spectral functions, the representation $q(x,t)$ of the solution. For the initial-boundary value problem of (1a) on the half-line, these maps are defined below.

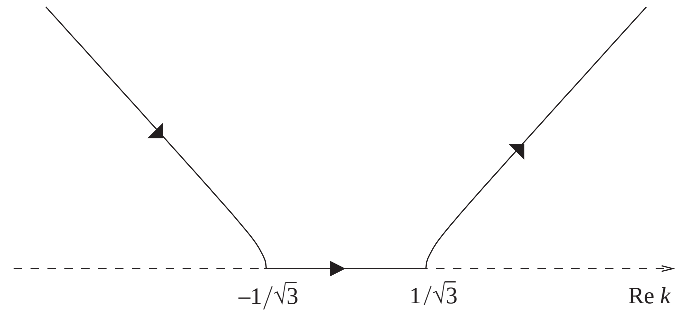

Figure 3. The contour ∂D_+ for equation (57).

Direct Map

$$\{q_0(x), q(0,t), \partial_x q(0,t), \ldots, \partial_x^{n-1} q(0,t)\} \mapsto \hat{q}_0(k), \tilde{g}(k),$$

where

$$(62) \qquad \hat{q}_0(k) = \int_0^\infty e^{-ikx} q_0(x)\,dx, \quad \operatorname{Im} k \le 0,$$

$$\tilde{g}(k) = \sum_{j=0}^{n-1} c_j(k)\tilde{g}_j(k),$$

$$(63) \qquad \tilde{g}_j(k) := \int_0^T e^{i\omega(k)t}(-i\partial_x)^j q(0,t)\,dt, \quad k \in \mathbb{C}.$$

Inverse Map

$$e^{-i\omega(k)t}\hat{q}_0(k), e^{-i\omega(k)t}\tilde{g}(k) \mapsto q(x,t),$$

with $q(x,t)$ given by (59).

(iii) For given boundary conditions (i.e., for a given subset of the boundary values), using the fact that the spectral functions satisfy a simple algebraic equation that we have called the *global relation*, we can determine the unknown part of $\tilde{g}(k)$. For equation (1a) on the half-line the relevant global relation is

$$(64) \qquad \hat{q}_0(k) + \tilde{g}(k) = e^{i\omega(k)T}\int_0^\infty e^{-ikx} q(x,T)\,dx, \quad \operatorname{Im} k \le 0.$$

For the IBV problem of (1a) formulated earlier, N of the functions $\{\partial_x^i q(0,t)\}_0^{n-1}$ are known, then the left-hand side of (64) involves $n - N$ unknowns. Furthermore, the right-hand side of (64) is also unknown. However, in spite of this ominous looking situation, it is actually possible to compute $\tilde{g}(k)$ by solving a system of $n - N$ linear algebraic equations. This is based on the crucial observation that the functions $\tilde{g}_l(k)$ defined by (63) remain invariant under the transforms in the complex k plane which leaves $\omega(k)$ invariant. As an illustrative example,

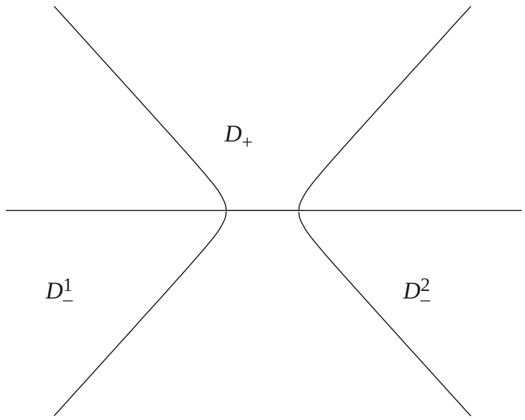

Figure 4. Domains D_+, D_-^1, and D_-^2 for (57).

we consider equation (57). In this case equation (64) becomes

$$\tilde{g}_2(k) + ik\tilde{g}_1(k) = (k^2 - 1)\tilde{g}_0(k) - \hat{q}_0(k) + e^{i\omega(k)T}\hat{q}_T(k), \quad \operatorname{Im} k \le 0, \tag{65}$$

where $\omega = k^3 - k$ and $\hat{q}_T(k)$ denotes the integral appearing on the right-hand side of (64). Replacing k in (65) by $\nu_1(k) \in D_-^1$ and $\nu_2(k) \in D_-^2$ (cf. Figure 4), we find the two equations

$$\begin{aligned} \tilde{g}_2(k) + i\nu_1(k)\tilde{g}_1(k) &= (\nu_1^2(k) - 1)\tilde{g}_0(k) - \hat{q}_0(\nu_1(k)) \\ &\quad + e^{i\omega(k)T}\hat{q}_T(\nu_1(k)), \quad k \in D_+, \\ \tilde{g}_2(k) + i\nu_2(k)\tilde{g}_1(k) &= (\nu_2^2(k) - 1)\tilde{g}_0(k) - \hat{q}_0(\nu_2(k)) \\ &\quad + e^{i\omega(k)T}\hat{q}_T(\nu_2(k)), \quad k \in D_+. \end{aligned} \tag{66}$$

If $\nu_1(k) \in D_-^1$ and $\nu_2(k) \in D_-^2$, then $k \in D_+$; thus the equations (66) are valid in D_+. These equations can be considered as two equations for the two unknown functions $\tilde{g}_1$ and $\tilde{g}_2$. Solving for these unknown functions and substituting the resulting expressions in $\tilde{g}(k)$, we find that $\tilde{g}(k)$ equals the right-hand side of (61) plus an expression involving $\hat{q}_T(\nu_1(k))$ and $\hat{q}_T(\nu_2(k))$. However, the product of this latter expression with $\exp[ikx - i\omega(k)t]$ gives rise to an expression which is bounded and analytic in D_+, and hence its contribution vanishes.

Equations (66) determine the t-transforms $\tilde{g}_1(k)$ and $\tilde{g}_2(k)$ of the unknown boundary values $q_x(0,t)$ and $q_{xx}(0,t)$ in terms of the given initial and boundary conditions ($q_0(x)$ and $g_0(t)$), as well as in terms of a function $\hat{q}_T(k)$ which is analytic for $\operatorname{Im} k < 0$ and of order $O(1/k)$ as $k \to \infty$. Thus, these equations define a generalized Dirichlet to Neumann map in the k-plane. This map is *not* unique, but it involves $\hat{q}_T(k)$ and hence is defined within an equivalence class.

It is also possible, using the global relation, to derive the generalized Dirichlet to Neumann map directly for the boundary values themselves; i.e., it is possible to express $\{\partial_x^j q(0,t)\}_N^{n-1}$ in terms of $\{g_l(t)\}_0^{N-1}$ and of $q_0(x)$. For simplicity we consider only the linearized version of the NLS, i.e., equation (1a) with $\omega = k^2$. Equation (64) becomes

$$i\int_0^T e^{ik^2 s} q_x(0,s)\,ds = k\tilde{g}_0(k) + \hat{q}_0(k) - e^{ik^2 T}\hat{q}_T(k), \quad \operatorname{Im} k \le 0. \tag{67}$$

In order to construct the Dirichlet to Neumann map, we must invert the integral appearing on the left-hand side of equation (67). This inversion is straightforward: multiplying this integral by $k\exp(-ik^2 t)/\pi$, $t < T$; integrating around the third quadrant of the complex k-plane; and using the FT, we find $iq_x(0,t)$. We note that $\exp[ik^2(T-t)]$ is bounded and analytic in the third quadrant, and $\hat{q}_T(k)$ is analytic for $\operatorname{Im} k < 0$; thus the unknown term $\hat{q}_T$ yields a zero contribution.

For more complicated initial boundary value problems, the analogous inversion is not trivial. For example, for the moving initial boundary value problem, $l(t) < x < \infty$, where $l(t)$ is a given smooth monotonic function, the relevant global relation involves the integral

$$\int_0^T e^{i[k^2 s - kl(s)]} f(s)\,ds, \quad f(s) = q_x(l(s), s).$$

This integral can be analytically inverted in terms of a linear Volterra integral equation by performing the spectral analysis of

$$\frac{\partial \mu}{\partial t} + i\Big[k^2 - k\frac{dl}{dt}\Big]\mu = f(t).$$

The NLS on the Half-Line

We first present the nonlinear version of the direct map, i.e., the nonlinear version of equations (62) and (63) with $n = 2$, $c_0 = -1$, and $c_1 = i$.

Direct Map

$$\{q_0(x), g_0(t), g_1(t)\} \mapsto \{a(k), b(k), A(k), B(k)\},$$
$$g_0(t) := q(0,t), \quad g_1(t) := \partial_x q(0,t).$$

The spectral functions $a(k)$ and $b(k)$ are defined in terms of $q_0(x)$ by

$$a(k) = \phi_2(0,k), \quad b(k) = \phi_1(0,k), \quad \operatorname{Im} k \ge 0, \tag{68}$$

where $(\phi_1(x,k), \phi_2(x,k))$ is the solution of the linear Volterra integral equation

$$\begin{aligned} \phi_1(x,k) &= \int_\infty^x e^{-2ik(x-x')} q_0(x')\phi_2(x',k)\,dx', \\ \phi_2(x,k) &= 1 + \int_\infty^x \bar{q}_0(x')\phi_1(x',k)\,dx', \qquad \operatorname{Im} k \ge 0. \end{aligned} \tag{69}$$

The spectral functions $A(k)$ and $B(k)$ are defined in terms of $g_0(t)$ and $g_1(t)$ by

(70) $A(k) = \overline{\Phi_2(T,\bar{k})}, \quad B(k) = -e^{4ik^2T}\Phi_1(T,k), \quad k \in \mathbb{C},$

where $(\Phi_1(t,k), \Phi_2(t,k))$ is the solution of the linear Volterra integral equation

(71)

$$\Phi_1(t,k) = \int_0^t e^{-4ik^2(t-t')}\big[-i|g_0(t')|^2\Phi_1(t',k) + (2kg_0(t') + ig_1(t'))\Phi_2(t',k)\big]\,dt',$$

$$\Phi_2(t,k) = 1 + \int_0^t \big[(2k\overline{g_0(t')} - i\,\overline{g_1(t')})\Phi_1(t',k) + i|g_0(t')|^2\Phi_2(t',k)\big]\,dt'.$$

Inverse Map

$$\left\{a(k), e^{-4ik^2t}b(k), A(k), e^{-4ik^2t}B(k)\right\} \mapsto q(x,t),$$

$$q(x,t) = -\frac{1}{\pi}\left\{\int_{\partial D_2}\overline{\Gamma(\bar{k})}e^{-i(4k^2t+2kx)}M_{11}^+\,dk + \int_{-\infty}^{\infty}\gamma(k)e^{-i(4k^2t+2kx)}M_{11}^+dk + \int_0^{\infty}|\gamma(k)|^2M_{12}^+dk\right\}, \quad (72)$$

where ∂D_2 denotes the boundary of the third quadrant of the complex k-plane and $M(x,t,k)$ satisfies the following 2×2 matrix RH problem:

(i) M is analytic in $k \in \mathbb{C}\setminus L$, where L denotes the union of the real and the imaginary axes.
(ii) $M = \operatorname{diag}(1,1) + O(1/k)$ as $k \to \infty$.
(iii) $M^- = M^+J$, $k \in L$ (cf. Figure 5), where

$$J_1 = \begin{bmatrix} 1 & 0 \\ \Gamma(k)e^{i(4k^2t+2kx)} & 1 \end{bmatrix}, \quad J_2 = J_3J_4^{-1}J_1,$$

$$J_3 = \begin{bmatrix} 1 & \overline{\Gamma(\bar{k})}e^{-i(4k^2t+2kx)} \\ 0 & 1 \end{bmatrix},$$

$$J_4 = \begin{bmatrix} 1 & -\gamma(k)e^{-i(4k^2t+2kx)} \\ \overline{\gamma(k)}e^{i(4k^2t+2kx)} & 1-|\gamma(k)|^2 \end{bmatrix},$$

and

$$\gamma(k) = \frac{b(k)}{\overline{a(\bar{k})}}, \quad \Gamma(k) = \frac{1}{a(k)}\left[\frac{\overline{A(\bar{k})}}{\overline{B(\bar{k})}}a(k) - b(k)\right]^{-1}.$$

The spectral functions satisfy the following global relation, which is the nonlinear analog of (64):

(73) $a(k)B(k) - b(k)A(k) = e^{4ik^2T}c_T(k), \quad \operatorname{Im} k \geq 0,$

where $c_T(k)$ is a function analytic for $\operatorname{Im} k \geq 0$ and of $O(1/k)$ as $k \to \infty$.

In order to determine $g_1(t)$ we must analyze the global relation (73). For simplicity we assume $q_0(x) = 0$, which implies $a = 1$ and $b = 0$. Then, using the definition of $B(k)$ (equations (70) and (71)), we find that equation (73) yields the following analog of equation (67):

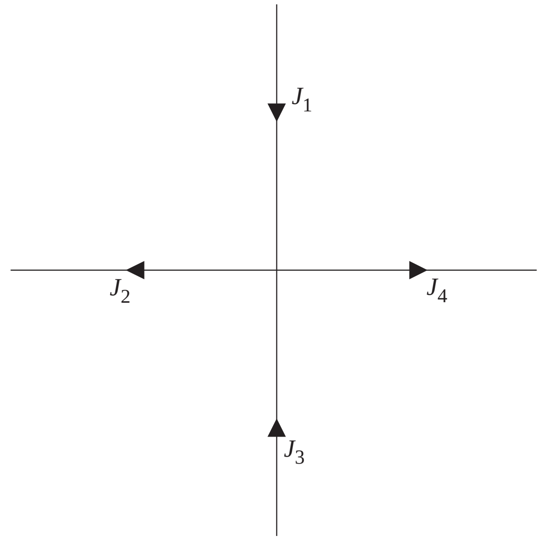

Figure 5. The contour L with jumps J_1, J_2, J_3, and J_4.

$$\int_0^T e^{4ik^2s}[i|g_0(s)|^2\Phi_1(s,k) - (2kg_0(s) + ig_1(s))\Phi_2(s,k)]ds = e^{4ik^2T}c_T(k), \ \operatorname{Im} k \geq 0.$$

In spite of the fact that the integral involving $g_1(s)$ is nonlinear (since Φ_2 is defined in terms of g_1), it is still possible to solve analytically the above equation for $g_1(t)$ in terms of $\{g_0(t), \Phi_1(t,k), \Phi_2(t,k)\}$. Then, substituting the resulting expression in the differential form of equations (71), we obtain a system of two nonlinear ODEs for Φ_1 and Φ_2, which yields (Φ_1, Φ_2) and hence $g_1(t)$.

The rigorous analysis of the Dirichlet problem of NLS [15, 16] involves the following: Given $q_0(x) \in S(\mathbb{R}^+)$, (ϕ_1, ϕ_2) are defined through the linear Volterra integral equation (69); thus $a(k)$ and $b(k)$ are well defined. Given these functions as well as the smooth function $g_0(t)$, we find that the procedure described earlier yields $g_1(t)$ through a system of two nonlinear ODEs. The function $g_1(t)$ is globally defined if g_0 is small or if there exists an *a priori* existence result for $q_x(0,t)$ from classical PDE techniques. For the focusing NLS, the identity $|\Phi_1|^2 + |\Phi_2|^2 = 1$ implies global existence without the smallness assumption. Having determined $g_1(t)$, we have defined globally the functions $A(k)$ and $B(k)$. The RH problem for M has a unique global solution, and hence $q(x,t)$ is globally defined. After establishing that the direct and inverse maps are well defined, we can establish that $q(x,t)$ satisfies NLS and that $q(x,0) = q_0(x)$ and $q(0,t) = g_0(t)$.

From the above result it follows that although the method for solving IBV problems for integrable nonlinear PDEs is conceptually similar to the method for solving linear PDEs, there exists the technical difficulty of characterizing the unknown boundary values through the solution of a system of nonlinear ODEs. There exist certain particular boundary conditions for which the nonlinear ODEs can be avoided and $A(k)$, $B(k)$ can be explicitly expressed in terms of $a(k)$, $b(k)$ and the given boundary condition. An example of such a boundary condition for the NLS is the homogeneous Robin problem $q_x(0,t) - c\,q(0,t) = 0$, where c is a real constant [16]. In this case

$$\frac{B(k)}{A(k)} = -\left(\frac{2k+ic}{2k-ic}\right)\frac{b(-k)}{a(-k)}. \tag{74}$$

The direct map defined by equations (68) and (70) is the nonlinearization of the corresponding direct map of the linear version of the NLS. Indeed, if the initial condition $q_0(x)$ is small, equations (69) imply that $\phi_1 \sim 0$ and $\phi_2 \sim 1$; thus equations (68) imply that $a \sim 1$ and $b(k)$ tends to the FT of $q_0(x)$. Similarly, if the boundary values $g_0(t) = q(0,t)$ and $g_1(t) = q_x(0,t)$ are small, equations (71) imply $\Phi_1 \sim 0$ and $\Phi_2 \sim 1$; thus equations (70) imply that $A \sim 1$ and that $B(k)$ tends to the time-transform of $2kg_0(t) + ig_1(t)$ (which is the spectral function associated with the linearized NLS). Similarly, the inverse map expressed by equation (72) is the nonlinearization of the inverse map expressed by equation (59). Indeed, if the spectral functions are small, the RH problem defining M implies that $M \sim \text{diag}(1,1)$, and then equation (72) implies that $q(x,t)$ tends to the expression defined by equation (59) (for the case of $\omega(k) = k^2$).

Conclusions

We have shown that Fourier transforms in one and two space dimensions can be nonlinearized. Nonlinear FTs can be used for the solution of the Cauchy problem of integrable nonlinear evolution PDEs in one and two spatial dimensions. Examples of nonlinear FTs in one and two spatial dimensions are presented above.

The derivation of nonlinear FTs is based on the spectral analysis of matrix linear eigenvalue problems. For example, the derivation of the nonlinear FTs presented above is based on the spectral analysis of the matrix eigenvalue equations (7a) and (25). This analysis is based on formulating an RH problem and a ∂-bar problem, respectively. These two problems can also be used for the derivation of the classical FTs: The spectral analysis of the scalar equation (9a) and the formulation of an RH problem give rise to the FT in one dimension, while the spectral analysis of the scalar eigenvalue equation (18) and the formulation of a ∂-bar problem give rise to the FT in two dimensions.

Integrable nonlinear PDEs have the distinctive feature that they can be written as the compatibility condition of two eigenvalue equations. The solution of their Cauchy problem is based on using one of these eigenvalue equations (the t-independent part) to derive a nonlinear FT. The other eigenvalue equation (the t-dependent part) determines the time evolution of the nonlinear Fourier data. Linear PDEs can also be written as the compatibility condition of two eigenvalue equations. For example, equation (1a) is the compatibility condition of the eigenvalue equations (4).

This provides the first unification: Both linear and integrable nonlinear evolution PDEs in one and two spatial dimensions can be written as the compatibility condition of two linear eigenvalue equations, called a Lax pair. Furthermore, the Cauchy problem of these equations can be analyzed by performing the spectral analysis of the t-independent part of the Lax pair. For linear equations this analysis yields the solution in terms of the FT, while for nonlinear equations it yields nonlinear generalizations of the associated linear formulas.

The formulation of linear PDEs in terms of Lax pairs appears to express a deeper kind of separability than the classical separation of variables. This claim is based on the following considerations: (i) It is the Lax pair formulation that generalizes to nonlinear integrable PDEs. The Lax pairs of nonlinear PDEs are non-Abelian versions of the corresponding Lax pairs of linear PDEs. (ii) The Lax pair formulation of linear PDEs provides a spectral approach to the solution of boundary value problems for linear PDEs. Actually, it provides the effective implementation, as well as the spectral interpretation, of the Ehrenpreis-Palamodov type integral representations. For example, using this approach, we can show that for linear evolution PDEs on the half-line the abstract Ehrenpreis-Palamodov type integral representation (58) takes the concrete form given by equation (59). In comparison with initial value problems, the analysis of IBV problems of linear PDEs uses two novel ideas: (a) In order to determine the direct and inverse maps, one uses the *simultaneous* spectral analysis of both parts of the Lax pair, as opposed to the case of the Cauchy problem, where one performs the spectral analysis of *only* the t-independent part of the Lax pair. (b) One eliminates the unknown boundary values by using the fact that the spectral functions satisfy a simple algebraic equation, called the *global relation*.

Our approach to IBV problems of integrable nonlinear evolution PDEs is conceptually similar to that of linear PDEs: (a) The simultaneous spectral analysis of the Lax pair yields a direct and an inverse map. (b) The spectral functions $a(k)$ and $b(k)$ are defined in terms of the given function $q_0(x)$. However, for the Dirichlet problem the spectral functions $A(k)$ and $B(k)$ depend on the known function $g_0(t)$ and on the *unknown* function $g_1(t)$. It is again possible to eliminate $g_1(t)$ using the global relation. But for nonlinear problems this step is in general *nonlinear*. Indeed, although for some particular boundary conditions it is possible to express A and B *explicitly* in terms of the known spectral functions and the known boundary conditions (see equation (74)), for general boundary conditions one must solve a system of two *nonlinear* ODEs in order to compute $g_1(t)$.

This provides a further unification: IBV problems for both linear and integrable nonlinear PDEs can be analyzed by (a) performing the simultaneous spectral analysis of the Lax pair and (b) using the global relation to eliminate the unknown boundary values. For linear PDEs this yields the effective implementation of the Ehrenpreis-Palamodov integral representation. For nonlinear equations, it provides the extension of this fundamental representation to integrable nonlinear PDEs.

For linear evolution PDEs in one spatial dimension, the solution of both the Cauchy and IBV problems can be expressed in terms of integrals involving *explicit exponential* (x, t) *dependence*. This makes it possible to study effectively the asymptotic properties of the solution. For example, one can use the steepest descent method to study the long-time behavior of the solution. For nonlinear evolution PDEs, in the case of the Cauchy problem the solution can be expressed in terms of an integral involving explicit exponential (x, t) dependence as well as a function $M(x, t, k)$ (see equation (16)). Although M has a complicated (x, t) dependence, it is defined in terms of an RH problem that has a jump matrix with *explicit exponential* (x, t) dependence. This allows one to obtain rigorous asymptotic results using the powerful and elegant nonlinearization of the steepest descent method of Deift and Zhou [17]. In this way, both the long-time asymptotics and the zero-dispersive limit [18] can be computed. An advantage of the new method for solving IBV problems is that it provides a similar representation for $q(x, t)$, where the associated integrals and RH problems, although now more complicated, still have explicit exponential (x, t) dependence. This again allows one to compute effectively the asymptotic behavior of the solution using the Deift-Zhou method [19].

In this article we have concentrated on IBV problems on the half-line. Similar results can be obtained for both linear and integrable nonlinear PDEs of the finite interval [20].

It was emphasized by I. M. Gel′fand and one of the authors [9] that the RH and ∂-bar formalism presented above provides a new approach to inverting linear integral transforms. This idea was used in [10], where it was shown that the spectral analysis of equation (43) (or equivalently of equation (49)) and the RH and ∂-bar formalisms yield the Radon transform pair. It is remarkable that the analysis of a slight generalization of this equation, namely of equation (54), gives rise to the attenuated Radon transform. This transform provides the mathematical basis of a new, powerful technique for the functional imaging of the brain called SPECT. This new approach can also be used to invert analytically other types of integrals. These include integrals characterizing the generalized Dirichlet to Neumann map for moving initial boundary value problems for evolution PDEs (in terms of linear Volterra integral equations), as well as integrals characterizing the Dirichlet to Neumann map for linear elliptic PDEs (in terms of linear Fredholm integral equations); see below.

The methodology for solving IBV problems for linear evolution PDEs can also be applied to linear elliptic PDEs. The analogs of the direct and inverse maps for the Laplace and Helmholtz equations in a convex polygon are given in [21]. The systematic analysis of the global relation yields the *explicit* generalized Dirichlet to Neumann map for a large class of problems, including several types of boundary value problems for the equilateral triangle (generalizing classical results of Lamé). However, elliptic PDEs in general polygonal domains present a new difficulty: the global relation *cannot* be solved in closed form, but it yields an *auxiliary* RH problem. For simple polygons this RH problem has a jump on the infinite line; thus it is equivalent to a Wiener-Hopf problem. This explains the central role played by the Wiener-Hopf technique in much earlier work.

For a convex domain the global relation reduces the characterization of the Dirichlet to Neumann map to the problem of inverting certain types of integrals. For example, if $q(z, \bar{z})$ satisfies

$$q_{z\bar{z}} + \alpha q = 0$$

in a convex domain D, where α is a real constant, then the differential form

$$W(z, \bar{z}, k) = e^{kz-(\alpha/k)\bar{z}}[q_z dz - \frac{\alpha}{k} q d\bar{z}], \quad k \in \mathbb{C},$$

is closed. The associated global relation is $\int_{\partial D} W = 0$. Hence, for the Dirichlet problem one must invert the integral

$$\int_0^S e^{kz(s)-(\alpha/k)\bar{z}(s)} f(s) ds, \qquad k \in \mathbb{C},$$

where s is the arclength parameter and S is the total arclength of the curve ∂D. This integral can be inverted through a linear Fredholm integral equation by performing the spectral analysis of an appropriate ODE.

Dedication. This article is dedicated to Peter Lax, whose decisive discovery of Lax pairs is the basis of *all* developments presented here.

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