Variations on the Excedance Statistic in Permutations

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Abstract

The distribution of permutations with respect to excedances is described by the Eulerian numbers. The reccurence generating the numbers is generalised to describe what happens to the excedance statistic when altering the permutation set. The discussion mainly deals with subsets of the natural numbers on the form $k[n]+j$, providing explicit formulas for the cases of $j=0$. Special attention is paid to excedances of permutations of only even or odd numbers, for which some explicit formulas and generating functions are proved, partly bijectively for the former. Also, further generalisations are discussed and a bijection to refined descents is given.

Keywords: permutation statistics, excedance, distribution, permutation set, generating function

Sammanfattning

Fördelningen av permutationer med avseende på överskott beskrivs av de eulerska talen. Rekursionen som genererar talen generaliseras för att beskriva vad som händer med överskottsstatisitiken när permutationsmängden ändras. Diskussionen behandlar främst delmängder av de naturliga talen på formen $k[n]+j$ och innehåller explicita uttryck för de fall då $j=0$. I synnerhet berörs överskott av permutationer av uteslutande udda eller jämna tal, för vilka några explicita uttryck och genererande funktioner bevisas, delvis bijektivt för de förra. Dessutom diskuteras ytterligare generaliseringar och en bijektion till generaliserade fall ges.

Nyckelord: permutationsstatistik, överskott, fördelning, permutationsmängd, genererande funktion

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1 Introduction

1.1 Background and Outline

The theory of permutation statistics has been studied for about a hundred years but never as intensely and with such great success as during the past few decades. The subject became an established discipline of mathematics after MacMahon's extensive study [6]. In his study, MacMahon considered four different statistics for a permutation, of which two were the now classical Eulerian statistics of the number of excedances and the number of descents. These statistics are closely related and have the very same distribution.

Until very recently, little research had been made into refinements of the excedance and descent statistics such as the occurrence of a number greater than some factor times its index, which has the same effect as using the classical definition of an excedance and varying the permutation set. Concurrently with this work, Kitaev and Remmel [4, 5] introduced the notion of descents of various types restricting the discussion to descents where the number causing the descent is equivalent to a number modulo k. We will investigate what happens to the excedance statistic when we vary the set to permute. The results have been found independently from Kitaev's and Remmel's work, but many of them can be shown to be equivalent to theirs.

The leading parts in this work will be played by the sets of positive even and odd numbers. In Section 2 we will see a general recurrence for the number of permutations with respect to excedances for any finite subset of $\mathbb{N}$ and derive explicit formulas for permutations with respect to even and odd numbers. For the sake of completeness, Section 3 will provide us with generating functions for the distributions of the permutations of even length. Ideas on how to find bijective proofs for the explicit formulas in Section 2 will be presented in Section 4, together with combinatorial arguments for parts of the distributions. The more general subsets $k[n] + d$ of $\mathbb{N}$ will enter the stage in Section 5, together with a discussion of how to generalise further to subsets of $\mathbb{Q}$. Also, a new application of the *transformation fondamentale* for refined excedances and descents will be presented and, finally, a discussion of similar refinements to other statistics will be given.

1.2 Preliminaries

There are some basic definitions and terms that are necessary for the following discussions.

Definition 1. Let S be a finite set, called the *alphabet*. A *word* is a finite sequence $a_1a_2\cdots a_n$ of elements in S.

Definition 2. Let S be a finite ordered set with n elements. A *permutation* π of S is a linear ordering of the elements of S. The permutation π can be denoted by a word $a_1a_2\cdots a_n$, where a_i is a letter in the alphabet S. If $S = \{b_1, b_2, \ldots, b_n\}$ then π is a mapping of the elements of S onto S itself, given by $\pi(b_i) = a_i$. Thus, π can also be regarded as a bijection $S \to S$.

A permutation π can be written in cyclic form. A cycle $(a_1a_2\cdots a_k)$ means that $\pi(a_i) = a_{i+1}$ for $i < k$ and $\pi(a_k) = a_1$. The inverse of π can be obtained by reversing every cycle in π.

The set of all permutations of S is denoted as $\mathcal{S}(S)$. This notation will also be used for sequences. If A is a sequence then $\mathcal{S}(A)$ is the set of all permutations of A considered as a set.

Remark. We will only be considering permutations of subsets of $\mathbb{Q}$ and almost exclusively subsets of $\mathbb{N}$.

Remark. Consider the set $\{1, 2, \ldots, n\}$. Further on we will denote this set as $[n]$. The set of all permutations of $[n]$, $\mathcal{S}([n])$, will be written in short as $\mathcal{S}_n$. There are $n!$ permutations of $[n]$.

Definition 3. An *excedance* of a permutation $\pi = a_1a_2\cdots a_n$ is an i such that $a_i > i$. A *weak excedance* of π is an i such that $a_i \geq i$.

Definition 4. A *descent* of a permutation $\pi = a_1a_2\cdots a_n$ is an i such that $a_i > a_{i+1}$.

Example. Let $\pi = 132$. Then 2 is an excedance of π since there is a 3 in the second position of π. Since the 3 is followed by a smaller number, 2 is also a descent of π.

1.3 Definitions and Terminology

There are some definitions and terms that will be used specifically for our discussions.

- Let X be a countable ordered set. Then X_n denotes the set consisting of the n smallest elements of X.

- The set $E = \{2, 4, 6, 8, \ldots\}$ and the set $O = \{1, 3, 5, 7, \ldots\}$. Thus, $E_n = \{2, 4, 6, \ldots, 2n\}$ and $O_n = \{1, 3, 5, \ldots, 2n-1\}$.

- The set $k[n] = \{k, 2k, 3k, \ldots, nk\}$. For convenience, we define $k[n] + d$ from the set $\{0, 1, 2, \ldots, n-1\}$ so that $k[n] + d = \{d, k+d, 2k+d, \ldots, k(n-1)+d\}$ for the rational number k and positive integers n and d, where $1 \leq d < k$.

- An element a_i of a permutation $\pi = a_1a_2\cdots a_n$ *causes* an excedance (or a descent) of π if i is an excedance (or a descent, respectively) of π.
- An excedance of a permutation $a_1a_2\cdots a_n$ on an index set $b_1b_2\cdots b_n$ is a b_i such that $a_i > b_i$.
- Unless explicitly stated, we will use the same definition for excedances on words as for permutations. Note that this is not the conventional definition (see Section 5.2.3).

Finally, the following definitions will be useful.

Definition 5. Let A be a subset of $\mathbb{N}$ with n elements. A number $i \in A$ is said to be *stable in A* if $i > n$. Otherwise it is said to be *unstable in A*.

Remark. The stable numbers in A are those which cause an excedance for any permutation $\pi \in \mathcal{S}(A)$.

In order to talk about distributions more easily, let us also introduce the following notation.

Definition 6. Let A be a finite set. The number of permutations of A with k excedances is denoted as $P(A, k)$.

2 Excedances of Permutations of Subsets of $\mathbb{N}$

2.1 A General Recurrence

In total, there is one permutation in $\mathcal{S}_3$ with no excedances, four with one excedance and one with two. What can be said in general about the statistics on excedances of permutations of different sets? There is a well-known answer to that question if the set we consider is $[n]$ for some positive integer n.

Proposition 2.1. *Let $A(n,k)$ be the number of permutations $\pi \in \mathcal{S}_n$ with k excedances. Then A satisfies the recurrence relation*

$$A(n,k) = (k+1)A(n-1,k) + (n-k)A(n-1,k-1), \tag{2.1}$$

subject to the initial conditions

$$A(0,k) = \begin{cases} 1 & \text{for } k = 0 \\ 0 & \text{otherwise} \end{cases}.$$

Remark. The numbers $A(n,k)$ are also known as the Eulerian numbers.

Proof. Take a permutation $\pi \in \mathcal{S}_{n-1}$ with k excedances. Define $f_i(\pi)$ for $i \in [n]$ as the bijective function which maps π to $\pi_i \in \mathcal{S}_n$, where π_i is created by adding n to π by placing it at index i and moving the element previously at index i to the last position. The permutation π_i has k excedances if i is an excedance in π, since i will still be an excedance and the new index will not, or if $i = n$. Otherwise, π_i has $k+1$ excedances, since i will be an excedance after inserting n. As π has k excedances there are $k+1$ positions where to insert n without causing a new excedance. Furthermore, there are $(n-1)-k$ indices in π which are not excedances, so f_i will map π to a permutation with $k+1$ excedances for $n-(k+1)$ of the indices. $\square$

Now that we can determine the number of permutations with a given number of excedances for the easily described finite subset $[n]$ of $\mathbb{Z}^+$, what can be said of other easily described finite subsets of $\mathbb{Z}^+$? We can note right away that as soon as the set we are permuting is a subset of $\mathbb{Z}^+$ not equal to $[n]$ we have a few numbers that are larger than the length of the permutation and thus always cause an excedance. This motivates the notion of stable and unstable numbers, introduced in Definition 5.

Let us begin by having a closer look at the previous proof. It appears that the recurrence formula can be generalised to arbitrary sequences of elements in $\mathbb{N}$ under a few restrictions.

Proposition 2.2. *Let A be a strictly increasing sequence of nonnegative integers. Let $f(n,k)$ be the number of permutations $\pi \in \mathcal{S}(A_n)$ with k excedances. If j is the number of excedances in π among the unstable numbers in A_n, then f satisfies the recurrence relation*

$$f(n,k) = (j+1)f(n-1,k) + (n-j)f(n-1,k-1),$$

subject to the initial conditions

$$f(0,k) = \begin{cases} 1 & \text{for } k = 0 \\ 0 & \text{otherwise} \end{cases}.$$

Remark. The sequence needs not be strictly increasing, but this restriction does not impose any limits on the sets we will be considering.

Proof. The difference when constructing $\pi_i \in \mathcal{S}(A_n)$ from $\pi \in \mathcal{S}(A_{n-1})$ in the previous proof between the case of proving the recurrence of the Eulerian numbers and this more general case is that the added number may be either an unstable or a stable number in A_n.

Due to the fact that A_n is strictly increasing we have that the added number is an unstable number in A_n only in the case of $A_n = 123\cdots n$, in

n \ k	1	2	3	4	5	6	7	8	9
1	1								
2	1	1							
3	0	4	2						
4	0	4	16	4					
5	0	0	36	72	12				
6	0	0	36	324	324	36			
7	0	0	0	576	2592	1728	144		
8	0	0	0	576	9216	20736	9216	576	
9	0	0	0	0	14400	115200	172800	57600	2880

Table 1: The number of permutations of E_n with k excedances.

which $j = k$ since there are only unstable numbers in A_n, and the result follows from Proposition 2.1.

In the case of adding a stable number the number of excedances in π_i will be the same as in π if and only if i is an excedance in π due to an unstable number in A_n (and not if i is the length of π_i). Otherwise the number of excedances will increase by 1.

According to the prerequisites of the proposition, j is the number of excedances in π_i among the unstable numbers in A_n. If the number of excedances in π_i is the same as in π then the number of excedances among the unstable numbers must have diminished by one in π_i, so there are $j+1$ excedances in π among the unstable numbers in A_n.

Moreover, there are $(n-1)-(j+1)$ indices in π which are not excedances nor stable numbers, so f_i will map π to a permutation with $j+1$ excedances for $n-(j+1)$ of the indices, including the index n. □

2.2 Distributions of Permutations of the Even and Odd Numbers

Now that we have found a nice general recurrence for the distribution of permutations with respect to excedances for a vast class of sequences we would of course like to see some explicit formulas for some of the members of the class. Working on the full class would be a vain attempt so we shall be doing something more feasible. It may be worthwhile having a glance at a set of even numbers or odd numbers. In fact, listing the distribution of permutations of O_n and E_n with respect to excedances yields interesting figures (see Table 1 and 2).

To start with, the following is true.

n \ k	0	1	2	3	4	5	6	7	8
1	1								
2	0	2							
3	0	2	4						
4	0	0	12	12					
5	0	0	12	72	36				
6	0	0	0	144	432	144			
7	0	0	0	144	1728	2592	576		
8	0	0	0	0	2880	17280	17280	2880	
9	0	0	0	0	2880	57600	172800	115200	14400

Table 2: The number of permutations of O_n with k excedances.

Lemma 2.3. *The number of permutations of E_n containing the maximum possible number of excedances (n) is*

$$\left\lfloor \frac{n}{2} \right\rfloor ! \left\lfloor \frac{n+1}{2} \right\rfloor !. \tag{2.2}$$

Proof. Disregarding the order of the stable numbers, there is a simple bijection between permutations of E_n with a maximum number of excedances and words of length $\lfloor \frac{n}{2} \rfloor$ with no excedances. As there are $\lfloor \frac{n}{2} \rfloor !$ such words and $\lfloor \frac{n+1}{2} \rfloor !$ permutations of the $\lfloor \frac{n+1}{2} \rfloor$ stable numbers, the result follows.

To construct the bijection, let f be the function that maps every $\pi \in \mathcal{S}(E_n)$ to a pair consisting of a word w of the same length as the number of unstable numbers, and a word depicting the order of the stable numbers. The number in position i in the word w denotes the position of $2i$ in π minus the number of smaller numbers to the left of $2i$ in π. f is clearly bijective. All numbers in π are excedances if and only if w has no excedances: A number which does not cause an excedance in π will obviously cause one in w. Consider the leftmost excedance in w, if there is one. If the position is i, then the position in π is at least $i+1$ added to the number of numbers to the left of the i-th position, which sums up to at least $2i$.

Example. $f(2846) = (12, 21)$.

□

Lemma 2.4. *The number of permutations of O_n containing the minimum possible number of excedances ($\lfloor \frac{n}{2} \rfloor$) is*

$$\left\lfloor \frac{n}{2} \right\rfloor ! \left\lceil \frac{n+1}{2} \right\rceil !. \tag{2.3}$$

n \ k	1	2	3	4	5	6	7	8	9	10
1	1									
2	1	1								
3	0	2	1							
4	0	1	4	1						
5	0	0	3	6	1					
6	0	0	1	9	9	1				
7	0	0	0	4	18	12	1			
8	0	0	0	1	16	36	16	1		
9	0	0	0	0	5	40	60	20	1	
10	0	0	0	0	1	25	100	100	25	1

Table 3: The number of permutations of E_n with k excedances, divided by the number in 2.2.

Proof. Let g be the bijective function that reverses a permutation $\pi \in \mathcal{S}_n$ and swaps the i-th largest unstable number in $\mathcal{S}_n$ with the i-th smallest for $1 \leq i \leq \lceil \frac{n}{2} \rceil$. Let us use the bijection f from the proof to Lemma 2.3 on the permutations $g(\pi) \in \mathcal{S}(O_n)$. The word w that f maps to will have length $\lceil \frac{n}{2} \rceil$. For odd n, none of the unstable numbers in π is an excedance if and only if no numbers in w are greater than their index, which gives us $\lceil \frac{n}{2} \rceil! = \lceil \frac{n+1}{2} \rceil!$ possible words. For even n, however, none of the unstable numbers in π is an excedance if and only if no numbers in w are greater than their index plus one, since the biggest unstable number is $n-1$ and we have two positions for where to place it without causing an excedance. The number of valid words is then $\left(\lceil \frac{n}{2} \rceil + 1\right)! = \lceil \frac{n+1}{2} \rceil!$.

Finally, there are $\lfloor \frac{n}{2} \rfloor$ stable numbers that may be permuted in any way. $\square$

Encouraged by these results, let us divide the number of permutations of E_n and O_n with respect to excedances with the numbers in 2.2 and 2.3, respectively. As a matter of fact, the result is an appealing picture, where some nice patterns emerge.

Take a closer look at every other line in Tables 3 and 4 and disregard the initial 0s in every line. The sequences of numbers occurring correspond to those of well-known ones.

Disregarding initial 0s, the lines in Table 3 where n is even constitute the entries of Pascal's triangle squared [9, sequence A008459].

The numbers in lines with n odd correspond to the number triangle $\binom{m}{k}\binom{m+1}{k+1}$, where $m = \lfloor \frac{n}{2} \rfloor$ [9, sequence A103371].

n \ k	0	1	2	3	4	5	6	7	8	9
1	1									
2	0	1								
3	0	1	2							
4	0	0	1	1						
5	0	0	1	6	3					
6	0	0	0	1	3	1				
7	0	0	0	1	12	18	4			
8	0	0	0	0	1	6	6	1		
9	0	0	0	0	1	20	60	40	5	
10	0	0	0	0	0	1	10	20	10	1

Table 4: The number of permutations of O_n with k excedances, divided by the number in 2.3.

As to Table 4, the sequence regarding n odd is the same as for Table 3 but with each line reversed. The sequence is described by $\binom{m}{k}\binom{m+1}{k}$.

Finally, and interestingly enough, the numbers in the odd lines in Table 4 are the Narayana numbers [9, sequence A001263].

Definition 7. The *Narayana numbers* $N(n,k)$ are defined by

$$N(n,k) = \frac{1}{n}\binom{n}{k}\binom{n}{k-1} = \frac{1}{k}\binom{n-1}{k-1}\binom{n}{k-1}.$$

2.3 Some Explicit Formulas

The discussion in the previous section suggests explicit formulas for the number of permutations of the even and odd numbers with respect to excedances. We will state the formulas more explicitly and perform inductive proofs for them. Based on the previous remarks, then, it seems reasonable to suspect that we have to deal with the following numbers.

Proposition 2.5. *Let $m = \left\lfloor \frac{n}{2} \right\rfloor$. The number of permutations of E_n with $k = \left\lfloor \frac{n+1}{2} \right\rfloor + j$ excedances for nonnegative even n is*

$$P(E_n,k) = \left(m!\binom{m}{j}\right)^2. \tag{2.4}$$

The number of permutations for positive odd n is

$$P(E_n,k) = m!(m+1)!\binom{m}{j}\binom{m+1}{j+1}. \tag{2.5}$$

Proof. By paying attention to the recurrence in Proposition 2.2 we have already paved the way for an inductive proof. The fact that we have a recurrence in two variables will not put any hardship on us since we have that $P(E_n, k) = 0$ for all $k > n$. Since we have two different formulas for the cases of n even and n odd, at least when not including floor functions, we will carry out an inductive step for each of the cases.

First of all, we have that Equation 2.4 is trivially true for $n = 0$, since

$$P(E_0, k) = \left(0!\binom{0}{j}\right)^2 = \begin{cases} 1 & \text{for } j = 0 \Leftrightarrow k = 0 \\ 0 & \text{otherwise} \end{cases} .$$

Let us assume that Equation 2.4 is true for a fix arbitrary even $n-1$. To facilitate the validation of the fact that Equation 2.5 applies to n, it seems sensible to first translate the recurrence in Proposition 2.2 into terms of m and j for odd n.

We have that $m = \frac{n-1}{2}$ for both n and $n-1$. There is one more stable number in a permutation of E_n than a permutation of E_{n-1} for n odd, why the number of excedances among the unstable numbers in E_{n-1} of permutations of E_{n-1} with k excedances is $j+1$. Accordingly, we have that

$$\begin{aligned}
P(E_n, k) &= (j+1)P(E_{n-1}, k) + (n-j)P(E_{n-1}, k-1) \\
&= (j+1)m!^2\binom{m}{j+1}^2 + (2m-j+1)m!^2\binom{m}{j}^2 \\
&= m!(m+1)!\binom{m}{j}\binom{m+1}{j+1}\cdot \\
&\quad \cdot \left[(j+1)\frac{1}{m+1}\frac{m-j}{j+1}\frac{m-j}{m+1} + (2m-j+1)\frac{1}{m+1}\frac{j+1}{m+1}\right] \\
&= m!(m+1)!\binom{m}{j}\binom{m+1}{j+1}\cdot \\
&\quad \cdot \frac{m^2 - 2mj + j^2 + 2mj - j^2 + j + 2m - j + 1}{(m+1)^2} \\
&= m!(m+1)!\binom{m}{j}\binom{m+1}{j+1}.
\end{aligned}$$

Now we know how to take the step from the even to the odd numbers. Let us assume, then, that Equation 2.5 is true for a fixed arbitrary odd $n-1$ (we know it is for $n = 2$ and if it is true for $n = 3$, then by the above it will be for $n = 4$ and so on). What remains is obviously to show Equation 2.4 for n.

There are $m = \frac{n}{2}$ unstable numbers in E_n. There is one more unstable number in E_n than E_{n-1} for n even, so the number of unstable numbers in

E_{n-1} is $m-1$. Since the number of unstable numbers increases when moving from E_{n-1} to E_n, the number of stable numbers does not. The recurrence in terms of m and j is thus

$$\begin{aligned} P(E_n,k) &= (j+1)P(E_{n-1},k) + (n-j)P(E_{n-1},k-1) \\ &= (j+1)(m-1)!m!\binom{m-1}{j}\binom{m}{j+1}+ \\ &\quad + (2m-j)(m-1)!m!\binom{m-1}{j-1}\binom{m}{j} \\ &= m!^2\binom{m}{j}^2\left[(j+1)\frac{1}{m}\frac{m-j}{m}\frac{m-j}{j+1} + (2m-j)\frac{1}{m}\frac{j}{m}\right] \\ &= m!^2\binom{m}{j}^2\frac{m^2-2mj+j^2+2mj-j^2}{m^2} \\ &= m!^2\binom{m}{j}^2, \end{aligned}$$

which is what we wanted. □

Corollary 2.6. *The number of permutations of E_n for all $n \in \mathbb{N}$ with k excedances is*

$$P(E_n,k) = \left\lfloor\frac{n}{2}\right\rfloor!\left\lfloor\frac{n+1}{2}\right\rfloor!\binom{\lfloor\frac{n}{2}\rfloor}{k-\lfloor\frac{n+1}{2}\rfloor}\binom{\lfloor\frac{n+1}{2}\rfloor}{k-\lfloor\frac{n}{2}\rfloor}. \tag{2.6}$$

Proof. The result follows immediately from Equations 2.4 and 2.5 since $\lfloor\frac{n+1}{2}\rfloor = \lfloor\frac{n}{2}\rfloor$ for even n and $\lfloor\frac{n+1}{2}\rfloor = \lfloor\frac{n}{2}\rfloor + 1$ for odd n. □

Proposition 2.7. *Let $m = \lfloor\frac{n}{2}\rfloor$. The number of permutations of O_n with $k = \lfloor\frac{n}{2}\rfloor + j$ excedances for positive even n is*

$$P(O_n,k) = m!m!\binom{m-1}{j}\binom{m+1}{j+1}. \tag{2.7}$$

The number of permutations for positive odd n is

$$P(O_n,k) = m!(m+1)!\binom{m}{j}\binom{m+1}{j}. \tag{2.8}$$

Proof. We will proceed as in the proof of Proposition 2.5. We have that $P(O_n,k) = 0$ for all $k \geq n$ and that Equation 2.8 is true for $n = 1$ since

$$P(O_1,k) = 0!1!\binom{0}{j}\binom{1}{j} = \begin{cases} 1 & \text{for } j = 0 \Leftrightarrow k = 0 \\ 0 & \text{otherwise} \end{cases}.$$

For the induction step, note that $\left\lfloor \frac{n-1}{2} \right\rfloor = \left\lfloor \frac{n}{2} \right\rfloor - 1 = m-1$ and $k - \left\lfloor \frac{n-1}{2} \right\rfloor = k - \left\lfloor \frac{n}{2} \right\rfloor + 1 = j+1$ for even n. It follows that for even n

$$\begin{aligned}
P(O_n, k) &= (j+1)P(O_{n-1}, k) + (n-j)P(O_{n-1}, k-1) \\
&= (j+1)(m-1)!m!\binom{m-1}{j+1}\binom{m}{j+1} + \\
&\quad + (2m-j)(m-1)!m!\binom{m-1}{j}\binom{m}{j} \\
&= m!^2\binom{m-1}{j}\binom{m+1}{j+1} \cdot \\
&\quad \cdot \left[(j+1)\frac{1}{m}\frac{m-j-1}{j+1}\frac{m-j}{m+1} + (2m-j)\frac{1}{m}\frac{j+1}{m+1}\right] \\
&= m!^2\binom{m-1}{j}\binom{m+1}{j+1} \cdot \\
&\quad \cdot \frac{m^2 - 2mj + j^2 - m + j + 2mj + 2m - j^2 - j}{m(m+1)} \\
&= m!^2\binom{m-1}{j}\binom{m+1}{j+1}.
\end{aligned}$$

For odd n we have that $\left\lfloor \frac{n-1}{2} \right\rfloor = \left\lfloor \frac{n}{2} \right\rfloor = m$ and $k - \left\lfloor \frac{n-1}{2} \right\rfloor = k - \left\lfloor \frac{n}{2} \right\rfloor = j$, so

$$\begin{aligned}
P(O_n, k) &= (j+1)P(O_{n-1}, k) + (n-j)P(O_{n-1}, k-1) \\
&= (j+1)m!^2\binom{m-1}{j}\binom{m+1}{j+1} + \\
&\quad + (2m-j+1)m!^2\binom{m-1}{j-1}\binom{m+1}{j} \\
&= m!(m+1)!\binom{m}{j}\binom{m+1}{j} \cdot \\
&\quad \cdot \left[(j+1)\frac{1}{m+1}\frac{m-j}{m}\frac{m-j+1}{j+1} + (2m-j+1)\frac{1}{m+1}\frac{j}{m}\right] \\
&= m!(m+1)!\binom{m}{j}\binom{m+1}{j} \cdot \\
&\quad \cdot \frac{m^2 - 2mj + j^2 + m - j + 2mj - j^2 + j}{m(m+1)} \\
&= m!(m+1)!\binom{m}{j}\binom{m+1}{j},
\end{aligned}$$

and we are done. □

Corollary 2.8. *The number of permutations of O_n for all $n \in \mathbb{Z}^+$ is*

$$P(O_n, k) = \left\lfloor \frac{n}{2} \right\rfloor ! \left\lfloor \frac{n+1}{2} \right\rfloor ! \binom{\lfloor \frac{n-1}{2} \rfloor}{k - \lfloor \frac{n}{2} \rfloor} \binom{\lfloor \frac{n}{2} \rfloor + 1}{k - \lfloor \frac{n-1}{2} \rfloor}. \tag{2.9}$$

With such neat explicit formulas for the distribution of permutations according to excedances in mind, we would probably be fairly optimistic about finding bijective proofs and generating functions as well. However, looking for the former has mostly resulted in vain attempt, while it will appear that we are more successful in the area of the latter. Moreover, if we are lucky, a generating function might eventually shed some light on how to find bijections.

3 Generating Functions

A naïve approach to finding generating functions for the distributions presented so far using standard methods is likely to lead to complicated formulas. Instead, there is a well-known simple formula for the Eulerian polynomials, which will not only fulfil the ambitions of providing a generating function for the sequence $\{P([n], k)\}_{k \geq 0}$ for any n, but also, as we will see, be a great inspiration when it comes to finding a generating function for $\{P(E_n, k)\}_{k \geq 0}$ and $\{P(O_n, k)\}_{k \geq 0}$ for even n.

3.1 Eulerian Polynomials

Let $e(\pi)$ denote the number of excedances in π. The polynomial

$$A_n(x) = \sum_{\pi \in \mathcal{S}_n} x^{1+e(\pi)}$$

is called the n-th *Eulerian polynomial*. If $A(n, k)$ denotes the coefficient to x^k, then obviously $A(n, k)$ is an Eulerian number. To be more specific, $A(n, k) = P([n], k - 1)$. There is a nice well-known formula for computing $A_n(x)$.

Proposition 3.1. *The Eulerian polynomial $A_n(x)$ satisfies the identity*

$$\frac{A_n(x)}{(1-x)^{n+1}} = \sum_{k \geq 0} k^n x^k. \tag{3.1}$$

Proof. We can get an identity of the form above by using the geometric series evaluation $\sum_{n\geq 0} x^n = \frac{1}{1-x}$ and repeatedly differentiating and multiplying by x. The resulting sequence of identities is as follows:

$$\begin{aligned}
\frac{x}{(1-x)^2} &= \sum_{k\geq 0} kx^k, \\
\frac{x+x^2}{(1-x)^3} &= \sum_{k\geq 0} k^2x^k, \\
\frac{x+4x^2+x^3}{(1-x)^4} &= \sum_{k\geq 0} k^3x^k, \\
&\vdots \\
\frac{\widehat{A}_n(x)}{(1-x)^{n+1}} &= \sum_{k\geq 0} k^nx^k, \\
&\vdots
\end{aligned}$$

where $\widehat{A}_n(x)$ is the numerator of the left-hand side in the n-th identity. Apparently, what remains to show is that $A_n(x) = \widehat{A}_n(x)$. First of all, $\widehat{A}_n(x)$ satisfies the recurrence

$$\begin{aligned}
&\widehat{A}_n(x) = (x-x^2)\frac{d}{dx}\widehat{A}_{n-1}(x) + nx\widehat{A}_{n-1}(x) \text{ for } n > 0, \\
&\widehat{A}_0(x) = x,
\end{aligned}$$

since the left-hand side of the n-th identity equals

$$\frac{(1-x)^n\frac{d}{dx}\widehat{A}_{n-1}(x) - (-1)n(1-x)^{n-1}\widehat{A}_{n-1}(x)}{(1-x)^{2n}}x.$$

The proof will be done if we can show that $\widehat{A}_n(x)$ has the same recurrence as $A_n(x)$. We already know that $A_n(x)/x$ satisfies the recurrence 2.1 ($A_n(x)$ is divided by x to adjust for the fact that the coefficient to x^k is the number of permutations with $k-1$ excedances). Let the coefficient to x^k in $\widehat{A}_n(x)$

be denoted by $\widehat{A}(n,k)$. Then we have that

$$\begin{aligned}
\frac{\widehat{A}_0(x)}{x} &= 1 \text{ and} \\
\frac{\widehat{A}_n(x)}{x} &= (1-x)\frac{d}{dx}\widehat{A}_{n-1}(x) + n\widehat{A}_{n-1}(x) \\
&= (1-x)\sum_{k\geq 0} k\widehat{A}(n-1,k)x^{k-1} + n\sum_{k\geq 0}\widehat{A}(n-1,k)x^k \\
&= \sum_{k\geq 0}\left[(k+1)\widehat{A}(n-1,k) + (n-k)\widehat{A}(n-1,k-1)\right]x^k,
\end{aligned}$$

which is the very same recurrence as the one for $A_n(x)/x$. □

Remark. An alternative proof can be found in Comtet [2, p. 245], which also deals more thoroughly with Eulerian numbers (pp. 240-246).

3.2 Permutations of Even Numbers of Even Length

Could Equation 3.1 lead us to a generating function for the permutations with stable numbers? As we have seen, the Eulerian numbers and the distributions of $P(E_n,k)$ and $P(O_n,k)$ have a similar structure, especially for even n. An interesting and perhaps not too far-fetched guess would be to substitute $A_n(x)$ in Equation 3.1 by a generating polynomial for the permutations of E_n and O_n, and expect some nice easily described polynomial to come out on the right hand side. Performing these actions on the polynomials of the asymmetrically distributed permutations, that is the polynomials generating $\{P(E_n,k)\}_k$ and $\{P(O_n,k)\}_k$ for odd n, does not appear to give any inspiring results. When it comes to the polynomials of the symmetrically distributed $P(E_n,k)$ for even n, however, a promising pattern emerges.

For example, for $n=1$ we have

$$\frac{x+x^2}{(1-x)^3} = x + 4x^2 + 9x^3 + 16x^4 + 25x^5 + \mathcal{O}(x^6),$$

where the coefficient to x^k seems to be k^2, and for $n=2$

$$\frac{4x+16x^2+4x^3}{(1-x)^5} = 4x + 36x^2 + 144x^3 + 400x^4 + 900x^5 + \mathcal{O}(x^6),$$

with the likely coefficent $k^2(k+1)^2$ of x^k.

This nice pattern continues. Let $e(\pi)$ denote the number of excedances in π. Let

$$F_{2n}(x) = \sum_{\pi\in\mathcal{S}(2[2n])} x^{1-n+e(\pi)},$$

that is, the polynomial where the coefficient to x^k is the number of permutations of the smallest $2n$ positive even numbers with $k+n-1$ excedances, where n is the number of stable numbers in $2[2n]$. We have the following result.

Proposition 3.2. *The generating function $F_{2n}(x)$ for the sequence $\{P(E_{2n}, k+n-1)\}_{k>0}$ satisfies, for all $n \in \mathbb{Z}^+$,*

$$\frac{F_{2n}(x)}{(1-x)^{2n+1}} = \sum_{k\geq 0} x^k \prod_{i=0}^{n-1} (k+i)^2. \tag{3.2}$$

Proof. Let us start by rewriting the expression into a sum of the form $\sum a_n x^n$:

$$\begin{aligned} F_{2n}(x) &= (1-x)^{2n+1} \sum_{k\geq 0} x^k \prod_{i=0}^{n-1} (k+i)^2 \\ &= \sum_{k\geq 0} \sum_{j\geq 0} \binom{2n+1}{j} (-1)^j x^{k+j} \prod_{i=0}^{n-1} (k+i)^2 \\ &= \sum_{k\geq 0} \sum_{j=0}^{k} \binom{2n+1}{j} (-1)^j x^k \prod_{i=0}^{n-1} (k-j+i)^2, \end{aligned}$$

where the second equality simply uses the binomial theorem and the third one involves a variable substitution of k for $k-j$ which, since we are only dealing with nonnegative k, introduces the restriction $k \geq j$.

The coefficients of x^{k+1} in $F_{2n}(x)$ are

$$\sum_{j=0}^{k+1} (-1)^j \binom{2n+1}{j} \prod_{i=0}^{n-1} (k-j+i+1)^2.$$

Thus, according to the definition of F_{2n} and Equation 2.4, it remains to show that

$$\begin{aligned} n!^2 \binom{n}{k}^2 &= \sum_{j=0}^{k+1} (-1)^j \binom{2n+1}{j} \prod_{i=0}^{n-1} (k-j+i+1)^2 \\ &= \sum_{j=0}^{k+1} (-1)^j \binom{2n+1}{j} \frac{(k-j+n)!^2}{(k-j)!^2} \\ &= \sum_{j=0}^{k+1} (-1)^j \binom{2n+1}{j} \binom{k-j+n}{n}^2 n!^2, \end{aligned}$$

which is equivalent to

$$\binom{n}{k}^2 = \sum_{i=0}^{k} (-1)^i \binom{2n+1}{i} \binom{n+k-i}{n}^2. \tag{3.3}$$

At first sight, this may look like an amenable identity which should easily be proven by way of the principle of inclusion and exclusion. Unfortunately, this has not appeared to be the case so far. Instead, we will verify it algebraically, something which may be expected to be a tedious task. But do not despair, there is an established algorithm, described by Petkovšek, Wilf and Zeilberger [8], which will leave the most troublesome parts to the computer.

Let k be fixed. If $n \geq k$, then we can divide the sum by the left hand side, and we have to show that a sum equals a constant:

$$\sum_{i=0}^{k} \frac{(-1)^i \binom{2n+1}{i} \binom{n+k-i}{n}^2}{\binom{n}{k}^2} = 1. \tag{3.4}$$

Paule's and Schorn's implementation [7] of Gosper's and Zeilberger's algorithms [8] produces an odd-looking rational function

$$R(n,i) = \frac{i(n+k-i+1)^2(4n^2+11n+i(-3n+2k-3)-2k^2-2k+7)}{(n+1)^3(2n-i+2)(2n-i+3)}. \tag{3.5}$$

Let $F(n,i)$ be the summand of the left-hand side in Equation 3.4. Let $G(n,i) = R(n,i)F(n,i)$. If we believe the implementation to be correct then we have that the following recurrence is satisfied:

I have discovered a truly horrible demonstration of this identity that this margin is too narrow to contain.

$$F(n+1,i) - F(n,i) = G(n,i+1) - G(n,i).$$

What we are actually interested in is the sum of $F(n,i)$ for all $i \in \{0,1,\ldots,k\}$. Summing over the equation gives us

$$\sum_{i=0}^{k} (F(n+1,i) - F(n,i)) = \sum_{i=0}^{k} (G(n,i+1) - G(n,i)),$$

where the right hand side telescopes to

$$G(n,k+1) - G(n,0) = R(n,0) \cdot 0 - 0 \cdot F(n,0) = 0.$$

Thus,

$$\sum_{i=0}^{k} F(n+1,i) = \sum_{i=0}^{k} F(n,i),$$

from which it follows that $\sum_{i=0}^{k} F(n,i)$ is independent of n, that is, constant. What remains is to check that the constant is 1, which we will do by verifying that the sum is 1 for the smallest possible value of n, which is k:

$$\begin{aligned}\sum_{i=0}^{k} F(k,i) &= \sum_{i=0}^{k} (-1)^i \binom{2k+1}{i}\binom{2k-i}{k}^2 \\ &= \underbrace{\sum_{i} (-1)^i \binom{2k+1}{i}\binom{2k-i}{k}^2}_{(\star)} + 1,\end{aligned}$$

and we get a sum whose value is far from obvious. In order to show that $(\star)$ equals 0 we can execute the algorithm once again, allowing k to vary.

Let $\widehat{F}(k,i)$ be the summand of $(\star)$. Let $\widehat{G}(k,i) = \widehat{R}(k,i)\widehat{F}(k,i)$ where

$$\widehat{R}(k,i) = \frac{i(2k-i+1)^2}{(k-i+1)^2(k+1)(2k-i+2)(2k-i+3)} r(k,i),$$

where

$$r(k,i) = -30k^3 - 101k^2 - 112k + i^2(-20k-23) + i(43k^2+98k+55) + 3i^3 + i^2 - 41.$$

According to the implementation of Gosper's and Zeilberger's algorithms we have the recursion

$$(k+1)\widehat{F}(k+1,i) - (k+1)\widehat{F}(k,i) = \widehat{G}(k,i+1) - \widehat{G}(k,i).$$

If we sum the equation over all integers k, the right hand side obviously telescopes to 0 and we get

$$\sum_i \widehat{F}(k+1,i) = \sum_i \widehat{F}(k,i).$$

Thus,

$$\sum_i \widehat{F}(k,i) = \sum_i \widehat{F}(0,i) = \sum_i (-1)^i \binom{1}{i}\binom{-i}{0}^2 = 1 - 1 = 0,$$

and we are done with the case where $n \geq k$. So, whatever happened to the case where $k > n$? Let us execute the algorithm a final time.

Suppose that $k > n$. We would like to see that $f(n) = 0$ where

$$f(n) = \sum_{i=0}^{k} \bar{F}(n,i)$$

and

$$\bar{F}(n,i) = (-1)^i \binom{2n+1}{i} \binom{n+k-i}{n}^2.$$

From the implementation of Gosper's and Zeilberger's algorithms again, we get the recurrence

$$(n-k+1)^2 \bar{F}(n+1,i) - (n+1)^2 \bar{F}(n,i) = \bar{G}(n,i+1) - \bar{G}(n,i),$$

with $\bar{G}(n,i) = R(n,i)\bar{F}(n,i)$, where R is the same as before (see Equation 3.5). The sum of the above over i from 0 to k equals 0 since both $R(n,0)$ and $\bar{F}(0,k+1)$ equal 0. Accordingly,

$$(n-k+1)^2 f(n+1) = (n+1)^2 f(n)$$

and consequently

$$f(n) = \frac{n^2}{(n-k)^2} f(n-1).$$

If we unwind the recurrence we will see that $f(n) = 0$ since the denominator is nonzero as $k > n$, and $f(0) = 0$. We finally have our sought-after result. □

Remark. Note that in the proof we found the noticeable identity 3.3. It may also be interesting to know that using the very same methods we can show that the infinite sum equals 0.

3.3 Permutations of Odd Numbers of Even Length

There is a symmetrical distribution also among the permutations of odd numbers, more specifically among those of even length.

As usual, let $e(\pi)$ denote the number of excedances in π. Moreover, let

$$G_{2n}(x) = \sum_{\pi \in \mathcal{S}(2[2n]-1)} x^{1-n+e(\pi)}.$$

It appears that the distribution for the odd numbers is just as willing to subject itself to a nice generating function as is the distribution for the even numbers.

For example, for $n = 1$ we have

$$\frac{2x}{(1-x)^3} = 2x + 6x^2 + 12x^3 + 20x^4 + 30x^5 + \mathcal{O}(x^6),$$

with the coefficient $k(k+1)$ of x^k. For $n = 2$,

$$\frac{12x + 12x^2}{(1-x)^5} = 12x + 72x^2 + 270x^3 + 600x^4 + 1260x^5 + \mathcal{O}(x^6),$$

which has the coefficient $k(k+1)^2(k+2)$. The general pattern is as follows.

Proposition 3.3. *The generating function $G_{2n}(x)$ for the sequence $\{P(O_{2n}, k+n-1)\}_{k>0}$ satisfies, for all $n \in \mathbb{Z}^+$,*

$$\frac{G_{2n}(x)}{(1-x)^{2n+1}} = \sum_{k\geq 0} k \left(\prod_{i=0}^{n-1}(k+i)^2\right)(k+n)x^k. \tag{3.6}$$

Proof. The proof will be somewhat sketchy, but fully follow the outline of the proof of Proposition 3.2.

Equation 3.6 may be rewritten as

$$G_{2n}(x) = \sum_{k\geq 0}\sum_{j=0}^{k}\binom{2n+1}{j}(-1)^j(k-j)\prod_{i=1}^{n-1}(k-j+1)^2\cdot(k+n-j)x^k.$$

Due to Equation 2.7, then, we want to show that

$$\begin{aligned} n!^2\binom{n-1}{k}\binom{n+1}{k+1} &= \\ &= \sum_{j=0}^{k+1}\binom{2n+1}{j}(-1)^j(k-j+1)\prod_{i=1}^{n-1}(k-j+i+1)^2(k+n-j+1) \\ &= \sum_{j=0}^{k+1}\binom{2n+1}{j}(-1)^j\frac{(k+n-j+1)!}{(k-j+1)!n!}\frac{(k+n-j)!}{(k-j)!n!}n!^2, \end{aligned}$$

which is equivalent to the identity

$$\binom{n-1}{k}\binom{n+1}{k+1} = \sum_{i=0}^{k}(-1)^i\binom{2n+1}{i}\binom{n+k-i+1}{n}\binom{n+k-i}{n}. \tag{3.7}$$

This equality can be proved by considering the cases $n > k$ and $n \leq k$, and closely following the methods for proving Equation 3.3. The proof is slightly lengthy, however, and does not introduce any new ideas, so it is not included here. □

4 Searching for a Bijective Proof

We have seen proofs for all results presented so far. What is missing is a more profound understanding of what the identities really say, or, rather, *why* they are true. Bijective proofs are really nice in that they usually provide us with a satisfactory explanation.

4.1 A Complete Understanding

We are usually quite optimistic about finding bijective functions between sets whose cardinalities are counted by binomial coefficients, as binomial coefficients are easily interpreted combinatorially. Nevertheless, there do not seem to be any obvious bijective functions explaining the identities in Section 2.3.

4.1.1 Some General Reasoning

The proofs of Lemmas 2.3 and 2.4 give us a way of interpreting the factorials as the number of permutations containing the minimum or maximum number of excedances possible. As the factorials are factors of the number of permutations with k excedances for any k, we could partition the set of all permutations of E_n or O_n into subsets of equal cardinality where one of the subsets is the set of permutations containing the minimum or maximum number, respectively, of excedances possible. This set is obviously an equivalence class given the equivalence relation of having the same number of excedances. If we can impose restrictions on the relation such that the quotient set of $\mathcal{S}(E_n)$ or $\mathcal{S}(O_n)$ by this relation is the desired partition, then we are done.

For example for E_n for even n, we need to find some permutation statistic which will divide the permutations with k excedances into $\binom{n/2}{k-n/2}^2$ equivalence classes, each with the cardinality $\frac{n}{2}!^2$.

A natural starting point might be to try to find good representatives of the equivalence classes and see that the number of such permutations for each number of excedances equals the binomial coefficients. In any sense, we can always feel free to divide the equations by the factorial counting the number of permutations of the stable numbers, as the order of these is irrelevant for the excedance statistic of the permutation. In fact, and quite obviously, we can replace a stable number by any other stable number without changing the excedance statistic.

There is a number of ways of interpreting the binomial coefficients in the formulas for $P(E_n, k)$ and $P(O_n, k)$. It might be the case that a natural interpretation of one of the binomial coefficients will be part of a combinatorial proof. An intuitive guess is probably that one of the coefficients counts the number of ways of choosing which numbers should be excedances among the unstable ones. Another one is to start out with a set of indices with as many elements as there are unstable numbers, perhaps based on a permutation where all of them are excedances, and then choosing which indices should contain unstable numbers, placing them in order to cause excedances.

Finally, it may be worth noting that Equations 3.3 and 3.7 supply us with equivalent formulas for Equations 2.4 and 2.7 that would perhaps subject themselves to a combinatorial argument based on the principle of inclusion and exclusion. Let $m = \left\lfloor \frac{n}{2} \right\rfloor$ and j be the number of excedances among the unstable numbers in the set we are permuting. For even positive n we have that

$$P(E_n, k) = m!^2 \sum_{i=0}^{j} (-1)^i \binom{2m+1}{i} \binom{m+j-i}{m}^2 \text{ and}$$

$$P(O_n, k) = m!^2 \sum_{i=0}^{j} (-1)^i \binom{2m+1}{i} \binom{m+j-i}{m} \binom{m+j-i+1}{m}.$$

4.1.2 A Bijection to Words

It may be helpful to use the bijection f constructed in the proof to Lemma 2.3 when trying to find a bijective proof for the identities in Section 2.3. f maps a permutation π to a word w. If π is a permutation of E_n, for example, we have that all indices in π are excedances if and only if w has no excedances. If there are excedances in w, then, how do they correspond to the excedance statistic of π? It is not quite as straightforward as we would hope for, with an excedance in w corresponding to a non-excedance in π. Let $w(i)$ be the number in the i-th position of w. The number $2n$ causes a non-excedance in π if and only if $w(n) \geq 2n - N_n$, where

$$N_i = N_{i-1} + \chi_{i-1}(2n - N_{i-1}) \text{ for } i > 0 \text{ and}$$
$$N_1 = 0,$$

where $\chi_i(n) = 1$ if $w(i) < n$ and 0 otherwise.

What is needed to be convinced is grasping the definitions. Assume that all numbers less than $2n$ have index at least $2n$ in π. Then $2n$ will cause a non-excedance in π if and only if $w(n) \geq 2n$. For every number smaller than $2n$ that is placed to the left of position $2n$ in π we can allow $w(n)$ to be reduced by one, since $w(n)$ is defined to be the free position in π in which $2n$ is inserted, after having inserted the even numbers smaller than $2n$. N_i depicts the number of numbers smaller than $2i$ that have an index less than $2n$ in π.

4.1.3 Other Statistics with the Same Distribution

The neat appearance of the formulas 2.4–2.5 and 2.7–2.8 suggests that the numbers occur in other contexts. Indeed, there are a few well-known structures counted by the binomial coefficients in the formulas, and they can be found in Sloane [9]. It would be nice to find a bijection from our permutations to any of these problems, and it could give us a hint for how to define the equivalence relations.

The binomial coefficients $\binom{n}{k}^2$ of $P(E_n, k)$ for even n also count the number of lattice paths from $(0,0)$ to (n,n) with steps $(1,0)$ and $(0,1)$, having k right turns. The lattice path is described by the coordinates of the k right turns, so we need to choose k first and k second coordinates, independently from each other, out of n. Of course, there are plenty more examples.

A result which is a little more intriguing is that the numbers we get after dividing $P(O_n, k)$ for even n with $P(O_n, j)$, where j is the minimum number of excedances possible in a permutation of O_n, are the Narayana numbers, which are defined on page 12. There are many interesting statistics having the Narayana distribution $N(n, k)$, such as:

- The number of 132-avoiding permutations of $[n]$ with $k-1$ excedances.
- The number of Dyck paths of length $2n$ with k peaks.

The first one seems to be closest at hand, establishing a connection between permutations of O_n and permutations of $[n]$ avoiding 132, based on the number of excedances. A permutation π is said to be 132-avoiding if there is no subsequence of length 3 of π with the smallest number first and the greatest one in the middle.

A Dyck path is a lattice path from $(0,0)$ to $(2n,0)$ with steps $(1,1)$ and $(1,-1)$ which never goes below the x-axis.

The sum over $N(n,k)$ for all $k \in \mathbb{N}$ is the n-th Catalan number, which is $C_n = \frac{1}{n+1}\binom{2n}{n}$.

4.2 Explaining Parts of the Distributions

Even though we have not seen any bijections explaining the formulas 2.4–2.9 fully, we can still say something about parts of their distributions.

4.2.1 The Symmetry of the Distributions

The numbers $P(E_n, k)$ and $P(O_n, k)$ have a symmetrical distribution with respect to k for even n. There is a bijective function which maps a permutation π in E_n or O_n with j excedances among the unstable numbers to a permutation in the same set with $n-j$ excedances.

Let f be a function on $\mathcal{S}(E_n)$ for any positive even n. Let π be any element in $\mathcal{S}(E_n)$ and $\pi(i)$ be the number at index i in π. If $f(\pi) = \bar{\pi}$ and

$$\bar{\pi}(n-i+1) = \begin{cases} n - \pi(i) + 2 & \text{if } \pi(i) \text{ is unstable} \\ \pi(i) & \text{if } \pi(i) \text{ is stable} \end{cases},$$

then f is the function we are looking for.

The function f is clearly bijective. The index $n-i+1$ in a permutation of length n is the i-th index in the reverse order. If $\pi(i)$ is the k-th smallest unstable number in E_n then $n - \pi(i) + 2$ is the k-th biggest. Thus, f maps a permutation with the k-th smallest unstable number in the i-th position to a permutation with the k-th biggest unstable number in the i-th position in the reverse order. But as the permutation has even length, the former number is an excedance if and only if the latter is not: There is only one position where 2 and n will cause an excedance or a non-excedance, respectively. For each greater or smaller number, respectively, in E_n there are two more positions where they will cause an excedance, respectively not cause one.

We can use quite the same function, getting the same result, for permutations of O_n, with the sole exception of excluding 1 from the unstable numbers in the previous discussion and include it where we refer to stable numbers instead, as the position of the number will not affect the excedance statistic.

Example. $f(8\ 2\ 4\ 10\ 12\ 6) = 2\ 12\ 10\ 4\ 6\ 8$.

Example. See Table 5.

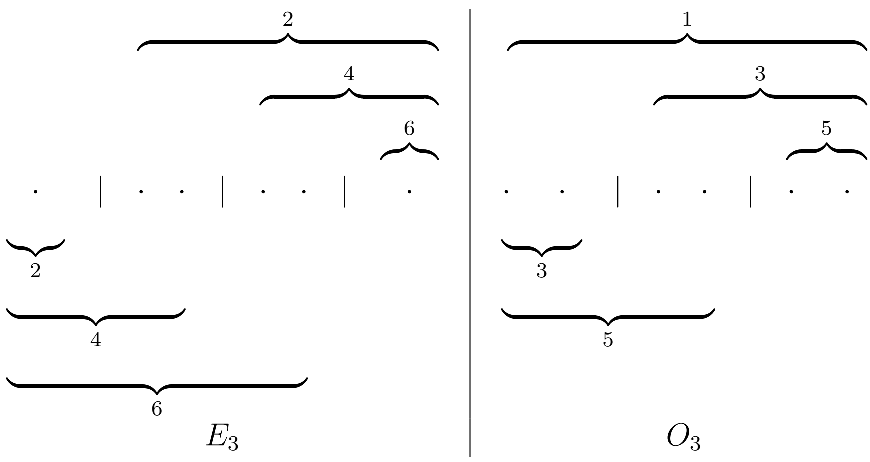

Table 5: An illustration of where the numbers of two sets can be placed not causing and causing an excedance.

4.2.2 Similarities between the Distributions

The lines in Tables 1 and 2 where the distribution of permutations is asymmetrical present us with another conspicuous pattern. The distribution of permutations of E_n for odd n is the same as the one for O_n, but reversed.

We can define a bijective function $g : \mathcal{S}(E_n) \to \mathcal{S}(O_n)$ for any positive odd n by applying the following to all indices i for any permutation $\pi \in \mathcal{S}(E_n)$. If $\pi(i)$ is the number at index i in π and $g(\pi) = \hat{\pi}$ then

$$\hat{\pi}(n-i+1) = \begin{cases} n - \pi(i) + 2 & \text{for } \pi(i) < n \\ \pi(i) + 1 & \text{for } n < \pi(i) < 2n \\ 1 & \text{for } \pi(i) = 2n \end{cases} .$$

The argument for why this is the desired bijection is similar to that of Section 4.2.1, so a small intuitive figure should suffice to get us at ease with the function. Keep in mind that g maps a permutation with j excedances among the unstable numbers to a permutation with $n - j - 1$ excedances, since a stable number will be replaced by the number 1.

The following are figures symbolising permutations of E_n and O_n, respectively. The subscripts are indices and the vertical lines are placed between positions where a specific number passes the border between causing and not causing an excedance:

$$\underset{1}{\cdot} \;\Big|\; \underset{2}{\cdot} \;\; \underset{3}{\cdot} \;\Big|\; \cdots \;\Big|\; \underset{n-1}{\cdot} \;\; \underset{n}{\cdot} \quad \text{resp.} \quad \underset{1}{\cdot} \;\; \underset{2}{\cdot} \;\Big|\; \cdots \;\Big|\; \underset{n-2}{\cdot} \;\; \underset{n-1}{\cdot} \;\Big|\; \underset{n}{\cdot}$$

Example. $g(2\,6\,8\,4\,10) = 13975$.

5 Generalisations

What we have seen so far is only the beginning of a mine of interesting problems and posers. For sure, we have found nice ways of characterising the distributions of permutations of $2[n]$ and $2[n] + 1$. What about the factor 3 with displacements 0, 1 and 2? What about the factor k for any $k \in \mathbb{N}$? We have already dealt with these cases and even more general than that, due to the recurrence in Proposition 2.2. Still, however, we have not seen any explicit expressions, other than those arising when considering the factor 2.

It needs not stop there. Once we have started to count excedances for permutations of other sets than $[n]$, it seems natural to consider other index sets than $[n]$. Moreover, in order to characterise the permutations more fully, we would also be interested in finding the distributions for other statistics than excedances.

We will have a look at some generalisations of what we have seen so far and discuss what is a natural generalisation to another statistic.

5.1 Other Subsets of $\mathbb{N}$

5.1.1 Permutation Sets

The sequences representing the distributions of permutations of every third number in $\{1, 2, \ldots, 3n\}$ have not got as straightforward a structure as the distributions of permutations of even and odd numbers. The number of permutations with the smallest or greatest number of excedances possible is not even always a divisor of the other numbers.

n \ k	0	1	2	3	4	5	6
				$3[n]$			
4	0	0	0	12	12		
5	0	0	0	0	72	48	
6	0	0	0	0	72	456	192
				$3[n]+1$			
4	0	0	6	18			
5	0	0	0	48	72		
6	0	0	0	0	360	360	
				$3[n]+2$			
4	0	0	0	18	6		
5	0	0	0	18	84	18	
6	0	0	0	0	192	456	72

Table 6: The number of permutations of different sets with k excedances.

However, for each permutation set in Table 6 one of the lines displays a symmetric distribution and each of the other two lines contains the same numbers as for one of the other sets, but reversed. This is an interesting observation but not a very surprising one, given the discussion in Section 4.2.

In general, for permutations of $k[n] + d$ for $1 \leq d \leq k - 1$, the smallest number, d, will cause an excedance in the first $d - 1$ positions (change d for k if $d = 0$). For every succeeding number (ordered by size), there are k more positions where that number will cause an excedance, compared to the number preceding it.

The symmetry, then, will occur when we have a symmetrical structure in how the numbers will affect the excedance statistic, that is when we have a symmetrical distribution for those n for which the number of positions where the greatest unstable number will not cause an excedance equals $d - 1$ (or $k - 1$ if $d = 0$), which is every k-th n. Bijections can be constructed in the same way as in Section 4.2.1.

For every k-th n there are i, where $1 \leq i \leq k$, positions for which the smallest unstable number will cause an excedance and for every k-th n there are i positions for which the greatest unstable number will not cause an excedance, which motivates that the same distribution with respect to k occurs in two sets (unless it is symmetrical), but reversely, for every n. Bijections are constructed in the same way as in Section 4.2.2.

Finally, if we would choose to include nonpositive numbers in the permutation sets (interpreting $k[n]$ as a set including 0, say), the distribution would only be affected by a displacement. A bijection can be constructed by mapping a permutation with a nonpositive number to a permutation with a stable number at the same index, leaving the rest of the permutation unchanged.

Even though the distributions follow an obscure pattern, we should be able to describe the number of permutations with the minimum and maximum number of excedances with a modest effort by reusing the ideas from Lemmas 2.3 and 2.4.

The number of unstable numbers in a set $k[n] + d$, where k, n and $d \leq k$ are positive integers ($k[n]+k = k[n]$), is $\lfloor \frac{n-d}{k} \rfloor + 1$, since the smallest unstable number is d, and there is another unstable number for each k-th position. We have the following results.

Proposition 5.1. *Let $A_n = k[n]+d$ for any positive integers k, n and $d \leq k$. Let $m = \lfloor \frac{n-d}{k} \rfloor + 1$. Let $\kappa = 1$ if $d = 1$ and $\kappa = 0$ otherwise. The number of permutations of A_n with no excedances among the unstable numbers is*

$$P(A_n, n - m - \kappa) = (n - m + \kappa)! \prod_{i=0}^{m-1} (d - 1 + \kappa + (k - 1)i). \tag{5.1}$$

The number of permutations of A_n with the maximum number of excedances is

$$P(A_n, n - \kappa) = (n - m)! \prod_{i=0}^{m-1} (n - k(m - 1) - (d - 1) + (k - 1)i). \tag{5.2}$$

Proof. Assume $d \neq 1$. Let f be the bijective function that maps every $\pi \in \mathcal{S}(A_n)$ to a pair consisting of a word w of the same length as the number of unstable numbers, and the order of the stable numbers. The number in position $i + 1$ in the word w denotes the position of $ki + d$ in π minus the number of smaller numbers to the left of $ki + d$ in π.

For π to contain no excedances among the unstable numbers, the first position in w needs to contain a number no greater than $d-1$. For each number we add to π, one position becomes occupied, so the number in position $i+1$ in w may be at most $d-1+(k-1)i$.

Let g be the bijective function that reverses a permutation and replaces the j-th smallest unstable number by the j-th greatest unstable number for all positive $j \leq m$. Let v be the first word in the pair $f \circ g(\pi)$ for any $\pi \in \mathcal{S}(A_n)$. For π to contain only excedances, the first position in v needs to contain a number no greater than $n-k(m-1)-(d-1)$, since the greatest unstable number in A_n is $d+k(m-1)$. Thus, the number in position $i+1$ in v may be at most $n-k(m-1)-(d-1)+(k-1)i$.

Finally, there are $n-m$ stable numbers whose positioning does not affect the number of excedances.

It remains to see what happens in the special case $d=1$. We modify the function f slightly by letting the position of 1 be part of the second word in the pair, as the position of the number will not affect the number of excedances. The first number in w, which is the position of the next smallest number, must not be greater than k if π is to contain no excedances among the unstable numbers. The number in position i may be at most $1+(k-1)i$.

As for the maximum number of excedances, $n-1$, the fact that $d=1$ does not affect the formula, since the greatest unstable number is still $d+k(m-1)$ and

$$(n-m+1)!\prod_{i=0}^{m-2}(n-k(m-1)+(k-1)i) = (n-m)!\prod_{i=0}^{m-1}(n-k(m-1)+(k-1)i).$$

□

The generality of Equation 5.1 somewhat obscures the simplicity of the formula for the case of $A_n = k[n]$.

Corollary 5.2. *Let $A_n = k[n]$ for any positive integers k and n, and let $m = \left\lfloor \frac{n}{k} \right\rfloor$. Then*

$$P(A_n, n-m) = m!(n-m)!(k-1)^m.$$

Kitaev and Remmel [5] gave a general formula for the number of permutations of $[n]$ with respect to descents, including only descents where the number causing the descent is an element of $k[n]$. Their proof is based on the same recursion as the one presented here in Proposition 2.2. Kitaev and Remmel presented the recursion with respect to descents in [5, Theorem 3], and the results in this and their work have been found independently from each other. We will see in Section 5.2.2 that the descent statistic is easily translated to the excedance statistic, even in this more general case, which means that their general formula is very useful also when considering excedances.

Proposition 5.3. *Let $A_{kn+j} = k[kn+j]$ for any $n \geq 0$ and $0 \leq j \leq k-1$. We have that*

$$P(A_{kn+j}, s) = ((k-1)n+j)! \sum_{r=0}^{\ell} (-1)^{\ell-r} \binom{(k-1)n+j-r}{r} \binom{kn+j+1}{\ell-r} \cdot \prod_{i=0}^{n-1} (r+1-j-(k-1)i),$$

where $\ell = s - n(k-1) - j$.

Proof. Note that ℓ denotes the number of excedances among the unstable numbers in A_{kn+j}. The result follows from Proposition 5.6 and Theorem 5 in [5], of which the proof is too lengthy to be included here. □

5.1.2 Index Sets

With an idea of how the excedance statistic will generalise to a wide range of permutation sets it may be worthwhile to consider what will happen if we change the index set instead. For example, we can let the *permutation* set be $[n]$ and the *index* set be E_n. Luckily, we need not brood on this generalisation as we have the following general result.

Proposition 5.4. *Let P and I be subsets of a totally ordered set such that $|P| = |I| = n$ for any $n \in \mathbb{Z}^+$. Let k be any integer such that $0 \leq k \leq n$. There are as many permutations of P with k weak excedances on the index set I as there are permutations of I with $n-k$ excedances on the index set P.*

Proof. Let $\pi \in P$ be a permutation with index set I and k weak excedances. Then π^{-1} is a permutation of I with index set P. It remains to show that π^{-1} has $n-k$ excedances. But if i is a weak excedance of π then $\pi(i) \geq i$, which is equivalent to the fact that $\pi(i)$ is not an excedance in π^{-1}. □

Once we have found the distribution of permutations with respect to weak excedances, which we will do in Section 5.2.1, we can extend the discussion in Section 5.1.1 to treat sets $q[n]$ where q is a rational number, since allowing other sets of integers than $[n]$ both for the permutation and index set is the very same thing as using $[n]$ as the index set and allowing the permutation set to consist of rational numbers.

5.2 Other Statistics

5.2.1 Weak Excedances

The connection between excedances and weak excedances is simple. A permutation $a_1a_2\dots a_n$ has k weak excedances if and only if the permutation $b_1b_2\dots b_n$ defined by $b_i = a_i + 1$ has k excedances. This means that if we find the distributions of permutations of $q[n] + d$, where q is a rational number and n and d are integers, with respect to excedances for all $0 \le d < q$ then we also know the distributions with respect to weak excedances.

The distribution of permutations of O_n with respect to weak excedances is the same as that for E_n with respect to excedances, and vice versa, but then with a displacement since it is the number of permutations of $\{0, 2, \dots, 2n-2\}$ with k weak excedances that is the same as the number of permutations of O_n with k excedances.

In general, the number of permutations of $q[n] + d$ for $0 \le d < q$ with k weak excedances is the same as that of $q[n]+(d+1)$ with k excedances, unless $d = q - 1$ for which case it is the same as that of $q[n]$ with k excedances, but then with a displacement.

5.2.2 Descents

There is a nice well-known bijection which establishes a close correspondence between excedances and descents, originally presented by Foata and Schützenberger [3].

Proposition 5.5. *[3, Thm. 1.12] There is a bijection $f : \mathcal{S}_n \to \mathcal{S}_n$, for any n, which maps a permutation π to a permutation $\hat{\pi}$ such that a number in π causes an excedance in π if and only if it causes a descent in $\hat{\pi}$.*

Remark. The bijection f is essentially the map referred to as the *transformation fondamentale* in [3].

Proof. For the construction of f, we will use some of the ideas presented by Stanley ([10], pp. 17f, 23), but do parts of the construction slightly differently.

First of all, we define a *standard representation* of a permutation on cyclic form by requiring that each cycle is written with its largest element first and the cycles are written in increasing order of their largest element. Define $\hat{\pi}$ to be the permutation we obtain by removing the brackets from the standard representation of π^{-1}. Then π can be recovered from $\hat{\pi}$ by inserting a left bracket before every left-to-right maximum (an element a_i such that $a_i > a_j$ for every $j < i$), and right brackets before every left bracket and at the end, and inverting the outcome.

Suppose that

$$\pi^{-1} = (a_1 a_2 \cdots a_{i_1})(a_{i_1+1} a_{i_1+2} \cdots a_{i_2}) \cdots (a_{i_{k-1}+1} a_{i_{k-1}+2} \cdots a_n)$$

is the standard representation of the inverse of a permutation π. Then $a_i > a_{i+1}$ if and only if a_i causes an excedance in π, and we are done. □

Example. Let $\pi = \underline{5}1\underline{6}4\underline{7}32$ (excedances are underlined). The standard representation of the inverse of π is (4)(63)(7512), which, after removing the brackets, results in the permutation $\hat{\pi} = 4\underline{6}3\underline{7}\underline{5}12$ (descents are underlined).

So, the Eulerian numbers constitute the distribution of permutations both with respect to excedances and descents. It appears that we have a similar relation for $\{P(X_n, k)\}_{k\geq 0}$, where X_n can equal E_n, O_n or any n-set of positive integers, for any n. Obviously, there is no bijection from $\mathcal{S}(X_n)$ to $\mathcal{S}(X_n)$ which will map a permutation with k excedances to a permutation with k descents, as there is a trivial bijection from $\mathcal{S}(X_n)$ to $\mathcal{S}_n$ preserving descents. We can, however, make use of Proposition 5.5 for $\mathcal{S}(X_n)$ if we refine the notion of a descent, similar to the definition in [5].

Definition 8. A *descent of type* X of a permutation $\pi = a_1 a_2 \cdots a_n$ is an i such that $a_i > a_{i+1}$ and $a_i \in X$.

Proposition 5.6. *Let X be a set of positive integers. There is a bijection $f : \mathcal{S}(X_n) \to \mathcal{S}_n$, for any n, which maps a permutation π to a permutation $\hat{\pi}$ such that an unstable number in X_n causes an excedance in π if and only if it causes a descent of type X in $\hat{\pi}$.*

Proof. Let us first define a bijective function $g : \mathcal{S}(X_n) \to \mathcal{S}_n$ which keeps the number of excedances among the unstable numbers in X_n for every permutation intact. Let g be the function that maps a permutation $a_1 a_2 \cdots a_n$ to a permutation $\pi' = b_1 b_2 \cdots b_n$ such that

- $b_i = a_i$ if a_i is an unstable number in X_n and
- b_i is the k-th smallest number in $[n] \setminus X_n$ if a_i is the k-th smallest stable number in X_n.

According to Proposition 5.5 there is a bijective function which will map π' to a permutation $\hat{\pi}$ such that a number in $[n]$ causes an excedance in π' if and only if it causes a descent in $\hat{\pi}$. As $[n] \cap X_n$ is the set of unstable numbers in X_n, we are done. □

Example. Let $\pi = 19\underline{5}37$ (excedance among unstable numbers in O_n underlined). Then $\pi' = 14\underline{5}32$ (number in O_n causing an excedance in π' underlined) and $\hat{\pi} = 1\underline{5}342$ (descent of type O_n underlined).

5.2.3 The Next Step

We have found ways of generalising the results to other statistics than excedances, with the same distributions, by finding bijections. How do we go further?

What we are looking at is permutations of various subsets of $\mathbb{N}$, in particular sets that can be described as $k[n] + d$ and even more specifically, we have mainly dealt with the even and odd numbers. What characterises even and odd numbers is the factor 2, so a generalisation could be to consider permutations of $[n]$ and imposing restrictions on the permutation statistic involving the factor 2. Indeed, we found a very close correspondence between excedances of permutations of even and odd numbers and descents with this restriction.

An example of a statistic which would be interesting to refine is the number of *inversions*. A pair of elements (i, j) is called an inversion in a permutation $a_1a_2 \cdots a_n$ if $i > j$ and $a_i < a_j$. A refinement could be to consider only inversions where one of the numbers in an inversion is even or to consider inversions where $a_i < 2a_j$.

If we want to stick to excedances, we could investigate the excedance statistic on words, using the classical definition of excedances of words (see [1]): Let $w = a_1a_2 \dots a_n$ and let $v = b_1b_2 \dots b_n$ be its non-decreasing rearrangement. An excedance of the word w is an i, where $1 \leq i \leq n$, such that $a_i > b_i$. We could, for example, keep v intact and multiply each number in w by 2.

To conclude, then, we have quite an exhausting characterisation of the distribution of permutations of even and odd numbers with respect to excedances, as to a recurrence relation, explicit expressions and generating functions, but are still lacking a bijective proof. We have also seen glimpses of what to expect from the excedance statistic on permutations of other sets. The prospects of generalising the results further are fairly good and there are quite a few related problems to explore.

References

[1] R. J. Clarke and D. Foata: Eulerian Calculus, I: Univariable Statistics, *European J. Combinatorics* **15**, 345-362, London: Academic Press Ltd. (1994)

[2] L. Comtet: *Advanced Combinatorics*, Dordrecht: D. Reidel Publishing Company (1974)

[3] D. Foata and M.-P. Schützenberger: Théorie géometrique des polynômes Eulériens, *Lecture Notes in Math.* **138**, Berlin: Springer Verlag (1970)

[4] S. Kitaev and J. Remmel: *Classifying Descents According to Parity*, preprint: Reykjavík University and University of California

[5] S. Kitaev and J. Remmel: *Classifying Descents According to Equivalence mod k*, preprint: Reykjavík University and University of California

[6] P. A. MacMahon: *Combinatory Analysis. 1-2*, Cambridge: Cambridge University Press (1915)

[7] P. Paule and M. Schorn: *The Paule/Schorn Implementation of Gosper's and Zeilberger's Algorithms*, `http://www.risc.uni-linz.ac.at/-research/combinat/software/PauleSchorn/`, Research Institute for Symbolic Computation (2006-05-08)

[8] M. Petkovšek, H. Wilf and D. Zeilberger: *A=B*, Wellesley: A K Peters (1996)

[9] N. J. A. Sloane and S. Plouffe: *The Encyclopedia of Integer Sequences*, San Diego: Academic Press (1995). Also available online: `http://-www.research.att.com/~njas/sequences/`, AT & T Labs Research

[10] R. P. Stanley: *Enumerative Combinatorics. 1*, Cambridge: Cambridge University Press (1997)