Multiaxial Fatigue Damage Criterion

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A multiaxial fatigue failure criterion is proposed based on the strain energy density damage law. The proposed criterion is hydrostatic pressure sensitive; includes the effect of the mean stress, and applies to materials which do not obey the idealized Masing type description. The material constants can be evaluated from two simple test results, e.g., uniaxial tension, and torsion fatigue tests. The predicted results are compared with biaxial tests and the agreement is found to be fairly good. A desirable feature of this criterion is its unifying nature for both short and long cyclic lives. It is also consistent with the crack initiation and propagation phases of the fatigue life, in the sense that both of these phases can be related to the strain energy density either locally or globally.

Introduction

There is a need for developing reliable methods to predict fatigue life under multiaxial stress conditions. It is desirable that such an endeavor be based on the general principles of mechanics, and the material constants contained in the failure criterion be evaluated, preferably, from the standard uniaxial tests.

Numerous proposals have been made to correlate fatigue test data. The multiplicity of the proposed criteria is an indication of the complexity of the problem and lack of an agreed upon unified approach. A review of various failure theories for the multiaxial fatigue is given by Ellyin and Valaire (1985) [1]. Most of the proposed criteria are either stress or strain based and are for the biaxial stress conditions.

For the initially isotropic materials, we can distinguish two main categories of stress based criterion. The first one is based on the analysis of the cylic stress invariants, and can be presented in a general form of [2, 3]

$$G(I_1, I_2, N_f) = 0, \tag{1}$$

where $I_1 = \sigma_1 + \sigma_2$; $I_2 = \sigma_1 \sigma_2$. In the second approach, it is assumed that the failure occurs at a given number of cycles when the maximum shear stress, modified by the normal stress acting on the plane of the maximum shear stress, reaches the failure stress in the pure shear [4–6]. The general form of this type of failure criterion may be expressed as,

$$\tau_{\max} + f(\sigma_n) = \tau_F. \tag{2}$$

The strain based failure criteria are generally used in the multiaxial low-cycle fatigue since strain amplitudes are controlled in these circumstances. The strain parameters frequently used are: the octahedral shear strain, the maximum shear strain, the normal strain, and the hydrostatic tension. By considering all other variables as fixed, the fatigue life can be expressed as a function of the in-plane strains, i.e.,

$$N_f = f(\epsilon_x, \epsilon_y). \tag{3}$$

A number of investigators have proposed a failure criterion based on the maximum shear strain and the normal strain acting on this shear plane [7–11],

$$N_f = g(\gamma_{\max}, \epsilon_n). \tag{4}$$

The above stress or strain based criterion do not account for the interaction between the stress and strain in a material deformation process. They cannot thus reflect the path dependence of the material response. Furthermore, some of these criteria do not have an invariant property with respect to the change of the coordinate system. In some of these criteria stress or strain components in different directions, e.g., shear and normal components are added in a manner that is inconsistent with the tensorial properties of these quantities.

Attempts were made to formulate an energy based criterion for multiaxial fatigue failure by Ellyin (1974), Ellyin and coworkers (1981) [12, 13]. By assuming that the cyclic plastic strain energy to fracture was related to the uniaxial fatigue failure strain energy, and that von Mises equivalence between the uniaxial and multiaxial conditions applies, a relationship of the form,

$$\Delta\bar{\sigma}\Delta\bar{\epsilon}(\text{or } \Delta\bar{\epsilon}^p) = KN_f^\alpha, \tag{5}$$

was derived. In the above relation $\Delta\bar{\sigma}$ and $\Delta\bar{\epsilon}$ are the equivalent stress and strain ranges based on the von Mises criterion. The two material parameters K and α were found to vary with the stress (strain) ratio. In more recent papers, Lefebvre and Ellyin (1984) [14], Ellyin and Kujawski (1984) [15] introduced a method whereby the absorbed plastic strain energy can be calculated from the knowledge of the cyclic stress-strain curve.

The concept of relating fatigue life and the plastic work during a cycle of loading was also studied by Garud (1979) [16].

Scope of the Paper

The aim of this investigation is to present a fatigue failure criterion for multiaxial stress states which is consistent with

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the frame indifference principle of constitutive laws, and the nonlinear fracture mechanics approach. The criterion should also apply to both low- and high-cycle regimes.

To gain an insight into such a criterion, we make the following observations. In the low- and high-cycle fatigue failures, the total fatigue life could be divided into two parts: the nucleation of "starting" cracks, and their subsequent propagation until failure occurs. For smooth specimens, the first process is primarily controlled by the amplitude of cyclic strain and/or stress of the bulk material. In the case of notched members, the crack nucleation process at the notch root is controlled by the nominal stress and/or strain and the notch geometry.

In the second phase, the crack propagation process is generally dominated by the local stress/strain fields ahead of the crack tip. In the single parameter description of this process, the controlling parameters become the range of stress (strain) intensity factor ΔK or the range of path-independent J-integral, ΔJ.

The challenge is therefore to define a damage parameter which can be used to describe both the initiation and propagation phases of the fatigue phenomenon. Ellyin and co-workers [17, 18] have recently introduced a strain energy density criterion which is consistent with the above stated objectives.

It has been shown in reference [17] that the total strain energy density, ΔW^t, can describe both the critical damage (demarcation between initiation and propagation phases) and fatigue lives, as well as cumulative damage sequence. The crack growth rate, da/dN, has also been derived from the plastic strain energy consideration [18]. Therefore, total strain energy density is shown to be a proper damage parameter for the uniaxial stress states. In this paper we will derive a relationship between the total strain energy density and number of cycles to failure for the case of the multiaxial states of stress.

Multiaxial Cyclic Strain Energy

During cyclic loading, energy is absorbed because of the plastic deformation. A major part of the mechanical input energy is converted into heat, and the remaining part causes damage. Thus, damage within the material resulting from the cyclic loading can be related to the energy input of the applied load. The fatigue resistance of a material can, therefore, be characterized in terms of its capacity to absorb the energy input. Consequently, to formulate a criterion in terms of the input energy, an estimate of the elastic and plastic parts of the strain energy during loading and unloading, is required.

In the analysis to follow, we assume that the material is initially isotropic and homogeneous. It is worthwhile to mention that a material which may initially exhibit anisotropic yielding (initial yield surface), would tend to become isotropic as a result of cyclic loading [19].

To calculate the input strain energy density we will proceed with calculating the elastic and plastic strain energy densities separately. This is made possible by the virtue of the separation of the total strain increment into the elastic and plastic parts will be adopted, i.e.,

$$d\epsilon_{ij} = d\epsilon_{ij}^e + d\epsilon_{ij}^p. \tag{6}$$

Elastic Strain Energy Density. The elastic SED is calculated from,

$$W^e = \int \sigma_{ij} d\epsilon_{ij}^e. \tag{7}$$

For an isotropic elastic material, the stress-strain relationship is given by:

$$\epsilon_{ij}^e = \frac{1+\nu}{E}\sigma_{ij} - \frac{\nu}{E}\sigma_{kk}\delta_{ij} \tag{8}$$

where ν is Poisson's ratio; E Young's modulus; δ_{ij} is the Kronecker delta, equal to unity when $i=j$ and zero otherwise, and a repeated index implies summation over its range, in this case $i, j = 1,2,3$.

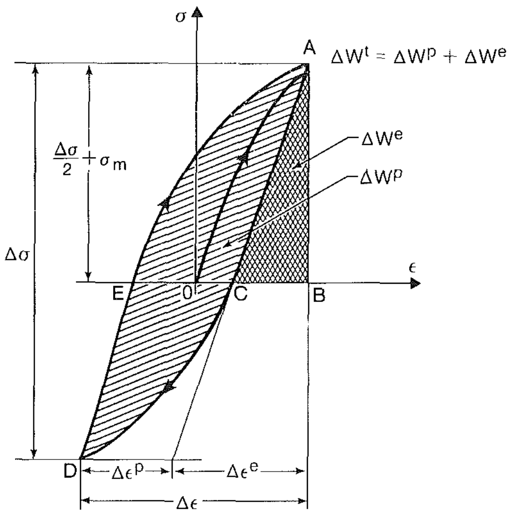

Fig. 1 Elastic and plastic strain energy densities for a uniaxial cyclic loading case

The cyclic elastic SED is calculated from (7) for the positive stress parts of the cycle (see Fig. 1). The limits of integration are therefore,

$$\Delta W^e = \int_{H(\sigma_i^{\min})\sigma_i^{\min}}^{H(\sigma_i^{\max})\sigma_i^{\max}} \sigma_i d\epsilon_i \quad (i=1,2,3) \tag{9}$$

where σ_i and ϵ_i are the principal stress and strain components, respectively, and $H(\sigma_i)$ is the Heaviside function [20] defined as:

$$\begin{aligned} H(\sigma_i) &= 1 \quad \text{for} \quad \sigma_i \geq 0, \\ H(\sigma_i) &= 0 \quad \text{for} \quad \sigma_i < 0. \end{aligned} \tag{10}$$

For the sake of simplicity of expressions, it is assumed that $\sigma_i^{\min} < 0$ and $\sigma_i^{\max} > 0$ (Fig. 1). Therefore, substituting from (8) into (9) and carrying out the integration we obtain

$$\Delta W^e = \frac{1}{2E}[(I_1^{\max})^2 - 2(1+\nu)I_2^{\max}], \tag{11}$$

where $I_1 = \sigma_1 + \sigma_2 + \sigma_3$, $I_2 = \sigma_1\sigma_2 + \sigma_2\sigma_3 + \sigma_3\sigma_1$ are the first and second invariants of the stress tensor. For the uniaxial stress state (11) reduces to,

$$\Delta W^e = \frac{1}{2E}(\sigma^{\max})^2, \tag{12}$$

which is the double cross-hatched area in Fig. 1.

In a series of tests on the thick-walled cylinders, Morrison, Crossland, and Parry [21, 22] observed that superimposed hydrostatic pressure has an influence on the fatigue life. The experimental results indicated that hydrostatic compression increased the fatigue life whereas the hydrostatic tension decreased it. Thus in the fatigue analysis, in contrast to the yielding of materials, the hydrostatic stress component plays a role, and has to be taken into account.

It is seen from equation (11) that the cyclic elastic SED contains the positive hydrostatic pressure components, $I_1^{\max}$. An alternative expression for ΔW^e can be written in terms of the equivalent (or effective) stress component, viz,

$$\Delta W^e = \frac{1+\nu}{3E}(\bar{\sigma}^{\max})^2 + \frac{1-2\nu}{6E}(\sigma_{kk}^{\max})^2, \tag{13}$$

where

$$\bar{\sigma}^2 = \frac{1}{2}[(\sigma_1-\sigma_2)^2 + (\sigma_2-\sigma_3)^2 + (\sigma_3-\sigma_1)^2] = 3J_2, \tag{14}$$

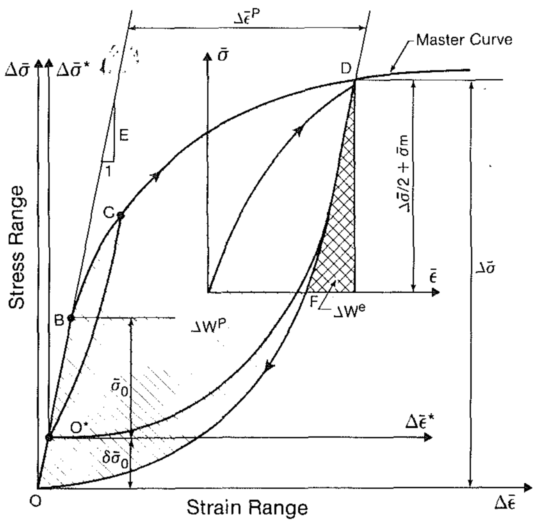

Fig. 2 The coordinate system for the master curve

J_2 being the second invariant of the deviatoric stress,

$$s_{ij} = \sigma_{ij} - \frac{1}{3}\sigma_{kk}\delta_{ij}. \tag{15}$$

Note that the first term on the RHS of (13) is the strain energy which causes distortion of an infinitesimal material element, and the second one is concerned with the positive (tensile) volumetric change.

At times it is more convenient to express (13) in terms of an alternating equivalent stress, $\bar{\sigma}_a$, with a zero mean, and a mean stress, $\bar{\sigma}_m$. The cyclic elastic SED then is given by:

$$\Delta W^e = \frac{1+\nu}{3\mathrm{E}}\left[(\bar{\sigma}_a + \bar{\sigma}_m)^2 + \frac{1-2\nu}{6\mathrm{E}}(\sigma_{kk}^{\max})^2\right]. \tag{16}$$

Plastic Strain Energy Density. The cyclic plastic strain energy density, ΔW^p, can be calculated from,

$$\Delta W^p = \int_{\text{cycle}} \sigma_{ij} d\epsilon^p_{ij}, \tag{17}$$

where the integration is carried over the closed cyclic loop. Due to the incompressibility of the material during the plastic deformation, we have $d\epsilon^p_{kk} = 0$; thus (17) can be written as:

$$\Delta W^p = \int_{\text{cycle}} s_{ij} d\epsilon^p_{ij}, \tag{18}$$

where the deviatoric stress, s_{ij}, is given by (15).

A stable cyclic stress-strain relation for the multiaxial stress states is given by Ellyin and Golos [23]. It is based on the generalization of the master curve concept. A master curve in the uniaxial case is defined as a unique curve which envelopes the upper (loading) branches of the hysteresis loops for different strain amplitudes. It is obtained by first making the compressive (minimum) stress point of all loops to coincide, and then translating each loop along the linear response portion until the upper branches match [14, 15]. Figure 2 shows the master curve and two stable hysteresis loops. The origin of the coordinate system for the master curve is taken to coincide with the minimum proportional range, i.e., point 0* in Fig. 2. The master curve is expressed by:

$$\frac{\Delta\epsilon}{2} = \frac{\Delta\epsilon^e}{2} + \frac{\Delta\epsilon^p}{2} = \frac{\Delta\sigma^*}{2\mathrm{E}} + \left(\frac{\Delta\sigma}{2K^*}\right)^{1/n^*} \tag{19}$$

where n^* and K^* are found from the best fit to the uniaxial test data. Herein after a quantity with superscript ( )* is measured with respect to the master curve origin, 0*.

The plastic strain components for a proportional or nearly proportional loading are given by [23],

$$\epsilon^p_{ij} = 3(2K^*)^{-1/n^*}(\bar{\sigma}^*)^{(1-n^*)/n^*} s^*_{ij} \tag{20}$$

Substituting (20) into (18) and noting that,

$$\bar{\sigma}^* = \left(\frac{3}{2} s^*_{ij} s^*_{ij}\right)^{1/2}, \quad \bar{\epsilon}^p = \left(\frac{2}{3}\epsilon^p_{ij}\epsilon^p_{ij}\right)^{1/2}, \tag{21}$$

we get,

$$\Delta W^p = \int_{\text{cycle}} \bar{\sigma}^* d\bar{\epsilon}^p \tag{22}$$

The equivalent stress $\bar{\sigma}^*$, and the equivalent plastic strain $\bar{\epsilon}^p$, are the range of stress and plastic strain (Fig. 2) and the prefix Δ will be added to denote them as such, i.e., $\Delta\bar{\sigma}^* \equiv \bar{\sigma}^*$, $\Delta\bar{\epsilon}^p \equiv \bar{\epsilon}^p$. Referring to Fig. 2, the area OBCDO is given by [15],

$$\Delta W^p = \frac{1-n^*}{1+n^*}(\Delta\bar{\sigma} - \delta\bar{\sigma}_0)\Delta\bar{\epsilon}^p + \delta\bar{\sigma}_0\Delta\bar{\epsilon}^p, \tag{23}$$

where $\delta\bar{\sigma}_0$ is the increase in the elastic range, a measure of cyclic strain hardening, and is given by,

$$\delta\bar{\sigma}_0 = \Delta\bar{\sigma} - \Delta\bar{\sigma}^* = \Delta\bar{\sigma} - 2K^*(\Delta\bar{\epsilon}^p/2)^{n^*}. \tag{24}$$

Note that in the above equation n^* is the hardening exponent of the master curve with the origin at 0*. For a Masing type material description, n^* is replaced by n', the hardening exponent of the cyclic curve, $K^* = K'$, and $\delta\bar{\sigma}_0 = 0$.

The plastic strain energy density, ΔW^p, equation (23) can be written in terms of the equivalent stress. Using (20) and (21), the equivalent plastic strain range is given by:

$$\Delta\bar{\epsilon}^p = 2(\Delta\bar{\sigma}^*/2K^*)^{1/n^*}. \tag{25}$$

Substituting above into (23) one gets,

$$\Delta W^p = \frac{2(1-n^*)(2K^*)^{-1/n^*}}{1+n^*}(\Delta\bar{\sigma}^*)^{(n^*+1)/n^*} + 2(2K^*)^{-1/n^*}\delta\bar{\sigma}_0(\Delta\bar{\sigma}^*)^{1/n^*}. \tag{26}$$

In particular case of Masing type of material description (26) reduces to [25],

$$\Delta W^p = \frac{2(1-n')(2K')^{-1/n'}}{1+n'}(\Delta\bar{\sigma})^{(1+n')/n'}. \tag{27}$$

Thus, the total damaging strain energy density, ΔW^t, is obtained by combining equation (16) and (26), i.e.

$$\Delta W^t = \Delta W^e + \Delta W^p, \tag{28}$$

It is to be noted that in calculating the plastic strain energy density, a constitutive law relating the plastic strain to the deviatoric stress (equation (20)) was used. This is a "deformation" type of plasticity theory and would predict similar results to that of the incremental theory in the case of proportional or nearly proportional loading paths [24].

Multiaxial Fatigue Failure Criterion for Proportional or Nearly Proportional Loading

A fatigue failure criterion can now be proposed relating the total cyclic strain energy density, ΔW^t, to the number of cycles to failure, N_f. In general terms, this criterion may be expressed as

$$\Delta W^t(I_1, J_2) = F(N_f). \tag{29}$$

In earlier investigations [12–17] it was shown that a particular functional form of RHS of (29) is a power law which is in good agreement with the experimental results. In this case (29) can be written as:

$$\Delta W^t = \kappa N_f^\alpha + C, \tag{30}$$

where κ, α, and C are material constants found from suitable test data.

For uniaxial case these constants, determined from the best fit to data, are given in Fig. 3, and elsewhere [17, 25] for a low alloy carbon steel A-516 Gr. 70. In the case of biaxial tests, κ would depend on a measure of controlled triaxial state. For example, one could define a strain ratio,

$$\rho=\frac{\epsilon_2}{(\epsilon_l-\epsilon_3)/2}, \tag{31}$$

for the strain-controlled tests. Thus, κ will be a function of ρ, and the simplest form of it is [26],

$$\kappa(\rho)=a\rho+b. \tag{32}$$

Therefore, the explicit form of the failure criterion is obtained by combining equations (30) and (32), i.e.,

$$\Delta W^t=(a\rho+b)N_f^\alpha+C. \tag{33}$$

Substituting for the elastic strain energy density, ΔW^e, from (16) and for that of plastic, ΔW^p, from (26), into (33), the explicit form of the multiaxial fatigue failure is given by:

$$\left[\frac{1+\nu}{3E}(\bar\sigma_a+\bar\sigma_m)^2+\frac{1-2\nu}{6E}(\sigma_{kk}^{\max})^2\right] + \left[\frac{2(1-n^*)(2K^*)^{-1/n^*}}{1+n^*}(\Delta\bar\sigma^*)^{(1+n^*)/n^*} + 2(2K^*)^{-1/n^*}\delta\bar\sigma_0(\Delta\bar\sigma^*)^{1/n^*}\right]=(a\rho+b)N_f^\alpha+C,$$
$$\bar\sigma_m<\bar\sigma_a, \sigma_i^{\min}<0 \tag{34}$$

Noting again that $\bar\sigma^2=3J_2$ and $\sigma_{kk}=I_1$, the LHS of the failure criterion (34) is a function of the first two invariants of the stress tensor and material properties (equation (29)). The RHS of (34) is also a scalar. Thus the stress criterion (34) is a frame indifference quantity; it is hydrostatic pressure sensitive and contains cyclic material properties which can be obtained from appropriate tests to be explained later on.

For the special case of the no-mean stress and the Masing type material description, (34) reduces to

$$\frac{1+\nu}{12E}[1+\psi\Delta\bar\sigma^{(1-n')/n'}]\Delta\bar\sigma^2+\frac{(1-2\nu)}{6E}(\sigma_{kk}^{\max})^2 =(a\rho+b)N_f^\alpha+C, \tag{35}$$

where

$$\psi=\frac{24E(1-n')(2K')^{-1/n'}}{(1+\nu)(1+n')}.$$

In particular, for the uniaxial loading $\Delta\bar\sigma=\Delta\sigma$, $\bar\sigma_a+\bar\sigma_m=\sigma_a+\sigma_m=\sigma^{\max}$, and $\sigma_{kk}^{\max}=\sigma^{\max}$, thus (34) reduces to,

$$\left[2\frac{(1+n^*)(2K^*)^{-1/n^*}}{1+n^*}(\Delta\sigma^*)^{(1+n^*)/n^*} + 2(2K^*)^{-1/n^*}\delta\sigma_0(\Delta\sigma^*)^{1/n^*}\right]+\frac{(\Delta\sigma/2+\sigma_m)^2}{2E}=\kappa_u N_f^\alpha+C_u. \tag{36}$$

where subscript u refers to the uniaxial loading conditions. The first term on the LHS of (36) in square brackets is the area of the hysteresis loop, and the second one is the area of the tensile elastic strain energy density, see Fig. 1. Referring to Fig. 2, the first term in the square brackets is the area enclosed by O*BCDO* (Masing description) and the second term in the square bracket is the area O*DFO (non-Masing).

Comparison With Experimental Results

To compare the predictions of the proposed criterion with the experimental results, data on the uniaxial cyclic stress-strain curve is required, from which E, and n' can be obtained. Alternatively, in the case of non-Masing material, the master curve equation has to be known so that n^* and $\delta\sigma_0$ can be determined. Furthermore, the stress amplitude and mean stress components have to be specified. To evaluate constants a and b on the RHS of equation (34), fatigue tests for two

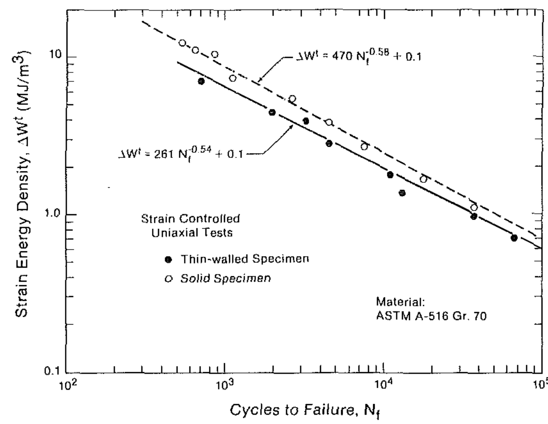

Fig. 3 The total strain energy density versus number of cycles to failure of thin-walled [8, 27] and solid round specimens [14] for uniaxial loading

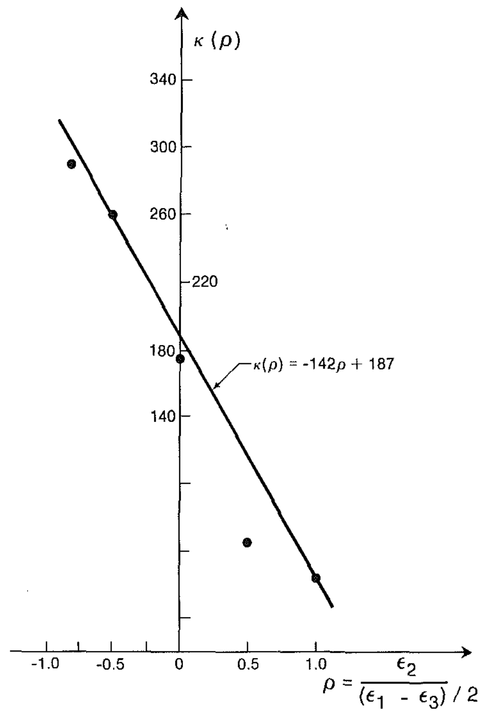

Fig. 4 Strain ratio function $\kappa(\rho)$ versus strain ratio for biaxial tests

strain ratios are required. One of these could be the uniaxial fatigue results where $\rho=-2\nu/(1+\nu)$ and the other one could be the torsional fatigue tests. Results for the above two tests are generally available for a wide range of materials or could be performed with relative ease.

For the low alloy carbon steel designated as A-516 Gr.70 which is widely used in the construction of the pressure vessels, uniaxial and biaxial fatigue tests results are available. The uniaxial test data are summarized in references [15, 17, 25] and biaxial tests in [27]. The latter tests were performed on the thin-walled tubes by subjecting them to a combination of alternating axial load, and constant external and alternating internal pressure. In this manner, fully reversed constant amplitude axial and circumferential strain-controlled tests were performed for six different strain ratios. The ratio of the circumferential strain amplitude, $\Delta\epsilon_t/2$, to the axial one, $\Delta\epsilon_a/2$, ($\rho^*=\Delta\epsilon_t/\Delta\epsilon_a$) varied from -1.25 to 1.0. In these tests, failure was defined when a crack caused a sudden decrease in the stress amplitude and oil penetrated through the thickness.

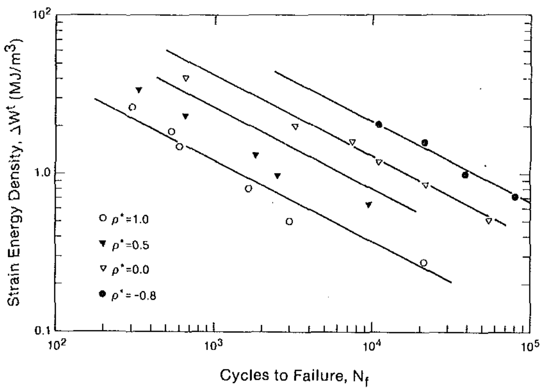

Fig. 5 The predicted strain energy density ΔW^t versus the number of cycles to failure, N_f, for various strain ratios, and the experimental data from reference [27]

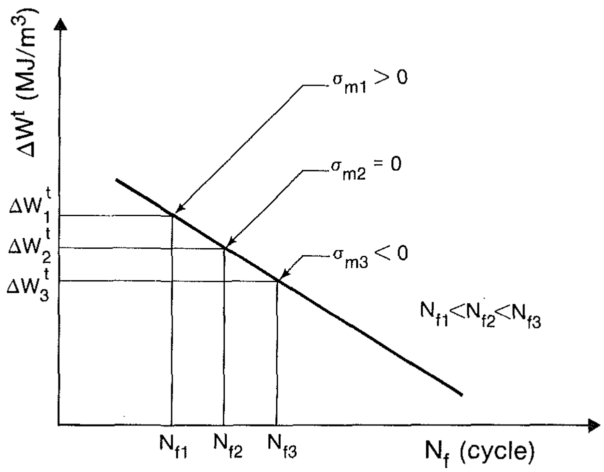

Fig. 6 The effect of mean-stress as predicted by the present theory

It is worthwhile to mention that for the high strains and positive strain ratios, additional bending strains are induced as a result of the barreling of the specimen at the gauge length, see reference [28]. Also, the buckling of the thinned central part of the specimen reduces the fatigue life of the specimen. In addition, the long rod through which the axial load is transmitted to the specimen induces bending strains due to the unavoidable eccentricity [28]. Therefore, a combination of the above mentioned factors have contributed to the lower fatigue lives obtained by these tests. For example, when the uniaxial test data of the solid circular specimens, reference [14] are fitted into equation (36), we obtain the exponent $\alpha = -0.58$, Fig. 3. On the other hand, thin-walled tubular specimen results give an $\alpha = -0.54$ (Fig. 3). Therefore, in comparing the prediction of the proposed criterion with the biaxial experimental results of reference [27] the trend is more important than the absolute values.

Referring to equation (30), the exponent α as mentioned above can be obtained from the uniaxial test data, Fig. 3. To obtain the parameter κ, we could use equation (32). As mentioned earlier, this will require two sets of test data with different strain ratios. One of these could be the abovementioned uniaxial tests, and equibiaxial test results, $\rho^* = 1$ were chosen for the other. Figure 4 shows $\kappa(\rho)$ versus the strain ratio, ρ. The solid line in Fig. 4 is equation (32) fitted to two points whereas solid circular points are the experimental data. It is noted that the experimental data can indeed be represented by a straight line,

$$\kappa(\rho) = -142\rho + 187. \tag{37}$$

Of course other triaxial parameter than ρ could be chosen and κ in (30) could be found in terms of the chosen parameter.

As mentioned earlier, the uniaxial test results can be used to determine the slope of equation (30) in a log-log scale. A least square fit of the thin-walled tubular specimen data yields the following expression,

$$\Delta W^t_{\text{uni}} = 261 N_f^{-0.54} + 0.1 \tag{38}$$

The constant C in (30) is associated with the fatigue (endurance) limit [17]. For fatigue lives $N_f \leq 10^5$, the effect of C is negligible and it can be set equal to zero (c.f., equation (38)).

The fatigue criterion (33) or (34, 35) is now completely determined, and its predictions can be compared with the experimental results for the other strain ratios. Figure 5 shows a comparison between the predicted equation (34) and experimental data for four strain ratios. In the case of experimental data, ΔW^t was obtained from measurement of the hysteresis loop plots in two directions. It is noted that with the exception of one strain ratio, $\rho^* = 0.5$, all other data fall very close to the predicted values. There was a considerable scatter in the experimental results for this strain ratio, and the need for additional tests were suggested by the authors, reference [27]. The agreement between the predicted and the experimental data is found to be fairly good, in spite of uncertainties involved in the actual magnitudes. Note that a certain amount of smoothing is achieved by obtaining the material properties in equation (32) through fitting the experimental data to two strain ratios.

Effect of Mean-Stress

The influence of the mean-stress on the fatigue life can be determined from equation (34). Figure 6 shows a plot of the total strain energy density ΔW^t versus number of cycles to failure in a log-log scale for three different mean stress values. For a fixed strain ratio, the coefficients on the RHS of equation (34) are known. For a given stress amplitude, the LHS will increase for the increasing value of the positive mean stress, thus requiring N_f to decrease – note that the exponent α is negative. Similarly, for an increasing negative mean value, the life will increase (see Fig. 6). This trend has been observed by the experimental observation, e.g., McDiarmid [29]. It is to be noted that accounting for the mean stress effect through equation (34) is limited to $\bar{\sigma}_m < \bar{\sigma}_a$ and strain-controlled condition.

Conclusions

An energy based criterion for the multiaxial fatigue failure has been proposed for proportional or nearly proportional deformation-controlled loading. The key element of this theory is that the damage caused as a result of cyclic loading, is a function of the mechanical input energy into the material. This has been substantiated by numerous tests. The proposed criterion includes the effect of hydrostatic pressure which is known to influence the fatigue life. Furthermore, the effect of mean-stress is also incorporated in the formulation, but has not yet been compared to the experimental data.

A desirous feature of this new criterion is that the constants related to the material properties can be evaluated from the simple tests. They are generally available in material handbooks or could be performed with the standard equipment.

The predictions of the proposed criterion are compared with the biaxial test results, and the agreement is found to be fairly good.

It is worth mentioning that the proposed criterion is applicable for both long and short fatigue lives. Also, the criterion is a frame independent one and thus conforms to the requirements of the continuum mechanics.

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