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Solving a class of linear and non-linear optimal control problems by homotopy perturbation method

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In this paper, we give an analytical approximate solution for non-linear quadratic optimal control problems and optimal control of linear systems using the homotopy perturbation method (HPM). Applying the HPM, the non-linear two-point boundary-value problem (TPBVP) and linear systems, derived from the Pontryagin’s maximum principle, are transformed into a sequence of linear time-invariant TPBVPs. Solving the proposed linear TPBVP sequence in a recursive manner leads to the optimal control law and the optimal trajectory in the form of rapid convergent series. Finally, a non-linear example and several linear examples are given to verify the reliability and efficiency of the proposed method.

Keywords: homotopy perturbation method; optimal control problems; Pontryagin’s maximum principle; Hamiltonian system.

1. Introduction

The investigation of the optimal control is of importance in modern control theory. One of the most active subjects in control theory is the optimal control which has a wide range of practical applications not only in all areas of physics but also in economy, aerospace, chemical engineering, robotic, etc. (Garrard & Jordan, 1997; Manousiouthakis & Chmielewski, 2002; Tang, 2005; Notsu et al., 2008). The theory and the application of optimal control for linear time-invariant systems have been developed perfectly. However, as for non-linear systems, synthesis problems that are solved by classic control theory lead to difficult computations. It is well-known that the non-linear optimal control problem (OCP) can be reduced to a two-point boundary-value problem (TPBVP), implementing the Pontryagin’s maximum principle (PMP). This TPBVP cannot be solved analytically, in general, and most researches have been devoted to find an approximate solution, for non-linear TPBVP’s (Ascher et al., 1995). In order to find an analytical approximate solution, some successive methods have been proposed, e.g. Tang (2005).

One familiar scheme is to determine the optimal control law using dynamic programming. This approach leads to the Hamilton–Jacobi–Bellman equation that is hard to solve in most cases.

Yousefi et al. (2010) used the original or basic Variational Iteration Method (VIM) for linear quadratic OCPs. They transfer the linear TPBVP obtained from PMP to an initial value problem (IVP) and then implement the basic VIM to get a feedback controller. In He (2008), optimal problems are solved in a more attractive way.

In recent years, the homotopy perturbation method (HPM), first proposed by He (1999, 2003b), has successfully been applied to solve many types of linear and non-linear functional equations. This method, which is a combination of homotopy in topology and classic perturbation techniques, provides us with a convenient way to obtain analytic or approximate solutions for a wide variety of problems arising in different fields. He used HPM to solve the Lighthill equation (He, 1999), the Duffing equation (He, 2003b) and the Blasius equation (He, 2003a), and then the idea found its way in sciences and...
has been used to solve non-linear wave equations (He, 2005), boundary-value problems (He, 2006), quadratic Riccati differential equations (Abbasbandy, 2006), integral equations (Golbabai & Keramati, 2008), Klein–Gordon and sine–Gordon equations (Odibat & Momani, 2007), initial value problems (Chowdhury & Hashim, 2007), non-linear evolution equations (Ganji et al., 2007) and many other problems. Biazar & Alizadeh (2010) performed decomposition of source terms in HPM. Hesameddini & Latifizadeh (2009) used an optimal choice of initial solutions in the HPM and Mohyud-Din et al. (2010) used coupling of He’s polynomials and Laplace transformation.

The main advantage of applying HPM is that the results are readily obtained and a few iterations are used. The significant merit of the analytic approach is to provide scientists with the general parametric relation between the dependent and independent variables, namely, displacement and time, respectively. Therefore, the related equations can be simply obtained, giving one the opportunity for further studies, for different cases and thereby different parameters. This paper concerns with a class of non-linear quadratic OCPs and linear OCP. First, HPM is employed for finding the optimal control of linear systems. As discussed earlier, non-linear quadratic OCPs, using PMP, leads to a nonlinear Two-Point Boundary Value Problem (TPBVP). Then applying a new method as shooting method for selection of the initial approximations, we solve this TPBVP by the HPM. We will apply He’s polynomials in order to make the solution procedure easier, more effective and more straightforward.

2. Optimality conditions for linear optimal control system

In the present work, we consider the following linear OCP:

\[
\begin{align*}
x(t) & = Ax(t) + Bu(t), \quad x(t_0) = x_0, \\
J &= \frac{1}{2} x(t_f)^T S x(t_f) + \frac{1}{2} \int_{t_0}^{t_f} (x^T P x + 2 x^T Q u + u^T R u) dt,
\end{align*}
\]

where \( x \in \mathbb{R}^n, \ u \in \mathbb{R}^m, \ A \in \mathbb{R}^{n \times n} \) and \( B \in \mathbb{R}^{m \times n} \). The control \( u(t) \) is an admissible control if it is piecewise continuous in \( t \) for \( t \in [t_0, t_f] \). Its values belong to a given closed subset \( U \) of \( \mathbb{R}^+ \). The input \( u(t) \) is derived by minimizing the quadratic performance index \( J \), where \( S \in \mathbb{R}^{n \times n}, \ P \in \mathbb{R}^{n \times n} \) and \( Q \in \mathbb{R}^{n \times m} \) are positive semi-definite matrices and \( R \in \mathbb{R}^{m \times m} \) is a positive definite matrix.

Consider the Hamiltonian for system (1) as

\[
H(x, u, \lambda, t) = \frac{1}{2} (x^T P x + 2 x^T Q u + u^T R u) + \lambda^T (A x + B u),
\]

where \( \lambda \in \mathbb{R}^n \) is a co-state vector.

According to the PMP (Datta and Mohan, 1995), we can obtain the following optimal control law

\[
u^* = -R^{-1} Q^x - R^{-1} B^T \lambda
\]

and the Hamiltonian system

\[
\begin{cases}
\dot{x} = [A - BR^{-1} Q^T]x - BR^{-1} B^T \lambda, \\
\dot{\lambda} = [-P + Q R^{-1} Q^T]x + [QR^{-1} B^T - A^T] \lambda,
\end{cases}
\]

with the condition \( x(t_0) = x_0 \). Since \( x(t_f) \) is indeterminate, then

\[
\lambda(t_f) = S x(t_f).
\]
By applying the terminal condition (5), we can write the solution of system (4) in the form

\[ x(t) = (F + GS)x(t_f), \]
\[ \lambda(t) = (L + MS)x(t_f), \]  

(6)

where \( F, G, L \) and \( M \) are \( n \times n \) matrices.

By taking \( Y(t, t_f) = (L + MS)(F + GS)^{-1} \), we have

\[ \lambda(t) = Y(t, t_f)x(t). \]  

(7)

Differentiating (7) with respect to \( t \) and summarizing lead to

\[ \dot{Y} = (YB + Q)R^{-1}(B^\top Y + Q^\top) - YA - A^\top - P. \]  

(8)

By considering (3) and (7), we can see the optimal control law as

\[ u^*(t) = -R^{-1}Q^\top x - R^{-1}B^\top Y(t, t_f)x(t). \]  

(9)

We now introduce the following variables:

\[ V(t) = F(t, t_f) + G(t, t_f)S, \]
\[ W(t) = L(t, t_f) + M(t, t_f)S. \]  

(10)

Substituting (10) into (6) and then into (4), we obtain the following system:

\[
\begin{cases}
\dot{V}(t) = [A - BR^{-1}Q^\top]V(t) - BR^{-1}B^\top W(t), \\
\dot{W}(t) = [-P + QR^{-1}Q^\top]V(t) + [QR^{-1}B^\top - A^\top]W(t),
\end{cases}
\]  

(11)

with conditions \( V(t_f) = I \) and \( W(t_f) = S \).

3. Non-linear quadratic OCPs and solution guidelines

Consider the non-linear dynamical system

\[
\begin{align*}
\dot{x}(t) &= f(t, x(t)) + g(t, x(t))u(t), \quad t \in [t_0, t_f], \\
x(t_0) &= x_0, \quad x(t_f) = x_f,
\end{align*}
\]  

(12)

with \( x(t) \in \mathbb{R}^n \) denoting the state variable, \( u(t) \in \mathbb{R}^m \) the control variable and \( x_0 \) and \( x_f \) the given initial and final states at \( t_0 \) and \( t_f \), respectively. Moreover, \( f(t, x(t)) \in \mathbb{R}^n \) and \( g(t, x(t)) \in \mathbb{R}^{n \times m} \) are two continuously differentiable functions in all arguments. Our aim is to minimize the quadratic objective functional

\[ J[x, u] = \frac{1}{2} \int_{t_0}^{t_f} (x^\top(t)Qx(t) + u^\top(t)Ru(t))dt, \]  

(13)
subject to the non-linear system (12), for $Q \in \mathbb{R}^{n \times n}$ and $R \in \mathbb{R}^{m \times m}$, positive semi-definite and positive definite matrices, respectively. Since the performance index (13) is convex, the following extreme necessary conditions are also sufficient for optimality (Geering, 2007):

$$
\dot{x} = f(t, x) + g(t, x)u^*,
$$

$$
\dot{\lambda} = -H_x(x, u^*, \lambda),
$$

$$
u^* = \arg \min_u H(x, u, \lambda),
$$

$$\begin{align*}
x(t_0) &= x_0, \\
x(t_f) &= x_f,
\end{align*}
$$

where $H(x, u, \lambda) = \frac{1}{2}[x^T Q x + u^T R u] + \lambda^T [f(t, x) + g(t, x)u]$ is referred to as the Hamiltonian. Equivalently, (14) can be written in the form

$$
\dot{x} = f(t, x) + g(t, x)[-R^{-1}g^T(t, x)\lambda],
$$

$$
\dot{\lambda} = - \left( Q x + \left( \frac{\partial f(t, x)}{\partial x} \right)^T \lambda + \sum_{i=1}^{n} \lambda_i [-R^{-1}g^T(t, x)\lambda] \frac{\partial g_i(t, x)}{\partial x} \right),
$$

$$\begin{align*}
x(t_0) &= x_0, \\
x(t_f) &= x_f,
\end{align*}
$$

where $\lambda(t) \in \mathbb{R}^n$ is the co-state vector with the $i$th component $\lambda_i(t), i = 1, \ldots, n$, and $g(t, x) = [g_1(t, x), \ldots, g_n(t, x)]^T$ with $g_i(t, x) \in \mathbb{R}^m, i = 1, \ldots, n$.

Also the optimal control law is obtained by

$$
u^* = -R^{-1}g^T(t, x)\lambda.
$$

For solving such a TPBVP in (15), we use a shooting-method-like procedure combined with the HPM. So, first we apply HPM to solve the following IVP:

$$
\dot{x} = f(t, x) + g(t, x)[-R^{-1}g^T(t, x)\lambda],
$$

$$
\dot{\lambda} = - \left( Q x + \left( \frac{\partial f(t, x)}{\partial x} \right)^T \lambda + \sum_{i=1}^{n} \lambda_i [-R^{-1}g^T(t, x)\lambda] \frac{\partial g_i(t, x)}{\partial x} \right),
$$

$$\begin{align*}
x(t_0) &= x_0, \\
\lambda(t_0) &= \alpha,
\end{align*}
$$

where $\alpha \in \mathbb{R}$ is an unknown parameter. Then it will be identified after sufficient iterations of HPM, as discussed hereinafter.

4. Basic idea of HPM

The principles of the HPM can be described as follows (He, 1999). Consider the following non-linear differential equation

$$
L(u) + N(u) - f(r) = 0, \quad r \in \Omega,
$$

subject to the boundary condition

$$
B \left( u, \frac{\partial u}{\partial n} \right) = 0, \quad r \in \Gamma,
$$
where $L$ is a linear operator, while $N$ is a non-linear operator, $B$ is a boundary operator, $f(r)$ is a known analytical function and $\Gamma$ is the boundary of domain $\Omega$.

By the homotopy technique, He constructed a homotopy $v(r, p): \Omega \times [0, 1]$, which satisfies

$$H(v, p) = (1 - p)[L(v) - L(u_0)] + p[A(v) - f(r)] = 0, \quad p \in [0, 1], \quad r \in \Omega,$$

(20)

where $p \in [0, 1]$ is an embedding parameter and $u_0$ is an initial guess approximation of (18), which satisfies the boundary conditions. It follows from (20) that

$$H(v, 0) = L(v) - L(u_0) = 0, \quad H(v, 1) = A(v) - f(r) = 0.$$

Here we assume that the solution of (20) is a power series in $p$:

$$v = v_0 + pv_1 + p^2v_2 + \cdots.$$  

(21)

Setting $p = 1$, we obtain the approximate solution of (18),

$$u = \lim_{p \to 1} v = v_0 + v_1 + v_2 + \cdots.$$  

(22)

The convergence of series (22) has been proved in He (2003b).

**THEOREM 1** Suppose $N(v)$ is a non-linear function and $v = \sum_{k=0}^{\infty} p^k v_k$, then we have

$$\frac{\partial^n}{\partial p^n} N(v)_{p=0} = \frac{\partial^n}{\partial p^n} N\left(\sum_{k=0}^{\infty} p^k v_k\right)_{p=0} = \frac{\partial^n}{\partial p^n} N\left(\sum_{k=0}^{n} p^k v_k\right)_{p=0}.$$

**Proof.** For more details, see Ghorbani (2009).

**THEOREM 2** The approximate solution of (18) obtained by the HPM can be expressed in He’s polynomials as

$$u(r) = f(r) + H_0(v_0) + H_1(v_0, v_1) + \cdots + H_n(v_0, v_1, \ldots, v_n),$$

where He’s polynomials are defined as follows:

$$H_j(v_0, v_1, v_2, \ldots, v_j) = -L^{-1}\left(\frac{1}{j!} \frac{\partial}{\partial p^j} N\left(\sum_{k=0}^{j} p^k v_k\right)\right), \quad j = 0, \ldots, n.$$

**Proof.** For more details, see Ghorbani (2009).

5. **Application of the HPM for non-linear quadratic OCPs**

In this section, we shall introduce a new reliable procedure for choosing the initial approximations in HPM to solve non-linear quadratic OCPs, which reduce to a TPBVP.
5.1 Selection of the initial approximations

As mentioned in He (1999, 2003a), \( \sum_{k=1}^{n} x_k(t) \) and \( \sum_{k=1}^{n} \dot{x}_k(t) \) tend to the exact solutions of (17), say \( \dot{x}(t, \alpha) \) and \( \dot{z}(t, \alpha) \), as \( n \to \infty \). Thus, for sufficiently large number of HPM iterations, \( N \), we can write \( \sum_{k=1}^{N} x_k(t) \approx \dot{x}(t, \alpha) \) and \( \sum_{k=1}^{N} \dot{x}_k(t) \approx \dot{z}(t, \alpha) \) or, more precisely, since \( \sum_{k=1}^{N} x_k \) and \( \sum_{k=1}^{N} \dot{x}_k \) are functions of both \( t \) and \( \alpha \), we can write \( \sum_{k=1}^{N} x_k(t, \alpha) \approx \dot{x}(t, \alpha) \) and \( \sum_{k=1}^{N} \dot{x}_k(t, \alpha) \approx \dot{z}(t, \alpha) \).

Note that we did not use the final-state condition \( x(t_f) = x_f \) until now. Considering this condition, \( \alpha \) should be identified such that \( \sum_{k=1}^{N} x_k(t_f, \alpha) \approx \dot{x}(t_f, \alpha) = x_f \). That is, \( \alpha \) should be a real root of \( \sum_{k=1}^{N} x_k(t_f, \alpha) - x_f = 0 \), which can be easily approximated by numerical methods such as Newton or Secant method.

Let us denote the approximated \( \alpha \) by \( \hat{\alpha} \). Therefore, the analytic approximate solution of (15) is

\[
x(t) \approx \sum_{k=1}^{N} x_k(t, \hat{\alpha}), \quad \dot{x}(t) \approx \sum_{k=1}^{N} \dot{x}_k(t, \hat{\alpha}),
\]

and the optimal control law can be obtained by (16).

5.2 Suboptimal control design

Consider the OCP of the non-linear system (12) with the quadratic cost function (13). Then, the \( N \)th-order suboptimal trajectory–control pair is obtained as follows:

\[
\begin{align*}
x^{(N)}(t) &= \sum_{i=0}^{N} x_i(t), \\
u^{(N)}(t) &= -R^{-1}g^\top(t, x) \sum_{i=0}^{N} \dot{x}_i(t).
\end{align*}
\]

The integer \( N \) in (24) is generally determined according to a concrete control precision. As we will present in the next subsection, every time \( x_i(t) \) and \( \dot{x}_i(t) \) are obtained from the linear TPBVP sequence, we let \( N = i \) and calculate \( x^{(N)}(t) \) and \( \dot{x}^{(N)}(t) \) from (24). Then the following quadratic performance index (QPI) can be calculated as

\[
J^{(N)} = \frac{1}{2} \int_{0}^{t_f} (x^{(N)}(t))^\top Q(x^{(N)}(t)) + (u^{(N)}(t))^\top R(u^{(N)}(t)))dt.
\]

The \( N \)th-order suboptimal trajectory–control pair in (24) has desirable accuracy if for two given positive constants \( \epsilon_1 > 0 \) and \( \epsilon_2 > 0 \), the following conditions hold jointly:

\[
\left| \frac{J^{(N)} - J^{(N-1)}}{J^{(N)}} \right| < \epsilon_1,
\]

\[
\|x(t_f) - x_f\| < \epsilon_2,
\]

where \( \| \cdot \| \) is a suitable norm on \( \mathbb{R}^n \) and \( x(t_f) \) is the value of the corresponding state trajectory at the final time \( t_f \). If the tolerance error bounds \( \epsilon_1 > 0 \) and \( \epsilon_2 > 0 \) be chosen small enough, the \( N \)th-order suboptimal control law will be very close to the optimal control law \( u^*(t) \), the value of QPI in (24) will be very close to its optimal value \( J^* \) and the boundary condition will be satisfied tightly.
5.3 An illustrative example

Consider the following non-linear OCP:

\[ \min J = \int_{0}^{1} u^2(t) dt \]

s.t. \[ \dot{x} = \frac{1}{2} x^2(t) \sin x(t) + u(t), \quad t \in [0, 1], \]

\[ x(0) = 0, \quad x(1) = 0.5. \] (28)

According to (12, 13), we have \( f(t, x(t)) = \frac{1}{2} x^2(t) \sin x(t) \), \( g(t, x(t)) = 1 \), \( Q = 0 \), \( R = 1 \), \( t_0 = 0 \) and \( t_f = 1 \). As mentioned in Section 5.1, we solve the following IVP:

\[ \dot{x} = \frac{1}{2} x^2(t) \sin x(t) - \frac{1}{2} \dot{\lambda}(t), \]

\[ \dot{\lambda} = -\lambda(t) x(t) \sin x(t) - \frac{1}{2} \dot{\lambda}(t) x^2(t) \cos x(t), \quad t \in [0, 1], \]

\[ x(0) = 0, \quad \lambda(0) = \alpha, \] (29)

where \( \alpha \in \mathbb{R} \) is an unknown parameter. Also the optimal control law is given by

\[ u^*(t) = -\frac{1}{2} \dot{\lambda}(t). \] (30)

To solve system (29) by HMP, we construct the following homotopy:

\[
\begin{aligned}
(1 - p)(\dot{v}(t) - \dot{x}_0(t)) + p(\dot{v}(t) - \frac{1}{2} v^2(t) \sin v(t) + \frac{1}{2} w(t)) &= 0, \\
(1 - p)(\dot{w}(t) - \dot{\lambda}_0(t)) + p(\dot{w}(t) + w(t) v(t) \sin v(t) + \frac{1}{2} w(t) v^2(t) \cos v(t)) &= 0,
\end{aligned}
\] (31)

where \( p \in [0, 1] \) is an embedding parameter and \( x_0 \) and \( \lambda_0 \) are initial approximations that satisfy the initial conditions.

Suppose the solution of (29) has the form

\[
\begin{aligned}
v &= x_0 + px_1 + p^2 x_2 + \cdots, \\
w &= \lambda_0 + p\lambda_1 + p^2 \lambda_2 + \cdots.
\end{aligned}
\] (32)

Substituting (32) into (31) and comparing coefficients of the terms with the identical powers of \( p \) lead to
\( p^0: \)
\[
\dot{w}_0(t) - \dot{x}_0(t) = 0, \quad v_0(0) = 0,
\]
\[
\ddot{w}_0(t) - \ddot{x}_0(t) = 0, \quad w_0(0) = \alpha,
\]

\( p^1: \)
\[
\dot{v}_1(t) + \dot{x}_0(t) - \frac{1}{2} v_0^2(t) \sin v_0(t) + \frac{1}{2} w_0(t) = 0, \quad v_1(0) = 0,
\]
\[
\dot{w}_1(t) + \dot{\lambda}_0(t) + w_0(t)v_0(t) \sin v_0(t) + \frac{1}{2} w_0(t)v_0^2(t) \cos v_0(t) = 0, \quad w_1(0) = 0,
\]

\( p^2: \)
\[
\dot{v}_2(t) - \frac{1}{2} \frac{\partial}{\partial p} N_1 \left( \sum_{k=0}^{1} p^k v_k, \sum_{k=0}^{1} p^k w_k \right)_{p=0} + \frac{1}{2} v_1(t) = 0, \quad v_2(0) = 0,
\]
\[
\dot{w}_2(t) + \frac{\partial}{\partial p} N_2 \left( \sum_{k=0}^{1} p^k v_k, \sum_{k=0}^{1} p^k w_k \right)_{p=0} + \frac{1}{2} \frac{\partial}{\partial p} N_3 \left( \sum_{k=0}^{1} p^k v_k, \sum_{k=0}^{1} p^k w_k \right)_{p=0} = 0, \quad w_2(0) = 0,
\]
\[
\vdots
\]

\( p^j: \)
\[
\dot{v}_j(t) - \frac{1}{2} \frac{1}{(j - 1)!} \frac{\partial^{j-1}}{\partial p^{j-1}} N_1 \left( \sum_{k=0}^{j-1} p^k v_k, \sum_{k=0}^{j-1} p^k w_k \right)_{p=0} + \frac{1}{2} w_{j-1}(t) = 0, \quad v_j(0) = 0,
\]
\[
\dot{w}_j(t) + \frac{1}{(j - 1)!} \frac{\partial^{j-1}}{\partial p^{j-1}} N_2 \left( \sum_{k=0}^{j-1} p^k v_k, \sum_{k=0}^{j-1} p^k w_k \right)_{p=0} + \frac{1}{2} \frac{1}{(j - 1)!} \frac{\partial^2}{\partial p^2} N_3 \left( \sum_{k=0}^{j-1} p^k v_k, \sum_{k=0}^{j-1} p^k w_k \right)_{p=0} = 0, \quad w_j(0) = 0,
\]

where \( \frac{1}{n!} \frac{\partial^n}{\partial p^n} N_i \left( \sum_{k=0}^{n} p^k v_k, \sum_{k=0}^{n} p^k w_k \right)_{p=0}, i = 1, 2, 3, \) are He’s polynomials relative to non-linear terms.
Solving the presented sequence of linear time-invariant TPBVPs in (33–36) with the initial approximations of $x_0$ and $\lambda_0$ leads to:

\[
x_0(t) = 0, \quad \lambda_0(t) = \alpha, \quad \lambda_4(t) = 0,
\]
\[
x_1(t) = -\frac{1}{2} \alpha t, \quad \lambda_1(t) = 0, \quad \lambda_5(t) = \frac{1}{192} \alpha^5 t^5,
\]
\[
x_n(t) = 0, \quad n \geq 2, \quad \lambda_2(t) = 0, \quad \vdots
\]
\[
\lambda_3(t) = -\frac{1}{8} \alpha^3 t^3.
\]

By considering the final-state condition, we should have $\sum_{k=0}^{i} x_k(1) = 0.5$, for $i = 1, \ldots, 5$. That is, $-\frac{1}{2} \alpha = 0.5$ or, equivalently, $\alpha = -1$. Therefore, we have:

\[
x_0(t) = 0, \quad \lambda_0(t) = -1, \quad \lambda_3(t) = \frac{1}{8} t^3,
\]
\[
x_1(t) = \frac{1}{2} t, \quad \lambda_1(t) = 0, \quad \lambda_4(t) = 0,
\]
\[
x_n(t) = 0, \quad n \geq 2, \quad \lambda_2(t) = 0, \quad \lambda_5(t) = -\frac{1}{192} t^5
\]
\[
\vdots
\]

and the associated optimal control is
Simulation curves of $u(t)$ and $x(t)$ for $n = 5$ are shown in Figs 1 and 2, respectively. Also, we compared the results of HPM with the solutions obtained using the collocation method (Ascher et al., 1995) and modal series (Jajarmi et al., 2011). Results of both methods are very close to each other as shown by Figs 1 and 2. This confirms that the proposed method yields excellent results.

In order to obtain an accurate enough suboptimal trajectory–control pair, we applied the strategy proposed in Section 5.2, with the tolerance error bounds $\epsilon_1 = 2 \times 10^{-3}$ and $\epsilon_2 = 2 \times 10^{-5}$. In this case, convergence is achieved after five iterations, i.e. $|J^{(5)} - J^{(4)}| = 1.6999575 \times 10^{-3} < 2 \times 10^{-3}$ and $\|x(t) - x_{t_1}\| = 4.1832292 \times 10^{-6} < 2 \times 10^{-5}$. A minimum of $J^{(5)} = 0.2353$ is obtained.

Problem (28) has also been solved by Rubio (1986) via the measure theory in which to find an acceptable solution, a linear programming problem with 1000 variables and 20 constraints should be solved. Results are listed in Tables 1 and 2.

### Table 1 Simulation results of example at different iteration times

| $i$ | $J(i)$ | $|J^{(i)} - J^{(i-1)}|$ | $\|x(t_i) - x_{t_1}\|$ |
|-----|--------|------------------------|------------------------|
| 1   | 0.25   | 1.6016512 $\times 10^{-2}$ |
| 2   | 0.25   | 1.6016512 $\times 10^{-2}$ |
| 3   | 0.2349 | 4.4742585 $\times 10^{-4}$ |
| 4   | 0.2349 | 4.4742585 $\times 10^{-4}$ |
| 5   | 0.2353 | 4.1832292 $\times 10^{-6}$ |

Simulation curves of $u(t)$ and $x(t)$ for $n = 5$ are shown in Figs 1 and 2, respectively. Also, we compared the results of HPM with the solutions obtained using the collocation method (Ascher et al., 1995) and modal series (Jajarmi et al., 2011). Results of both methods are very close to each other as shown by Figs 1 and 2. This confirms that the proposed method yields excellent results.

In order to obtain an accurate enough suboptimal trajectory–control pair, we applied the strategy proposed in Section 5.2, with the tolerance error bounds $\epsilon_1 = 2 \times 10^{-3}$ and $\epsilon_2 = 2 \times 10^{-5}$. In this case, convergence is achieved after five iterations, i.e. $|J^{(5)} - J^{(4)}| = 1.6999575 \times 10^{-3} < 2 \times 10^{-3}$ and $\|x(1) - 0.5\| = 4.1832292 \times 10^{-6} < 2 \times 10^{-5}$. A minimum of $J^{(5)} = 0.2353$ is obtained.

Problem (28) has also been solved by Rubio (1986) via the measure theory in which to find an acceptable solution, a linear programming problem with 1000 variables and 20 constraints should be solved. Results are listed in Tables 1 and 2.
TABLE 2 Results of the proposed method and measure theoretical method

<table>
<thead>
<tr>
<th>Method</th>
<th>Performance index value</th>
<th>Final-state error</th>
</tr>
</thead>
<tbody>
<tr>
<td>HPM</td>
<td>0.2353</td>
<td>4.2 \times 10^{-6}</td>
</tr>
<tr>
<td>Measure theory method</td>
<td>0.2425</td>
<td>4.3 \times 10^{-3}</td>
</tr>
<tr>
<td>Modal series method</td>
<td>0.2353</td>
<td>2.8 \times 10^{-5}</td>
</tr>
</tbody>
</table>

6. Application of the HPM for linear optimal control system

In this section, to illustrate the effectiveness of the HPM we shall consider two examples of optimal control of linear systems. In the following examples, we assume, for the sake of simplicity, \( Q = 0 \) and \( k(t) = R^{-1}B^T Y(t) \), and according to (9), we have \( u^*(t) = -k(t)x(t) \). Other numerical methods for approximating \( k(t) \) based on orthogonal functions are available in Datta and Mohan (1995).

EXAMPLE 3 Consider a single-input scalar system as follows:

\[
\dot{x} = -2x(t) + u(t),
\]

\[
J = \frac{1}{2}x^2(1) + \frac{1}{2} \int_0^1 (x^2(t) + u^2(t)) dt.
\]

According to system (1), we have \( A = -2, B = 1, S = 1, P = 1, Q = 0, R = 1 \) and \( t_1 = 1 \). By using system (11), we have

\[
\begin{align*}
\dot{V}(t) &= -2V(t) - W(t), \\
\dot{W}(t) &= -V(t) + 2W(t).
\end{align*}
\]

The exact solution of \( k(t) \) is

\[
k(t) = \frac{\sqrt{5} \cosh \sqrt{5}(1 - t) - \sinh \sqrt{5}(1 - t)}{\sqrt{5} \cosh \sqrt{5}(1 - t) + 3 \sinh \sqrt{5}(1 - t)}.
\]

To solve system (39) by HPM, we construct the following homotopy:

\[
\begin{cases}
(1 - p)(\dot{V}^*(t) - \dot{V}_0(t)) + p(\dot{V}^*(t) + 2V^*(t) + W^*(t)) = 0, \\
(1 - p)(\dot{W}^*(t) - \dot{W}_0(t)) + p(\dot{W}^*(t) + V^*(t) - 2W^*(t)) = 0.
\end{cases}
\]

In Fig. 3, the approximate value for \( k(t) \), obtained from HPM with \( n = 15 \), and the following exact value are plotted. Table 3 gives the results from the HPM and exact solution of Example 6.1 and illustrates the absolute error.
Figure 3. Comparison of the exact solution with the HPM solution.

Table 3. Comparison of the error of the 15th-order HPM, exact solution

<table>
<thead>
<tr>
<th>t</th>
<th>HPM</th>
<th>Exact solution</th>
<th>Absolute error</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0</td>
<td>$2.4353 \times 10^{-001}$</td>
<td>$2.4353 \times 10^{-001}$</td>
<td>$1.1364 \times 10^{-009}$</td>
</tr>
<tr>
<td>0.2</td>
<td>$2.5437 \times 10^{-001}$</td>
<td>$2.5437 \times 10^{-001}$</td>
<td>$5.0452 \times 10^{-011}$</td>
</tr>
<tr>
<td>0.4</td>
<td>$2.8111 \times 10^{-001}$</td>
<td>$2.8111 \times 10^{-001}$</td>
<td>$7.8126 \times 10^{-013}$</td>
</tr>
<tr>
<td>0.6</td>
<td>$3.4786 \times 10^{-001}$</td>
<td>$3.4786 \times 10^{-001}$</td>
<td>$1.5543 \times 10^{-015}$</td>
</tr>
<tr>
<td>0.8</td>
<td>$5.1975 \times 10^{-001}$</td>
<td>$5.1975 \times 10^{-001}$</td>
<td>$8.8818 \times 10^{-016}$</td>
</tr>
<tr>
<td>1.0</td>
<td>$1.0000 \times 10^{+000}$</td>
<td>$1.0000 \times 10^{+000}$</td>
<td>$4.4409 \times 10^{-015}$</td>
</tr>
</tbody>
</table>

Example 4. Consider a single-input scalar system as follows:

\[
\dot{x} = -x(t) + u(t), \tag{41}
\]

\[
J = \frac{1}{2} \int_{0}^{1} (x^2(t) + u^2(t))dt. \tag{42}
\]

According to system (1), we have $A = -1$, $B = 1$, $S = 0$, $Q = 1$, $R = 1$ and $t_f = 1$, then

\[
\dot{V}(t) = -V(t) - W(t),
\]

\[
\dot{W}(t) = -V(t) + W(t). \tag{43}
\]
The exact solution of \( k(t) \) is

\[
k(t) = -\frac{(1 + \sqrt{2}\beta) \cosh(\sqrt{2}t) + (\sqrt{2} + \beta) \sinh(\sqrt{2}t)}{\cosh(\sqrt{2}t) + \beta \sinh(\sqrt{2}t)},
\]

where

\[
\beta = -\frac{\cosh(\sqrt{2}) + \sqrt{2} \sinh(\sqrt{2})}{\sqrt{2} \cosh(\sqrt{2}) + \sinh(\sqrt{2})}.
\]

In Fig. 4, the approximate value for \( k(t) \), obtained from HPM with \( n = 15 \), and the following exact value are plotted. Table 4 gives the results from the HPM and exact solution of Example 6.2 and illustrates the absolute error.

**Fig. 4.** Comparison of the exact solution with the HPM solution.

**Table 4** Comparison of the error of the 15th-order HPM approximate solution with exact solution

<table>
<thead>
<tr>
<th>( t )</th>
<th>HPM</th>
<th>Exact solution</th>
<th>Absolute error</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0</td>
<td>( 3.8582 \times 10^{-001} )</td>
<td>( 3.8582 \times 10^{-001} )</td>
<td>( 1.2143 \times 10^{-012} )</td>
</tr>
<tr>
<td>0.2</td>
<td>( 3.6460 \times 10^{-001} )</td>
<td>( 3.6460 \times 10^{-001} )</td>
<td>( 4.2799 \times 10^{-014} )</td>
</tr>
<tr>
<td>0.4</td>
<td>( 3.2801 \times 10^{-001} )</td>
<td>( 3.2801 \times 10^{-001} )</td>
<td>( 2.2204 \times 10^{-016} )</td>
</tr>
<tr>
<td>0.6</td>
<td>( 2.6588 \times 10^{-001} )</td>
<td>( 2.6588 \times 10^{-001} )</td>
<td>( 4.4409 \times 10^{-016} )</td>
</tr>
<tr>
<td>0.8</td>
<td>( 1.6306 \times 10^{-001} )</td>
<td>( 1.6306 \times 10^{-001} )</td>
<td>( 7.7716 \times 10^{-016} )</td>
</tr>
<tr>
<td>1.0</td>
<td>( 1.8221 \times 10^{-016} )</td>
<td>( 1.3634 \times 10^{-015} )</td>
<td>( 1.1812 \times 10^{-015} )</td>
</tr>
</tbody>
</table>
7. Conclusions

In this paper, we have successfully developed HPM and He’s polynomials for solving non-linear quadratic OCPs. Then we employed HPM to finding optimal control of linear systems. Applying the HPM, the optimal control law and the optimal trajectory are determined in the form of rapid convergent series with easy computable terms. The proposed method avoids directly solving the non-linear TPBVP. Therefore, in view of computational complexity, the proposed method is more practical than the other approximate methods.

Matlab has been used for computations in this paper.

REFERENCES


