# On the Connectivity and Superconnectivity of Bipartite Digraphs and Graphs

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**Abstract**

In this work, first, we present sufficient conditions for a bipartite digraph to attain optimum values of a stronger measure of connectivity, the so-called superconnectivity. To be more precise, we study the problem of disconnecting a maximally connected bipartite (di)graph by removing nontrivial subsets of vertices or edges. Within this framework, both an upper-bound on the diameter and Chartrand type conditions to guarantee optimum superconnectivities are obtained. Secondly, we show that if the order or size of a bipartite (di)graph is small enough then its vertex connectivity or edge-connectivity attain their maximum values. For example, a bipartite digraph is maximally edge-connected if $\delta^+(x)+\delta^-(y) \geq \lceil \frac{n+1}{2} \rceil$ for all pair of vertices $x, y$ such that $d(x, y) \geq 4$. This result improves some of the above known conditions given by Dankelmann and Volkmann in [10] for the undirected case.

## 1 Introduction

The study of some parameters related to the connectivity of (di)graphs has recently proved to be of some interest in the design of reliable and fault-tolerance interconnection of communication networks. The fiability of a network is of prime importance to network designers. Many (di)graphs theoretical parameters have been used to describe the fiability of communication. The most frequently used are the vertex-connectivity and edge-connectivity. These parameters present a difficulty: they do not take into account what remains after the (di)graph is disconnected. One can ask what is the size of the largest remaining group within which mutual communication can still occur. To deal with this problem a number of other parameters have recently been introduced that attempt to cope with this difficulty, including superconnectivity [3, 6, 14, 20], extraconnectivity [4, 12], etc.

These facts have encouraged many authors to find sufficient conditions for a graph or digraph to have high values of the above mentioned parameters. Most of these conditions are stated in terms of other usual parameters in network design, such as the number of vertices, minimum and maximum degrees, diameter and girth.

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In this paper we are primarily concerned with the study of superconnected bipartite digraphs. With this aim, new parameters $\kappa_1$ and $\lambda_1$ that measure the quantity of superconnectivity of the digraph, are defined. First of all, we get sufficient conditions on the diameter in terms of the parameter $\ell$, which imply that the bipartite digraph is optimal superconnected. For graphs these conditions are formulated in terms of the girth. Upper-bounds on the diameter to assure optimum connectivities can be found in [11, 13, 16]. These results play an important role to state new upper bounds on the order for a bipartite digraph to have optimal superconnectivity. Afterthat, we study the standard connectivity and we show that if the order is small enough, then the connectivity is maximum. From these results we derive conditions on the minimum degree in terms of the order, which extend those given by Volkmann [25] and Dankelmann and Volkmann [10] for graphs. They will be referred as Chartrand-type conditions, because these conditions are of the same type as the well-known result given by Chartrand in [9].

The remaining of this section is devoted to recall some concepts and to explain the new ones. Thus, $G$ stands for a (finite) simple digraph, that is, without loops or multiple edges, with set of vertices $V = V(G)$ and set of (directed) edges $E = E(G)$. If $G$ is bipartite we will write $V = U_0 \cup U_1$, where $U_0$ and $U_1$ denote the partite sets of vertices. The cardinalities $n = |V(G)|$ and $m = |E(G)|$ are respectively the *order* and *size* of $G$. For any edge $(x, y) \in E$, we say that $x$ is its *initial* vertex, and $y$ its *final* vertex. For any pair of vertices $x, y \in V$, a path from $x$ to $y$ is called an *$x \to y$ path.* A digraph $G$ is said to be *(strongly) connected* when for any pair of vertices $x, y \in V$ there always exists an $x \to y$ path. The *distance* from $x$ to $y$ is denoted by $d(x, y)$, and $D = \max_{x,y\in V}\{d(x, y)\}$ stands for the *diameter* of $G$. The distance from $x$ to $U \subset V$, denoted by $d(x, U)$, is the minimum over all the distances $d(x, u)$, $u \in U$. The distance from $U$ to $x$, $d(U, x)$, is defined analogously. Given a set of edges $A \subset E$ we define $d(x, A) = \min_{(u,v)\in A} d(x, u)$ and $d(A, x) = \min_{(u,v)\in A} d(v, x)$. Let $\Gamma^-(x)$ and $\Gamma^+(x)$ denote the sets of vertices adjacent to and from $x$ respectively. Their cardinalities are the *in-degree* of $x$, $\delta^-(x)$, and *out-degree* of $x$, $\delta^+(x)$ respectively. The *minimum degree* $\delta$ [*maximum degree* $\Delta$] of $G$ is the minimum [maximum] over all the in-degrees and out-degrees of the vertices of $G$. We will always assume that our digraphs are connected, hence $\delta \geq 1$.

Alternatively, we will use the following notation involving the sets of edges: $\omega^-(x)$ and $\omega^+(x)$ denote respectively the sets of edges adjacent to and from $x$. Their cardinalities are respectively the *in-degree* of $x$, $\delta^-(x)$, and *out-degree* of $x$, $\delta^+(x)$.

Given a subset of vertices $C$, let $\Gamma^+(C) = \bigcup_{x\in C} \Gamma^+(x)$ and $\Gamma^-(C) = \bigcup_{x\in C} \Gamma^-(x)$. The *positive* and *negative boundaries* of $C$ are $\partial^+C = \Gamma^+(C) \setminus C$ and $\partial^-C = \Gamma^-(C) \setminus C$, respectively. The corresponding concepts for edges are the *positive* and *negative edge-boundaries*, $\omega^+C = \{(x, y) \in E : x \in C$ and $y \in V \setminus C\}$ and $\omega^-C = \{(x, y) \in E : x \in V \setminus C$ and $y \in C\}$. Moreover, note that $\omega^+C = \omega^-(V \setminus C)$.

Clearly, if $C \cup \partial^+C \neq V$ $[C \cup \partial^-C \neq V]$ then $\partial^+C$ $[\partial^-C]$ is a cutset of

$G$. Similarly, if $C$ is a proper (nonempty) subset of $V$, then $\omega^+C$ $[\omega^-C]$ is an edge cutset. Hence, by using these concepts, the *(vertex) connectivity* and *edge-connectivity* of $G$ can be respectively defined as

$$\kappa = \min\{|\partial^+ C| \ : \ C \subset V, \, C \cup \partial^+ C \neq V \text{ or } |C| = 1\};$$
$$\lambda = \min\{|\omega^+ C| \ : \ C \subset V, \, C \neq \emptyset, V\}.$$

It is well-known that, for any digraph $G$, $\kappa \leq \lambda \leq \delta$, see [17]. Hence, $G$ is said to be *maximally connected* when $\kappa = \lambda = \delta$, and *maximally edge-connected* if $\lambda = \delta$.

In order to study the connectivity of digraphs, a new parameter related to the number of shortest paths was used in [11] (see also [16]):

**Definition 1.1** *For a given digraph $G$ with diameter $D$, let $\ell = \ell(G)$, $1 \leq \ell \leq D$, be the greatest integer such that, for any $x, y \in V$,*

*(a) if $d(x,y) < \ell$, the shortest $x \to y$ path is unique and there are no $x \to y$ paths of length $d(x,y) + 1$;*

*(b) if $d(x,y) = \ell$, there is only one shortest $x \to y$ path.*

In recent years, several results relating the connectivity of a (di)graph with the aforementioned parameters, $n$, $m$, $\Delta$, $\delta$, $\ell$ and $D$ have been given. See the survey of Bermond, Homobono and Peyrat [5], and [22] for more details.

Concerning bipartite digraphs less work have been done until now. Since in a bipartite digraph, between any two vertices there are no paths whose lengths differ by one, the definition of the parameter $\ell$ can be simplified by saying that it is the greatest integer such that, for any pair of vertices $x, y \in V$ at distance $d(x,y) \leq \ell$, the shortest $x \to y$ path is unique.

The results for bipartite digraphs to be maximally connected involving this parameter and the diameter were given in [13].

$$\begin{aligned} \kappa = \delta \quad & \text{if } D \leq 2\ell; \\ \lambda = \delta \quad & \text{if } D \leq 2\ell + 1. \end{aligned} \tag{1}$$

Similar concepts and results apply for graphs. In this case, all the introduced concepts are unsigned. Thus, for example, given a subset of vertices $C$, $\Gamma(C) = \bigcup_{x \in C} \Gamma(x)$. The *boundary* of $C$ is $\partial C = \Gamma(C) \setminus C$. The *edge-boundary*, of $C$ is $\omega C = \{(x,y) \in E \ : \ x \in C \text{ and } y \in V \setminus C\}$. Obviously, the same definitions of parameter $\ell$ applies for graphs. It turns out that the parameter $\ell$ is tightly related to the girth $g$ of $G$. Indeed, one can readily check that $\ell = \lfloor (g-1)/2 \rfloor$. So that, in bipartite case $g = 2\ell + 2$.

## 2 Superconnectivity

Superconnectivity is a stronger measure of connectivity, introduced by Boesch and Tindell in [6], whose study has deserved some attention in the last years. Given a subset of vertices $C$, let us say that $\partial^+C$ $[\partial^-C]$ is *nontrivial* if $\partial^+C$

$[\partial^- C]$ does not contain a set $\Gamma^+(x)$ or $\Gamma^-(x)$, for each $x \in C$. Notice that if $|\partial^+ C| < \delta$ then $\partial^+ C$ is nontrivial. Analogously, $\omega^+ C$ $[\omega^- C]$ is said to be *nontrivial* if $\omega^+ C$ $[\omega^- C]$ does not contain a set $\omega^+(x)$ or $\omega^-(x)$, for each $x \in C$. A maximally connected digraph is called *super-$\kappa$* if all the minimum disconnecting sets are trivial. Similarly, a maximally edge-connected digraph is called *super-$\lambda$* if all the minimum edge-disconnecting sets are trivial. Some results about superconnectivity can be found in Hamidoune, LLadó and Serra [20], Lesniak [21] and Fàbrega and Fiol [11, 15].

Hence, by using these concepts, we define new parameters

$$\kappa_1 = \min\{|\partial^+ C| \ : \ \partial^+ C \text{ nontrivial}, C \cup \partial^+ C \neq V \ \};$$
$$\lambda_1 = \min\{|\omega^+ C| \ : \ \omega^+ C \text{ nontrivial}\}.$$

Notice that if $\kappa_1 \leq \delta$ then $\kappa_1 = \kappa$. When $\kappa_1 > \delta$ (that is to say, all the disconnecting set of order $\delta$ must be trivial) the digraph must be super-$\kappa$ and we define the *(vertex) superconnectivity* of the digraph as the value of $\kappa_1$. Analogously, if $\lambda_1 \leq \delta$ then $\lambda_1 = \lambda$. When $\lambda_1 > \delta$ the digraph must be super-$\lambda$ and we define the *edge-superconnectivity* of the digraph as the value of $\lambda_1$.

Following Hamidoune [18, 19], a subset $C$ of vertices of a maximally connected digraph $G$ is a *positive 1-fragment* of $G$ if $\partial^+ C$ is nontrivial, $|\partial^+ C| = \kappa_1$ and $\overline{C} \neq \emptyset$, where $\overline{C} = V \setminus (C \cup \partial^+ C)$. Analogously it is defined a *negative 1-fragment*. Note that $C$ is a positive 1-fragment if and only if $\overline{C}$ is a negative 1-fragment. The set of vertices $C$ is called a *positive $\alpha_1$-fragment* of $G$ if $\omega^+ C$ is nontrivial and $|\omega^+ C| = \lambda_1$. Similarly, it is defined a *negative $\alpha_1$-fragment*.

If the bipartite digraph $G$ contains a digon then $\Gamma^+(x) \cap \Gamma^+(y) = \emptyset$. So, $\Gamma^+(x) \cup \Gamma^+(y) \setminus \{x, y\}$ could be an example of disconnecting nontrivial set of at least $2\delta - 2$ vertices. Then, we claim that a good lower bound for $\kappa_1, \lambda_1$ is $2\delta - 2$. Thus, $G$ is said to be *optimal superconnected* if $\kappa_1 \geq 2\delta - 2$ and similarly, *optimal edge-superconnected* if $\lambda_1 \geq 2\delta - 2$. In what follows we state sufficient conditions on the diameter to assure that the digraph is optimal superconnected. We begin by considering the super-vertex-connectivity.

The following concepts were introduced in [1]. The *positive deepness* of a subset of vertices $C$ is $\mu(C) = \max_{x \in C} d(x, \partial^+ C)$. Similarly, the *negative deepness* of $C$ is $\mu'(C) = \max_{x \in C} d(\partial^- C, x)$. The *positive valley* of $C$ is the set of vertices $x \in C$ such that $d(x, \partial^+ C) = \mu(C)$. The negative valley is defined in a similar way.

Let us consider two vertices $x \in C$, $y \in \overline{C}$. We need to introduce the following notation. Let $p^+(x) = \{f \in \partial^+ C \ : \ d(x, f) = d(x, \partial^+ C)\}$ and $p^-(y) = \{f \in \partial^+ C \ : \ d(f, y) = d(\partial^+ C, y)\}$ denote the set of vertices belonging to the positive boundary of $C$ at minimum distance from $x$ and to $y$, respectively.

The following lemma will be useful in our next results.

**Lemma 2.1** *Let $G$ be a connected bipartite digraph with parameter $\ell$ and minimum degree $\delta$, and $C$ denotes a subset of vertices. Let vertices $x, y$ belonging to the positie valley of $C$ and negative valley of $\overline{C}$, respectively. Then,*

*(a) for each pair of vertices $x_i, x_j \in \Gamma^+(x)$ it is satisfied that $p^+(x_i) \cap p^+(x_j) = \emptyset$, if $\mu(C) \leq \ell - 1$;*

(b) *for each pair of vertices* $y_i, y_j \in \Gamma^-(y)$ *it is satisfied that* $p^-(y_i) \cap p^-(y_j) = \emptyset$, *if* $\mu'(\overline{C}) \le \ell - 1$.

**Proof.** Let us assume that $f \in p^+(x_i) \cap p^+(x_j)$. Then, there are two distinct paths from vertex $x$ to $f$, namely, $xx_i \to f$ and $xx_j \to f$ of length $\mu(C)$ or $\mu(C)+1$, contradicting the definition of parameter $\ell$, since $\mu(C) \le d(x,f) \le \mu(C)+1 \le \ell$. The reasoning is similar for $\overline{C}$. ∎

We observe that if $x$ is a vertex belonging to the positive valley of $C$ with $k$ out-neighbors belonging also to the positive valley of $C$, then $|p^+(x)| \ge \delta - k$. We will use this fact in the following lemma.

**Lemma 2.2** *Let $G$ be a maximally connected bipartite digraph with parameter $\ell$ and minimum degree $\delta \ge 3$. Let $C$ denote a subset of vertices such that $|\partial^+ C| \le 2\delta - 3$, and let vertices $x, y$ belonging to the positive valley of $C$ and negative valley of $\overline{C}$, respectively. Then,*

(a) $\mu(C) \ge \ell$ *if there exists a vertex* $z \in \Gamma^+(x)$ *into the positive valley of* $C$*;*

(b) $\mu'(\overline{C}) \ge \ell$ *if there exists a vertex* $z \in \Gamma^-(y)$ *into the negative valley of* $\overline{C}$.

**Proof.** The proof is by contradiction. Let $z \in \Gamma^+(x)$ a vertex belonging to the positive valley of $C$ and assume that $\mu = \mu(C) \le \ell - 1$. Notice that $\Gamma^+(x) \cap \Gamma^+(z) = \emptyset$ because the digraph is bipartite. By Lemma 2.1 each pair of vertices $x_i, x_j \in \Gamma^+(x)$ satisfy $p^+(x_i) \cap p^+(x_j) = \emptyset$ and, for the same reason, each pair of vertices $z_i, z_j \in \Gamma^+(z)$ satisfy $p^+(z_i) \cap p^+(z_j) = \emptyset$. As $\left(\cup_{x_i \in \Gamma^+(x)} p^+(x_i) \cup_{z_j \in \Gamma^+(z)} p^+(z_j)\right) \subset \partial^+(C)$ and $|\partial^+ C| \le 2\delta - 3$, it follows that there exist at least two vertices in $\Gamma^+(x) \setminus \{z\}$ and two vertices in $\Gamma^+(z)$, say, $x_r, z_r$ $r = 1, 2$, such that $p^+(x_r) \cap p^+(z_r) \ne \emptyset$. This implies that there are two different paths from vertex $x$ to each vertex $f_r \in p^+(x_r) \cap p^+(z_r)$, namely, the path $xx_r \to f_r$ of length $\mu$ or $\mu+1$ and the path $xzz_r \to f_r$, of length $\mu+1$ or $\mu+2$. Since we are assuming that $\mu \le \ell - 1$, then the path $zz_r \to f_r$ must have length $\mu+1$ and so, the path $xx_r \to f_r$ has length $\mu$ because the digraph is bipartite. Therefore, $|p^+(x)| \ge 2$ and also $|p^+(z)| \ge 2$, because vertex $z$ satisfies the same hypothesis as vertex $x$. Let $\{z_1, z_2, \ldots, z_k\} \subset \Gamma^+(z)$ denote the out-neighbors of $z$ belonging to the valley of $C$. Then $|p^+(z)| \ge \delta - k$ and we have shown that $k \ge 2$. Furthermore, $|p^+(z_i)| \ge 2$ for $1 \le i \le k$, because all these vertices satisfy the same hypothesis as vertex $z$. Then, $k \le \delta - 3$, since otherwise, by Lemma 2.1 we get that $2\delta - 3 \ge |\partial^+ C| \ge \sum_{z_j \in \Gamma^+(z)} |p^+(z_j)| \ge \delta + k \ge 2\delta - 2$, which is a contradiction (see Fig. 1.) Hence $|p^+(z)| \ge 3$ and also $|p^+(z_i)| \ge 3$ for $1 \le i \le k$. Thus, $2\delta - 3 \ge |\partial^+ C| \ge \delta + 2k$, that is $2k \le \delta - 3$. As $k \ge 2$ the lemma would be proved if $\delta \le 6$. Besides, $k \le \delta - 5$, otherwise $2\delta - 3 \ge |\partial^+ C| \ge 3\delta - 8$ which is impossible for $\delta \ge 7$. In a finite number of steps we prove that $k = 2$ and hence $|p^+(z_i)| \ge \delta - 2$ for $1 \le i \le k$. Then $2\delta - 3 \ge |\partial^+ C| \ge 3(\delta - 2)$ which is a contradiction.

The argument is similar for $\overline{C}$. ∎

Now, we are ready to give a lower bound to the deepness of an 1-fragment.

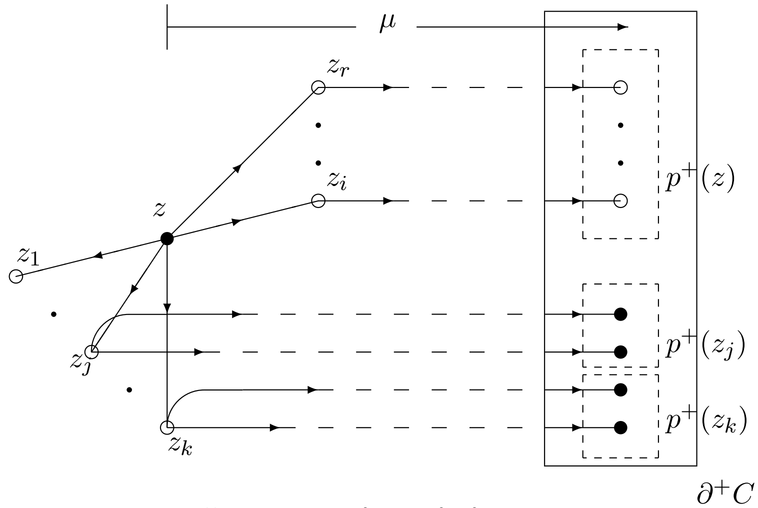


Figure 1: Illustration of proof of Lemma 2.2.

**Lemma 2.3** *Let $G$ be a maximally connected bipartite digraph with parameter $\ell$ and minimum degree $\delta \geq 3$. Let $C$ denote a positive 1-fragment. Then,*
*$\mu(C) \geq \ell$ and $\mu'(\overline{C}) \geq \ell$ if $\kappa_1 \leq 2\delta - 3$.*

**Proof.** Assume that $\mu = \mu(C) \leq \ell - 1$. By Lemma 2.2 each vertex of the positive valley of $C$ has no out-neighbours belonging to the positive valley of $C$. So $\mu > 1$ since $\partial^+ C$ is a nontrivial set, and by Lemma 2.1 each vertex of the positive valley of $C$ satisfies that $|p^+(x)| = \sum_{x_i \in \Gamma^+(x)} |p^+(x_i)| \geq \delta$. Therefore, there exists some $x_1 \in \Gamma^+(x)$ such that $|p^+(x_1)| = 1$, otherwise, $|\partial^+ C| \geq |p^+(x)| \geq 2\delta$, a contradiction. Let $y_2, y_3 \ldots, y_\delta$ be $\delta - 1$ out-neighbors of $x_1$ such that $\mu - 1 \leq d(y_i, \partial^+ C) \leq \mu$. As $|\partial^+ C| \leq 2\delta - 3$, it follows that there exist at least two vertices in $\Gamma^+(x) \setminus \{x_1\}$ and two vertices in $\Gamma^+(x_1)$, say, $x_r, y_r$ $r = 2, 3$, such that $p^+(x_r) \cap p^+(y_r) \neq \emptyset$. This implies that there are two different paths from vertex $x$ to each vertex $f_r \in p^+(x_r) \cap p^+(y_r)$, namely, the shortest path $xx_r \to f_r$ of length $\mu$, and the path $xx_1y_r \to f_r$, whose length must be at least $\mu + 2$, because $\mu \leq \ell - 1$. So, the path $x_1 y_r \to f_r$ has length $\mu + 1$ and hence $y_2, y_3$ belong to the positive valley of $C$. Then vertices $y_r$ satisfy the same hypothesis as vertex $x$, so that $|p^+(y_r)| \geq \delta$. Therefore there must exist at least three vertices $f'_t \in \partial^+ C$, $1 \leq t \leq 3$, at distance $\mu$ from both vertices $y_2, y_3$. Thus, the paths $x_1 y_2 \to f'_t$ and $x_1 y_3 \to f'_t$ are different and have length $\mu + 1$, which contradicts the definition of parameter $\ell$, unless $d(x_1, f'_t) = \mu - 1$, but this is impossible because $|p^+(x_1)| = 1$.

The result is analogous for $\overline{C}$. ■

In the following theorems we obtain sufficient conditions on the diameter to assure optimal superconnectivities for a bipartite digraph.

**Theorem 2.1** *Let $G$ be a bipartite digraph with $\delta \geq 3$, parameter $\ell$, and superconnectivity $\kappa_1$. Then $G$ is super-$\kappa$ and $\kappa_1 \geq 2\delta - 2$ if $D \leq 2\ell - 1$.*

**Proof.** By (1) the bipartite digraph $G$ is maximally connected because $D \leq 2\ell - 1$. Let $C$ a positive 1-fragment such that $\partial^+ C$ is nontrivial and $\delta \leq |\partial^+ C| \leq 2\delta - 3$. Let $x$, $y$ be two vertices belonging to the positive valley of $C$ and the negative valley of $\overline{C}$ respectively. Then by Lemma 2.3 we have that $D \geq d(x,y) \geq d(x, \partial^+ C) + d(\partial^+ C, y) = \mu(C) + \mu(\overline{C}) \geq 2\ell$, which is a contradiction. Therefore, the digraph is super-$\kappa$ and $\kappa_1 \geq 2\delta - 2$. ∎

Now, let us consider the edge-superconnectivity. With respect to $\alpha_1$-fragments, the *deepness* of a positive $\alpha_1$-fragment $C$ is $\nu(C) = \max_{x \in C} d(x, \omega^+ C)$. The *deepness* of a negative $\alpha_1$-fragment $C$ is defined analogously, $\nu'(C) = \max_{x \in C} d(\omega^- C, x)$.

In the next lemma we find the minimum deepness of $\alpha_1$-fragments.

**Lemma 2.4** *Let $G$ be a bipartite digraph maximally edge-connected, with parameter $\ell$ and minimum degree $\delta \geq 3$. Let $C$ denote a positive $\alpha_1$-fragment of $G$. Then, $\nu(C) \geq \ell$ and $\nu'(\overline{C}) \geq \ell$ if $\lambda_1 \leq 2\delta - 3$.*

**Proof.** Let us denote by $F = \{x \in C : (x,y) \in \omega^+ C\}$ and $F' = \{y \in \overline{C} : (x,y) \in \omega^+ C\}$. Then $\nu = \nu(C) = \max_{x \in C} d(x, F)$, and $\nu' = \nu'(\overline{C}) = \max_{x \in \overline{C}} d(F', x)$. The values $2 \leq \nu(C) = \nu \leq \ell - 1$ are proved impossible reasoning as in the proof of Lemma 2.3, since $|F| \leq |\omega^+ C| \leq 2\delta - 3$. It suffices to deal with the case $\nu = 0$, and $\nu = 1$. First, we study the case $\nu = 0$ in which case $F = C$. As $\omega^+ C$ is nontrivial and $G$ has no loops we have that for each vertex $x \in F$ there exists a vertex $y \in \Gamma^+(x) \cap F$, $y \neq x$. Denote by $E^+(x) = \{(x, f') \in \omega^+ C\}$ and by $E^+(H) = \bigcup_{x \in H} E^+(x)$, where $H$ is a subset of $F$. We have that $|E^+(z)| \geq 1$ for any $z \in F$ and hence, $|E^+(x)| + |E^+(\Gamma^+(x) \cap F \setminus \{y\})| \geq \delta - 1$. Besides, as the digraph $G$ is bipartite $\Gamma^+(x) \cap \Gamma^+(y) = \emptyset$, which implies that, $|\omega^+ C| \geq |E^+(x)| + |E^+(\Gamma^+(x) \cap F \setminus \{y\})| + |E^+(y)| + |E^+(\Gamma^+(y) \cap F \setminus \{x\})| \geq 2\delta - 2$, which is a contradiction.

Now suppose that $1 = \nu \leq \ell - 1$, that is $\ell \geq 2$. By Lemma 2.2 we have that $\Gamma^+(x) \subset F$ for each vertex $x$ of the positive valley of $C$. So, each $x_i \in \Gamma^+(x)$ has at least two out-neighbors into $C \setminus F$. Otherwise we would have that $|\omega^+ C| \geq 2\delta - 2$, which is a contradiction. In effect, suppose that there exists a vertex $x_i \in \Gamma^+(x)$ with at most one out-neighbor into the valley of $C$. Then, $|E^+(x_i)| + |E^+(\Gamma^+(x_i) \cap F)| \geq \delta - 1$. Besides, $\Gamma^+(x_i) \cap \Gamma^+(x) = \emptyset$ because de digraph is bipartite. Hence, $|\omega^+ C| \geq |E^+(x_i)| + |E^+(\Gamma^+(x_i) \cap F)| + |E^+(\Gamma^+(x) \setminus \{x_i\})| \geq 2\delta - 2$. Therefore, each vertex $x_i \in \Gamma^+(x)$ has two out-neighbors $y_1, y_2$ belonging to $C \setminus F$. From Lemma 2.2 we have that $\Gamma^+(y_j) \subset F$. As $|F| \leq 2\delta - 3$ it follows that $y_1, y_2$ have at least a common out-neighbor $z \in F$, and thus, we find two different paths of length 2, namely, $x_i y_1 z$ and $x_i y_2 z$, which is a contradiction because $\ell \geq 2$. ∎

From the above lemma we now have:

**Theorem 2.2** *Let $G$ be a bipartite digraph with parameter $\ell$ and minimum degree $\delta \geq 3$. Then $G$ is super-$\lambda$ and $\lambda_1 \geq 2\delta - 2$ if $D \leq 2\ell$.* ∎

The above results can be stated for graphs. Since a bipartite graph $G$ has even girth, it must be $\ell = (g-2)/2$. Hence, from Theorems 2.1 and 2.2, we can derive the following corollary.

**Corollary 2.1** *Let $G$ be a bipartite graph with girth $g$ and minimum degree $\delta \geq 3$. Then,*

*(a) $G$ is super-$\kappa$ and $\kappa_1 \geq 2\delta - 2$ if $D \leq g-3$;*

*(b) $G$ is super-$\lambda$ and $\lambda_1 \geq 2\delta - 2$ if $D \leq g-2$.* ■

From this result each bipartite graph with $g \geq 6$ and $D \leq 3$ (respectively $D \leq 4$) is super-$\kappa$ and has optimal superconnectivity (respectively super-$\lambda$ and has optimal edge-superconnectivity.) Furthermore, it can be shown that some of the largest known bipartite $(\Delta, D)$ graphs given in [8] by Delorme and Bond are super-$\kappa$ and super-$\lambda$ and have superconnectivities $\kappa_1, \lambda_1 \geq 2\delta - 2$.

# 3 Connectivity and superconnectivity of bipartite digraphs and graphs

In this section we give some upper bounds on the order or size of a bipartite digraph that guarantee maximum connectivities or superconnectivities. With this aim we consider $s$-geodetic bipartite digraphs. Recall that a bipartite digraph $G$ with diameter $D$ is said to be *$s$-geodetic* if for any two (not necessarily different) vertices $x$, $y$, there is at most one $x \to y$ path of length at most $s$. Of course, if $d(x,y) \leq s$ there exists exactly one such a path. Note that $1 \leq s \leq \min\{D, g-1\}$, since $G$ has no loops, where $g$ stands for the girth of the digraph. Moreover, $s \leq \ell$, and then, the results of the above section still hold by considering $s$ instead of $\ell$. We can found some results about $s$-geodetic digraphs in [23].

First of all we give some new notation. In general, for any integer $k \geq 0$, let $\Gamma_k^+(x) = \{v \in V : d(x,v) \leq k\}$ and $\Gamma_k^-(x) = \{v \in V : d(v,x) \leq k\}$ be respectively the set of vertices at distance at most $k$ from and to $x$; and $\delta_k^+(x)$, $\delta_k^-(x)$ their cardinalities. We will also use the following similar notation involving the sets of edges whose initial and final vertices are at a given distance from and to $x$: $\Omega_k^+(x) = \{(u,v) \in E : d(x,u) \leq k\}$, $\Omega_k^-(x) = \{(u,v) \in E : d(v,x) \leq k\}$, $\epsilon_k^+(x) = |\Omega_k^+(x)|$ and $\epsilon_k^-(x) = |\Omega_k^-(x)|$.

In the next lemma we give a lower bound for the minimum number of vertices which are at distance at most $s+1$ from/to a given vertex $x$. This bound is denoted by $n_B(\delta, s+1)$.

**Lemma 3.1** *Let $G$ be a $s$-geodetic bipartite digraph, $s \leq D-1$, with minimum degree $\delta > 1$. Then, for each $x \in V$,*

$$\delta_{s+1}^-(x),\ \delta_{s+1}^+(x) \geq n_B(\delta, s+1) = \begin{cases} 2\dfrac{\delta^{s+2}-\delta}{\delta^2-1} & \text{if } s \text{ is odd;} \\[2ex] 2\dfrac{\delta^{s+2}-1}{\delta^2-1} & \text{if } s \text{ is even.} \end{cases}$$

**Proof.** Let $G = (V, A)$, $V = U_0 \cup U_1$, a bipartite digraph. Assume that $|\Gamma^+_{s+1}(x) \cap U_0| \geq |\Gamma^+_{s+1}(x) \cap U_1|$. Note that the minimum number of vertices at distance $i$ from vertex $x$ is at least $\delta^i$, $0 \leq i \leq s$, since the bipartite digraph is $s$-geodetic. We need to distinguish the following cases:

- If $s = 2r$ and $x \in U_1$ then,

  $$|\Gamma^+_{s+1}(x) \cap U_1| = |\Gamma^+_s(x) \cap U_1| \geq \sum_{i=0}^{r} \delta^{2i} = \frac{\delta^{2(r+1)} - 1}{\delta^2 - 1} = \frac{\delta^{s+2} - 1}{\delta^2 - 1}.$$

  If $x \in U_0$ then, for each $x' \in \Gamma^+(x)$ we have

  $$|\Gamma^+_{s+1}(x) \cap U_1| \geq |\Gamma^+_s(x') \cap U_1| \geq \frac{\delta^{s+2} - 1}{\delta^2 - 1}, \text{ since } x' \in U_1.$$

- If $s = 2r + 1$, and $x \in U_0$ as before, we have that

  $$|\Gamma^+_{s+1}(x) \cap U_1| = |\Gamma^+_s(x) \cap U_1| \geq \sum_{i=0}^{r} \delta^{2i+1} = \delta\frac{\delta^{2(r+1)} - 1}{\delta^2 - 1} = \frac{\delta^{s+2} - \delta}{\delta^2 - 1}.$$

  If $x \in U_1$ the reasoning is analogous by considering a vertex $x' \in \Gamma^+(x)$.

The claimed result follows from $|\Gamma^+_{s+1}(x) \cap U_0| \geq |\Gamma^+_{s+1}(x) \cap U_1|$.

The same above lower bound also holds for $\delta^-_{s+1}(x)$. ∎

Now, a subset $C$ of vertices of a strongly connected digraph $G$ is a *positive fragment* of $G$ if $|\partial^+ C| = \kappa$ and $\overline{C} \neq \emptyset$, where $\overline{C} = V \setminus (C \cup \partial^+ C)$. Analogously, $C$ is a *negative fragment* if $|\partial^- C| = \kappa$ and $\overline{C} \neq \emptyset$, where now $\overline{C} = V \setminus (\partial^- C \cup C)$. Note that $C$ is a positive fragment if and only if $\overline{C}$ is a negative one. The set of vertices $C$ is called a *positive $\alpha$-fragment* of $G$ if $|\omega^+ C| = \lambda$ and, similarly, $C$ is a *negative $\alpha$-fragment* if $|\omega^- C| = \lambda$. As in the proof of Lemma 2.4 let us denote by $F = \{x \in C : (x, y) \in \omega^+ C\}$ and $F' = \{y \in \overline{C} : (x, y) \in \omega^+ C\}$.

In [1] it was proved the following lower bounds for the deepness of a fragment or $\alpha$-fragment of a $s$-geodetic digraph.

$$\begin{aligned} &\text{If } \kappa < \delta, \text{ then } \mu(C) \geq s \text{ and } \mu'(\overline{C}) \geq s; \\ &\text{if } \lambda < \delta, \text{ then } \nu(C) \geq s \text{ and } \nu'(\overline{C}) \geq s. \end{aligned} \tag{2}$$

Moreover, in [2] it was proved that for $s$-geodetic bipartite digraphs with $\delta = 2$ the deepness are lower bounded by $s + 1$ instead of $s$. This result will be used in Lemma 3.3. From now on, we will assume that $\delta \geq 2$, since otherwise the digraph is trivially maximally connected.

Making use of the particular properties of bipartite digraphs, the results of (2) can be slightly modified.

**Lemma 3.2** *Let $G$ be a $s$-geodetic bipartite digraph with minimum degree $\delta$ and connectivities $\kappa$ and $\lambda$. Let $C$ be a positive fragment or $\alpha$-fragment of $G$.*

*(a) If $\kappa < \delta$, then $\mu(C) \geq s$ and $\mu'(\overline{C}) \geq s$. Moreover, for any $x, y$ belonging to the valley of $C$ and $\overline{C}$ respectively, there exist $u, v \in \partial^+ C$ such that $d(x,u) \geq s+1$, $d(x, \partial^+ C \setminus \{u\}) \geq s$, $d(v,y) \geq s+1$, $d(\partial^+ C \setminus \{v\}, y) \geq s$.*

*(b) If $\lambda < \delta$, then $\nu(C) \geq s$ and $\nu'(\overline{C}) \geq s$. Moreover, for any $x, y$ belonging to the valley of $C$ and $\overline{C}$ respectively, there exist $u \in F$, $v \in F'$ such that $d(x,u) \geq s+1$, $d(x, F \setminus \{u\}) \geq s$, $d(v,y) \geq s+1$, $d(F' \setminus \{v\}, y) \geq s$.*

**Proof.** (a) The first part is the same result as (2). If $\mu(C) \geq s+1$ the result is straightforward. (Note that this is the case when $\delta = 2$.) So, assume that $\mu(C) = s$ and that there exists a vertex $x$ belonging to the positive valley of $C$ such that $d(x,u) = s$ for any $u \in \partial^+ C$. Then, there exists a vertex $z \in \Gamma^+(x)$ belonging to the valley of $C$. Otherwise, as $|\partial^+ C| = \kappa < \delta$ we would have two different paths from $x$ to some vertex $u \in \partial^+ C$ of length $s$, which contradicts the definition of being $s$-geodetic. Thus, the path $xz \to u$ has length $s+1$, which leads to a contradiction, because $d(x,u) = s$ and the digraph is bipartite. The reasoning is similar for $\overline{C}$.

(b) The proof is analogous to case (a). ■

A similar result applies for negative fragments.

When the $s$-geodetic bipartite digraph has not optimum connectivities the above lemmas allow us to state the minimum order or size for it.

**Lemma 3.3** *Let $G$ be a $s$-geodetic bipartite digraph with order $n$ and size $m$, minimum and maximum degrees $\delta$ and $\Delta$ respectively, and connectivities $\kappa$ and $\lambda$. Let $C$ be a positive fragment or $\alpha$-fragment of $G$.*
*(a) If $\kappa < \delta$ then there exist two vertices $x, y$ such that $d(x,y) \geq 2s+1$ and*

$$\begin{aligned} n &\geq \delta^+_{s+1}(x) + \delta^-_{s+1}(y) - \kappa, && \text{if } s \geq 2, \delta = 2; \\ n &\geq \delta^+_{s+1}(x) + \delta^-_{s+1}(y) - \kappa(\Delta+1), && \text{if } s \geq 2, \delta \geq 3; \\ n &\geq 2[\delta^+(x) + \delta^-(y)] - \kappa, && \text{if } s = 1. \end{aligned}$$

*(b) If $\lambda < \delta$ then there exist two vertices $x, y$ such that $d(x,y) \geq 2s+2$ and*

$$\begin{aligned} &m \geq \epsilon^+_{s+1}(x) + \epsilon^-_{s+1}(y) - \lambda(\Delta+1), && \text{if } s \geq 2; \\ &m \geq 2[\delta^+(x)^2 + \delta^-(y)^2] \text{ and } n \geq 2[\delta^+(x) + \delta^-(y)], && \text{if } s = 1. \end{aligned}$$

**Proof.** (a) We can assume that $\mu'(\overline{C}) \geq \mu(C)$ (if not consider the converse digraph.) By Lemma 3.2(a), $\mu'(\overline{C}) \geq \mu(C) \geq s$.

• If $\mu(C) \geq s+1$, (in fact this is the case when $\delta = 2$) then for any pair of vertices $x, y$ belonging to the valley of $C$ and $\overline{C}$ respectively, we have that $d(x,y) \geq 2s+2$ and $\Gamma^+_{s+1}(x) \subset C \cup \partial^+ C$, $\Gamma^-_{s+1}(y) \subset \partial^+ C \cup \overline{C}$. Hence, $\Gamma^+_{s+1}(x) \cap \Gamma^-_{s+1}(y) \subset \partial^+ C$. Therefore,

$$n = |C| + |\partial^+ C| + |\overline{C}| \geq |\Gamma^+_{s+1}(x) \cup \Gamma^-_{s+1}(y)| \geq \delta^+_{s+1}(x) + \delta^-_{s+1}(y) - \kappa.$$

• If $\mu(C) = s$ then by Lemma 3.2(a) there are vertices of both partite sets into the positive valley of $C$. This fact allow us to consider two vertices $x$, $y$ of different partite sets belonging to the valley of $C$ and $\overline{C}$, respectively. Hence, $d(x, y) \geq 2s + 1$ and $\Gamma^+_{s+1}(x) \subset C \cup \partial^+ C \cup \left(\Gamma^+(\partial^+ C) \cap \overline{C}\right)$. As $\mu'(\overline{C}) \geq s$, we also have that $\Gamma^-_{s+1}(y) \subset (\Gamma^-(\partial^+ C) \cap C) \cup \partial^+ C \cup \overline{C}$. Hence, $\Gamma^+_{s+1}(x) \cap \Gamma^-_{s+1}(y) \subset (\Gamma^-(\partial^+ C) \cap C) \cup \partial^+ C \cup \left(\Gamma^+(\partial^+ C) \cap \overline{C}\right)$. Once more again, by Lemma 3.2(a) we can consider a partition of $\partial^+ C$ into two nonempty subsets, $T = \{u \in \partial^+ C : d(x, u) = s\}$ and $T' = \{u \in \partial^+ C : d(x, u) \geq s + 1\}$. Let us consider a vertex $z \in \Gamma^+_{s+1}(x) \cap \Gamma^-_{s+1}(y)$, which implies $d(x, z) \geq s$ and $d(z, y) \geq s$. Otherwise, we would have $2s+1 \leq d(x, y) \leq d(x, z)+d(z, y) \leq 2s$, a contradiction. Therefore, $z \notin \left(\Gamma^+(T') \cap \overline{C}\right)$, since if not $s + 1 \geq d(x, z) = d(x, t') + 1 = s + 2$, for some $t' \in T'$, a contradiction. Furthermore, $z \notin (\Gamma^-(T) \cap C)$, since otherwise, as $d(x, z) \geq s$ vertex $z$ is not in the shortest paths from vertex $x$ to each vertex of $T$. Hence, as the digraph is bipartite the only possibility is that $d(x, z) = s+1$ and then $s+1 \geq d(z, y) = s+2$, because $x$ and $y$ belong to different partite sets, again a contradiction. Then $\Gamma^+_{s+1}(x) \cap \Gamma^-_{s+1}(y) \subset (\Gamma^-(T') \cap C) \cup \partial^+ C \cup \left(\Gamma^+(T) \cap \overline{C}\right)$. By Lemma 3.2(a) we have that $1 \leq t' = |T'| \leq \kappa-1$, and so $|\Gamma^+_{s+1}(x) \cap \Gamma^-_{s+1}(y)| \leq |\,(\Gamma^-(T') \cap C) \cup \partial^+ C \cup \left(\Gamma^+(T) \cap \overline{C}\right)| \leq \kappa + (\kappa - t')\Delta + t'\Delta = \kappa(\Delta+1)$. Therefore,

$$n = |C| + |\partial^+ C| + |\overline{C}| \geq |\Gamma^+_{s+1}(x) \cup \Gamma^-_{s+1}(y)| \geq \delta^+_{s+1}(x) + \delta^-_{s+1}(y) - \kappa(\Delta + 1).$$

When $s = 1$ the above bound may be improved.

• If $\mu(C) \geq 2$ then for any $x, y$ into the valley of $C$ and $\overline{C}$ respectively, we get that

$$n \geq |\Gamma^+_2(x) \cup \Gamma^-_2(y)| \geq \delta^+_2(x) + \delta^-_2(y) - \kappa \geq 2[\delta^+(x) + \delta^-(y)] - \kappa.$$

• If $\mu(C) = 1$, we consider a vertex $x \in C$ such that $\delta^+(x) \leq \delta^+(z)$ for all $z \in C$. Let $t = |\partial^+ C \cap \Gamma^+(x)|$, $1 \leq t \leq \kappa - 1$, then $\delta^+(x) - t$ vertices must belong to $C$. Moreover, each one of these vertices cannot be adjacent to the vertices of $\partial^+ C \cap \Gamma^+(x)$, since the digraph is bipartite. So, $C$ contains at least $\delta^+(x) - \kappa + t - 1$ vertices different from the above ones. Hence, $|C| \geq 1+\delta^+(x)-t+\delta^+(x)-\kappa+t-1 = 2\delta^+(x)-\kappa$. Analogously, if we consider a vertex $y \in \overline{C}$ such that $y$ belongs to a different partite set from $x$ and $\delta^-(y) \leq \delta^-(z)$ for all $z \in \overline{C}$ we get that $|\overline{C}| \geq 2\delta^-(y) - \kappa$. Then,

$$n \geq |\Gamma^+_2(x) \cup \Gamma^-_2(y)| \geq 2[\delta^+(x) + \delta^-(y)] - \kappa.$$

(b) The proof is analogous by using Lemma 3.2(b) ■

As a consequence of the above lemma we formulate, in the next theorem, sufficient conditions for a $s$-geodetic bipartite digraph, $s \geq 2$, to have optimum connectivities.

**Theorem 3.1** *Let $G$ be a $s$-geodetic bipartite digraph, $s \geq 2$, with order $n$, size $m$, minimum and maximum degrees $\delta$ and $\Delta$ respectively, and connectivities $\kappa$ and $\lambda$.*

*(a) If* $\delta^+_{s+1}(x) + \delta^-_{s+1}(y) \geq n + (\delta - 1)(\Delta + 1) + 1$ *for all pair of vertices* $x, y$ *such that* $d(x, y) \geq 2s + 1$*, then* $\kappa = \delta$*;*

*(b) if* $\epsilon^+_{s+1}(x) + \epsilon^-_{s+1}(y) \geq m + (\delta - 1)(\Delta + 1) + 1$ *for all pair of vertices* $x, y$ *such that* $d(x, y) \geq 2s + 2$*, then* $\lambda = \delta$*.*

**Proof.** To prove case (a), assume $\kappa < \delta$. Then, if $x$ and $y$ are vertices given by the above lemma, we would have $\delta^+_{s+1}(x) + \delta^-_{s+1}(y) \leq n + \kappa(\Delta + 1) \leq n + (\delta - 1)(\Delta + 1)$, contradicting the hypothesis. Case (b) is proved analogously. ■

Note that this result extends conditions (1) since if the diameter $D \leq 2s$ $[D \leq 2s + 1]$, then there are no vertices at distance at least $2s + 1$ $[2s + 2]$.

Since a bipartite digraph is always $s$-geodetic with $s \geq 1$ we deduce the following theorem.

**Theorem 3.2** *Let* $G$ *be a bipartite digraph with order* $n$*, size* $m$*, minimum degree* $\delta$*, and connectivities* $\kappa$ *and* $\lambda$*.*

*(a) If* $2[\delta^+(x) + \delta^-(y)] \geq n + \delta$ *for all pair of vertices* $x, y$ *such that* $d(x, y) \geq 3$*, then* $\kappa = \delta$*;*

*(b) if* $2[\delta^+(x)^2 + \delta^-(y)^2] \geq m + 1$ *for all pair of vertices* $x, y$ *such that* $d(x, y) \geq 4$*, then* $\lambda = \delta$*;*

*(c) if* $\delta^+(x) + \delta^-(y) \geq \lceil \frac{n+1}{2} \rceil$ *for all pair of vertices* $x, y$ *such that* $d(x, y) \geq 4$*, then* $\lambda = \delta$*.* ■

Notice that from condition (a) we can deduce $n \leq 3\delta \Longrightarrow \kappa = \delta$, which was given in [24] for the undirected case. Whereas condition (c) show that the one given by Volkmann in [25], $n \leq 4\delta - 1 \Longrightarrow \lambda = \delta$, can be relaxed to guarantee maximum edge-connectivity. Furthermore, (c) is an improvement of condition given in [10]: $d(x) + d(y) \geq \lceil \frac{n+1}{2} \rceil$ for all non adjacent vertices $x$ and $y$, then $\lambda = \delta$, which was only given for the undirected case.

We have constructed two families of bipartite digraphs which prove that conditions (a) and (c) of Theorem 3.2 are best possible for all values of the minimum degree $\delta$. In Figure 2 and Figure 3 we show such constructions. In them, each line represents a digon, that is, $(x, y), (y, x)$ are two directed edges of $G$, and a line between boxes denotes all the digons between the vertices of different boxes. Note that the family of Figure 2 has minimum degree $\delta \geq 3$, order $n = 3\delta + 1$, the vertices $x, y$ marked in it verifies $2[\delta^+(x) + \delta^-(y)] = n + \delta - 1$ and $\kappa = \delta - 1$. On the other hand, the family of Figure 3 has minimum degree $\delta \geq 3$, order $n = 4\delta$, the vertices $x, y$ marked in it verifies $\delta^+(x) + \delta^-(y) = \lceil \frac{n+1}{2} \rceil - 1$ and $\lambda = \delta - 1$

Keeping in mind that in a $s$-geodetic bipartite digraph the number $n_B(\delta, s + 1)$ is a lower bound for both $\delta^+_{s+1}(x)$ and $\delta^-_{s+1}(y)$, and, moreover, $\epsilon^+_{s+1}(x) \geq \delta\delta^+_{s+1}(x)$ and $\epsilon^-_{s+1}(x) \geq \delta\delta^-_{s+1}(x)$, the following theorem holds.

**Theorem 3.3** *Let* $G$ *be a* $s$*-geodetic bipartite digraph,* $s \geq 2$*, with order* $n$*, size* $m$*, minimum and maximum degrees* $\delta$ *and* $\Delta$ *respectively, and connectivities* $\kappa$ *and* $\lambda$*.*

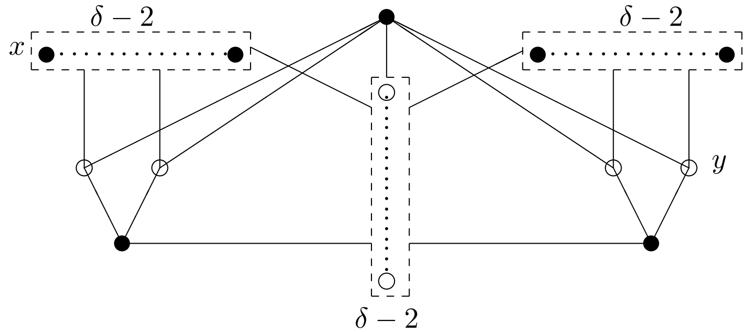


Figure 2: A bipartite digraph with $n = 3\delta + 1$ and $\kappa = \delta - 1$

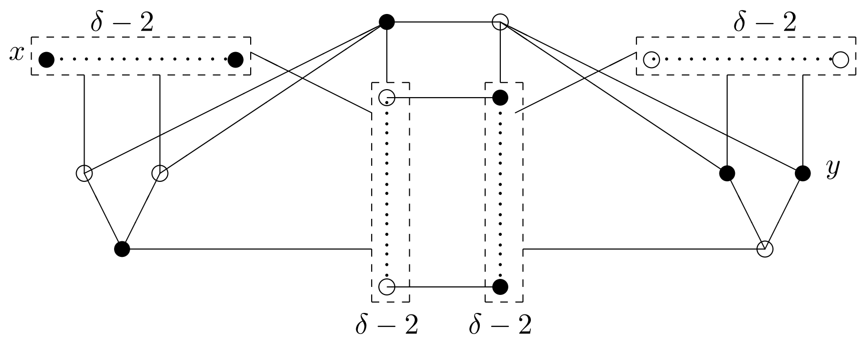


Figure 3: A bipartite digraph with $n = 4\delta$ and $\lambda = \delta - 1$

*(a)* $\kappa = \delta$ *if* $n \le 2p_B(\delta, s+1) - (\delta - 1)(\Delta + 1) - 1$;

*(b)* $\lambda = \delta$ *if* $m \le 2\delta p_B(\delta, s+1) - (\delta - 1)(\Delta + 1) - 1$. ■

From these results we can establish the following conditions of Chartrand-type for a bipartite digraph to be maximally connected.

**Corollary 3.1** *Let $G$ be a $d$-regular $s$-geodetic bipartite digraph, $d \ge 3$, $s \ge 2$, with order $n$, size $m$, and connectivities $\kappa$ and $\lambda$.*

$$\text{(a) } \kappa = d \quad \text{if} \quad d \ge \begin{cases} \left\lceil \sqrt{\frac{n-4}{3}} \right\rceil, & s = 2; \\ \left\lceil \sqrt[3]{\frac{6}{23}n - 2} \right\rceil, & s = 3; \\ \left\lceil \sqrt[s]{\frac{n}{4} - 8} \right\rceil, & s \ge 4. \end{cases}$$

$$(b)\ \lambda = d \quad if \quad d \geq \begin{cases} \left\lceil \sqrt[3]{\frac{4}{15}m - 2} \right\rceil, & s = 2; \\ \left\lceil \sqrt[4]{\frac{m}{4} - 6} \right\rceil, & s = 3; \\ \left\lceil \sqrt[s+1]{\frac{m}{4} - 27} \right\rceil, & s \geq 4. \end{cases} \quad \blacksquare$$

The above results can be stated for graphs. Notice that a bipartite graph is always $s$-geodetic with $2s + 2 = g$ where $g$ denotes the girth. Moreover, the minimum number of vertices of a bipartite graph with minimum degree $\delta \geq 3$ which are at distance at most $s+1$ from or to a given vertex, is lower bounded by $p(\delta, g) = 2\frac{(\delta-1)^{g/2} - 1}{\delta - 2}$ (see Bollobás [7].) Also we have that $\epsilon_{s+1}(x) \geq (\delta/2)p(\delta, g)$. The following results are analogous to Theorems 3.1 and Corollary 3.1 and they give sufficient conditions for a $s$-geodetic bipartite graph with $s \geq 2$ to have maximum connectivities. When $s = 1$, the results are the same as in the directed case, except for the bound over the size $m$, which is divided by 2, since each digon is now an edge.

**Theorem 3.4** *Let $G$ be a $s$-geodetic bipartite graph, $s \geq 2$ with order $n$, size $m$, minimum and maximum degrees $\delta \geq 3$ and $\Delta$ respectively, and connectivities $\kappa$ and $\lambda$.*

*(a) If $\delta_{s+1}(x) + \delta_{s+1}(y) \geq n + (\delta-1)\Delta + 1$ for all pair of vertices $x, y$ such that $d(x, y) \geq 2s + 1$, then $\kappa = \delta$;*

*(b) if $\epsilon_{s+1}(x) + \epsilon_{s+1}(y) \geq m + (\delta-1)\Delta + 1$ for all pair of vertices $x, y$ such that $d(x, y) \geq 2s + 2$, then $\lambda = \delta$.* $\blacksquare$

**Corollary 3.2** *Let $G$ be a $d$-regular ($d \geq 3$) bipartite graph, with girth $g \geq 6$, order $n$, size $m$, and connectivities $\kappa$ and $\lambda$.*

$$(a)\ \kappa = d \quad if \quad d \geq \begin{cases} \left\lceil \sqrt{\frac{n}{3} - 1} \right\rceil + 1, & s = 2; \\ \left\lceil \sqrt[s]{\frac{n}{4} - 4} \right\rceil + 1, & s \geq 3. \end{cases}$$

$$(b)\ \lambda = d \quad if \quad d \geq \left\{ \left\lceil \sqrt[s+1]{\frac{m}{2} - 5} \right\rceil + 1. \right. \quad \blacksquare$$

Finally we can derive analogous results to the above ones for the superconnectivities of a $s$-geodetic bipartite digraph. We omit most of the proof because they are totally analogous to the previous ones. Moreover we must keep in mind the results of Section 2 on the deepness.

**Lemma 3.4** *Let $G$ be a $s$-geodetic bipartite digraph maximally connected with order $n$ and size $m$, minimum and maximum degrees $\delta \geq 3$ and $\Delta$, respectively and superconnectivities $\kappa_1$ and $\lambda_1$. Let $C$ be a positive 1-fragment or $\alpha_1$-fragment of $G$.*

*(a) If $\kappa_1 < 2\delta - 2$ then there exist two vertices $x, y$ such that $d(x,y) \geq 2s$ and*

$$n \geq \delta^+_{s+1}(x) + \delta^-_{s+1}(y) - \kappa_1(2\Delta+1), \text{ if } s \geq 2;$$
$$n \geq 2[\delta^+(x) + \delta^-(y)] - \kappa_1, \text{ if } s = 1.$$

*(b) If $\lambda_1 < 2\delta - 2$ then there exist two vertices $x, y$ such that $d(x,y) \geq 2s+1$ and*

$$m \geq \epsilon^+_{s+1}(x) + \epsilon^-_{s+1}(y) - \lambda_1(2\Delta+1), \text{ if } s \geq 2;$$
$$n \geq 2[\delta^+(x) + \delta^-(y) - 1], \text{ if } s = 1.$$

**Proof.** (a) By Lemma 2.3 we can assume that $\mu'(\overline{C}) \geq \mu(C) \geq s$. Consider two vertices $x, y$ belonging to the valley of $C$ and $\overline{C}$ respectively. We have that $d(x,y) \geq 2s$ and $\Gamma^+_{s+1}(x) \subset C \cup \partial^+C \cup \left(\Gamma^+(\partial^+C) \cap \overline{C}\right)$, $\Gamma^-_{s+1}(y) \subset (\Gamma^-(\partial^+C) \cap C) \cup \partial^+C \cup \overline{C}$. Hence, $\Gamma^+_{s+1}(x) \cap \Gamma^-_{s+1}(y) \subset (\Gamma^-(\partial^+C) \cap C) \cup \partial^+C \cup \left(\Gamma^+(\partial^+C) \cap \overline{C}\right)$. Therefore,

$$n = |C| + |\partial^+C| + |\overline{C}| \geq |\Gamma^+_{s+1}(x) \cup \Gamma^-_{s+1}(y)| \geq \delta^+_{s+1}(x) + \delta^-_{s+1}(y) - \kappa_1(2\Delta+1).$$

When $s = 1$ the above bound can be improved. In effect,

• if $\mu(C) \geq 2$ then for any $x, y$ into the valley of $C$ and $\overline{C}$ respectively, we get that $n \geq |\Gamma^+_2(x) \cup \Gamma^-_2(y)| \geq \delta^+_2(x) + \delta^-_2(y) - \kappa_1 \geq 2[\delta^+(x) + \delta^-(y)] - \kappa_1$;

• if $\mu(C) = 1$, then for each $x \in C$ it is $\Gamma^+(x) \cap C \neq \emptyset$ because $\partial^+C$ is a nontrivial set. Let $x \in C$ be such that $\delta^+(x) \leq \delta^+(w)$ for all $w \in C$. Then $|C| \geq \delta^+(x) + 1 - |\partial^+(x)|$ and $|\partial^+(x)| + |\partial^+(z)| \leq \kappa_1$, where $z \in \Gamma^+(x) \cap C$ since the digraph is bipartite. Furthermore, when $|C|$ is minimum, that is, when $|C| = \delta^+(x) + 1 - |\partial^+(x)|$, it must be $|\partial^+(z)| \geq \delta^+(z) - 1 \geq \delta^+(x) - 1$ and therefore $|\partial^+(x)| \leq \kappa_1 - \delta^+(x) + 1$. Hence, $|C| \geq 2\delta^+(x) - \kappa_1$. Analogously, $|\overline{C}| = 2\delta^-(y) - \kappa_1$ and hence, $n \geq 2[\delta^+(x) + \delta^-(y)] - \kappa_1$.

(b) The proof is analogous by using Lemma 2.4(b) ■

As a consequence of the above lemma we get sufficient conditions for a $s$-geodetic bipartite digraph with $s \geq 2$, to have optimum superconnectivities.

**Theorem 3.5** *Let $G$ be a $s$-geodetic bipartite digraph, $s \geq 2$, with order $n$, size $m$, minimum and maximum degrees $\delta \geq 3$ and $\Delta$ respectively, and superconnectivities $\kappa_1$ and $\lambda_1$.*

*(a) If $\delta^+_{s+1}(x) + \delta^-_{s+1}(y) \geq n + (2\delta-3)(2\Delta+1) + 1$ for any pair of vertices $x, y$ such that $d(x,y) \geq 2s$, then $\kappa_1 \geq 2\delta - 2$;*

*(b) if $\epsilon^+_{s+1}(x) + \epsilon^-_{s+1}(y) \geq m + (2\delta-3)(2\Delta+1) + 1$ for any pair of vertices $x, y$ such that $d(x,y) \geq 2s+1$, then $\lambda_1 \geq 2\delta - 1$.*

As in the case of the standard connectivity, this result extends Theorems 2.1 and 2.2 since if the diameter $D \leq 2s-1$ [$D \leq 2s$], then there are no vertices at distance at least $2s$ [$2s+1$].

Next, by considering the lower bound for $\delta^+_{s+1}(x)$ and $\delta^-_{s+1}(x)$, we obtain the following result whose proof is analogous to the one of Theorem 3.3.

**Theorem 3.6** *Let $G$ be a $s$-geodetic bipartite digraph with order $n$, size $m$, minimum and maximum degrees $\delta \geq 3$ and $\Delta$ respectively, and superconnectivities $\kappa_1$ and $\lambda_1$.*

$$(a)\; \kappa_1 \geq 2\delta - 2 \;\; if \; \begin{cases} n \leq 2\delta + 3 & and\; s = 1 \\ n \leq 2p_B(\delta, s+1) - (2\delta-3)(2\Delta+1) - 1 & and\; s \geq 2; \end{cases}$$

$$(b)\; \lambda_1 \geq 2\delta - 2 \;\; if \; \begin{cases} n \leq 4\delta - 2 & and\; s = 1 \\ m \leq 2\delta p_B(\delta, s+1) - (2\delta-3)(2\Delta+1) - 1 & and\; s \geq 2. \end{cases}$$

■

From Lemma 3.4 we might derive analogous results to Theorems 3.5 and 3.6 for a $s$-geodetic bipartite digraph to be super-$\kappa$ or super-$\lambda$. It suffices to take $\kappa_1 = \delta$ or $\lambda_1 = \delta$. For instance, when $s = 1$ the bipartite digraph is super-$\kappa$ if $n \leq 3\delta - 1$ and super-$\lambda$ if $n \leq 4\delta - 2$. Solving this inequality for $\delta$ we obtain that if $\delta \geq \lceil \frac{n+2}{4} \rceil$ the digraph is super-$\lambda$. This condition coincides, except for $n \equiv 2(4)$, with the one given by Fiol in [14], $\delta \geq \lfloor \frac{n+2}{4} \rfloor + 1$. Furthermore, in [3, 14] it was proved that if $D = 3$ and $n > 2(\Delta + \delta)$, then the bipartite digraph is super-$\lambda$. By joining the two results it is clear that, with the exception of $n = 4d - 1$ or $n = 4d$, all $d$-regular bipartite digraphs with $D = 3$ are super-$\lambda$.

Now we can derive the following Chartrand-type conditions for a $d$-regular bipartite digraph maximally connected to be superconnected.

**Corollary 3.3** *Let $G$ be a $d$-regular $s$-geodetic bipartite digraph, $d \geq 3$, with order $n$, size $m$, and superconnectivities $\kappa_1$ and $\lambda_1$.*

$$(a)\; \kappa_1 \geq 2d - 2 \;\; if \;\; d \geq \begin{cases} \frac{n-6}{4}, & s = 2; \\ \left\lceil \sqrt[3]{\frac{n}{3}} - 6 \right\rceil, & s = 3; \\ \left\lceil \sqrt[s]{\frac{n}{4}} - 4 \right\rceil, & s \geq 4. \end{cases}$$

$$(b)\; \lambda_1 \geq 2d - 2 \;\; if \;\; d \geq \begin{cases} \left\lceil \sqrt[3]{\frac{m}{3}} - 6 \right\rceil, & s = 2; \\ \left\lceil \sqrt[4]{\frac{m}{4}} - 3 \right\rceil, & s = 3; \\ \left\lceil \sqrt[s+1]{\frac{m}{4}} - 24 \right\rceil, & s \geq 4. \end{cases} \quad ■$$

For graphs we can state analogous results to Theorems 3.5 and 3.6.

**Theorem 3.7** *Let $G$ be a $s$-geodetic bipartite graph, $s \geq 2$, with order $n$, size $m$, minimum and maximum degrees $\delta \geq 3$ and $\Delta$ respectively, and superconnectivities $\kappa_1$ and $\lambda_1$.*

*(a) If $\delta_{s+1}(x) + \delta_{s+1}(y) \geq n + (2\delta - 3)(\Delta + 1) + 1$ for any pair of vertices $x, y$ such that $d(x, y) \geq 2s$, then $\kappa_1 \geq 2\delta - 2$;*

*(b) if* $\epsilon_{s+1}(x) + \epsilon_{s+1}(y) \geq m + (2\delta - 3)(\Delta + 1) + 1$ *for any pair of vertices* $x, y$ *such that* $d(x, y) \geq 2s + 1$*, then* $\lambda_1 \geq 2\delta - 1$. ■

**Theorem 3.8** *Let* $G$ *be a bipartite graph,* $g \geq 6$ *with order* $n$*, size* $m$*, minimum and maximum degrees* $\delta \geq 3$ *and* $\Delta$ *respectively, and superconnectivities* $\kappa_1$ *and* $\lambda_1$*.*

*(a)* $\kappa_1 \geq 2\delta - 2$ *if* $n \leq 2p(\delta, g) - (2\delta - 3)(\Delta + 1) - 1$*;*

*(b)* $\lambda_1 \geq 2\delta - 2$ *if* $m \leq \delta p(\delta, g) - (2\delta - 3)(\Delta + 1) - 1$. ■

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