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Stokes Matrices for the Quantum Cohomologies of Grassmannians

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1 Introduction

Gromov-Witten invariants of homogeneous spaces contain enumerative information such as the number of nodal rational curves of a given degree passing through a given set of points in general position. The theory of Frobenius manifold allows a systematic treatment of these invariants. A Frobenius manifold is a complex manifold whose tangent bundle has a holomorphic bilinear form and an associative commutative product with certain compatibility conditions. From these compatibility conditions, it follows that there is a function on the Frobenius manifold, called the potential, whose third derivatives give the structure constants of the product.

Given a symplectic manifold X, one can endow a Frobenius structure on its total cohomology group $H^*(X;\mathbb{C})$. In this case, the holomorphic bilinear form is given by the Poincaré pairing and the potential is the generating function of the genus-zero Gromov-Witten invariants. The product structure in this case is called the quantum cohomology ring. It is a deformation of the cohomology ring parametrized by $H^*(X;\mathbb{C})$ itself.

Given a Frobenius manifold, one can construct the following isomonodromic family of ordinary differential equations on $\mathbb{P}^1$:

$$\frac{\partial \Phi}{\partial \hbar} = -\left(\frac{1}{\hbar^2}U + \frac{1}{\hbar}V\right)\Phi, \tag{1.1}$$

$$\hbar\frac{\partial \Phi}{\partial t_\alpha} = \frac{\partial}{\partial t_\alpha} \circ \Phi, \quad \alpha = 0, \dots, N-1. \tag{1.2}$$

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Here, Φ is the unknown function on $\mathbb{P}^1$ times the Frobenius manifold taking value in the tangent bundle of the Frobenius manifold, $\hbar$ is the coordinate on $\mathbb{P}^1$, N is the dimension of the Frobenius manifold, and $\{t_\alpha\}_{\alpha=0}^{N-1}$ is the *flat coordinate* of the Frobenius manifold. The circle denotes the product on the tangent bundle and U, V are certain operators acting on sections of the tangent bundles. See Dubrovin [3] for details. Note that z in [3] is $1/\hbar$ in this paper. Equation (1.1) is an ordinary differential equation on $\mathbb{P}^1$ with a regular singularity at infinity and an irregular singularity at the origin, and (1.2) gives its isomonodromic deformation. If a point on the Frobenius manifold is semisimple, that is, if there are no nilpotent elements in the product structure on the tangent space at this point, one can define the *monodromy data* of (1.1) at this point, consisting of the monodromy matrix at infinity, the Stokes matrix at the origin, and the connection matrix between infinity and the origin. These data do not depend on the choice of a semisimple point because of the isomonodromicity.

The following conjecture, originally due to Kontsevich, developed by Zaslow [12], and formulated into the following form by Dubrovin [4], reveals a striking connection between the Gromov-Witten invariants and the derived category of coherent sheaves.

Conjecture 1.1. The quantum cohomology of a smooth projective variety X is semisimple if and only if the bounded derived category $D^b \operatorname{coh}(X)$ of coherent sheaves on X is generated as a triangulated category by an exceptional collection $(\mathcal{E}_i)_{i=1}^N$. In such a case, the Stokes matrix S for the quantum cohomology of X is given by

$$S_{ij} = \sum_k (-1)^k \dim \operatorname{Ext}^k\left(\mathcal{E}_i, \mathcal{E}_j\right). \tag{1.3}$$

□

An exceptional collection appearing above is the following.

Definition 1.2. (1) An object $\mathcal{E}$ in a triangulated category is exceptional if

$$\operatorname{Ext}^i(\mathcal{E}, \mathcal{E}) = \begin{cases} \mathbb{C} & \text{if } i = 0, \\ 0 & \text{otherwise.} \end{cases} \tag{1.4}$$

(2) An ordered set of objects $(\mathcal{E}_i)_{i=1}^N$ in a triangulated category is an exceptional collection if each $\mathcal{E}_i$ is exceptional and $\operatorname{Ext}^k(\mathcal{E}_i, \mathcal{E}_j) = 0$ for any $i > j$ and any k.

To our knowledge, Conjecture 1.1 was previously known to hold only for projective spaces [5, 7]. The main result in this paper is as follows.

Theorem 1.3. Conjecture 1.1 holds for the Grassmannian $\mathrm{Gr}(r, n)$ of r-dimensional subspaces in $\mathbb{C}^n$. □

The proof consists of explicit computations on both sides of (1.3). The computation on the left-hand side relies on the following two results. The first is a conjecture of Hori and Vafa [8], proved by Bertram, Ciocan-Fontanine and Kim [2], describing the solution of (1.1) for the Grassmannian $\mathrm{Gr}(r, n)$ in terms of that of the product of projective spaces $(\mathbb{P}^{n-1})^r$. The second is the Stokes matrix for the quantum cohomology of projective spaces obtained by Dubrovin [5] for the projective plane and by Guzzetti [7] in any dimensions. By combining these two results, we can compute the Stokes matrix for the quantum cohomology of the Grassmannian.

On the right-hand side, we have an exceptional collection generating $D^b \operatorname{coh}(\mathrm{Gr}(r, n))$ by Kapranov [9]. It consists of equivariant vector bundles on $\mathrm{Gr}(r, n)$, and Ext-groups between them can be computed by the Borel-Weil theory.

Both of the above computations can be carried out for any r and n, and Conjecture 1.1 reduces to the combinatorial identity in Corollary 4.3.

2 Stokes matrix from the Hori-Vafa conjecture

We begin with the discussion of the Stokes matrix. Fix a semisimple point on a Frobenius manifold. The differential equation (1.1) has a regular singularity at infinity and an irregular singularity at the origin, and the Stokes matrix is the monodromy data for the irregular singularity at the origin, defined as follows: first, fix a formal fundamental solution Φ_{formal} of the form

$$\Phi_{\text{formal}}(\hbar) = \Psi R(\hbar) \exp\left[\frac{U}{\hbar}\right], \tag{2.1}$$

where

$$U = \operatorname{diag}(u_1, \ldots, u_N), \tag{2.2}$$

$\{u_i\}_{i=1}^N$ is the *canonical coordinate*, Ψ is the coordinate transformation matrix from the flat coordinate to the normalized canonical coordinate, and $R(\hbar) = (1 + R_1\hbar + R_2\hbar^2 + \cdots)$ is a formal series satisfying

$$R^t(\hbar) R(-\hbar) = 1. \tag{2.3}$$

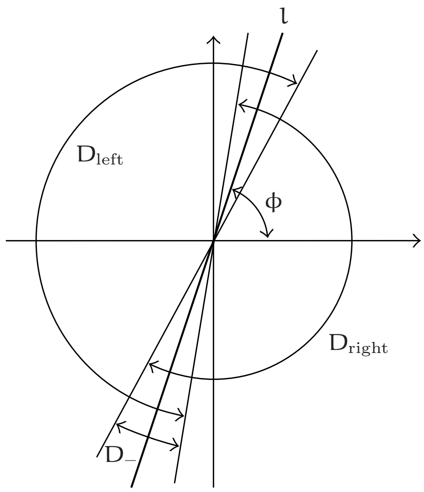

Figure 2.1

Here, $\bullet^t$ denotes the transpose of a matrix. By [5, Lemma 4.3], such $R(\hbar)$ exists uniquely. Here we have taken the local trivialization of the tangent bundle given by the normalized canonical coordinate and regarded Φ as an $n \times n$ matrix-valued function.

Definition 2.1. For $0 \leq \phi < \pi$, a straight line $l = \{\hbar \in \mathbb{C}^\times \mid \arg(\hbar) = \phi,\ \phi - \pi\}$ passing through the origin is called *admissible* if the line through u_k and $u_{k'}$ is not orthogonal to l for any $k \neq k'$.

Fix such a line, and choose a small enough number $\epsilon > 0$ so that any line passing through the origin with angle between $\phi - \epsilon$ and $\phi + \epsilon$ is admissible (see Figure 2.1). Define

$$\begin{aligned} D_{\mathrm{right}} &= \left\{\hbar \in \mathbb{C}^\times \mid \phi - \pi - \epsilon < \arg(\hbar) < \phi + \epsilon\right\}, \\ D_{\mathrm{left}} &= \left\{\hbar \in \mathbb{C}^\times \mid \phi - \epsilon < \arg(\hbar) < \phi + \pi + \epsilon\right\}, \\ D_{-} &= \left\{\hbar \in \mathbb{C}^\times \mid \phi - \pi - \epsilon < \arg(\hbar) < \phi - \pi + \epsilon\right\}. \end{aligned} \tag{2.4}$$

Since the singularity at the origin is irregular, the formal solution $\Phi_{\mathrm{formal}}(\hbar)$ does not converge. Nevertheless, by [5, Theorem 4.2], there exist unique solutions $\Phi_{\mathrm{right}}(\hbar)$ and $\Phi_{\mathrm{left}}(\hbar)$, defined on the angular domains D_{right} and D_{left}, respectively, which asymptote to the same formal solution:

$$\Phi_{\mathrm{right\,/\,left}} \sim \Phi_{\mathrm{formal}} \qquad \text{as } \hbar \longrightarrow 0 \text{ in } D_{\mathrm{right\,/\,left}}. \tag{2.5}$$

Since these two solutions satisfy the same linear differential equation on D_-, there exists a matrix S independent of $\hbar$ such that

$$\Phi_{\mathrm{right}}(\hbar) = \Phi_{\mathrm{left}}(\hbar) S, \quad \hbar \in D_-. \tag{2.6}$$

This matrix S is called the Stokes matrix. Although locally on the Frobenius manifold this Stokes matrix does not depend on the choice of a semisimple point by [5], isomonodromicity theorem (second part), it undergoes a discrete change as we vary the point on the Frobenius manifold so that it crosses the point where the line l we have fixed at the beginning is not admissible any more. This change in the Stokes matrix is described by an action of the braid group B_N (the number of strands is the dimension of the Frobenius manifold).

In the case of the projective space $\mathbb{P}^{n-1}$, semisimplicity of the quantum cohomology is well known. The solution to (1.1), (1.2) has an integral representation by Givental.

Theorem 2.2 (Givental [6]). Let

$$W(x_1, \dots, x_{n-1}) = x_1 + \cdots + x_{n-1} + \frac{e^t}{x_1 \cdots x_{n-1}} \tag{2.7}$$

be a function on $(\mathbb{C}^\times)^{n-1}$ depending on a parameter $t \in \mathbb{C}$ and choose a basis $\{\Gamma_i\}_{i=1}^n$ of the space of flat sections of the relative homology bundle (the flat bundle on the $\hbar$-plane whose fiber over $\hbar \in \mathbb{C}^\times$ is $H_{n-1}((\mathbb{C}^\times)^{n-1}, \mathfrak{Re}(W/\hbar) = -\infty)$). Let $\{p^\alpha\}_{\alpha=0}^{n-1}$ be the basis of $H^*(\mathbb{P}^{n-1}; \mathbb{Z})$ such that $p^\alpha \in H^{2\alpha}(\mathbb{P}^{n-1}; \mathbb{Z})$. For $k = 1, \dots, n$, define a cohomology-valued function I_k by

$$I_k = \sum_{\alpha=0}^{n-1} p^\alpha \int_{\Gamma_k} \left(\hbar \frac{d}{dt}\right)^\alpha \exp\left[\frac{W(x_1, \dots, x_{n-1})}{\hbar}\right] \frac{dx_1 \cdots dx_{n-1}}{x_1 \cdots x_{n-1}}. \tag{2.8}$$

Then $(I_k)_{k=1}^n$ gives a fundamental solution to (1.1), (1.2) where t is the coordinate of $H^2(\mathbb{P}^{n-1}; \mathbb{C})$ and all the other flat coordinates are set to zero. □

Note that since the relative homology bundle has a monodromy, Γ_k's (and hence I_k's) cannot be defined globally. The above integral representation is related to the Stokes matrix in the following way: fix ϕ and ϵ such that any line passing through the origin with angle between $\phi - \epsilon$ and $\phi + \epsilon$ is admissible. There are n critical points and their critical values are the canonical coordinate $\{u_i\}_{i=1}^n$. Order these critical points so that $\mathfrak{Re}[\exp(-\sqrt{-1}\phi)u_i] > \mathfrak{Re}[\exp(-\sqrt{-1}\phi)u_j]$ if $i < j$. Take the Lefschetz thimble (the descending Morse cycle for a suitable choice of a Riemannian metric on $(\mathbb{C}^\times)^{n-1}$) for $\mathfrak{Re}(W/\hbar)$ at $\hbar = \exp[\sqrt{-1}(\phi - \pi/2)]$ starting from the ith critical point of W and extend it to a flat section of the relative homology bundle on D_{right}. We call this section $\Gamma_{i,\text{right}}$ and let $I_{i,\text{right}}$ be the integral as in (2.8) with $\Gamma_{i,\text{right}}$ as the integration cycle. Now form the row vector $(I_{i,\text{right}})_{i=1}^n$ and think of it as an $n \times n$ matrix by regarding an element in the cohomology group as a column vector by the normalized canonical coordinate. Then we can see that $(I_{i,\text{right}})_{i=1}^n$ asymptotes on D_{right} to the formal solution of the form (2.1) as $\hbar \to 0$ by the saddle-point method. In the same way, starting from the Lefschetz thimble at $\hbar = \exp[\sqrt{-1}(\phi + \pi/2)]$, we obtain a solution $(I_{i,\text{left}})_{i=1}^n$ defined on D_{left} which asymptotes to the same formal solution as $(I_{i,\text{right}})_{i=1}^n$. Since the integrand is single-valued, the monodromy of I_i's comes solely from the monodromy of the integration cycles and the Stokes matrix is given by

$$\Gamma_{i,\text{right}} = \sum_{j=1}^{n} \Gamma_{j,\text{left}} S_{ji}. \tag{2.9}$$

The Stokes matrix for the quantum cohomology of $\mathbb{P}^{n-1}$ has been computed by Dubrovin [5] for $n \leq 3$ and by Guzzetti [7] for general n. See also [11].

Theorem 2.3 (Dubrovin, Guzzetti). The Stokes matrix S for the quantum cohomology of the projective space $\mathbb{P}^{n-1}$ is given by

$$S_{ij} = \binom{n-1+j-i}{j-i} \tag{2.10}$$

up to the braid group action. Here, $\binom{n}{r}$ is the binomial coefficient. □

Since $(\mathcal{O}_{\mathbb{P}^{n-1}}(i))_{i=0}^{n-1}$ is an exceptional collection generating $D^b \operatorname{coh} \mathbb{P}^{n-1}$ by Beĭlinson [1] and

$$\binom{n-1+j-i}{j-i} = \sum_k (-1)^k \dim \operatorname{Ext}^k \left(\mathcal{O}_{\mathbb{P}^{n-1}}(i), \mathcal{O}_{\mathbb{P}^{n-1}}(j)\right), \tag{2.11}$$

Conjecture 1.1 holds for projective spaces.

Now we move on to the Grassmannian case. Let $\mathrm{Gr}(r,n)$ be the Grassmannian of r-dimensional subspaces in $\mathbb{C}^n$. The semisimplicity of the quantum cohomology in this case is also known. The following theorem is proved by Bertram, Ciocan-Fontanine, and Kim (see the proof of [2, Theorem 3.3]).

Theorem 2.4 (Bertram-Ciocan-Fontanine-Kim)**.** For a choice of a basis $\{\phi_\alpha\}_{\alpha=0}^{N-1}$ of $H^*(\mathrm{Gr}(r,n);\mathbb{C})$ where $N = \binom{n}{r} = \dim H^*(\mathrm{Gr}(r,n);\mathbb{C})$, there exists a set $\{\varphi_\alpha(x_{1,1},\dots,x_{r,n-1}; t,\hbar)\}_{\alpha=0}^{N-1}$ of functions of $(x_{1,1},\dots,x_{r,n-1}) \in (\mathbb{C}^\times)^{r(n-1)}$, $t \in \mathbb{C}$, and $\hbar \in \mathbb{C}^\times$ such that

$$\left(\sum_{\alpha=0}^{N-1} \phi_\alpha \int_{\Gamma_{k_1}\times\cdots\times\Gamma_{k_r}} e^{W_r/\hbar}\varphi_\alpha(x_{1,1},\dots,x_{r,n-1};t,\hbar) \times \prod_{j=1}^{r} \frac{dx_{j,1}\cdots dx_{j,n-1}}{x_{j,1}\cdots x_{j,n-1}}\right)_{1\le k_1<k_2<\cdots<k_r\le n} \tag{2.12}$$

forms a fundamental solution to (1.1), (1.2) where t is the coordinate of $H^2(\mathrm{Gr}(r,n);\mathbb{C})$ and all the other flat coordinates are set to zero. Here,

$$W_r(x_{1,1},\dots,x_{r,n-1}) = \sum_{j=1}^{r}\left(x_{j,1}+\cdots+x_{j,n-1}+\frac{e^t}{x_{j,1}\cdots x_{j,n-1}}\right) \tag{2.13}$$

is the sum of r copies of the function W appearing in Theorem 2.2 and $\Gamma_{k_1}\times\cdots\times\Gamma_{k_r}$'s are sections of the relative homology bundle for W_r obtained as the products of the basis of sections Γ_k's of the relative homology bundle for W in Theorem 2.2. □

By construction, $\varphi_\alpha(x_{1,1},\dots,x_{r,n-1};t,\hbar)$ is antisymmetric with respect to the exchange of $(x_{i,1},\dots,x_{i,n-1})$ and $(x_{j,1},\dots,x_{j,n-1})$ for any $1 \le i < j \le r$. Therefore, if we define $H^*(\mathrm{Gr}(r,n);\mathbb{C})$-valued functions $I_K(t,\hbar)$ for $K=(k_1,\dots,k_r)$, $1\le k_i\le n$, $i=1,\dots,r$ by

$$I_K = \sum_{\alpha=0}^{N-1} \phi_\alpha \int_{\Gamma_{k_1}\times\cdots\times\Gamma_{k_r}} e^{W/\hbar}\varphi_\alpha(x_{1,1},\dots,x_{r,n-1};t,\hbar)\prod_{j=1}^{r}\frac{dx_{j,1}\cdots dx_{j,n-1}}{x_{j,1}\cdots x_{j,n-1}}, \tag{2.14}$$

then I_K is totally antisymmetric in $k_1,\dots,k_r$. Hence it follows that if we put

$$\Gamma_K = \frac{1}{r!}\sum_{\sigma\in\mathfrak{S}_r} \operatorname{sgn}\sigma\, \Gamma_{k_{\sigma(1)}}\times\cdots\times\Gamma_{k_{\sigma(r)}}, \tag{2.15}$$

where $\mathfrak{S}_r$ is the symmetric group of degree r and $\operatorname{sgn}\sigma$ is the signature of σ, then we have

$$I_K = \sum_{\alpha=0}^{N-1} \phi_\alpha \int_{\Gamma_K} e^{W/\hbar} \varphi_\alpha(x_{1,1}, \dots, x_{r,n-1}; t, \hbar) \prod_{j=1}^{r} \frac{dx_{j,1} \cdots dx_{j,n-1}}{x_{j,1} \cdots x_{j,n-1}}. \tag{2.16}$$

We can use the above result to compute the Stokes matrix for the quantum cohomology of $\mathrm{Gr}(r, n)$ from that of $\mathbb{P}^{n-1}$ as follows. By Theorem 2.3, there exists a choice $\{\Gamma_{i,\mathrm{right}}\}_{i=1}^n$ and $\{\Gamma_{i,\mathrm{left}}\}_{i=1}^n$ of bases of flat sections of the relative homology bundle for W on D_{right} and D_{left}, respectively, such that

$$\Gamma_{i,\mathrm{right}} = \sum_{j=1}^{n} \Gamma_{j,\mathrm{left}} S_{ji} \tag{2.17}$$

on D_- for $S_{ij} = \binom{n-1+j-i}{j-i}$. Then, since Γ_K is just the antisymmetrization of the product of Γ_k's, the monodromy for Γ_K is given by

$$\Gamma_{K,\mathrm{right}} = \sum_{1 \le l_1 < l_2 < \cdots < l_r \le n} I_{L,\mathrm{left}}\, S_{L,K}, \tag{2.18}$$

where $K = (k_1, \dots, k_r)$, $L = (l_1, \dots, l_r)$, and

$$\begin{aligned} S_{L,K} &= \det\left(S_{l_j,k_i}\right)_{1 \le i,j \le r} \\ &= \det\left(\binom{n + l_i - k_j - 1}{l_i - k_j}\right)_{1 \le i,j \le r}. \end{aligned} \tag{2.19}$$

3 Derived category of coherent sheaves

In this section, we use the presentation

$$\mathrm{Gr}(r, n) = \mathrm{GL}_n(\mathbb{C})/P \tag{3.1}$$

of the Grassmannian as a homogeneous space, where

$$P = \left\{ \begin{pmatrix} A & B \\ 0 & D \end{pmatrix} \mid A \in \mathrm{GL}_r(\mathbb{C}),\ B \in M_{r,n-r}(\mathbb{C}),\ D \in \mathrm{GL}_{n-r}(\mathbb{C}) \right\} \tag{3.2}$$

is a parabolic subgroup of $GL_n(\mathbb{C})$. A representation of $GL_r(\mathbb{C})$ gives a representation of P through the projection $P \ni \left(\begin{smallmatrix} A & B \\ 0 & D \end{smallmatrix}\right) \mapsto A \in GL_r(\mathbb{C})$, hence a $GL_n(\mathbb{C})$-equivariant bundle on $Gr(r,n)$ associated to the principal P-bundle $GL_n(\mathbb{C}) \to Gr(r,n)$. Let $\mathcal{E}_\rho$ denote the equivariant bundle on $Gr(r,n)$ corresponding to a representation ρ of $GL_r(\mathbb{C})$ in this way.

Let

$$\Lambda = \left\{ (\lambda_1, \ldots, \lambda_r) \in \mathbb{Z}^r \mid n - r \geq \lambda_1 \geq \cdots \geq \lambda_r \geq 0 \right\} \tag{3.3}$$

be a set of weights of $GL_r(\mathbb{C})$. Given a weight λ, let ρ_λ denote the irreducible representation of $GL_r(\mathbb{C})$ with highest weight λ. We abbreviate $\mathcal{E}_{\rho_\lambda}$ as $\mathcal{E}_\lambda$.

Theorem 3.1 [9]. $\{\mathcal{E}_\lambda\}_{\lambda\in\Lambda}$ is an exceptional collection generating $D^b\,\mathrm{coh}(Gr(r,n))$. □

Kapranov also proved that $\mathrm{Ext}^k(\mathcal{E}_\lambda, \mathcal{E}_\mu) = 0$ for any $\lambda, \mu \in \Lambda$ and any $k \neq 0$. $\mathrm{Hom}(\mathcal{E}_\lambda, \mathcal{E}_\mu)$ is calculated as follows. Decompose the tensor product $\rho_\lambda^\vee \otimes \rho_\mu$ of the dual representation of ρ_λ and ρ_μ into the direct sum of irreducible representations

$$\rho_\lambda^\vee \otimes \rho_\mu = \bigoplus_\nu \rho_\nu^{\oplus \widetilde{N}_{\lambda\mu}^\nu}. \tag{3.4}$$

Here, $\widetilde{N}_{\lambda\mu}^\nu$ is the multiplicity of ρ_ν in $\rho_\lambda^\vee \otimes \rho_\mu$ and ν runs over all weights of $GL_r(\mathbb{C})$. Define

$$N_{\lambda\mu}^\nu = \begin{cases} \widetilde{N}_{\lambda\mu}^\nu & \text{if } \nu_r \geq 0, \\ 0 & \text{otherwise.} \end{cases} \tag{3.5}$$

For a weight $\lambda \in \mathbb{Z}^r$ of $GL_r(\mathbb{C})$, let R_λ be the irreducible representation of $GL_n(\mathbb{C})$ with highest weight $(\lambda_1, \ldots, \lambda_r, 0, \ldots, 0) \in \mathbb{Z}^n$. Then

$$\begin{aligned} \mathrm{Hom}\,(\mathcal{E}_\lambda, \mathcal{E}_\mu) &= H^0(\mathcal{E}_\lambda^\vee \otimes \mathcal{E}_\mu) \\ &= H^0\left(\mathcal{E}_{\rho_\lambda^\vee \otimes \rho_\mu}\right) \\ &= \bigoplus_\nu H^0(\mathcal{E}_\nu)^{\oplus \widetilde{N}_{\lambda\mu}^\nu} \\ &= \bigoplus_\nu R_\nu^{\oplus N_{\lambda\mu}^\nu}, \end{aligned} \tag{3.6}$$

where the last equality follows from the Borel-Weil theory.

4 A combinatorial identity

The content of this section is due to A. N. Kirillov. Fix two integers r, n such that $r < n$. Let $A = \{(k_1, \dots, k_r) \in \mathbb{Z}^r \mid 1 \le k_1 < \cdots < k_r \le n\}$. A and Λ defined in the previous section are bijective by the correspondence

$$\Lambda \ni (\lambda_i)_{i=1}^r \longmapsto (\lambda_{r-i+1} + i)_{i=1}^r \in A. \tag{4.1}$$

For n variables $x = (x_1, \dots, x_n)$, let $s_\lambda(x) = \det(h_{\lambda_i - i + j}(x))_{1 \le i,j \le n}$ be the Shur function, where $h_i(x)$ is the complete symmetric function (the sum of all monomials of degree i). For generalities on symmetric functions, see, for example, [10]. Define integers $c^\lambda_{\mu\nu}$'s by

$$s_\mu(x) s_\nu(x) = \sum_\lambda c^\lambda_{\mu\nu} s_\lambda(x) \tag{4.2}$$

and the skew Shur function $s_{\lambda/\mu}(x)$ by

$$s_{\lambda/\mu}(x) = \sum_\nu c^\lambda_{\mu\nu} s_\nu(x). \tag{4.3}$$

Then

$$s_{\lambda/\mu}(x) = \det\left(h_{\lambda_i - \mu_j - i + j}(x)\right)_{1 \le i,j \le n}. \tag{4.4}$$

Lemma 4.1. Let μ, ν, and λ be partitions such that $\mu_1 \le \nu_r$. Define $\mu^c = (\mu_1 - \mu_r, \mu_1 - \mu_{r-1}, \dots, \mu_1 - \mu_2, 0)$ and $\widetilde{\nu} = (\nu_1 - \mu_1, \nu_2 - \mu_1, \dots, \nu_r - \mu_1)$. Then

$$c^\nu_{\lambda\mu^c} = c^\lambda_{\mu\widetilde{\nu}}. \tag{4.5}$$

□

Proof.

$$\begin{aligned} c^\nu_{\lambda\mu^c} &= \dim \mathrm{Hom}_{GL_r(\mathbb{C})}\left(\rho_\lambda \otimes \rho_{\mu^c}, \rho_\nu\right) \\ &= \dim \mathrm{Hom}_{GL_r(\mathbb{C})}\left(\rho_0, \rho_\lambda^\vee \otimes \rho_{\mu^c}^\vee \otimes \rho_\nu\right) \\ &= \dim \mathrm{Hom}_{GL_r(\mathbb{C})}\left(\rho_0, \rho_\lambda^\vee \otimes \left(\rho_\mu^\vee \otimes \det{}^{\otimes \mu_1}\right)^\vee \otimes \rho_\nu\right) \\ &= \dim \mathrm{Hom}_{GL_r(\mathbb{C})}\left(\rho_0, \rho_\lambda \otimes \rho_\mu^\vee \otimes \det{}^{\otimes \mu_1} \otimes \rho_\nu^\vee\right) \\ &= \dim \mathrm{Hom}_{GL_r(\mathbb{C})}\left(\rho_0, \rho_\lambda \otimes \rho_\mu^\vee \otimes \rho_{\widetilde{\nu}}^\vee\right) \\ &= \dim \mathrm{Hom}_{GL_r(\mathbb{C})}\left(\rho_\mu \otimes \rho_{\widetilde{\nu}}, \rho_\lambda\right), \end{aligned} \tag{4.6}$$

where ρ_0 is the trivial representation and det is the determinant representation (the irreducible representation with highest weight $(1, \dots, 1) \in \mathbb{Z}^r$). ■

Theorem 4.2. $s_{\lambda/\mu}(x) = \sum_{\nu} N^{\nu}_{\mu\lambda} s_{\nu}(x)$. □

Proof.

$$\sum_{\nu} N^{\nu}_{\mu\lambda} s_{\nu}(x) = \sum_{\nu} c^{\nu}_{\lambda\mu^c} s_{\tilde{\nu}}(x)$$
$$= \sum_{\tilde{\nu}} c^{\lambda}_{\mu\tilde{\nu}} s_{\tilde{\nu}}(x) \tag{4.7}$$
$$= s_{\lambda/\mu}(x).$$
■

By substituting $x_1 = \cdots = x_n = 1$ in Theorem 4.2 and using $h_r(1, \ldots, 1) = \binom{n+r-1}{r}$, we obtain the following.

Corollary 4.3. For $\lambda, \mu \in \Lambda$, let $k = (\lambda_{r-i+1} + i)_{i=1}^{r}$, $l = (\mu_{r-i+1} + i)_{i=1}^{r}$. Then

$$\det\left(\binom{n + l_i - k_j - 1}{l_i - k_j}\right)_{1 \le i,j \le r} = \sum_{\nu} N^{\nu}_{\lambda\mu} \dim R_{\nu}. \tag{4.8}$$
□

The left-hand side is the component of the Stokes matrix from (2.19) and the right-hand side is the Euler number in the derived category of coherent sheaves from (3.6). This proves Conjecture 1.1 in the case of Grassmannians.

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