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Spanning Even Subgraphs of 3-edge-connected Graphs

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Abstract

By Petersen's theorem, a bridgeless cubic graph has a 2-factor. H. Fleischner extended this result to bridgeless graphs of minimum degree at least three by showing that every such graph has a spanning even subgraph. Our main result is that, under the stronger hypothesis of 3-edge-connectivity, we can find a spanning even subgraph in which every component has at least five vertices. We show that this is in some sense best possible by constructing an infinite family of 3-edge-connected graphs in which every spanning even subgraph has a 5-cycle as a component.

1 Introduction

All graphs considered are finite and may contain loops and multiple edges. We refer to graphs without loops and multiple edges as *simple graphs*. A graph is said to be *even* if every vertex has positive even degree. All notation and terminology not explained in this paper is given in [5].

Petersen [12] showed that every bridgeless cubic graph has a 2-factor. Fleischner [6] extended this result to graphs of minimum degree at least three.

Theorem 1 ([6]). *Every bridgeless graph with minimum degree at least* 3 *has a spanning even subgraph.*

[1]This research was carried out while the second author was visiting Queen Mary, University of London.

We are concerned with finding spanning even subgraphs in which all components are relatively large. In this context, we proved the following result in [7].

Theorem 2. *Every bridgeless simple graph G with minimum degree at least 3 has a spanning even subgraph in which each component has at least four vertices.*

The same conclusion does not hold for general graphs. Consider a bridgeless graph H with minimum degree at least 3, which contains a 3-edge cut $\{e_1, e_2, e_3\}$. Let G be obtained from H by inserting either a vertex incident to a loop, or two vertices joined by a multiple edge, or a triangle with one edge replaced by a multiple edge, into each edge e_i, $1 \leq i \leq 3$, see Figure 1. Then every spanning even subgraph

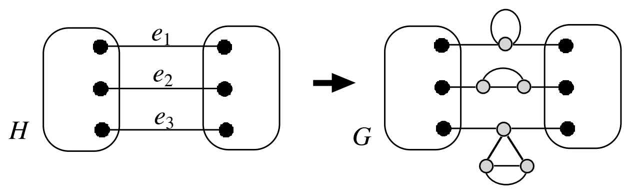

Figure 1:

of G contains at least one of the inserted loops, multiple edges, or triangles.

We will show, however, that both Theorems 1 and 2 can be extended for 3-edge-connected graphs.

Theorem 3. *Let G be a 3-edge-connected graph with n vertices. Then G has a spanning even subgraph in which each component has at least $\min\{n, 5\}$ vertices.*

The Petersen graph is an example of a 3-edge-connected graph in which every spanning even subgraph has a component with five vertices. We give an infinite family of such graphs in Section 4.

2 Notation and Preliminary Results

The set of all the neighbours of a vertex x in a graph G is denoted by $N_G(x)$, or simply $N(x)$, and degree of x by $d_G(x)$, or $d(x)$. The set of edges incident to a vertex v is denoted by $E(v)$. For a connected subgraph H of G, we denote by G/H

the graph obtained from G by contracting every edge in H and use $[H]$ to denote the vertex of G/H corresponding to H. The maximum and minimum degrees of G are denoted by $\Delta(G)$ and $\delta(G)$, respectively. We refer to the number of vertices in a graph as its *order*. We consistently use n to denote the order of a graph G and extend this notation using subscripts and superscripts. Thus we denote the order of a graph G_1' by n_1'. We use $\sigma(G)$ to represent the minimum order of a component of G.

An edge-cut E_0 in a graph G is said to be *essential* if at least two components of $G - E_0$ contain edges. The graph G is *essentially* k-edge-connected if all essential edge-cuts of G have at least k edges.

Given two distinct edges $e_1 = vx_1, e_2 = vx_2$ incident to a vertex v in a graph G, let $G_v^{e_1,e_2}$ be the graph obtained from $G - \{e_1, e_2\}$ by adding a new vertex v' and new edges $e_1' = x_1v'$ and $e_2' = x_2v'$. We say that $G_v^{e_1,e_2}$ has been obtained by *splitting* the vertex v. We will need the following result on splitting in k-edge-connected graphs due to Mader.

Lemma 4 ([11]). *Let G be a k-edge-connected graph, $v \in V(G)$ with $d(v) \geq k+2$. Then there exist edges $e_1, e_2 \in E(v)$ such that $G_v^{e_1,e_2}$ is homeomorphic to a k-edge-connected graph.*

3 Even Subgraphs

We first prove a slight strengthening of Theorem 1.

Theorem 5. *Suppose G is a bridgeless graph with $\delta(G) \geq 3$ and $f_1, f_2 \in E(G)$. Then G has a spanning even subgraph X with $f_1, f_2 \in E(X)$.*

Proof. We proceed by contradiction. Suppose the theorem is false and choose a counterexample G such that $\Delta = \Delta(G)$ is as small as possible and, subject to this condition, the number of vertices of G of degree Δ is as small as possible. Clearly G is 2-edge-connected.

We first show that $\Delta = 3$. Suppose $\Delta \geq 4$ and choose a vertex $v \in V$ with $d(v) = \Delta$. By Lemma 4 we can choose two edges $e_1 = x_1v, e_2 = x_2v \in E$ incident to v such that the graph $G_v^{e_1,e_2}$ is 2-edge-connected, see Figure 2(i). Thus the

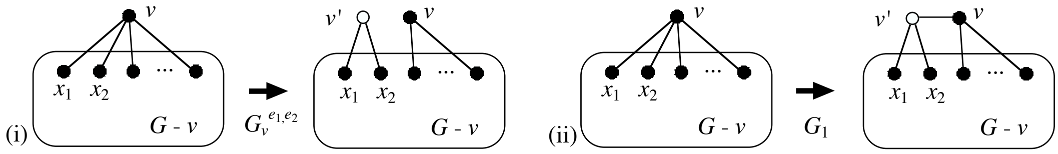

Figure 2:

graph G^* obtained from $G_v^{e_1,e_2}$ by adding the new edge vv' is 2-edge-connected, see Figure 2(ii). If $f_i \notin \{e_1, e_2\}$ then put $f_i^* = f_i$ for each $i \in \{1, 2\}$. Otherwise, when $f_i = e_j \in \{e_1, e_2\}$ put $f_i^* = v'x_j$. By induction G^* has a spanning even subgraph X^*, with $f_1^*, f_2^* \in X^*$. If $vv' \notin X^*$, then $x_1v', x_2v' \in X^*$ and we let $X = (X^* - v') \cup \{x_1v, x_2v\}$. On the other hand, if $vv' \in X^*$, then relabelling if necessary, we have $x_1v' \in X^*$ and $x_2v' \notin X^*$ and we let $X = (X^* - v') \cup \{x_1v\}$. In both cases X is a spanning even subgraph of G with $f_1, f_2 \in X$. This contradicts the choice of G.

Thus G is 3-regular. By a well known strengthening of Petersen's Theorem, see for example Plesník [13], G has a 2-factor X with $f_1, f_2 \in X$. This again contradicts the choice of G. □

Notice that we cannot obtain a similar strengthening of Theorem 3. In the graph drawn in Figure 3, every spanning even subgraph which contains e_1, e_2 has a 4-cycle as a component.

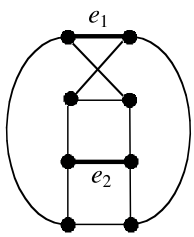

Figure 3:

We will show, however, that we can find an even subgraph X with $\sigma(X) \geq 5$ which contains two specified edges e_1, e_2 in a 3-connected graph G as long as e_1, e_2

are both incident to a common vertex of degree three. Indeed, we need this stronger statement for our inductive proof.

Theorem 6. *Let G be a 3-edge-connected graph with n vertices, u_2 be a vertex of G with $d(u_2) = 3$, and $e_1 = u_1u_2, e_2 = u_2u_3$ be edges of G. (We allow the possibility that $u_1 = u_3$.) Then G has a spanning even subgraph X with $\{e_1, e_2\} \subset E(X)$ and $\sigma(X) \geq \min\{n, 5\}$.*

Proof. Suppose the theorem is false and choose a counterexample G such that:
(a) $\Delta = \Delta(G)$ is as small as possible;
(b) subject to (a), the number of vertices of degree Δ in G is as small as possible;
(c) subject to (a) and (b), $|E(G)|$ is as small as possible.

Claim 1. $\Delta \leq 4$.

Proof. Suppose $\Delta \geq 5$ and let x be a vertex with $d(x) = \Delta$. By Lemma 4, there exist two edges $f, f' \in E(x)$ such that the graph G' obtained from $G_x^{f,f'}$ by suppressing x' is 3-edge-connected. Note that, since $d(u_2) = 3$, $u_2 \neq x$. Let e_1', e_2' be the edges of G' corresponding to e_1, e_2, respectively. By induction, G' has a spanning even subgraph X' such that $\{e_1, e_2\} \subset E(X')$ and $\sigma(X') \geq \min\{n, 5\}$. Then X' readily gives rise to the required even subgraph of G. □

Claim 2. *G is essentially 4-edge-connected.*

Proof. Suppose that $\{f_1, f_2, f_3\}$ is an essential 3-edge-cut in G. Let G_1', G_2' be the two components of $G - \{f_1, f_2, f_3\}$ and let $G_1 = G/G_2'$ and $G_2 = G/G_1'$. We denote by f_i^j the edge in G_j corresponding to f_i for $1 \leq i \leq 3$ and $1 \leq j \leq 2$.

By symmetry, we may assume that $E(G_2') \cap \{e_1, e_2\} = \emptyset$. Let e_1', e_2' be the edges of G_1 corresponding to e_1, e_2, respectively. By induction, G_1 has a spanning even subgraph X_1 such that $\{e_1', e_2'\} \subset E(X_1)$ and $\sigma(X_1) \geq \min\{n_1, 5\}$. By symmetry, we may suppose that:

$$E(X_1) \cap \{f_1^1, f_2^1, f_3^1\} = \{f_1^1, f_2^1\}.$$

By induction, G_2 has a spanning even subgraph X_2 such that $\{f_1^2, f_2^2\} \subset E(X_2)$ and $\sigma(X_2) \geq \min\{n_2, 5\}$. Then $((X_1 - [G_1']) \cup (X_2 - [G_2']) \cup \{f_1, f_2\}$ is the required spanning even subgraph of G. □

Claim 3. *No edge of G is adjacent to two vertices of degree four.*

Proof. Suppose there is an edge $f = xy$ adjacent to two vertices of degree four. Then $G_1 = G - f$ is 3-edge-connected by Claim 2. Since $d(u_2) = 3$, $f \notin \{e_1, e_2\}$. By induction, G_1 has a spanning even subgraph X such that $\{e_1, e_2\} \subset X$ and $\sigma(X) \geq \min\{n_1, 5\}$. Then X is the required subgraph of G. □

Claim 4. *G is simple and hence $P = u_1u_2u_3$ is a path.*

Proof. This follows immediately from Claims 1, 2 and 3. □

Claim 5. *Let x be a vertex of G of degree 4 and $f, f' \in E(x)$. Then the graph G' obtained from $G_x^{f,f'}$ by adding the edge xx' is 3-edge-connected.*

Proof. This follows immediately from Claim 2. □

Claim 6. *G is cubic.*

Proof. Suppose x has degree 4 in G. Let $N(x) = \{z_1, z_2, z_3, z_4\}$ and $f_i = xz_i$. See Figure 4(i). Let G_1 be the graph obtained from $G_x^{f_2,f_3}$ by adding the edge xx'. Since

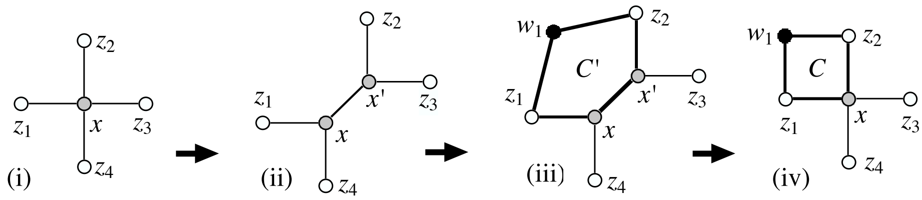

Figure 4:

$d(u_2) = 3$, $u_2 \neq x$ and hence $P \subset G_1$. By induction, G_1 has a spanning even subgraph X_1' such that $P \subset X_1'$ and $\sigma(X_1') \geq 5$. Then $xx' \in E(X_1')$; otherwise $E(X_1')$ induces the required subgraph of G. Let C_1' be the component of X_1' passing through xx'. Since $X_1 = X_1'/xx'$ is a spanning even subgraph of G containing P,

$C_1 = C'/xx'$ has exactly four vertices; otherwise X_1 would be the required subgraph of G. Since G is simple C_1 is a 4-cycle. If C_1 contained three vertices in $N(x)$, say z_1, z_2, z_3 then, since each neighbour of x has degree three by Claim 3, the edges joining C_1 and $G - C_1$ would be a 3-edge-cut of G, which would contradict Claim 2, or G would be a wheel on five vertices. Since the theorem holds for the wheel on five vertices, we deduce that C_1 contains exactly two vertices in $N(x)$, say z_1, z_2. Let $C_1 = xz_1w_1z_2x$. See Figure 4(iii)-(iv). Since $P \subset X_1$ we have,

$$P \subset C_1 \text{ or } E(C_1) \cap E(P) = \emptyset. \tag{1}$$

Let G'_2 be the graph obtained from $G_x^{f_3,f_4}$ by adding the edge xx''. See Figure 5(i). We may apply the above argument to G'_2, and relabel z_1 and z_2, and z_3 and z_4 if

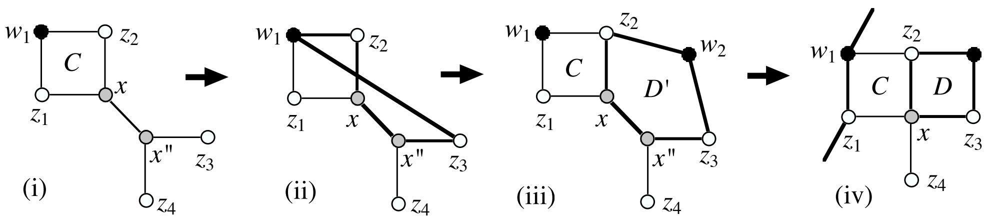

Figure 5:

necessary, to deduce that G has a spanning even subgraph X_2 with $P \subset X_2$, and such that $C_2 = xz_2z_3w_2x$ is a component of G_2. If $w_1 = w_2$, then since $z_1x, z_1w_1 \notin E(X_2)$ and $d_G(z_1) = 3$ we would have $z_1 \notin V(X_2)$. This would contradict the fact that X_2 is a spanning even subgraph of G. Thus $w_1 \neq w_2$. See Figure 5(ii). Since $z_1x, z_2w_1 \notin E(X_2)$ we have $z_1x, z_2w_1 \notin E(P)$. Now (1) implies that $E(C_1) \cap E(P) = \emptyset$. Since $d_G(z_1) = 3$ and $z_1x \notin E(X_2)$, the component of X_2 containing z_1 passes through the edge z_1w_1. Hence $(X_2 - \{z_1w_1, z_2x\}) \cup \{w_1z_2, z_1x\}$ is the required subgraph of G. □

Claim 7. *G is triangle-free.*

Proof. This follows immediately from Claims 2 and 6. □

Claim 8. *G contains no 4-cycles.*

Proof. Suppose $C = x_1x_2x_3x_4x_1$ is a 4-cycle in G. For $1 \leq i \leq 4$, let y_i be the neighbour of x_i in $G - C$. Let $G^* = (G - \{x_3, x_4\}) \cup \{x_1y_3, x_2y_4\}$.

Suppose G^* has a 2-edge-cut E_0. Then E_0 must contain the edge x_1x_2. See Figure 6. Let f be the other edge in E_0. By Claim 2, neither $E_1 = \{x_1y_1, x_3y_3, f\}$

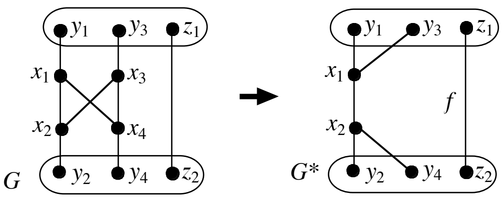

Figure 6:

nor $E_2 = \{x_2y_2, x_4y_4, f\}$ are essential 3-edge-cuts of G. This implies that G is isomorphic to the complete bipartite graph $K_{3,3}$, and it can easily be checked that G has the desired subgraph.

Thus G^* is 3-edge-connected. Consider the following three cases.

Case 1 $E(C) \cap E(P) = \emptyset$.

By induction, G^* has a 2-factor F^* such that $P \subset F^*$ and $\sigma(F^*) \geq \min\{n^*, 5\}$.

Suppose F^* passes through the edge x_1x_2. By symmetry, we may assume that F^* passes through x_1y_1 and x_2y_2. See Figure 7(iii). Then $(F^* - \{x_1x_2\}) \cup x_1x_4x_3x_2$

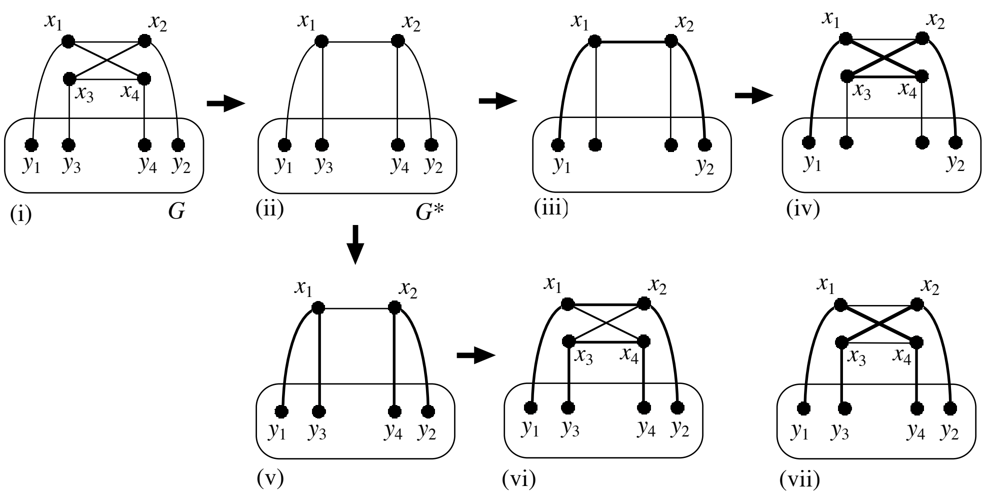

Figure 7:

is the required 2-factor of G. See Figure 7(iv).

Thus $x_1x_2 \notin E(F^*)$. Then $F = F^* \cup \{x_1x_2, x_3x_4\}$ is a 2-factor of G passing through P. See Figure 7(v)-(vi). Let D_1, D_2 be the cycles of F passing through x_1x_2 and x_3x_4, respectively. (We allow the possibility $D_1 = D_2$.) If neither D_1 nor D_2 is a 4-cycle, then F is the required 2-factor of G. Hence either D_1 or D_2 is a 4-cycle, so $D_1 \neq D_2$ and $F' = F^* \cup \{x_1x_4, x_3x_2\}$ is the required 2-factor of G.

Case 2 $P \subset C$.

By symmetry, we may assume that $P = x_1x_4x_3$. We specify the path $P^* = x_1x_2y_2$ in G^*. By induction, G^* has a 2-factor F^* such that $P^* \subset F^*$ and $\sigma(F^*) \geq \min\{n^*, 5\}$. If $x_1y_1 \in E(F^*)$, then $(F^* - \{x_1x_2\}) \cup x_1x_4x_3x_2$ is the required 2-factor of G. See Figure 7(iii)-(iv). Thus $x_1y_3 \in E(F^*)$, and $(F^* - \{x_1x_2\}) \cup x_2x_1x_4x_3$ is the required 2-factor of G. See Figure 8.

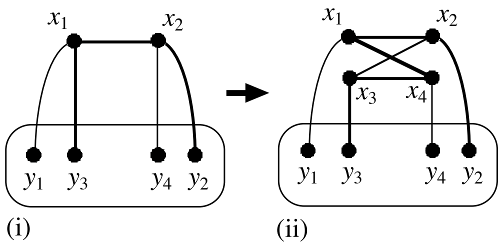

Figure 8:

Case 3 $|E(C) \cap E(P)| = 1$.

By symmetry, we may assume that $P = y_1x_1x_2$. We specify the path $P^* = y_1x_1y_3$ in G^*. By induction, G^* has a 2-factor F^* such that $P^* \subset F^*$ and $\sigma(F^*) \geq \min\{n^*, 5\}$. See Figure 7(v). Then $F = F^* \cup \{x_1x_2, x_3x_4\}$ is a 2-factor of G with $P \subset F$. See Figure 7(vi). Since G is a counterexample to the theorem, F must contain a 4-cycle C'. Since $\sigma(F^*) \geq \min\{n^*, 5\}$, C' passes through x_1x_2 or x_3x_4. If the first alternative holds then C' is a cycle of G with $P \subset C'$. If the second alternative holds then C' is a 4-cycle of G with $E(P) \cap E(C') = \emptyset$. We can now obtain a contradiction by returning to Case 1 or 2 with C replaced by C'. □

We can now complete the proof of the theorem. By the above mentioned strengthening of Petersen's theorem, G has a 2-factor F with $P \subset F$. Since G

has girth at least 5, $\sigma(F) \geq 5$. □

Proof of Theorem 3

We use induction on the number of edges of G. If $G-e$ is 3-edge-connected for some $e \in E(G)$ then we are through by induction. Thus $G-e$ is not 3-edge-connected for all $e \in E(G)$. By a result of Mader [10], G has a vertex u_2 of degree 3. We can now choose a pair of edges incident to u_2 and apply Theorem 6. □

4 Closing Remarks

The construction illustrated in Figure 9 shows that there exists an infinite family

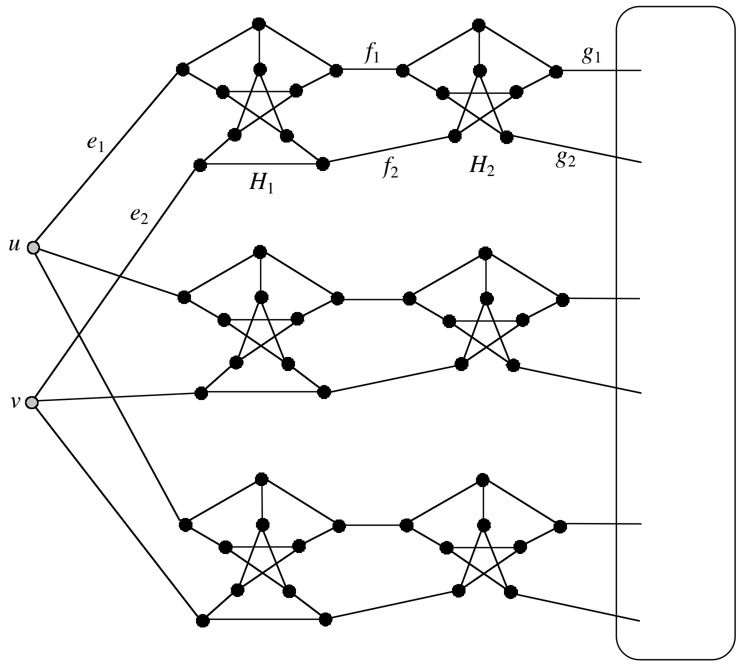

Figure 9:

of 3-edge-connected, essentially 4-edge-connected graphs G in which every spanning even subgraph has a component with at most five vertices. To see this let X be an even subgraph of G. Since u, v have degree three in G we have $d_X(u) = 2 = d_X(v)$. Hence, by symmetry we may suppose that X contains at most one edge from $\{e_1, e_2\}$. If X contains exactly one edge from $\{e_1, e_2\}$, then X must also contain exactly one of f_1, f_2 and exactly one of g_1, g_2. The fact that every 2-factor of the Petersen graph contains a 5-cycle now implies that $X \cap H_2$ contains a 5-cycle. Thus we may assume that $E(X) \cap \{e_1, e_2\} = \emptyset$. Then either $E(X) \cap \{f_1, f_2\} = \emptyset$ or $\{f_1, f_2\} \subset E(X)$. In both cases we have that $X \cap H_1$ contains a 5-cycle. Thus $\sigma(X) \leq 5$.

Jaeger [8] showed that every 4-edge-connected graph has a spanning connected even subgraph, and Zhan [14] showed that the same conclusion holds for 3-edge-connected, essentially 7-edge-connected graphs. Chen and Lai [4] conjecture that every 3-edge-connected, essentially 5-edge-connected graph has a spanning connected even subgraph. An affirmative answer to either of the following problems would be significantly weaker than their conjecture.

Problem 7. *Does there exist an unbounded function $f : \mathbb{N} \to \mathbb{N}$ such that every 3-edge-connected, essentially 6-edge-connected graph G has a spanning even subgraph X with $\sigma(X) \geq f(n)$?*

Problem 8. *Does there exist a constant $c > 0$ such that every 3-edge-connected, essentially 6-edge-connected graph has a connected even subgraph with at least cn vertices?*

(It follows from a result of the first named author [3, Theorem 1] that every 3-edge-connected graph has a connected even subgraph with $O(n^c)$ vertices, where $c = \log_2(1 + \sqrt{5}) - 1\,(\simeq 0.69)$. On the other hand, Bondy and Simonovits [1] have constructed an infinite family of 3-edge-connected cubic graphs with no cycle of length greater than n^c, where $c = \log 8/\log 9\,(\simeq 0.96)$. Their graphs are not essentially 4-edge-connected and Bondy conjectures, see [3, Conjecture 1], that there exists a constant $c > 0$ such that every essentially 4-edge-connected cubic graph G has a cycle of length at least cn.)

One could also ask whether a 'highly connected' cubic graph must contain a 2-factor with no short cycles. Since cubic graphs cannot be essentially k-edge-connected for any $k \geq 5$, we need to introduce another measure of connectivity. We say that a graph G is *cyclically k-edge-connected* if at most one component of $G - S$ contains a cycle for all $S \subseteq E(G)$ with $|S| \leq k - 1$. Kochol [9, Theorem10.5] has constructed an infinite family of cyclically 6-edge-connected cubic graphs in which every 2-factor has at least $\lfloor n/118 \rfloor$ components, and hence contains a cycle of length at most 118.

Problem 9. *Is there a value of* k *for which there exist an unbounded function* $g : \mathbb{N} \to \mathbb{N}$ *such that every cyclically* k*-edge-connected cubic graph* G *has a 2-factor* X *with* $\sigma(X) \geq g(n)$*?*

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