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STABLY SPLITTING BG

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ABSTRACT. In the early nineteen eighties, Gunnar Carlsson proved the Segal conjecture on the stable cohomotopy of the classifying space BG of a finite group G. This led to an algebraic description of the ring of stable self-maps of BG as a suitable completion of the "double Burnside ring". The problem of understanding the primitive idempotent decompositions of the identity in this ring is equivalent to understanding the stable splittings of BG into indecomposable spectra. This paper is a survey of the developments of the last ten to fifteen years in this subject.

1. The Segal conjecture

The nineteen eighties saw a number of major breakthroughs in homotopy theory. Among the most spectacular are the proofs of the nilpotence conjecture by Devinatz, Hopkins and Smith [10], the Sullivan conjecture by Haynes Miller [24], and the Segal conjecture by Gunnar Carlsson [7]. Two of these three concern the role of finite groups in homotopy theory: the Sullivan conjecture is "unstable", while the Segal conjecture is "stable" with respect to suspension. This report is about the consequences of the Segal conjecture for the stable splittings of the classifying space of a finite group.

We begin by setting the scene. Around 1960, Atiyah [3] calculated the K-theory of the classifying space of a finite group. There is a natural map from the character ring $\mathcal{R}(G)$ to $K^0(BG)$ which sends a complex representation V to the corresponding vector bundle $EG\times_G V \to BG$. Atiyah proved that this map induces an isomorphism

$$\mathcal{R}(G)\hat{}_I \xrightarrow{\cong} K^0(BG)$$

where the completion

$$\mathcal{R}(G)\hat{}_I = \varprojlim_n \mathcal{R}(G)/I^n$$

is with respect to the augmentation ideal $I = \mathsf{Ker}(\dim : \mathcal{R}(G) \to \mathbb{Z})$. He also showed that $K^1(BG) = 0$.

The Segal conjecture is the corresponding statement for stable cohomotopy, in which the representation ring is replaced by the Burnside ring $A(G)$. This is the

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Grothendieck ring of finite permutation representations of G, with addition corresponding to disjoint union and multiplication corresponding to Cartesian product with diagonal G-action.

We first explain stable maps and stable cohomotopy. If X and Y are pointed CW complexes, let $[X;Y]$ denote the homotopy classes of basepoint preserving maps $X \to Y$. Then there is a suspension homomorphism

$$[X;Y] \to [SX;SY] \cong [X;\Omega SY].$$

If X is finite, then passing to the limit we have

$$\lim_{m\to\infty}[S^mX;S^mY] = \lim_{m\to\infty}[X;\Omega^m S^m Y] = [X;\lim_{m\to\infty}\Omega^m S^m Y].$$

We write $\Omega^\infty S^\infty Y$ for $\lim_{m\to\infty}\Omega^m S^m Y$. Taking $Y = S^n$ (the n-sphere) and adding a disjoint basepoint to an unbased space X to give a based space X_+, the right-hand side of the above equation defines a generalized cohomology theory

$$\pi^n_{\mathrm{s}}(X) = [X_+;\lim_{m\to\infty}\Omega^m S^{m+n}]$$

called stable cohomotopy. Notice also that this formula makes sense for n negative, so that π^*_{s} is a $\mathbb{Z}$-graded theory.

For an arbitrary (not necessarily finite) CW complex X, taking homotopy classes of maps does not commute with passing to the limit, but the right-hand side above is the appropriate definition to ensure that π^*_{s} is a generalized cohomology theory. In general, therefore, we define the stable maps from X to Y to be

$$\{X;Y\} = [X;\Omega^\infty S^\infty Y].$$

There is a natural way to add stable maps, using a suspension coordinate, so that $\{X;Y\}$ is an abelian group. There is also a natural way to compose stable maps, so that for example $\{X;X\}$ is a ring and π^*_{s} is a *multiplicative* cohomology theory. This also allows us to make CW complexes and stable maps into a category. Any stable map from X to Y induces a map in cohomology $f^* : \tilde{H}^*(Y;R) \to \tilde{H}^*(X;R)$ with coefficients in, say, an abelian group R, so cohomology is a functor on CW complexes and stable maps.

We next describe the mapping telescope construction. For actual maps of spaces $f : X \to X$, this is easy to describe: we take a disjoint union of copies of $X \times I$ indexed by the natural numbers and identify the right-hand end of each copy with its image under f in the left-hand end of the next copy. If f is idempotent (i.e., $f \circ f = f$), then the cohomology of the mapping telescope is equal to the image of f^* in the cohomology of X.

For stable maps, we need to extend the category in order to be able to make the corresponding construction. The category of CW complexes and stable maps is a full subcategory of the category of spectra. The spectra corresponding to CW complexes are called suspension spectra. The homotopy category of spectra is called the **stable homotopy category**. We refer the interested reader to Adams [1] for full details, but suffice it to say here that the mapping telescope construction has an obvious analog in the category of spectra and allows us to form the mapping telescope of a stable map. The mapping telescope of a map between suspension spectra is not necessarily a suspension spectrum, but it is at least a "connective" spectrum—it has no homotopy in negative degrees.

A stable splitting of X as a wedge sum $Y \vee Z$ (i.e., an isomorphism between X and $Y \vee Z$ in the stable homotopy category) corresponds to an idempotent element

e of $\{X;X\}$. Conversely, any idempotent $e \in \{X;X\}$ provides a splitting of X as a wedge sum of spectra, via the mapping telescope construction. The wedge summands Y and Z in a stable splitting are equivalent to the mapping telescopes of e and $1-e$ respectively.

If $p : X \to Y$ is a finite covering, then there is a stable map $\mathsf{Tr}_p : Y_+ \to \Omega^\infty S^\infty X_+$ called the transfer map. For example, if H is a subgroup of finite index in a group G, then there is a finite covering $BH \to BG$, and the corresponding transfer map $\mathsf{Tr}_{H,G} \in \{BG_+;BH_+\}$ induces the usual transfer map in cohomology, $\mathsf{Tr}^*_{H,G} : H^*(BH;R) \to H^*(BG;R)$ (note that $\tilde{H}^*(X_+;R) = H^*(X;R)$). Composing $\mathsf{Tr}_{H,G}$ with the map $B\rho_H : BH_+ \to B\{1\}_+ = S^0$ (where ρ_H maps H to the trivial group) gives an element of stable cohomotopy $B\rho_H \circ \mathsf{Tr}_{H,G} \in \pi^0_{\mathrm{s}}(BG)$. There is thus a natural map $A(G) \to \pi^0_s(BG)$ sending the permutation representation (G/H) to the stable map $B\rho_H \circ \mathsf{Tr}_{H,G}$. This is a ring homomorphism, and the **Segal conjecture** states that this map induces an isomorphism of rings $A(G)\hat{\ }_I \to \pi^0_s(BG)$ between the completion of $A(G)$ with respect to the augmentation ideal I and degree zero stable cohomotopy. Furthermore, it states that $\pi^n_{\mathrm{s}}(BG) = 0$ for $n > 0$. Of course, even for the trivial group, the stable cohomotopy in negative degrees (namely, the stable homotopy of spheres) is extremely complicated, but a suitable form of the conjecture (see page 190 of Carlsson [7]) gives the stable cohomotopy in negative degrees, in terms of the stable homotopy groups of classifying spaces of subquotients of G of the form $N_G(K)/K$, $K \leq G$.

The Segal conjecture was proved first for the cyclic group of order two by Lin [19], then for the cyclic groups of odd prime order by Gunawardena [13], for general finite cyclic groups by Ravenel [29], for elementary abelian 2-groups by Gunnar Carlsson [6], for odd elementary abelian groups by Adams, Gunawardena and Miller [2], and finally for general finite groups by Gunnar Carlsson [7]. An unstable proof along entirely different lines can be found in Lannes [17].

2. The double Burnside ring

Now consider the more general problem of computing $\{BG_+;BH_+\}$ for finite groups G and H. In the case $H = \{1\}$, this is the stable cohomotopy $\pi^0_{\mathrm{s}}(BG) \cong A(G)\hat{\ }_I$. For a more general H, the appropriate algebraic gadget is the Grothendieck group $A(G,H)$ of finite sets with a commuting G-action and free H-action (for short, H-free $G \times H$-sets). Given such a set X, the principal H-bundle $EG \times_G X \to EG \times_G (X/H)$ is classified by a map $EG \times_G (X/H) \to BH$. Composing this with the transfer for the finite covering $EG \times_G (X/H) \to BG$ gives us a stable map

$$BG_+ \to \Omega^\infty S^\infty (EG \times_G (X/H))_+ \to \Omega^\infty S^\infty BH_+$$

in $\{BG_+;BH_+\}$. We therefore have a natural map $A(G,H) \to \{BG_+;BH_+\}$ sending the set X to this stable map. Following ideas of Haynes Miller, Adams conjectured in [1] that this map induces an isomorphism $A(G,H)\hat{\ }_I \to \{BG_+;BH_+\}$, where I is the augmentation ideal in $A(G)$. The action of $A(G)$ on $A(G,H)$ is given by taking Cartesian products, with diagonal G-action. It was proved by Lewis, May and McClure [18] that this conjecture follows from the Segal conjecture.

The effect of knowing that we have an isomorphism

$$A(G,H)\hat{\ }_I \to \{BG_+;BH_+\}$$

is that topological questions about stable maps are converted into algebraic questions. To make this an effective tool, we need to know how to compose maps.

Composition

$$\{BH_+; BK_+\} \times \{BG_+; BH_+\} \to \{BG_+; BK_+\}$$

corresponds to the operation

$$A(H, K) \times A(G, H) \to A(G, K)$$

sending a K-free $H \times K$-set Y and an H-free $G \times H$-set X to the K-free $G \times K$-set $(X \times Y)/H$, where H acts diagonally on $X \times Y$. This defines a category, with groups as objects, where the morphisms from G to H are given by $A(G, H)$. Any functor on groups which admits appropriately behaved transfer maps extends to a functor on this category.

More explicitly, $A(G, H)$ has a basis given as follows. Given a subgroup $G' \leq G$ and a homomorphism $\phi : G' \to H$, there is a finite H-free $G \times H$-set $X_{G',\phi} = (G \times H)/\Delta_{G',\phi}$, where

$$\Delta_{G',\phi} = \{(x, \phi(x)),\ x \in G'\} \leq G \times H.$$

Every transitive H-free $G \times H$-set is of this form, and so $A(G, H)$ has a basis corresponding to the conjugacy classes of pairs (G', ϕ). We write $\zeta_{G',\phi}$ for the basis element of $A(G, H)$ corresponding to $X_{G',\phi}$. An explicit double coset formula for multiplying basis elements is given in Benson and Feshbach [4].

This multiplication makes $A(G, G)$ into a noncommutative Noetherian ring, called the double Burnside ring. Stable splittings

$$BG_+ \cong X_1 \vee \cdots \vee X_n$$

correspond to decompositions

$$1 = e_1 + \cdots + e_n$$

in $A(G, G)\hat{_I}$ of the identity element as a sum of orthogonal idempotents. Here, two idempotents e and f are said to be orthogonal if $ef = fe = 0$. The space $X_i = e_i(BG_+)$ is formed as the mapping telescope $\mathrm{Tel}(BG_+ \xrightarrow{e_i} BG_+ \xrightarrow{e_i} \cdots)$. Thus in order to understand stable splittings of BG_+, we need to understand the algebraic structure of $A(G, G)$ and the effect of I-adic completion.

At this stage, it is worth getting rid of the disjoint basepoint. Choosing a basepoint in BG gives us a stable equivalence $BG_+ \cong S^0 \vee BG$. So the problem of splitting BG_+ is equivalent to the problem of splitting BG. Denote by $\tilde{A}(G, G)$ the quotient of $A(G, G)$ by the two-sided ideal given by the linear span of the basis elements $\zeta_{G',\phi}$ with $\mathsf{Im}(\phi) = \{1\}$. Then the map

$$\tilde{A}(G, G)\hat{_I} \to \{BG; BG\}$$

is an isomorphism.

The next step is to work one prime at a time. If $|G| = p_1^{\alpha_1} \dots p_s^{\alpha_s}$, then stably, BG splits as a wedge sum of p_i-local spectra

$$BG \cong BG_{p_1} \vee \cdots \vee BG_{p_s}.$$

Furthermore, the transfer map displays BG_p as a stable wedge summand of BP, where P is a Sylow p-subgroup of G. Regarding G as a P-free $P \times P$-set via the action given by $(x, y) : g \to xgy^{-1}$ and writing $[G]$ for the corresponding element of $\tilde{A}(P, P)\hat{_I} \cong \{BP; BP\}$, we have $BG_p = [G].BP$ (the mapping telescope $\mathrm{Tel}(BP \xrightarrow{[G]} BP \xrightarrow{[G]} \cdots))$. So it makes sense to concentrate on the case of a

p-group P. In this case, the I-adic completion is not quite the same as the p-adic completion on $A(P,P)$, but it is the same on $\tilde{A}(P,P)$:

$$\tilde{A}(P,P)\hat{}_p \cong \tilde{A}(P,P)\hat{}_I \cong \{BP;BP\},$$

and

$$\{BG_p;BG_p\} \cong [G]\tilde{A}(P,P)\hat{}_p[G].$$

Set $\tilde{A}_p(P,P) = \mathbb{F}_p \otimes_{\mathbb{Z}} \tilde{A}(P,P)\hat{}_p$. Then the idempotent refinement theorem (see for example Curtis and Reiner [9], Theorem 6.7) does not apply directly to $\mathbb{F}_p \otimes_{\mathbb{Z}} A(P,P)\hat{}_p$, but it does apply to $\tilde{A}_p(P,P)$. It implies that any decomposition of the identity as a sum of orthogonal idempotents in $\tilde{A}_p(P,P)$ lifts to such a decomposition in $\tilde{A}(P,P)\hat{}_p$ and that any two such lifts are conjugate. It follows that stable p-local splittings of BG correspond to orthogonal idempotent decompositions of the identity in the finite-dimensional $\mathbb{F}_p$-algebra $\mathbb{F}_p \otimes_{\mathbb{Z}} \{BG_p;BG_p\} \cong [G]\tilde{A}_p(P,P)[G]$.

The homotopy theoretic consequences of these algebraic properties of idempotents in $\{BG_p, BG_p\}$ are as follows. First, we have a Krull–Schmidt theorem: BG_p splits essentially uniquely as a finite wedge of indecomposable pieces. The homotopy types of indecomposable stable wedge summands are in one-one correspondence with the isomorphism classes of simple modules for the algebra $[G]\tilde{A}_p(P,P)[G]$. The multiplicity of a given stable homotopy type as a wedge summand is equal to the dimension of the corresponding simple module over its endomorphism ring, which is a finite field of characteristic p. Unfortunately, the representation theory of these algebras is not easy to study, though the papers of Martino and Priddy [21] and Benson and Feshbach [4] make some progress in this direction. The paper [21] gives an explicit formula for the multiplicity of a wedge summand as the rank of a certain matrix defined in terms of subgroups and conjugations, while the paper [4] attempts a more abstract description of the simple modules. For further work in this area, see Martino and Priddy [22, 23], Nishida [26], Priddy [27, 28]. Some explicit calculations appear in Dietz [11], Dietz and Priddy [12], Martino and Priddy [20]. Much remains to be done.

In the next section, we begin by explaining the abelian case, as a lot of the complications of the general case are not present here. The problem is not completely solved even in this case, but rather, it reduces to a well-known problem in modular representation theory—the determination of the simple modules in characteristic p for the finite general linear groups $GL(n,\mathbb{F}_p)$. In the remaining sections, we explain some of the ideas of Nishida [26], Martino and Priddy [21] and Benson and Feshbach [4] in the general case.

3. Abelian groups

It is worth discussing a particular case in some detail. If P is a finite abelian p-group, then denote by $\mathcal{E}(P)$ the multiplicative semigroup of endomorphisms of P and by $R\mathcal{E}(P)$ its semigroup algebra over a coefficient ring R. Note that the zero endomorphism of P is not equal to the zero element of $R\mathcal{E}(P)$, but it spans a two-sided ideal isomorphic to R. We write $R\tilde{\mathcal{E}}(P)$ for the quotient by this ideal.

There is an obvious embedding, $i : \mathbb{Z}\mathcal{E}(P) \hookrightarrow A(P,P)$, which passes down to embeddings $\tilde{\imath} : \mathbb{Z}\tilde{\mathcal{E}}(P) \hookrightarrow \tilde{A}(P,P)$ and $\tilde{\imath}_p : \mathbb{F}_p\tilde{\mathcal{E}}(P) \hookrightarrow \tilde{A}_p(P,P)$. Following work of C. Witten [30] and S. Mitchell [25], J. Harris [14] and G. Nishida [26] have independently shown that there is a surjective ring homomorphism $\rho : \tilde{A}_p(P,P) \to \mathbb{F}_p\tilde{\mathcal{E}}(P)$

whose composite with $\tilde{\imath}_p$ is the identity map on $\mathbb{F}_p\tilde{\mathcal{E}}(P)$. Furthermore, the kernel of ρ is a nilpotent two-sided ideal in $\tilde{A}_p(P,P)$. Again applying the idempotent refinement theorem, one finds that stable splittings of BP are in one-one correspondence with orthogonal idempotent decompositions of the identity in $\mathbb{F}_p\tilde{\mathcal{E}}(P)$ and that the homotopy types of indecomposable stable wedge summands are in one-one correspondence with the simple $\mathbb{F}_p\tilde{\mathcal{E}}(P)$-modules. Thus, for example, the indecomposable stable wedge summands of $B(\mathbb{Z}/p)^n_+$ correspond to simple $\mathbb{F}_p\mathsf{Mat}(n,\mathbb{F}_p)$-modules.

Having reduced to the theory of matrix representations of finite semigroups in this way, we can draw on the literature on finite semigroup representations (Clifford and Preston [8], Howie [16]) to reduce to finite group representation theory. The details may be found in Harris [14], Harris and Kuhn [15]. If S is a finite multiplicative semigroup with zero element (for example $S = \mathcal{E}(P)$) and k is a field, there is a natural bijection between isomorphism classes of irreducible kS-modules and

$$\coprod_G \{\text{isomorphism classes of irreducible } kG\text{-modules}\}.$$

Here, G runs over the equivalence classes of maximal subgroups of S. Two maximal subgroups G and G' of S are equivalent if $SGS = SG'S$ (this implies $G \cong G'$). Given a simple kG-module M, the corresponding kS-module is described as follows. First we form the $kSGS$-module

$$\tilde{M} = kGS \otimes_{k(G\cup\{0\})} M$$

and then we extend $\tilde{M}$ to a kS-module by letting $s \in S$ act as $s.1_G \in SGS$.

As an example, the simple $\mathbb{F}_p\mathsf{Mat}(n,\mathbb{F}_p)$-modules (and hence the indecomposable stable wedge summands of $B(\mathbb{Z}/p)^n_+$) correspond to

$$\coprod_{m=0}^{n} \{\text{simple } GL(m,\mathbb{F}_p)\text{-modules}\}.$$

Although the simple $GL(m,\mathbb{F}_p)$-modules are classified by their highest weights, it remains an unsolved problem to describe their dimensions over $\mathbb{F}_p$, which determine the multiplicities of the corresponding indecomposable stable wedge summands of $B(\mathbb{Z}/p)^n_+$. There remains also the interesting and largely unsolved problem of understanding the cohomology of the wedge summands. Some surprising work of Carlisle and Kuhn [5] relates the representation theoretic indexing of the factors to their Morava K-theories, via freeness over finite subalgebras of the Steenrod algebra.

4. Dominant summands

One feature of the elementary abelian case generalizes easily to arbitrary finite p-groups. Namely, it is apparent from the description given in the last section that an indecomposable stable wedge summand of $B(\mathbb{Z}/p)^n_+$ "comes from" a subgroup $(\mathbb{Z}/p)^m$ for a uniquely determined value of $m \le n$. The generalization to arbitrary finite groups is Nishida's theory of dominant summands [26]. If P is a finite p-group, Nishida defines J_P to be the two-sided ideal in $\{BP;BP\} \cong \tilde{A}(P,P)\hat{}_p$ generated by the maps which factor through the classifying space of a proper subgroup of P, and he shows that

$$\{BP;BP\}/J_P \cong \mathbb{Z}\hat{}_p\mathrm{Out}(P),$$

the group ring over the p-adic integers of the outer automorphism group of P. The ideal J_P is not in general nilpotent.

Reduction modulo J_P induces a one-one correspondence between conjugacy classes in $\{BP; BP\}$ of primitive idempotents not lying in J_P and conjugacy classes of primitive idempotents in $\mathbb{Z}_p^\wedge \mathrm{Out}(P)$. These in turn are in one-one correspondence with the isomorphism classes of irreducible R-modules, where R is the group algebra $\mathbb{F}_p\mathrm{Out}(P)$.

An indecomposable stable wedge summand $X = e.BP$ of BP is said to be *dominant* if the corresponding primitive idempotent $e \in \{BP; BP\}$ does not lie in J_P. This is equivalent to the statement that X does not have the homotopy type of a summand of BQ for any proper subgroup Q of P. The dominant summand corresponding to the trivial one-dimensional representation of R is called the *principal* dominant summand. It turns out that every indecomposable stable wedge summand of $BG_p^\wedge$ (G now a general finite group) has the homotopy type of a dominant summand of BQ for a uniquely determined isomorphism class of p-subgroups Q of G.

There is no sense in which Q is determined up to *conjugacy* in G, but we can do better than just naming the isomorphism class. By analogy with modular representation theory, we say that Q is a *vertex* of $X = e.BG_p^\wedge$ if the map $Bi : BQ \to BG$ induced by the inclusion $i : Q \to G$ induces a homotopy equivalence between a dominant summand of BQ and an indecomposable summand of $BG_p^\wedge$ isomorphic to X. The corresponding simple R-module S (where $R = \mathbb{F}_p\mathrm{Out}(Q)$) is called the *source* of X. The question of the extent to which these are determined by X in G is investigated in Section 5 of [4].

5. Multiplicities

The discussion of dominant summands in the last section gives rise to two related questions. Which dominant summands of BQ appear as summands in BG, as Q runs over the p-subgroups of G? And what are their multiplicities in BG?

To some extent, we have the answers to these questions. Martino and Priddy [21] show that the multiplicity of the dominant summand of BQ corresponding to the simple R-module S (where $R = \mathbb{F}_p\mathrm{Out}(Q)$) appears with multiplicity in BG equal to the rank of a certain matrix $(\bar{W}_{\alpha\beta})$ whose entries lie in the field $k = \mathrm{End}_R(S)$. To describe the entries $\bar{W}_{\alpha\beta}$, we introduce some notation. We write $\mathrm{Split}(Q)$ for the set of conjugacy classes of triples consisting of a subgroup $Q_\alpha \cong Q$ in G, a subgroup $P_\alpha \geq Q_\alpha$, and a split surjection $q_\alpha : P_\alpha \to Q_\alpha$. Two such triples $(q_\alpha : P_\alpha \to Q_\alpha)$ and $(q_\beta : P_\beta \to Q_\beta)$ are said to be conjugate if there is a pair of conjugations $c_u : P_\alpha \to P_\beta$ and $c_v : Q_\alpha \to Q_\beta$ ($u, v \in G$) satisfying $c_u \circ q_\alpha = q_\beta \circ c_v$.

For each such conjugacy class, we choose a representative and a fixed isomorphism between Q_α and Q. We set

$$\bar{W}_{\alpha\beta} = \sum_x 1 \otimes (q_\beta \circ c_x) \in \mathbb{F}_p \otimes_{\mathbb{Z}_p^\wedge} \{BP; BP\}/J_P \cong R$$

where x runs over a set of representatives for the orbits of P_β on $N_G(Q_\alpha, P_\beta)$ (the set of elements of G conjugating Q_α into P_β). This gives us an $n \times n$ matrix over R, where $n = |\mathrm{Split}(Q)|$. The action of R on S gives a map $R \to \mathrm{Mat}_m(k)$ where $m = \dim_k(S)$, and via this map we interpret $(\bar{W}_{\alpha\beta})$ as an $mn \times mn$ matrix over k. The theorem of Martino and Priddy [21] is that the multiplicity of the corresponding dominant stable summand of BQ as a stable summand of BG is equal to the rank of this matrix.

6. Representation theory of $A_p(G,G)$

A different approach to the determination of the stable summands of BG_+ is taken by Benson and Feshbach [4]. They give a construction for the simple modules for $A_p(G,G)$ (which, as we have indicated, parametrize the isomorphism types of indecomposable stable wedge summands of $(BG_+)\hat{}_p$) without determining the dimensions (and hence the multiplicities). Their starting point is the construction of a faithful integral representation of $A(G,G)$, which they call the *coadjoint module*, whose submodule structure is easier to analyse than that of the regular representation or its dual.

The idea is as follows. If $\mathcal{H}$ is any collection of subgroups of a finite group G, closed under conjugation and intersections, then we write $A(G,\mathcal{H})$ for the subring of the (usual) Burnside ring $A(G)$ spanned by the G-sets where the point stabilizers are in $\mathcal{H}$. For each conjugacy class of $H \in \mathcal{H}$, taking H-fixed points gives a ring homomorphism $A(G,\mathcal{H}) \to \mathbb{Z}$. It is well known that the sum of these maps embeds $A(G,\mathcal{H})$ as a subring of finite index in a direct product of copies of the ring $\mathbb{Z}$.

Replace G by $G \times H$, and take for $\mathcal{H}$ the collection of subgroups $\Delta_{G',\phi}$ described in Section 2. Then $A(G \times H, \mathcal{H})$ may be identified with the double Burnside ring $A(G,H)$. Write $f_{G',\phi}$ for the corresponding map $A(G,H) \to \mathbb{Z}$, and write $M(G,H)$ for the free abelian group whose basis elements consist of the $f_{G',\phi}$. If $\zeta_{G'',\phi'}$ is a basis element of $A(G,G)$, then composition

$$A(G,H) \xrightarrow{-\circ\zeta_{G'',\phi'}} A(G,G) \xrightarrow{f_{G',\psi}} \mathbb{Z}$$

defines a map

$$A(G,G) \times M(G,H) \to M(G,H)$$

making $M(G,H)$ into an $A(G,G)$-module. An explicit formula for the above composition is given in Proposition 3.1 of [4]. In particular, if we set $G = H$, we obtain the coadjoint $A(G,G)$-module $M(G,G)$, which is contained in the dual of the regular representation $\mathrm{Hom}_{\mathbb{Z}}(A(G,G),\mathbb{Z})$ as a subgroup of finite index.

Next, we assume that $G = P$ is a p-group and reduce modulo p. We obtain an $A_p(G,G)$-module $M_p(G,H)$, and in case $G = H$, this must contain every simple module with nonzero multiplicity. The advantage over the regular representation (or its dual) is that $M_p(G,H)$ admits an obvious filtration by the conjugacy classes of pairs (G',ϕ) (as usual, $G' \le G$ and $\phi : G' \to H$). We partially order these pairs by writing $(G',\phi) \succeq (G'',\phi')$ if there is a surjective homomorphism $\alpha : G' \to G''$ which extends to a (not necessarily surjective) homomorphism $G'C_G(G') \to G''$ and if there is an element $g \in G$ such that $\phi = c_g \circ \phi' \circ \alpha$. If $(G',\phi) \succeq (G'',\phi')$ and $(G'',\phi') \succeq (G',\phi)$, then we write $(G',\phi) \sim (G'',\phi')$ and say that (G',ϕ) has the same *type* as (G'',ϕ'). It is shown in Proposition 4.4 of [4] that if $f_{G'',\phi'}$ appears with nonzero multiplicity modulo p in the image of $f_{G',\phi}$ under the coadjoint action of an element of $A(G,G)$ then $(G',\phi) \succeq (G'',\phi')$. Thus we have a filtration of $M_p(G,H)$ by types of pairs (G',ϕ). The filtered quotients are written $\bar{L}(G,H)_{G',\phi}$. This admits commuting actions of $A_p(G,G)$ and $\mathbb{F}_p\mathrm{Out}(H)$. So if S is a simple right $\mathbb{F}_p\mathrm{Out}(H)$-module, we may form the tensor product $S \otimes_{\mathbb{F}_p\mathrm{Out}(H)} \bar{L}(G,H)$. One can define a submodule $\mathcal{M}$ of this tensor product in such a way that the quotient is either zero or a simple $A_p(G,G)$-module $\bar{L}(G,H,S)$. The modules $\bar{L}(G,H,S)$ form a set of representatives for the isomorphism classes of simple $A_p(G,G)$-modules. The quotient is zero precisely when the dominant summand of BH_+ corresponding to S does not appear as a summand of BG_+, and when it does appear, $\bar{L}(G,H,S)$ is the simple $A_p(G,G)$-module corresponding to this summand. In particular, its dimension over its endomorphism ring gives the multiplicity of the summand.

For the general finite group G with Sylow p-subgroup P, we form as before the element $[G]$ of $A_p(P,P)$ and set $\bar{L}(G,H,S) = [G].\bar{L}(P,H,S)$. This is again equal to the simple $[G].A_p(P,P).[G]$-module corresponding to the occurrences of the dominant summand of BH_+ corresponding to S as a summand of BG_+, if there are any, and zero otherwise.

7. Concluding remarks

We may liken the current state of the subject to the current state of the modular representation theory of the finite symmetric groups. In this case, the reduction modulo p of a Specht module S^λ has defined on it a symmetric bilinear form b. The modules $S^\lambda/\mathrm{Rad}(b)$ are either zero or simple. In the latter case, they are written D^λ. It is known exactly which cases give rise to nonzero modules D^λ in this way, but the only formula we have for the dimension of D^λ is as the rank of a certain matrix with entries in $\mathbb{F}_p$. The decomposition numbers are not known.

In the case of the finite groups of Lie type in the defining characteristic, the problem is similar but with the Specht modules replaced by the Weyl modules and the bilinear form replaced by a contravariant form.

In the case of $A_p(G,G)$, again we know the simple modules, and we have a formula for the dimension as the rank of a certain matrix. These tell us the indecomposable stable wedge summands of the classifying space and their multiplicities. Is it possible that there is some kind of bilinear form, or contravariant form, or something similar on $S \otimes_{\mathbb{F}_p \mathrm{Out}(H)} \bar{L}(G,H)$ whose radical is the submodule $\mathcal{M}$ mentioned at the end of the last section?

What is the precise relationship between the matrices of Martino and Priddy and the modules of Benson and Feshbach? Is it possible to reduce the entire multiplicity question down to questions about modular representations of finite groups which are as comprehensible as in the abelian case? It seems that there is still much to be done in this area.

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