

The subgroup structure of finite classical groups in terms of geometric configurations

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1 Introduction

There are four classes of Classical Group. Perhaps it is most straightforward to name a significant group within each class and to then describe the various related groups in the class. Four significant groups, then, are $GL_n(q)$, $Sp_n(q)$, $O_n(q)$ and $U_n(q)$. We begin by describing these groups in some detail. This paper is a survey of progress towards a geometric description of the subgroup structure of the classical groups. We shall describe Aschbacher's Theorem in some detail, even though Aschbacher's approach is clearly not geometrical, for Aschbacher's Theorem demonstrates very largely the structure that one should expect to find. It is unrealistic to expect to furnish a proof of Aschbacher's Theorem that is entirely geometric, but it is a reasonable hope that nearly all the maximal subgroups of classical groups can be described geometrically and that maximality can be proven using geometric means. Maximal here means nothing more than maximal relative to inclusion.

The general linear group, $GL_n(q)$, consists of the invertible linear transformations of an n -dimensional vector space $V = V(n, q)$ over the finite field $GF(q)$, where $q = p^e$ with p prime. Above this sits $\Gamma L_n(q)$, the group of all invertible semi-linear transformations of V . Below $GL_n(q)$ sits $SL_n(q)$, the set of all linear transformations of V having determinant 1; $SL_n(q)$ is a characteristic subgroup of $\Gamma L_n(q)$. The centre of $\Gamma L_n(q)$ is the group Z of non-zero scalar linear transformations; this also the centre of $GL_n(q)$. The quotients $P\Gamma L_n(q) = \Gamma L_n(q)/Z$, $PGL_n(q) = GL_n(q)/Z$ and $PSL_n(q) = Z.SL_n(q)/Z$ act naturally on the projective space $PG(n-1, q)$. In this context, $P\Gamma L_n(q)$ may be viewed as the set of all collineations of $PG(n-1, q)$. For some people, this is the natural geometric group. For others, $PGL_n(q)$ is the natural geometric group because its elements can be represented as matrices. From another perspective, $PSL_n(q)$ is sometimes regarded as the most important group in the class because it is usually simple.

The symplectic group, $Sp_n(q)$, is the group of all elements of $GL_n(q)$ preserving a non-degenerate alternating form; the non-degeneracy leads to n being even. There are various ways to describe $Sp_n(q)$ in terms of matrices. One is to consider the matrix $J_S = \begin{bmatrix} 0 & I \\ -I & 0 \end{bmatrix}$. Then $Sp_n(q)$ consists of the matrices A such that $AJ_SA^T = J_S$. Above $Sp_n(q)$ sit $\Gamma Sp_n(q)$ consisting of the invertible semi-linear transformations that preserve the alternating form up to a scalar, and $GSp_n(q) = \Gamma Sp_n(q) \cap GL_n(q)$. A feature of $Sp_n(q)$ is that its elements already have determinant 1 so it is a subgroup of $SL_n(q)$. Each group has a corresponding projective group, namely its image in $P\Gamma L_n(q)$. In geometric terms, $P\Gamma Sp_n(q)$ preserves a symplectic polarity. The geometric objects are the absolute subspaces with respect to this polarity, although $P\Gamma Sp_n(q)$ may be viewed as the group of collineations preserving the set of absolute lines. The group $PSP_n(q)$ is usually simple.

The unitary group, $U_n(q)$, is defined when q is a square and is the group of all

elements of $GL_n(q)$ preserving a non-degenerate Hermitian form. In matrix terms, $U_n(q)$ consists of the matrices A such that $A\bar{A}^T = I$, where \bar{A} is obtained from A by raising each entry to the power \sqrt{q} . Above $U_n(q)$ sit $\Gamma U_n(q)$ consisting of the invertible semi-linear transformations that preserve the Hermitian form up to a scalar, and $GU_n(q) = \Gamma U_n(q) \cap GL_n(q)$. Below $U_n(q)$ sits $SU_n(q) = U_n(q) \cap SL_n(q)$. Each group has a corresponding projective group, namely its image in $PGL_n(q)$. In geometric terms, $P\Gamma U_n(q)$ preserves a Hermitian polarity or, put another way, is the group of an Hermitian surface $\mathcal{H}(n-1, q)$ (the set of absolute points). The geometric objects are the subspaces lying on $\mathcal{H}(n-1, q)$. The group $PSU_n(q)$ is usually simple. There are two schools of thought regarding notation for the unitary group: for geometers, the Hermitian surface lies in $PG(n-1, q)$ and so $PSU_n(q)$ is the appropriate terminology; for group theorists, the motivation might come from Chevalley groups or algebraic groups, and then $PSU_n(\sqrt{q})$ or something similar is deemed appropriate. Here we follow the geometric school.

The orthogonal group, $O_n(q)$, is the most difficult to describe. It is the group of all elements of $GL_n(q)$ preserving a non-singular quadratic form Q whose associated symmetric bilinear form B is non-degenerate. The description depends to some extent on the parity of q . Suppose first that q is odd. Then $B(x, y) = Q(x + y) - Q(x) - Q(y)$ and $Q(x) = B(x, x)/2$, so preservation of Q and preservation of B are equivalent; in matrix terms, $O_n(q)$ consists either of the matrices A such that $AA^T = I$ or of the matrices A such that $AJ_OA^T = J_O$, where J_O is the diagonal matrix with entries $1, 1, \dots, 1, \lambda$ for some non-square λ . These amount to the same thing if n is odd, but when n is even the two possibilities give rise to different types of geometric structure (depending on n and q) and the groups are sometimes written as $O_n^\epsilon(q)$ with $\epsilon = \pm 1$ (we return to this shortly). Above $O_n(q)$ sit $\Gamma O_n(q)$ consisting of the invertible semi-linear transformations that preserve the quadratic form up to a scalar, and the general orthogonal group $GO_n(q) = \Gamma O_n(q) \cap GL_n(q)$. Below $O_n(q)$ sit the special orthogonal group $SO_n(q) = O_n(q) \cap SL_n(q)$ and the commutator subgroup $\Omega_n(q)$ of $O_n(q)$ that has index 2 in $SO_n(q)$. Each group has a corresponding projective group, namely its image in $PGL_n(q)$. If q is even, then life is more complicated. The bilinear form B is now an alternating form, and Q cannot be recovered from B ; an immediate consequence is that n must be even if B is to be non-degenerate. Any element of $GL_n(q)$ preserving Q also preserves B , and this leads to the inclusion $O_n(q) \leq Sp_n(q)$. It is possible, but not straightforward, to determine a matrix criterion for elements of $O_n(q)$, but such a criterion is not used in practice. As with the case where q is odd, there are two canonical forms for a quadratic form, leading to two types of orthogonal group. The fact that $O_n(q) \leq Sp_n(q)$ means that all the elements of $O_n(q)$ have determinant 1. Here the special orthogonal group is defined as the kernel of the *Dickson invariant*; in practical terms $SO_n(q)$ is the subgroup of $O_n(q)$ consisting of the elements having fixed space with even dimension (c.f., [25]); further, $SO_n(q) = \Omega_n(q)$ with just one exception. In geometric terms, the points where Q takes the value 0 are termed *singular* and the set of such points is a quadric. When q is odd, these are precisely the set of absolute points of the polarity corresponding to B . In any case $P\Gamma O_n(q)$ preserves the quadric. The geometric objects are the subspaces of $PG(n-1, q)$ lying on the quadric. For n odd, the quadric is termed *parabolic*. For n even, one type is termed *hyperbolic* (when there are $(n/2) - 1$ -dimensional subspaces of $PG(n-1, q)$

lying on the quadric, and we write $O_n^+(q)$) and the other *elliptic* (in which case there are $(n/2) - 2$ -dimensional subspaces but no $(n/2) - 1$ -dimensional subspaces lying on the quadric, and we write $O_n^-(q)$). The group $P\Omega_n(q)$ is usually simple for $n \geq 3$. However, there is one notable exception when $n = 4$ and the quadric is hyperbolic; in this case $P\Omega_4^+(q)$ is isomorphic to $PSL_2(q) \times PSL_2(q)$.

The approach we shall take to describing the subgroups structure is to consider the smallest groups in each class of classical groups and to consider the projective groups in most cases. Thus we shall be looking at the subgroup structure of $PSL_n(q)$, $PSp_n(q)$, $P\Omega_n(q)$ and $PSU_n(q)$. The reason for this is that these are usually the cases where results are hardest to prove, with other groups following relatively easily. There will be occasions when it is more convenient to describe subgroups of $SL_n(q)$, $Sp_n(q)$, $\Omega_n(q)$ and $SU_n(q)$, but this does not affect the maximality.

We shall begin, in the next section, by describing the structure of low-dimensional groups. Then we shall describe Aschbacher's Theorem in some detail, with a geometric interpretation, and relate it the low-dimensional results. After that we shall discuss current progress towards a geometric description of the subgroups of the classical groups in general.

2 Low dimensional groups

The first geometrical description of a subgroup structure appeared more than one hundred years ago. The subgroup structure of $PSL_2(q)$ was determined by E.H. Moore [44] and A. Wiman [51], although Dickson's treatment based on these two papers is often quoted as a source. Not too many years later Mitchell determined the maximal subgroups of $PSL_3(q)$ and then $PSp_4(q)$, both for q odd, and Hartley determined the maximal subgroups of $PSL_3(q)$ for q even. In the process, both Mitchell and Hartley determined the maximal subgroups of $PSU_3(q)$.

2.1 $n = 2$

L.E. Dickson's book ([12]) on linear groups investigates finite classical groups in considerable detail. The terminology appears a little strange nowadays: $GL_n(q)$ and $SL_n(q)$ are written $GLH(n, p^e)$ and $SLH(n, p^e)$, the General and Special Linear Homogeneous groups, and $PSL_n(q)$ is written $LF(n, p^e)$, the Linear Fractional Group. In Chapter 12, Dickson gives a complete determination of the subgroups of $LF(2, p^e)$, based on the work of Moore and Wiman. Mitchell gives another treatment in [42].

Theorem 2.1 [12] *The subgroups of $PSL_2(q)$ are as follows:*

- (a) *a single class of $q + 1$ conjugate abelian groups of order q ;*
- (b) *a single class of $q(q + 1)/2$ conjugate cyclic groups of order d for each divisor d of $q - 1$ for q even and $(q - 1)/2$ for q odd;*
- (c) *a single class of $q(q - 1)/2$ conjugate cyclic groups of order d for each divisor d of $q + 1$ for q even and $(q + 1)/2$ for q odd;*
- (d) *for q odd, a single class of $q(q^2 - 1)/(4d)$ dihedral groups of order $2d$ for each divisor d of $(q - 1)/2$ with $(q - 1)/(2d)$ odd;*

- (e) for q odd, two classes each of $q(q^2 - 1)/(8d)$ dihedral groups of order $2d$ for each divisor $d > 2$ of $(q - 1)/2$ with $(q - 1)/(2d)$ even;
- (f) for q even, a single class of $q(q^2 - 1)/(2d)$ dihedral groups of order $2d$ for each divisor d of $q - 1$;
- (g) for q odd, a single class of $q(q^2 - 1)/(4d)$ dihedral groups of order $2d$ for each divisor d of $(q + 1)/2$ with $(q + 1)/(2d)$ odd;
- (h) for q odd, two classes each of $q(q^2 - 1)/(8d)$ dihedral groups of order $2d$ for each divisor $d > 2$ of $(q + 1)/2$ with $(q + 1)/(2d)$;
- (i) for q even, a single class of $q(q^2 - 1)/(2d)$ dihedral groups of order $2d$ for each divisor d of $q + 1$;
- (j) a single class of $q(q^2 - 1)/24$ conjugate four-groups when $q \equiv \pm 3 \pmod{8}$;
- (k) two classes each of $q(q^2 - 1)/48$ conjugate four-groups when $q \equiv \pm 1 \pmod{8}$;
- (l) a number of classes of conjugate abelian groups of order q_0 for each divisor q_0 of q ;
- (m) a number of classes of conjugate groups of order $q_0 d$ for each divisor q_0 of q and for certain d depending on q_0 , all lying inside a group of order $q(q - 1)/2$ for q odd and $q(q - 1)$ for q even;
- (m) two classes each of $[q(q^2 - 1)]/[2q_0(q_0^2 - 1)]$ groups $PSL_2(q_0)$, where q is an even power of q_0 , for q odd;
- (o) a single class of $[q(q^2 - 1)]/[q_0(q_0^2 - 1)]$ groups $PSL_2(q_0)$, where q is an odd power of q_0 , for q odd;
- (p) a single class of $[q(q^2 - 1)]/[q_0(q_0^2 - 1)]$ groups $PSL_2(q_0)$, where q is a power of q_0 , for q even;
- (q) two classes each of $[q(q^2 - 1)]/[2q_0(q_0^2 - 1)]$ groups $PGL_2(q_0)$, where q is an even power of q_0 , for q odd;
- (r) two classes each of $q(q^2 - 1)/48$ conjugate S_4 when $q \equiv \pm 1 \pmod{8}$;
- (s) two classes each of $q(q^2 - 1)/48$ conjugate A_4 when $q \equiv \pm 1 \pmod{8}$;
- (t) a single class of $q(q^2 - 1)/24$ conjugate A_4 when $q \equiv \pm 3 \pmod{8}$;
- (u) a single class of $q(q^2 - 1)/12$ conjugate A_4 when q is an even power of 2;
- (v) two classes each of $q(q^2 - 1)/120$ conjugate A_5 when $q \equiv \pm 1 \pmod{10}$;

After close study of the inclusions here, we can write down the following corollary. We exclude the cases $q = 2, 3$ where $PSL_2(q)$ is not simple.

Corollary 2.2 *The maximal subgroups of $PSL_2(q)$ are as follows:*

- (a) *dihedral groups of order $q - 1$ for $q \geq 13$ odd and $2(q - 1)$ for q even: each stabilizes a pair of points (a hyperbolic quadric);*
- (b) *dihedral groups of order $q + 1$ for $q \neq 7, 9$ odd and $2(q + 1)$ for q even: each stabilizes a pair of imaginary points (i.e., points of $PG(1, q^2)$ that don't lie in $PG(1, q)$, conceivably thought of as an elliptic quadric);*
- (c) *a group of order $q(q - 1)/2$ for q odd and $q(q - 1)$ for q even: each stabilizes a point;*
- (d) *$PSL_2(q_0)$, where q is an odd prime power of q_0 , for q odd, or a prime power of q_0 , for q even: each stabilizes a sub-line;*
- (e) *$PGL_2(q_0)$, where $q = q_0^2$, for q odd: each stabilizes a sub-line;*
- (f) *S_4 when $q \equiv \pm 1 \pmod{8}$, with either q prime, or $q = p^2$ and $3 < p \equiv \pm 3 \pmod{8}$;*
- (g) *A_4 when $q \equiv \pm 3 \pmod{8}$, with $q > 3$ prime;*
- (h) *A_5 when $q \equiv \pm 1 \pmod{10}$, with either q prime, or $q = p^2$ and $p \equiv \pm 3 \pmod{10}$.*

A second corollary is the enumeration of maximal subgroups of $PGL_2(q)$ for q odd, because this group lies inside $PSL_2(q^2)$. We do not need to consider q even because in that case $PGL_2(q) = PSL_2(q)$.

Corollary 2.3 *For $q > 3$ odd, the maximal subgroups of $PGL_2(q)$ are as follows:*

- (a) *dihedral groups of order $2(q - 1)$ for $q > 5$ (the stabilizer of a pair of points);*
- (b) *dihedral groups of order $2(q + 1)$ (the stabilizer of a pair of imaginary points);*
- (c) *a group of order $q(q - 1)$ (the stabilizer of a point);*
- (d) *$PSL_2(q)$;*
- (e) *$PGL_2(q_0)$, where q is a prime power of q_0 (the stabilizer of a sub-line);*
- (f) *S_4 when $q \equiv \pm 3 \pmod{8}$, with $q > 3$ prime;*

2.2 $n = 3$

The collineation groups of the finite projective plane were determined by Mitchell in [42] for q is odd and later by Hartley in [26] for q is even. Mitchell's treatment is perhaps more geometrical than Dickson's in the previous section because there is more room. He defines five canonical transformations: fixing a triangle, fixing two points and two lines, fixing a line and a point on the line, fixing all points on a line and all lines through a point off the line (i.e., an homology), and fixing all points on a line and all lines through a point on the line (i.e., an elation). He then considers in detail how such transformations may lie in a group, alone or in combination. His

theorem picks out the maximal subgroups and goes a long way towards identifying all other subgroups. Hartley (a student of Mitchell) employed a similar approach for even q , but phrases his theorem slightly differently. Mitchell and Hartley use the term *hyperorthogonal group* for $PSU_3(q)$ and write it as $HO(3, q)$.

Theorem 2.4 (Mitchell) *Suppose that q is odd. The following is a list of subgroups of $PSL_3(q)$ (with μ the hcf of 3 and $q - 1$). A subgroup of $PSL_3(q)$ either fixes a point, a line or a triangle (so is a subgroup of (a), (b) or (c) below), or is one of the groups in (d)-(k):*

- (a) *the stabilizer of a point, having order $q^3(q + 1)(q - 1)^2/\mu$;*
- (b) *the stabilizer of a line, having order $q^3(q + 1)(q - 1)^2/\mu$;*
- (c) *the stabilizer of a triangle, having order $6(q - 1)^2/\mu$;*
- (d) *the stabilizer of an imaginary triangle (i.e., a triangle with co-ordinates in $GF(q^3)$), having order $3(q^2 + q + 1)/\mu$;*
- (e) *the stabilizer of a conic, having order $q(q^2 - 1)$;*
- (f) *$PSL_3(q_0)$, where q is a power of q_0 ;*
- (g) *$PGL_3(q_0)$, where q is a power of q_0^3 and 3 divides $q_0 - 1$;*
- (h) *$PSU_3(q_0^2)$, where q is a power of q_0^2 ;*
- (i) *$PU_3(q_0^2)$, where q is a power of q_0^6 and 3 divides $q_0 + 1$;*
- (j) *the Hessian groups of orders 216 (where 9 divides $q - 1$), 72 and 36 (where 3 divides $q - 1$);*
- (k) *groups of order 168 (when -7 is a square in $GF(q)$), 360 (when 5 is a square in $GF(q)$ and there is a non-trivial cube root of unity), 720 (when q is an even power of 5), and 2520 (when q is an even power of 5).*

The groups of orders 168, 360, 720 and 2520 are isomorphic to $PSL_3(2)$, A_6 , $A_{6.2}$ and A_7 respectively, each is almost simple. The groups of orders 216 and 72 are of symplectic type and are isomorphic to $PU_3(4)$ and $PSU_3(4)$ respectively, with the group of order 36 a subgroup of the latter. The stabilizer of an imaginary triangle is the normalizer of a Singer cyclic subgroup.

Theorem 2.5 (Hartley) *When q is even, the maximal subgroups of $PSL_3(q)$ are given in the following list, where μ is the hcf of 3 and $q - 1$ and is thus 1 precisely when q is non-square:*

- (a) *the stabilizer of a point, a line, a triangle or an imaginary triangle, as for q odd (and with the same orders);*
- (b) *$PSL_3(q_0)$, where q is a prime power of q_0 ;*
- (c) *$PGL_3(q_0)$, when $q = q_0^3$ and q_0 is square;*

- (d) $PSU_3(q)$, where q is square;
- (e) $PU_3(q_0^2)$, where $q = q_0^6$ and q_0 is non-square;
- (f) groups of order 360 (when $q = 4$).

The group of order 360 is isomorphic to A_6 .

Having broadly identified all the subgroups of $PSL_3(q)$ for q odd, Mitchell is able to pick out the subgroups that lie in $PSU_3(q)$.

Theorem 2.6 (Mitchell) *Suppose that q_0 is odd and $q = q_0^2$. The following is a list of subgroups of $PSU_3(q)$ (with ν the hcf of 3 and $q_0 + 1$). A subgroup of $PSU_3(q)$ either fixes a point and a line, or a triangle (so is a subgroup of (a), (b) or (c) below), or is one of the groups in (d)-(i):*

- (a) the stabilizer of the centre and axis of a group of elations (i.e., the stabilizer of a point on $\mathcal{H}(2, q)$ together with its polar line), having order $q_0^3(q_0 + 1)(q_0 - 1)/\nu$;
- (b) the stabilizer of the centre and axis of an homology (i.e., a point not on $\mathcal{H}(2, q)$ together with its polar line), having order $q_0(q_0 + 1)^2(q_0 - 1)/\nu$;
- (c) the stabilizer of a triangle, having order $6(q_0 - 1)^2/\nu$;
- (d) the stabilizer of an imaginary triangle, having order $3(q_0^2 - q + 1)/\nu$;
- (e) the stabilizer of a conic, having order $q_0(q_0^2 - 1)$;
- (f) $PSU_3(q_1)$, where q is an odd power of q_1 ;
- (g) $PU_3(q_1)$, where q is an odd power of q_1^3 and 3 divides $\sqrt{q_1} + 1$;
- (h) the Hessian groups of orders 216 (where 9 divides $q_0 + 1$), 72 and 36 (where 3 divides $q_0 + 1$);
- (i) groups of order 168 (when -7 is a non-square in $GF(q_0)$), 360 (when 5 is a square in $GF(q_0)$ but there is not a non-trivial cube root of unity), 720 (when q_0 is an odd power of 5), and 2520 (when q_0 is an odd power of 5).

As with $PSL_3(q)$, Hartley only lists the maximal subgroups of $PSU_3(q)$.

Theorem 2.7 (Hartley) *Suppose that q_0 is even and $q = q_0^2$. The maximal subgroups of $PSU_3(q)$ are given in the following list (where ν is the hcf of 3 and $q_0 + 1$):*

- (a) the stabilizer of a point and a line, a triangle or an imaginary triangle, as for q odd (and with the same orders);
- (b) $PSU_3(q_1)$, where q is an odd prime power of q_1 ;
- (c) $PU_3(q_1)$, where $q = q_1^3$ and $\sqrt{q_1}$ is non-square;
- (d) groups of order 36 (when $q = 4$).

2.3 $n = 4$

In considering $n = 4$, Mitchell's approach to $PSL_3(q)$ proved difficult to extend to $PSL_4(q)$. However he did manage to deal with the slightly smaller group $PSp_4(q)$, identifying all the maximal subgroups. In Dickson's notation, this is the group $A(4, p^e)$.

Theorem 2.8 (Mitchell) *Assume that $p > 2$. The maximal subgroups of $PSp_4(q)$ are as follows:*

- (a) $PSp_4(q_0)$, where q is an odd prime power of q_0 ;
- (b) $PGSp_4(q_0)$, where $q = q_0^2$;
- (c) the stabilizer of a point and a plane, having index $q^3 + q^2 + q + 1$;
- (d) the stabilizer of a parabolic congruence, having index $q^3 + q^2 + q + 1$;
- (e) the stabilizer of a hyperbolic congruence, having index $q^2(q^2 + 1)/2$;
- (f) the stabilizer of an elliptic congruence, having index $q^2(q^2 - 1)/2$;
- (g) the stabilizer of a quadric, having index $q^3(q^2 + 1)(q + 1)/2$ (for $q > 3$);
- (h) the stabilizer of a quadric, having index $q^3(q^2 + 1)(q - 1)/2$ (for $q > 3$);
- (i) the stabilizer of a twisted cubic, having index $q^3(q^4 - 1)$ (for $p > 3, q > 7$);
- (j) groups of orders 1920 (for q prime and $\equiv \pm 1 \pmod{8}$), 960 (for q prime and $\equiv \pm 3 \pmod{8}$), 720 (for q prime and $\equiv \pm 1 \pmod{12}$), 360 (for q prime, $\neq 7$ and $\equiv \pm 5 \pmod{12}$) and 2520 (for $q = 7$).

The groups of orders 1920 and 960 are groups of symplectic type. The groups of orders 360, 720 and 2520 are isomorphic to A_6 , $A_6.2$ and A_7 respectively. A parabolic congruence consists of a self-polar line together with all of the self-polar lines that meet it, so the stabilizer of a parabolic congruence is the stabilizer of a self-polar line. A hyperbolic congruence is the set of self-polar lines that meet a pair of skew polar lines, so the stabilizer of a hyperbolic congruence is the stabilizer of a pair of skew polar lines. An elliptic congruence is a regular spread. The first quadric listed is determined by a pair of skew self-polar lines that is stabilized by the subgroup. The stabilizer of the second quadric in $Sp_4(q)$ has structure $U_2(q^2).2$. These geometric configurations can be described nicely in terms of the Klein quadric that we discuss in the next section. Indeed Hirschfeld defines these configurations in this manner in [27].

Mwene published two papers on the maximal subgroups of $PSL_4(q)$. The first ([45]), on even q , is geometric and group-theoretic on equal measure. The second ([46]), on odd q , is much more algebraic, without a complete determination of maximal subgroups: we don't list the results here.

Theorem 2.9 (Mwene) *The maximal subgroups of $PSL_4(q)$ for q even are as follows:*

- (a) *the stabilizer of a point, having order $q^6(q^3 - 1)(q^2 - 1)(q - 1)$;*
- (b) *the stabilizer of a plane, having order $q^6(q^3 - 1)(q^2 - 1)(q - 1)$;*
- (c) *the stabilizer of a line, having order $q^6(q - 1)^3(q + 1)^2$;*
- (d) *the stabilizer of a tetrahedron, having order $24(q - 1)^3$ (for $q > 4$);*
- (e) *the stabilizer of a pair of mutually skew lines, having order $2q^2(q - 1)^3(q + 1)^2$ (for $q > 2$);*
- (f) *the stabilizer of a pair of mutually skew imaginary lines, that is, lines in $PG(3, q^2)$, having order $2q^2(q^2 - 1)(q^2 + 1)(q + 1)$;*
- (g) *$PSL_4(q_0)$ where q is a prime power of q_0 ;*
- (h) *$PSp_4(q)$;*
- (i) *$PSU_4(q)$ when q is a square;*
- (j) *A_7 (for $q = 2$).*

2.4 Klein quadric and exceptional isomorphisms

Lines of $PG(3, q)$ are represented as points of $PG(5, q)$. This is a particular case of a construction of the Grassmannian of lines. The representation can be given explicitly in terms of Plücker co-ordinates: given a co-ordinate system for $PG(3, q)$, choose any two points, (x_0, x_1, x_2, x_3) and (y_0, y_1, y_2, y_3) , on a given line and calculate the values $p_{ij} = x_i y_j - x_j y_i$; the point $(p_{01}, p_{02}, p_{03}, p_{12}, p_{31}, p_{23})$ lies on the quadric \mathcal{K} given by $X_0 X_5 + X_1 X_4 + X_2 X_3 = 0$. The fundamental property of this representation of lines of $PG(3, q)$ is that two lines intersect if and only if the corresponding points lie on a line on \mathcal{K} . A consequence is that a plane on \mathcal{K} corresponds either to the set of lines of $PG(3, q)$ passing through a point or to the set of lines of $PG(3, q)$ lying in a plane, and we thus have two families of planes on \mathcal{K} ; two planes in one family are either equal or meet in a point; two planes from different families are either disjoint or meet in a line. From this it can be deduced that a line on \mathcal{K} corresponds to the set of lines of $PG(3, q)$ lying in a plane and passing through a common point.

The geometric construction leads to a group isomorphism between $PSL_4(q)$ and $P\Omega_6^+(q)$. It is possible to examine the Plücker co-ordinate construction and find that the lines of a general linear complex of $PG(3, q)$ correspond to points of \mathcal{K} lying in a non-isotropic hyperplane of $PG(5, q)$, and from this there arises an isomorphism between $PSp(4, q)$ and $P\Omega(5, q)$. There is a natural embedding of $PSL_2(q^2)$ in $PSp_4(q)$ permuting a regular spread ($q^2 + 1$ lines) of the general linear complex; it turns out that these lines correspond to points of \mathcal{K} lying in a 3-dimensional subspace of $PG(5, q)$, and this leads to an isomorphism between $PSL_2(q^2)$ and $P\Omega_4^-(q)$. If q is a square, and we consider just the lines of a Hermitian surface $\mathcal{H}(3, q)$, then the corresponding points lie in a subgeometry $PG(5, \sqrt{q})$ of $PG(5, q)$

and the intersection of $PG(5, \sqrt{q})$ with $\mathcal{Q}^+(5, q)$ is an elliptic quadric $\mathcal{Q}^-(5, \sqrt{q})$. From this there arises an isomorphism between $PSU_4(q)$ and $P\Omega_6^-(\sqrt{q})$. Finally for q odd, consider a regulus \mathcal{R} in $PG(3, q)$ together with its opposite regulus \mathcal{R}' . Then the lines in \mathcal{R} correspond to $q+1$ points on \mathcal{K} that are pairwise non-orthogonal but that are all orthogonal to the $q+1$ points corresponding to the lines in \mathcal{R}' . It follows that \mathcal{R} and \mathcal{R}' determine orthogonal, non-degenerate planes, π and π' of $PG(5, q)$. The subgroup of $PSL_4(q)$ that permutes the lines in \mathcal{R} but fixes each line in \mathcal{R}' acts as $PSL_2(q)$, but at the same time the corresponding subgroup of $P\Omega_6^+(q)$ acts as the identity on π' and as $P\Omega(3, q)$ on π : $PSL_2(q)$ is isomorphic to $P\Omega_3(q)$.

Returning for a moment to $PSp_4(q)$, we have noted that it is the group of a general linear complex whose lines correspond to points of \mathcal{K} lying in a non-isotropic hyperplane of $PG(5, q)$. Let us suppose that P is the point of $PG(5, q) \setminus \mathcal{K}$ such that the hyperplane is P^\perp , and write $\mathcal{Q}_4 = P^\perp \cap \mathcal{K}$ (a parabolic quadric). A parabolic congruence is the set of lines in $PG(3, q)$ corresponding to the points of \mathcal{Q}_4 polar to a point of \mathcal{Q}_4 . Consider a point $R \notin \mathcal{K}$ such that the line PR is non-isotropic. If PR is a secant line, then it contains two non-polar points of \mathcal{K} neither of which lie in \mathcal{Q}_4 , so they correspond to polar non-isotropic lines of $PG(3, q)$; the points of $\mathcal{Q}_4 \cap R^\perp$ are polar to both points of PR and the corresponding lines of $PG(3, q)$ form a hyperbolic congruence. If PR is an external line, then $\mathcal{Q}_4 \cap R^\perp$ is an elliptic quadric with q^2+1 pairwise non-polar points, so corresponds to q^2+1 skew self-polar lines of $PG(3, q)$, i.e., a regular spread. If l is a secant line in P^\perp , then $l^\perp \cap \mathcal{Q}_4$ consists of $q+1$ points of a conic, each polar to the two points of \mathcal{Q}_4 lying in l , so they correspond to $q+1$ self-polar lines of $PG(3, q)$, each meeting a pair of skew, self-polar lines: the $q+3$ lines described all lie on a quadric in $PG(3, q)$. If l is an external line in P^\perp , then $l^\perp \cap \mathcal{Q}_4$ again consists of $q+1$ points of a conic, but no longer polar to a pair of points in \mathcal{Q}_4 ; however, if one worked over $GF(q^2)$ it would be possible to see these points as corresponding to lines on a quadric in $PG(3, q^2)$ and so there is a sense in which the stabilizer in $PSp_4(q)$ of this set of lines stabilizes a quadric.

If we now look at the results of Dickson (for $n=2$) and Mitchell (for $n=4$) we can deduce the following results.

Corollary 2.10 *The maximal subgroups of $P\Omega_3(q)$ ($q > 3$ odd) acting on a conic \mathcal{C} are as follows:*

- (a) *the stabilizer of a point external to \mathcal{C} for $q \geq 13$;*
- (b) *the stabilizer of a point internal to \mathcal{C} for $q \neq 7, 9$;*
- (c) *the stabilizer of a point of \mathcal{C} ;*
- (d) *$P\Omega_3(q_0)$, where q is an odd prime power of q_0 ;*
- (e) *$PSO_3(q_0)$, where $q = q_0^2$;*
- (f) *S_4 when $q \equiv \pm 1 \pmod{8}$, with either q prime, or $q = p^2$ and $3 < p \equiv \pm 3 \pmod{8}$;*
- (g) *A_4 when $q \equiv \pm 3 \pmod{8}$, with $q > 3$ prime;*

- (h) A_5 when $q \equiv \pm 1 \pmod{10}$, with either q prime, or $q = p^2$ and $p \equiv \pm 3 \pmod{10}$.

Corollary 2.11 *The maximal subgroups of $P\Omega_4^-(q)$ are as follows:*

- (a) *the stabilizer of a secant line for $q > 3$;*
 (b) *the stabilizer of a spread of lines, half of which are secant and the other half external;*
 (c) *the stabilizer of a point on the elliptic quadric;*
 (d) $P\Omega_4^-(q_0)$, *where q is an odd prime power of q_0 ;*
 (e) *the stabilizer of a point off \mathcal{Q} ;*
 (f) S_4 *when q is prime and $3 < q \equiv \pm 3 \pmod{8}$;*
 (g) A_5 *when q is prime and $q \equiv \pm 3 \pmod{10}$.*

$PSL_2(q^2)$ acts on the $q^2 + 1$ lines of the regular spread of $PG(3, q)$ as though the lines were points of $PG(1, q^2)$, so a pair of lines in the spread corresponds to a pair of non-orthogonal points on \mathcal{Q} , i.e., to a secant line (and its orthogonal complement which is an external line).

A point of \mathcal{Q} corresponds to a line of the spread.

Corollary 2.12 *The maximal subgroups of $P\Omega_5(q)$ for odd q are as follows:*

- (a) $P\Omega_5(q_0)$, *where q is an odd prime power of q_0 ;*
 (b) $PSO_5(q_0)$, *where $q = q_0^2$;*
 (c) *the stabilizer of a line on \mathcal{Q} ;*
 (d) *the stabilizer of a point on \mathcal{Q} ;*
 (e) *the stabilizer of a point off \mathcal{Q} orthogonal to a hyperbolic quadric;*
 (f) *the stabilizer of a point off \mathcal{Q} orthogonal to an elliptic quadric;*
 (g) *the stabilizer of a secant line (for $q > 3$);*
 (h) *the stabilizer of an external line (for $q > 3$);*
 (i) *the stabilizer of a partial ovoid of size $q + 1$ (for $p > 3, q > 7$), isomorphic to $PSL_2(q)$;*
 (j) *groups of orders 1920 (for q prime and $\equiv \pm 1 \pmod{8}$), 960 (for q prime and $\equiv \pm 3 \pmod{8}$), 720 (for q prime and $\equiv \pm 1 \pmod{12}$), 360 (for q prime, $\neq 7$ and $\equiv \pm 5 \pmod{12}$) and 2520 (for $q = 7$).*

The subgroups of orders 1920 and 960 stabilize a pentagon of pairwise orthogonal points off \mathcal{Q} . A line on \mathcal{Q} corresponds to a point together with its polar plane in $PG(3, q)$, a point on \mathcal{Q} corresponds to a self-polar line in $PG(3, q)$ which in turn determines a parabolic congruence, a point off \mathcal{Q} corresponds to a hyperbolic or elliptic congruence, a secant line contains two points of \mathcal{Q} and corresponds to a pair of skew self-polar lines of $PG(3, q)$, an external line is orthogonal to a conic that corresponds to a partial spread of $PG(3, q)$ determined by a subgroup $U_2(q^2)$ of $PSp_4(q)$.

Corollary 2.13 *The maximal subgroups of $P\Omega_6^+(q)$ for q even are as follows:*

- (a) *the stabilizer of a plane on \mathcal{K} ;*
- (b) *the stabilizer of a point on \mathcal{K} ;*
- (c) *the stabilizer of three pairwise orthogonal secant lines spanning $PG(5, q)$ (for $q > 4$);*
- (d) *the stabilizer of a secant line (for $q > 2$);*
- (e) *the stabilizer of an external line;*
- (f) *$P\Omega_6^+(q_0)$ where q is a prime power of q_0 ;*
- (g) *the stabilizer of a point off \mathcal{K} ;*
- (h) *$P\Omega_6^-(q_0)$ where $q = q_0^2$;*
- (i) *A_7 (for $q = 2$).*

A plane on \mathcal{K} corresponds to either a point or a plane of $PG(3, q)$, a point on \mathcal{K} corresponds to a line of $PG(3, q)$, the three pairwise orthogonal secant lines each contain two points of \mathcal{K} corresponding to opposite lines of a tetrahedron in $PG(3, q)$, a secant line contains two points of \mathcal{K} that correspond to a pair of skew lines of $PG(3, q)$, the stabilizer of an external line fixes a pair of points on $\mathcal{Q}_6^+(5, q^2)$ (i.e., imaginary points), the stabilizer of a point off \mathcal{K} corresponds to $PSp_4(q)$, $P\Omega_6^-(q_0)$ is isomorphic to $PSU_4(q)$.

3 Aschbacher's Theorem

In the early 1980s, several papers appeared determining some classes of maximal subgroups of classical groups. We shall refer to these later. In 1984, Michael Aschbacher published his paper, "On the maximal subgroups of the finite classical groups" ([1]), in which he identified eight classes of subgroups $\mathcal{C}_1 - \mathcal{C}_8$ and showed that an arbitrary subgroup either lies in a maximal subgroup belonging to one of $\mathcal{C}_1 - \mathcal{C}_8$ or satisfies several significant constraints. All but one of the classes can be described geometrically, but the proof of Aschbacher's Theorem is group-theoretic. It is worth spending a short time on the shape of the proof and the way in which the eight classes arise.

In the statement of Aschbacher's Theorem, the groups studied are *almost simple*, meaning that they lie between a simple group and its automorphism group, and this

means that the classical groups in this context are subgroups of $P\Gamma_n(q)$. However the description of the classes of subgroups is substantially in terms of matrix groups acting on vector spaces. For the purposes of exposition, we shall regard the classical group G as being one of $SL_n(q)$, $Sp_n(q)$, $\Omega_n(q)$ and $SU_n(q)$, and we shall write \bar{G} for the projective equivalent. Thus \bar{G} is non-abelian simple except for: $PSL_2(2) = PSp_2(2)$, $PSL_2(3) = PSp_2(3)$, $PSp_4(2)$, $P\Omega_2(q)$, $P\Omega_3(3)$, $P\Omega_4^+(q)$ and $PSU_3(4)$. These cases are thus omitted from the following discussion. We shall consider a subgroup M of G which is assumed to contain the centre of G . We shall write A for the non-degenerate alternating form given by $A(x, y) = xJ_Sy^T$ preserved by $Sp_n(q)$ (with J_S as in the introduction) and C for the non-degenerate Hermitian form given by $C(x, y) = x\bar{y}^T$ preserved by $SU_n(q)$; the non-degenerate symmetric bilinear form B and the quadratic form Q preserved by $\Omega_n(q)$ are as given in the introduction. Given any subspace U of $V = V(n, q)$, the *orthogonal complement* (with respect to a form $(,)$) is given by $U^\perp = \{v \in V : (u, v) = 0 \text{ for all } u \in U\}$ (in projective terms, this is the *polar space*). To say that a form is non-degenerate simply means that $V^\perp = \{0\}$. A subspace U is said to be *totally isotropic* (self-polar in projective terms) if $U \subseteq U^\perp$ and *non-isotropic* if $U \cap U^\perp = \{0\}$; when we have a quadratic form, U is *totally singular* if $Q(u) = 0$ for all $u \in U$, we shall use this term to be synonymous with ‘totally isotropic’ for the alternating and Hermitian forms.

The first possibility is that M is reducible, i.e., it stabilizes (globally) a non-trivial proper subspace U . If G has an associated form, then M also stabilizes U^\perp and thus $U \cap U^\perp$. It follows that M lies in the stabilizer of a smaller subspace unless $U \cap U^\perp = U$ or $\{0\}$. This means that we need only consider totally isotropic or non-isotropic subspaces. In the case of $\Omega_n(q)$ with q even, a totally isotropic subspace that is not totally singular has a unique totally singular subspace of codimension 1 that would also be stabilized by M , and that means that we need only consider 1-dimensional ‘non-singular’, totally isotropic subspaces. There is a further complication when U is non-isotropic, where there is the possibility that G contains elements that switch U and U^\perp ; in this case U is said to be *isometric* to U^\perp and the stabilizer of U is properly contained in the stabilizer of $\{U, U^\perp\}$. The class \mathcal{C}_1 consists of subgroups that stabilize a non-isotropic subspace not isometric to its orthogonal complement, a totally singular subspace or a 1-dimensional non-singular, totally isotropic subspace.

If M is not reducible, then clearly it is irreducible. One might consider the possibility that M is irreducible but not absolutely irreducible, but it turns out that it is more fruitful to consider a normal subgroup N of M .

It is possible that M is irreducible, but that there is a non-trivial subgroup N that is reducible (non-trivial here means that N is not contained in the centre of G). Two quite different possibilities arise. The first is that we can choose an appropriate subspace U stabilized by N and find that the images of U under M form a set whose direct sum is V . In this case M stabilizes this set of subspaces and is an *imprimitive* subgroup of G . If G has an associated form, then the subspaces are all either non-isotropic and pairwise orthogonal, or totally singular with n even and such that U can be chosen to have dimension $n/2$. The class \mathcal{C}_2 consists of the stabilizers of such direct sum decompositions. The second possibility is that if we choose U as small as possible (so that N acts irreducibly on U), say of dimension m , then V can be expressed as the direct sum of subspaces $U_1, U_2, \dots, U_{n/m}$ fixed by N in such a way that the elements of N can be written as block diagonal matrices $diag(P, P, \dots, P)$

with P an $m \times m$ matrix. Let F be the set of $m \times m$ matrices that commute with all the P s arising from N . Then, by Schur's Lemma, F is a division ring and then, by Wedderburn's Theorem, F is a field. We can see that F effectively contains $GF(q)$ as the set of scalar matrices: let r be the degree of F over $GF(q)$. If $r > 1$, then $N \leq SL_{n/r}(F)$ and M can be shown to normalize $SL_{n/r}(F)$; we find here that F lies in the class \mathcal{C}_3 that we describe more fully in the next paragraph. If $r = 1$, then the matrices in M have the form $A \otimes B$, where $A \in GL_{n/m}(q)$ and $B \in GL_m(q)$, so that M stabilizes a tensor product structure. If there is a form on V , then each factor of the tensor product admits a form. In terms of maximal subgroups, it is only necessary to consider groups arising from $GL_{n_1} \otimes GL_{n_2}$ ($n_1 < n_2$), $U_{n_1} \otimes U_{n_2}$ ($n_1 < n_2$), $Sp_{n_1} \otimes O_{n_2}^\epsilon$ (q odd, $n_2 \geq 3$), $Sp_{n_1} \otimes Sp_{n_2}$ ($n_1 < n_2$) and $O_{n_1}^{\epsilon_1} \otimes O_{n_2}^{\epsilon_2}$ ($(n_1, \epsilon_1) \neq (n_2, \epsilon_2)$, q odd, $n_2 \geq 3$) as subgroups of $SL_n(q)$, $SU_n(q)$, $Sp_n(q)$, $\Omega_n(q)$ and $\Omega_n^{\epsilon}(q)$ respectively; some of these restrictions are present in order to avoid the possibility of a larger group that includes switching of the factors. The class of groups just described is \mathcal{C}_4 . In geometric terms, the structure in $PG_{n-1}(q)$ on which such groups act is a *Segre variety*, sometimes written S_{n_1, n_2} .

It is possible that M is irreducible, and that there is a non-trivial subgroup N that is irreducible but not absolutely irreducible (this includes the case where M is not absolutely irreducible). If we consider the set C of $n \times n$ matrices that commute with N (in the manner of the previous paragraph), then C is a field containing $GF(q)$. The group N is *absolutely irreducible* if $C = GF(q)$. In the current setting, where N is not absolutely irreducible, C is isomorphic to $GF(q^r)$ for some $r \geq 2$ and $m = n/r$ is an integer; if an appropriate basis is chosen for V , then the elements in N become $m \times m$ block matrices, where the blocks are $r \times r$ matrices representing elements of $GF(q^r)$. Conjugation by elements of M then corresponds to conjugation by an element of N , extended by an automorphism of $GF(q^r)$ that fixes $GF(q)$. In other words M lies in $(G \cap GL_{n/r}(q^r)).Gal_{GF(q)}(GF(q^r))$. If r is composite, say $r = st$ with $s, t > 1$, then M lies inside the larger group $(G \cap GL_{n/s}(q^s)).Gal_{GF(q)}(GF(q^s))$. Hence there is a restriction that r must be prime; we now have the class \mathcal{C}_3 that preserve an overfield structure. In geometric terms, if $n > r$ then V can be viewed as an n/r -dimensional vector space over $GF(q^r)$, the set of 1-dimensional subspaces of which become a set of r -dimensional subspaces of $V(n, q)$ preserved by M . Thus M stabilizes a spread of $(n/r) - 1$ -subspaces of $PG(n - 1, q)$. If $n = r$, then $GL_{n/r}(q^r)$ is just the (cyclic) multiplicative group of $GF(q^n)$, in other words it is the Singer cyclic subgroup of $GL_n(q)$; in this case M normalizes the Singer cyclic subgroup of G .

If M is irreducible, then the remaining possibility is that every non-trivial normal subgroup N is absolutely irreducible. In this case we can choose N such that \bar{N} (the image of N in \bar{G}) is a minimal normal subgroup of \bar{M} . It is known that such a subgroup \bar{N} is one of: non-abelian simple; a product of isomorphic non-abelian simple groups; an elementary abelian r -group for some prime r distinct from p . Let us take the elementary abelian r -group case: this is the case where there appears little geometric structure in $PG(n - 1, q)$, despite the name (N is an extraspecial r -group of symplectic type); this is the class \mathcal{C}_6 .

If \bar{N} is a product of (at least two) isomorphic non-abelian simple groups, then N becomes a central product of isomorphic groups $N_1 N_2 \dots N_r$ and each N_i has to be reducible. The setting is quite similar to that for \mathcal{C}_4 described above, but regarding

N as the absolutely irreducible subgroup of G and N_1 the reducible normal subgroup. We find that N preserves a tensor product structure with r factors (that are isometric when a form is present) permuted by M : this is the class \mathcal{C}_7 . In geometric terms, this class corresponds once more to a Segre variety, $S_{m,m,\dots,m}$ (where $m = (n/r) - 1$).

Finally consider the possibility that \bar{N} is non-abelian and simple. Two examples come to mind. The first is that N is defined over a subfield of $GF(q)$: these are the subgroups in the class \mathcal{C}_5 and in geometric terms are groups that preserve a sub-geometry. The other is that N cannot be defined over a subfield but that it fixes a form: this is the class \mathcal{C}_8 . These examples do not exhaust the possibilities for N , but they do significantly restrict the characteristics of such a group if it has not already been addressed. We thus arrive at Aschbacher's Theorem:

Theorem 3.1 (*Aschbacher*) *Let F_0 be a simple classical group and suppose that $F_0 \leq F \leq \text{Aut}(F_0)$. If $F_0 \cong P\Omega^+(8, q)$ assume that F contains no triality automorphism. Let H be a proper subgroup of F such that $F = HF_0$. Then either H is contained in one of $\mathcal{C}_1 - \mathcal{C}_8$ or the following hold:*

- (a) $H_0 \leq H \leq \text{Aut}(H_0)$ for some non-abelian simple group H_0 ;
- (b) Let L be the full covering group of H_0 and let V be the natural vector space on which L acts (such that the projective image of L is precisely H_0), then L is absolutely irreducible on V ;
- (c) The representation of L on V is defined over no proper subfield of $GF(q)$;
- (d) If L fixes a form on V , then F_0 is the group $PSL_n(q)$, $PSp_n(q)$, $P\Omega_n(q)$ or $PSU_n(q)$ corresponding to the form.

In geometrical terms, Aschbacher's Theorem states that a subgroup of a simple classical group acting on $PG(n-1, q)$ stabilizes one of the following configurations: a subspace, a set of m subspaces of dimension $(n/m) - 1$ spanning $PG(n-1, q)$, a spread of subspaces, a Segre variety, a sub-geometry, a polar space or a quadric, or it normalizes a Singer cycle, or it normalizes an r -group of symplectic type, or it is almost simple.

Liebeck and Seitz ([41]) have produced an alternative proof of Aschbacher's Theorem as a corollary to a version of the theorem for algebraic groups, but this does not amount to a geometric proof.

4 Progress towards a geometric classification for \mathcal{C}_1 to \mathcal{C}_8

In this section we chart the progress towards proving maximality or otherwise of subgroups in the classes \mathcal{C}_1 to \mathcal{C}_8 in a geometric manner. It is appropriate to note the work of Shangzhi Li who has obtained extensive results by matrix methods, but also to state that these are not geometrical. It is also appropriate to note that Aschbacher's Theorem itself can be used, because an overgroup of a prospective subgroup would itself either lie inside a maximal subgroup of one of the eight classes or be almost simple. This is precisely the approach taken in [40], where knowledge of the representations of finite groups of Lie type, of symmetric and alternating groups, and of sporadic simple groups enables Kleidman and Liebeck to obtain

a classification of the maximal subgroups among the classes \mathcal{C}_1 to \mathcal{C}_8 . It should be noted, however, that the statement of their result is for $n \geq 13$, with smaller cases referred to a book by Kleidman that has not, to date, been published. A statement of Kleidman's results appears in his thesis ([39]), but no proof is furnished and one or two errors have been identified. In his paper, Aschbacher identifies more restrictions than we have noted in the previous section and Kleidman and Liebeck cover similar ground; we give details, where appropriate, below. Many of the papers cited below state results over arbitrary commutative fields, but we restrict statements to finite fields. To date there have been no geometrical proofs of maximality for subgroups in \mathcal{C}_4 or \mathcal{C}_7 , the stabilizers of Segre varieties, although it is reasonable to anticipate such proofs. The normalizers of symplectic type groups (\mathcal{C}_6) are harder to fathom geometrically and they tend to be relatively small subgroups, so a complete geometric proof of maximality appears some way off (there is a partly geometric proof in one case, but it requires a deep result from Group Theory).

4.1 \mathcal{C}_1

The groups in this section are stabilizers of subspaces. We denote by G the classical group $SL_n(q)$, $Sp_n(q)$, $\Omega_n^\epsilon(q)$ or $SU_n(q)$ and by H the stabilizer of an r -dimensional subspace U of $V(n, q)$. We have already noted that in the cases $G \neq SL_n(q)$, the subspace has to be totally isotropic or non-isotropic, and that in the case $G = \Omega_n^\epsilon(q)$, the subspace has to be either totally singular, or 1-dimensional with q even. We denote the Witt dimension by ν , this being the dimension of a maximal totally singular subspace of $V(n, q)$. The maximal subgroups in this class are completely determined. The corresponding results for \tilde{G} follow very easily from the theorems below.

Theorem 4.1 [30] *If $G = SL_n(q)$, then H is a maximal subgroup of G .*

Theorem 4.2 [30] *If $G = Sp_n(q)$, $\Omega_n^\epsilon(q)$ or $SU_n(q)$, and if U is totally singular, then H is a maximal subgroup of G , except when $G = \Omega_n^\epsilon(q)$ and $(n, \nu) = (2, 1)$, $n = 2\nu = 2r + 2$ or $(n, \nu, q) = (4, 2, 2)$.*

In the first exceptional case, there are only two totally singular 1-dimensional subspaces and G fixes them both, so $H = G$. In the second case, U lies in two $r + 1$ -dimensional totally singular subspaces and H fixes both of them, so H lies in the stabilizer of a larger subspace. The third case is one that we exclude from our discussion because $\Omega_4^+(2)$ is not simple, although H is in fact maximal when $r = 2$. Note that in [30] results are stated for $SO_n(q)$ rather than $\Omega_n(q)$, but the methods are readily adapted to $\Omega_n(q)$.

Theorem 4.3 [31] *If $G = Sp_n(q)$, $\Omega_n^\epsilon(q)$ or $SU_n(q)$, and if U is non-isotropic but not isometric to U^\perp , then H is a maximal subgroup of G , except in the following cases:*

(a) $G = \Omega_n^\epsilon(q)$ and

$$(n, r, \nu, \nu_1, \nu_2, q) = (\geq 6, 2, \nu, 1, \nu_2, 2), (6, 2, 2, 0, 2, 2), (3, 1, 1, 0, 1, \leq 11), \\ (3, 1, 1, 0, 0, \leq 9), (4, 2, 1, 0, 1, 3), (5, 2, 2, 0, 1, 3).$$

(b) $G = SU_n(q)$ and $(n, r, q) = (3, 1, 4)$.

It happens again that [31] gives results for $SO_n(q)$ rather than $\Omega_n(q)$. However the methods can be adapted and further exceptions arise only when $n = 3$ (and here $\Omega_3(q)$ is isomorphic to $PSL_2(q)$). In fact the arguments in [31] can be simplified significantly for finite fields by noting that H is self-normalizing and generated by elements leaving fixed a subspace of codimension 2.

Theorem 4.4 [32] *If $G = \Omega_n^\epsilon(q)$ with q even and if U is non-singular with $r = 1$, then H is a maximal subgroup of G , except when $(n, q, \nu) = (4, 2, 2)$.*

In the exceptional case, there is a unique anisotropic 2-dimensional subspace W containing U , and H preserves the direct sum decomposition $W \oplus W^\perp$. Thus H lies inside a subgroup of class \mathcal{C}_2 (and inclusion is proper).

4.2 \mathcal{C}_2

In this section H is the stabilizer of a direct sum decomposition into subspaces of dimension r . If $G = Sp_n(q)$, $\Omega_n^\epsilon(q)$ or $SU_n(q)$, then the subspaces have to be isometric and either totally isotropic or non-isotropic: we write them as U_1, U_2, \dots, U_s . Moreover if U_i is totally isotropic, then $s = 2$; if U_i is non-isotropic, then U_1, U_2, \dots, U_s are pairwise orthogonal. The maximal subgroups in this class are almost completely determined and the corresponding results for \bar{G} follow very easily.

Theorem 4.5 [28, 29, 33] *If $G = SL_n(q)$, then H is a maximal subgroup of G , except when $(n, r, q) = (4, 2, 3)$, $(n, 2, 2)$, $(\text{odd} > 3, 1, \text{odd})$ or $(\geq 3, 1, 2 \text{ or } 4)$.*

Theorem 4.6 [35] *If $G = Sp_n(q)$ (with q odd), $\Omega_n^+(q)$ (with n a multiple of 4) or $SU_n(q)$, and if U_i is totally isotropic, then H is a maximal subgroup of G , except when $G = Sp_n(q)$ and $(n, q) = (4, 3)$, $G = \Omega_n^+(q)$ and $(n, q) = (4, \leq 11)$ or $(8, 2)$, and $G = SU_n(q)$ and $(n, q) = (4, 4 \text{ or } 9)$.*

We note that [35] deals with $SO_n^+(q)$ rather than $\Omega_n^+(q)$, but that the methods are readily adapted for $n > 4$. In the isomorphism from $P\Omega_4^+(q)$ to $PSL_2(q) \times PSL_2(q)$, the copies of $PSL_2(q)$ act independently on the two sets of $q+1$ totally singular lines. The subgroup H may be seen to correspond to $PSL_2(q) \times D_{q-1}$ and is therefore maximal precisely when D_{q-1} is maximal in $PSL_2(q)$. We should also explain that when $G = Sp_n(q)$ and q is even, H lies inside an orthogonal group, and when $G = \Omega_n^+(q)$ with r odd, H stabilizes each of U_1 and U_2 .

Theorem 4.7 [31, 33, 34, 36, 24] *If $G = Sp_n(q)$, $\Omega_n^\epsilon(q)$ or $SU_n(q)$, and $r \geq 2$, and if each U_i is non-isotropic, then H is a maximal subgroup of G , except when*

- (a) $G = Sp_n(q)$ and $(n, r, q) = (4, 2, 3)$ or $(n, 2, 2)$;
- (b) $G = \Omega_n^\epsilon(q)$ and $(r, \nu_1, q) = (2, 1, \leq 5)$, $(3, 1, 3)$, or $(4, 2, 2)$, $(n, r) = (4, 2)$, or $(r, \nu_1, q) = (2, 0, 3)$.
- (c) $G = SU_n(q)$ and $(r, q) = (2, 4)$.

We have excluded here the stabilizer of a simplex (i.e., when $r = 1$) when $G = \Omega_n^\epsilon(q)$ or $SU_n(q)$. In [24], Dye has results for $O_n(q)$ and $U_n(q)$ but the methods are not readily applicable when $\Omega_n^\epsilon(q)$ and $SU_n(q)$. In [31] and [34] (but not in [36]) results are stated for $SO_n(q)$ rather than $\Omega_n(q)$, but again the methods are readily adapted to $\Omega_n(q)$.

4.3 \mathcal{C}_3

In this section we describe the subgroups in terms of their actions on $PG(n-1, q)$: each stabilizes a spread of subspaces. Throughout the section, $n = rm$ with r prime and $m \geq 2$, and H is the stabilizer of a spread of $r-1$ -spaces. Complete results are available for some groups but not for others. Little has been done from a geometric viewpoint regarding the normalizers of Singer cyclic subgroups.

The first group considered is $PSL_n(q)$. All maximal subgroups in the class are determined and are given by the following result.

Theorem 4.8 [20] *H is the normalizer of $PSL_m(q^r)$ and is a maximal subgroup of $PSL_n(q)$.*

Next we consider $PSp_n(q)$. There is one set of subgroups corresponding to the $PSL_n(q)$ case, but there is an additional set of subgroups arising from the embedding of $U_m(q^2)$ in $Sp_n(q)$ when $r = 2$. In the first case, the subspaces in the spread are all totally isotropic, but in the second case the spread is a mixed spread of totally isotropic and non-isotropic lines. All maximal subgroups in the class are determined and are given as follows.

Theorem 4.9 [17, 18, 19] *Suppose that n is a multiple of 4 when $r = 2$. Then H is the normalizer of $PSp_m(q^r)$ is a maximal subgroup of $PSp_n(q)$.*

Theorem 4.10 [4] *For $m \geq 2$ and $n = 2m$, the H is the image of the normalizer of $U_m(q^2)$ in $Sp_n(q)$ and is a maximal subgroup of $PSp_n(q)$ except for $(m, q) = (2, 3)$.*

Now consider $P\Omega_n(q)$. There are results when $r = 2$, although it is known (from [40] for example) that there are maximal subgroups for all prime $r > 2$ corresponding to the first set below (i.e., $\Omega_m(q^r) \leq \Omega_n(q)$). In the first result, the spread is a mixed spread of totally isotropic and non-isotropic lines. In the second result, the spread of $PG(n-1, q)$ is again a mixed spread but the totally singular lines partition the quadric so that the subgroup may be seen as the stabilizer of a spread of a quadric.

Theorem 4.11 [6] *Suppose that $n = 2m$ with $m \geq 3$ and q odd. Then H is the normalizer of $P\Omega_m(q^2)$ and is a maximal subgroup of $P\Omega_n(q)$.*

Theorem 4.12 [21] *Suppose that $n = 2m$ with $m \geq 3$. Then H is the image of the normalizer of $U_m(q^2) \cap \Omega_n(q)$ in $\Omega_n(q)$ and is a maximal subgroup of $P\Omega_n(q)$.*

There are currently no results for $PSU_n(q)$, although it is known (from [40] for example) that there are maximal subgroups for all prime $r > 2$.

4.4 \mathcal{C}_5

There are not many results for groups in this class. Here we take q_0 such that $q = q_0^r$: an immediate restriction is that r must be prime. It is known that there maximal subgroups for all $r \geq 2$ of the same type, and a result for $PSL_n(q)$ has been announced by Cossidente and Siciliano to the effect that the normalizer of $PSL_n(q_0)$ is maximal in $PSL_n(q)$. The only published results are the following, covering the only exceptions to the ‘same type’ pattern. They both arise from commuting polarities of $PG(n-1, q)$, studied extensively in [48]. In the first case, the commuting polarities are symplectic and unitary and in the second case they are orthogonal and unitary. In both cases the subgeometry consists of the points fixed by the product of the polarities.

Theorem 4.13 [7] *Suppose that n is even and $q = q_0^2$. Then the normalizer of $PSp_n(q_0)$ is a maximal subgroup of $PSU_n(q)$.*

Theorem 4.14 [5] *Suppose that $n \geq 3$ and that $q = q_0^2$ is odd. The normalizer of $P\Omega_n^\epsilon(q_0)$ is a maximal subgroup of $PSU_n(q)$, except when $(n, q) = (3, 3)$ or $(3, 5)$ and when $(n, q, \epsilon) = (4, 3, +)$.*

4.5 \mathcal{C}_8

The groups in this class are essentially stabilizers of forms, sometimes up to a scalar factor. In practice this means the normalizer of one classical group lying inside another. With one exception this amounts to studying $PSp_n(q)$, $P\Omega_n^\epsilon(q)$ or $PSU_n(q)$ as subgroups of $PSL_n(q)$. The geometric structure can be viewed as the polar space structure: the set of totally isotropic or totally singular subspaces. In the cases of $P\Omega_n^\epsilon(q)$ or $PSU_n(q)$ this equivalent to stabilizing the set of points of a quadric or Hermitian variety; in the case of $PSp_n(q)$ it is equivalent to stabilizing the self-polar lines. In the following the given subgroup is the stabilizer of the relevant polarity. The maximal subgroups in this class are completely determined.

Theorem 4.15 [23, 37, 38]

- (a) *For $n > 2$ and even, the normalizer of $PSp_n(q)$ is a maximal subgroup of $PSL_n(q)$.*
- (b) *When q is odd, the normalizer of $P\Omega_n(q)$ is a maximal subgroup of $PSL_n(q)$ except when $n = 2$ and $q \leq 11$.*
- (c) *The normalizer of $PSU_n(q)$ is a maximal subgroup of $PSL_n(q)$.*

The only other situation that needs to be addressed is when q is even and one considers $P\Omega_n^\epsilon(q) \leq PSL_n(q)$. Here the bilinear form is an alternating form and $PO_n(q)$ is a subgroup of $PSp_n(q)$, and the subgroup of $PSp_n(q)$ that we need is the stabilizer of a quadric.

Theorem 4.16 [22] *When q is even, the normalizer of $P\Omega_n^\epsilon(q)$ is a maximal subgroup of $PSp_n(q)$, except when $n = 2$ and $\epsilon = -$.*

In fact Dye only considers quadratic forms with positive Witt index. However, the only remaining case is that of the normalizer of $P\Omega_2^-(q)$, i.e., $D_{2(q+1)}$, as a subgroup of $PSL_2(q)$, and this is maximal unless $q = 2$ (where the two groups are equal).

5 The Class \mathcal{S} : subgroups not lying inside groups in $\mathcal{C}_1 - \mathcal{C}_8$

By Aschbacher's Theorem, a subgroup of a simple finite classical group that does not lie inside one of the classes \mathcal{C}_1 to \mathcal{C}_8 is almost simple and its pre-image in $SL_n(q)$ is absolutely irreducible. Such a subgroup lies in a class that Aschbacher denotes by \mathcal{S} . However, there arises within \mathcal{S} a class of subgroups that is worth picking out and labelling as \mathcal{C}_9 . Early attention paid to these groups in the context of \mathcal{S} appears in [49], although not in a geometric fashion. The existence of the groups comes out of Steinberg's Tensor Product Theorem ([50]). Nevertheless, the groups can be described relatively easily in geometric terms: they stabilize the intersection of a Segre variety and a sub-geometry. In [8], the terms *Hermitian Veronesean* and *Twisted Hermitian Veronesean* are used for these sets in the context of $PG(8, q)$.

5.1 The Class \mathcal{C}_9 : Twisted Tensor Products

Consider the group $G = GL_m(q^t)$ acting on an m -dimensional vector space V_1 over $GF(q^t)$ and let $\psi : GF(q^t) \rightarrow GF(q^t)$, $x \mapsto x^q$, be the Frobenius automorphism of $GF(q^t)$. Given a basis, x_1, x_2, \dots, x_m for V_1 , for any matrix A and any vector $v \in V_1$, we denote by A^ψ and v^ψ the matrix and vector obtained by raising each coefficient to the power q . The tensor product $V_t = V_1 \otimes V_1 \otimes \dots \otimes V_1$ (with t components) admits an action whereby $v_1 \otimes v_2 \otimes \dots \otimes v_t$ is mapped to $v_1 A \otimes v_2 A^\psi \otimes \dots \otimes v_t A^{\psi^{t-1}}$. We thus have a representation ρ of G on V_t that happens to be absolutely irreducible.

This representation of G can be written over $GF(q)$, which means to say that there is a basis for V_t with respect to which the elements of G act as matrices of $GL_n(q)$ (where $n = \dim V_t$). In order to see this, let $\phi : V_t \rightarrow V_t$ be the map given by $\lambda u_1 \otimes u_2 \otimes \dots \otimes u_t \rightarrow \lambda^q u_t \otimes u_1 \otimes \dots \otimes u_{t-1}$, with each u_i being one of x_1, x_2, \dots, x_m , extended linearly over $GF(q)$. The set V of all vectors in V_t that are fixed by ϕ is fixed by G and is a $GF(q)$ -subspace of V_t . Moreover $GF(q)$ -linearly independent vectors in V are linearly independent over $GF(q^t)$ and we conclude that V has dimension $n = m^t$ over $GF(q)$. Let $\Omega = \{1, 2, \dots, t\}$ and let $c = (1234 \dots t)$, a cyclic permutation of Ω ; we can consider the action of c on the set of partitions of Ω into m (possibly empty) subsets. For each orbit, Δ say, of c on these partitions, choose an element \mathcal{P} of Δ (i.e., a partition of Ω into m subsets, $\mathcal{O}_1, \mathcal{O}_2, \dots, \mathcal{O}_m$ say) and a vector $u = u_1 \otimes u_2 \otimes \dots \otimes u_t$, with each u_i being one of v_1, v_2, \dots, v_m and equalling v_j if and only if $i \in \mathcal{O}_j$. Let s be the length of Δ . Then the vectors $\sum_{j=1}^s \phi^{j-1}(\lambda u)$, as λ ranges over $GF(q^s)$, span a $GF(q)$ -subspace $V(\Delta)$ of V_t of dimension s , fixed by ϕ vectorwise. A basis for $GF(q^s)$ over $GF(q)$ gives rise to a basis for $V(\Delta)$. The direct sum of such subspaces gives a $GF(q)$ -subspace of dimension $n = m^t$ and this is precisely V .

This representation of G is not necessarily faithful, for scalar matrices in $GL_m(q^t)$ can act trivially on V_t . However the associated representation of $PGL_m(q^t)$ on $PG(n-1, q)$ is faithful. In geometric terms, $PG(n-1, q)$ is a subgeometry of

$PG(n-1, q^t)$ fixed by $PGL_m(q^t)$. For any $0 \neq v \in V_1$, consider a vector $\mathbf{v} = v \otimes v^\psi \otimes \cdots \otimes v^{\psi^{t-1}} \in V_t$. We see that \mathbf{v} is fixed by ϕ so lies in V . Moreover $\lambda \mathbf{v} = (\lambda + \lambda^q + \cdots + \lambda^{q^{t-1}}) \mathbf{v}$ with $\lambda + \lambda^q + \cdots + \lambda^{q^{t-1}} \in GF(q)$. Hence points of $PG(m-1, q^t)$ are mapped to points of $PG(m-1, q)$ and the image of $PG(m-1, q^t)$ is a configuration fixed by $PGL_m(q^t)$. Moreover, whenever there is a configuration of points in $PG(m-1, q^t)$ fixed by a subgroup H of $PGL_m(q^t)$, there is a corresponding configuration in $PG(m-1, q)$ fixed by H . We shall say more on possible configurations shortly. If $g \in GL_m(q^t)$ has determinant 1, then the same is true for $\rho(g) \in GL_n(q)$, so $PSL_m(q^t)$ is embedded in $PSL_n(q)$.

Suppose that f_1 is a non-degenerate form on V_1 , being one of: alternating, symmetric bilinear or Hermitian. Then f_1 determines a matrix A_1 so that $f_1(u, v) = uA_1v^T$ and we can regard f_1^ψ as being the form corresponding to A_1^ψ . In most cases the basis for V_1 can be chosen in a natural manner so that A_1 has coefficients in $GF(q)$ and then $A_1^\psi = A_1$. Now let f be the form on V_t given by

$$f(u_1 \otimes \cdots \otimes u_t, w_1 \otimes \cdots \otimes w_t) = \prod_{i=1}^t f_1^{\psi^{i-1}}(u_i, w_i),$$

where u_i, w_i are arbitrary vectors in V_1 . It is not difficult to show that f is non-degenerate. If f_1 is symmetric bilinear, or if f_1 is alternating and t is even, then f is symmetric bilinear. If f_1 is alternating and t is odd, then f is alternating. If q is even, then the terms alternating and symmetric bilinear are equivalent; here there is a unique quadratic form Q on V_t such that f is the bilinear form associated with Q and such that $Q(u_1 \otimes \cdots \otimes u_t) = 0$ for all $u_i \in V_1$. It is not difficult to show that the restrictions of f and Q to V take values in $GF(q)$: the vectors \mathbf{v} span V and $f(\mathbf{u}, \mathbf{v}) = \prod_{i=1}^t f_1^{\psi^{i-1}}(u^{\psi^{i-1}}, v^{\psi^{i-1}}) = \prod_{i=1}^t f_1(u, v)^{\psi^{i-1}} \in GF(q)$. If H is a subgroup of $GL_m(q^t)$ preserving f_1 , then H preserves f and Q . Thus we have embeddings $PO_m(q^t)$ in $PO_n(q)$, $PSp_m(q^t)$ in $PO_n(q)$ when t is even and/or q is even, and $PSp_m(q^t)$ in $PSp_n(q)$ when t and q are odd. When q is even we have $PSp_m(q^t) \leq PO_n(q) \leq PSp_n(q)$. Given points of $PG(m-1, q^t)$ represented by vectors u, v we see that $f(\mathbf{u}, \mathbf{v}) = 0$ if and only if $f_1(u, v) = 0$. It follows that a partial ovoid in a polar space in $PG(m-1, q^t)$ is embedded as a partial ovoid of a polar space in $PG(m-1, q)$, and that a subgroup of $PGL_m(q^t)$ fixing the former is embedded as a subgroup of $PGL_m(q)$ fixing the latter.

If we now consider $U_m(q^{2t})$, then f is given by

$$f(u_1 \otimes \cdots \otimes u_t, w_1 \otimes \cdots \otimes w_t) = \prod_{i=1}^t f_1^{\psi^{2(i-1)}}(u_i, w_i).$$

This is an Hermitian form on V_t . This time V is defined over $GF(q^2)$. The restriction of f to V takes values in $GF(q^2)$. However this restriction is an Hermitian form only if t is odd; if t is even, then the restriction becomes a symmetric bilinear form and $PU_m(q^{2t})$ becomes a subgroup of $O_n(q^2)$. Steinberg's Tensor Product Theorem leads us to believe that for t even $\rho(U(n, q^t))$ is not absolutely irreducible. Indeed for the case $t = 2$ it is known that $\rho(U(n, q^2))$ is reducible, for it follows from [11, Theorem 43.14] that $\rho(U(n, q^2))$ fixes all vectors in a 1-dimensional subspace of $V(n^2, q^2)$; moreover the restriction of the Hermitian form f to $V_q = W$ is actually a symmetric

bilinear form so $\rho(U(n, q^2))$ is a subgroup of $O(n^2, q)$ (for q odd) or $Sp(n^2, q)$ (for q even).

If t is composite, say $t = rs$, then we can consider initially the map from $GL_m((q^r)^s)$ to $GL_{m^s}(q^r)$ followed by the map from $GL_{m^s}(q^r)$ to $GL_n(q)$. This explains most of the restrictions below to t prime. The same argument applied to the map from $GL_m((q^r)^{2s})$ to $GL_{m^s}(q^{2r})$ followed by the map from $GL_{m^s}(q^{2r})$ to $GL_n(q^2)$ explains the unitary restrictions.

Now we consider a twisted version of the above. Given a matrix A , we denote by A^* the matrix $(A^T)^{-1}$. For t even, there is an action on V_t (taking V_1 as being defined over $GF(q^{2t})$) whereby $v_1 \otimes v_2 \otimes \cdots \otimes v_t$ is mapped to $v_1 A \otimes v_2 A^{*\psi} \otimes v_3 A^{\psi^2} \otimes \cdots \otimes v_t A^{*\psi^{t-1}}$. Once again we have a representation, this time denoted ρ^* , of $GL_m(q^{2t})$ on V_t that happens to be absolutely irreducible and realizable over $GF(q^2)$. If $t > 2$, say $t = 2s$ with $s > 1$, then we have $(A \otimes A^{\psi^2} \otimes \cdots \otimes v_t A^{*\psi^{2s-2}}) \otimes (A \otimes A^{\psi^2} \otimes \cdots \otimes v_t A^{*\psi^{2s-2}})^{*\psi} \in GL_{m^s}(q^4) \leq GL_n(q^2)$. Thus $t = 2$ is the only case that we need to consider. We actually consider the action of G given by $v_1 \otimes v_2$ being mapped to $v_1 A^* \otimes v_2 A^\psi$. A non-degenerate Hermitian form is defined by $(u \otimes v, w \otimes z) = (uz^{\psi T}, (w^\psi v^T))$ and this is preserved by $\rho^*(g) = (g^T)^{-1} \otimes g^\psi$ for all $g \in G$. Hence $\rho^*(G) \leq U_n(q^2)$.

We have observed in [9] that $PSL(2, 2^t)$ fixes an ovoid in $PG(2^t - 1, 2)$ which is also a polygon admitting A_{2^t+1} as an automorphism group, so that $PSL(2, 2^t) < A_{2^t+1} < P\Omega^+(2^t, 2)$ and that the alternating group is the full stabilizer of the ovoid. The representation here of the alternating group is precisely that arising from the fully deleted permutation module. In the case of the embedding of $P\Omega(3, 3^t)$ in $P\Omega(3^t, 3)$ (see [8]), the configuration is a rational curve and a partial ovoid in $PG(3^t - 1, 3)$. The partial ovoid has A_{3^t+1} as its stabilizer, acting on the points of the partial ovoid, and once again the representation is equivalent to that arising from the deleted permutation module.

In summary the inclusions considered as \mathcal{C}_9 subgroups are as follows, with t prime and $n = m^t$:

- (a) $PSL_m(q^t) \leq PSL_n(q)$ with $m \geq 3$;
- (b) $PSp_m(q^t) \leq PSp_n(q)$ with m even, t odd and q odd;
- (c) $PSp_m(q^t) \leq P\Omega_n^+(q)$ with m even, t odd and q even;
- (d) $PSp_m(q^2) \leq P\Omega_n^\epsilon(q)$ with m even and $\epsilon = (-1)^{m/2}$;
- (e) $P\Omega_m(q^t) \leq P\Omega_n(q)$ with m odd and q odd;
- (f) $P\Omega_m(q^t) \leq P\Omega_n(q)$ with m even, $m \geq 6$, t odd and q odd;
- (g) $P\Omega_m(q^2) \leq P\Omega_n^\epsilon(q)$ with m even, $m \geq 6$, q odd and $\epsilon = (-1)^{m/2}$;
- (h) $PSU_m(q^{2t}) \leq PSU_n(q^2)$ with $m \geq 3$ and t odd;
- (i) $PSL_m(q^4) \leq PSU_n(q^2)$ with $m \geq 3$ and $t = 2$.

Most of the subgroups listed above are maximal. This is proved by Schaffer in the theorem given below, but the proof is very far from geometrical, being based on the representation theory of the groups involved. We should add that Schaffer

has the restriction tq odd in (e) above, although there appears no reason to exclude $t = 2$.

Theorem 5.1 [47] *The subgroups listed above are maximal except in the following cases, with a maximal overgroup as indicated.*

- (a) $PSp_2(2^t) \leq A_{2^{t+1}} \leq P\Omega_{2^t}^+(2)$ with t odd;
- (b) $PSp_2(q^t) \leq PSp_{2t}(q) \leq P\Omega_{2^t}^+(q)$ with t odd and q even;
- (c) $PSp_4(q^t) \leq PSp_{4t}(q) \leq P\Omega_{4^t}^+(q)$ with q even;
- (d) $P\Omega_3(3^t) \leq A_{3^{t+1}} \leq P\Omega_{3^t}(3)$.

As noted above, points of $PG(m-1, q^t)$ are mapped to points of $PG(n-1, q)$ and so various configurations in $PG(m-1, q^t)$ are mapped to configurations in $PG(n-1, q)$. In particular, partial ovoids of a polar space in $PG(m-1, q^t)$ are mapped to partial ovoids of a polar space in $PG(n-1, q)$ having the same size. In what might be regarded as a degenerate case of this, when $m = 2$ the points of a projective line are mapped to points of a partial ovoid of a quadric in $PG(2^t-1, q)$. When q is even, the size of this partial ovoid meets the Blokhuis-Moorhouse bound (see [2]). In terms of the embedding of groups, this is the excepted case $PSp_2(q^t) \leq P\Omega_{2^t}^+(q)$ with t odd and q even. This embedding is studied in detail in [9], along with the embedding of $PSp_4(q^t)$ in $PSp_{4t}(q) \leq P\Omega_{4^t}^+(q)$ with q even. This latter case sees an ovoid of $PG(3, q^t)$ mapped to a partial ovoid of $PG(4^t-1, q)$, again the size of this partial ovoid meets the Blokhuis-Moorhouse bound. In [10] we shall show that the intermediate subgroups noted by Schaffer for these two cases can be described geometrically.

5.2 Other groups in \mathcal{S}

It is difficult to assess the general pattern of almost simple groups that might appear as subgroups of a simple finite classical group and that lie in \mathcal{S} but not \mathcal{C}_9 . When listing these groups we often give the simple group rather than its slightly larger normalizer, and we omit most details as to which values of q apply.

Among the small dimensional groups for which complete details were given in Section 2 (at least of maximal subgroups) we have:

- $A_5 \leq PSL_2(q)$;
- $A_5 \leq P\Omega_3(q)$;
- $PSL_3(2)$, A_6 , A_7 all subgroups of $PSL_3(q)$;
- $PSL_3(2)$, A_6 , A_7 all subgroups of $PSU_3(q)$;
- A_6 , A_7 , $PSL_2(q)$ all subgroups of $PSp_4(q)$ for odd q ;
- $A_7 \leq PSL_4(2)$;
- $A_5 \leq P\Omega_4^-(q)$;

- $A_6, A_7, PSL_2(q)$ all subgroups of $P\Omega_5(q)$;
- $A_7 \leq P\Omega_6^+(2)$.

We have already commented on the status of the lists in Kleidman's thesis ([39]). However the lists do provide useful evidence. In the lists below, we give the simple groups rather than their normalizers, and note that these go as far as $n = 11$.

The first list comprises specific simple groups rather than families of simple groups, and we omit the classical groups they belong to. Suffice to say that these are all subgroups of $PSL_n(q)$ for some $n \leq 11$ and in many cases for infinitely many values of q .

- A_m for $m = 6, 7, 9, 10, 11, 12, 13$;
- $PSL_2(r)$ for $r = 8, 11, 13, 17, 19, 23$;
- $PSL_3(3); PSL_3(4)$;
- $PSU_3(9); PSU_4(4); PSU_4(9); PSU_5(4)$;
- $PSp_6(2)$;
- $M_{11}, M_{12}, M_{22}, M_{24}$;
- $J_2; J_3$;
- $P\Omega_8^+(2)$;
- $Sz(8)$.

The second list comprises families of simple groups that appear to arise as subgroups in \mathcal{S} , excluding those known to lie in \mathcal{C}_9 .

- $PSL_3(q) \leq PSL_6(q)$;
- $PSL_3(q), PSL_4(q), PSL_5(q)$ all subgroups of $PSL_{10}(q)$;
- $PSU_3(q) \leq PSU_6(q)$;
- $PSU_3(q), PSU_4(q), PSU_5(q)$ all subgroups of $PSU_{10}(q)$;
- $Sz(q) \leq PSp_4(q)$ for q even;
- $G_2(q), PSL_2(q)$ both subgroups of $PSp_6(q)$;
- $PSL_2(q) \leq PSp_{10}(q)$;
- $G_2(q) \leq P\Omega_7(q)$;
- $PSL_3(q), PSU_3(q^2), P\Omega_7(q), P\Omega_8^-(\sqrt{q}), {}^3D_4(\sqrt[3]{q})$ all subgroups of $P\Omega_8^+(q)$;
- $PSL_3(q), PSU_3(q^2)$ both subgroups of $P\Omega_8^-(q)$;
- $PSL_2(q) \leq P\Omega_9(q)$;

- $PSp_4(q) \leq P\Omega_{10}^-(q)$;
- $PSp_4(q) \leq P\Omega_{10}^+(q)$.
- $PSL_2(q) \leq P\Omega_{11}(q)$;

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